Solutions

Q1. In the Course Notes, read the following three proofs.

- Pascal's Identity (PI)
- Binomial Theorem, Version 1 (BT1)
- Binomial Theorem, Version 2 (BT2)

Then answer the following questions.

(a) In the proof of Pascal's Identity, where do we use the condition $m < n$?

(b) In the proof of Binomial Theorem, Version 1 (BT1), we find the phrase "Note how we have changed the bounds of summation." What was the purpose in changing the bounds of summation?

(c) In the proof of Binomial Theorem, Version 2 (BT2), where do we use the fact that $0^0 = 1$?

(d) Is the following statement true or false? No justification required.

For all $a \in \mathbb{Z}$ and all $n \in \mathbb{N}$, $a \mid [(2+a)^n - 2^n]$.

Solution(s).

(a) In the first line of the proof we apply the definition of *binomial coefficient* to $\binom{n-1}{m-1}$ and $\binom{n-1}{m}$. To apply the definition in these two situations we need $m-1 \leq n-1$ and $m \leq n-1$, respectively. The first inequality is equivalent to $m \leq n$ and the second is equivalent to $m < n$, since m and n are integers. Therefore, we need the condition that $m < n$.

(b) We have changed the bounds of summation so that the powers of x match in the two sums. This matching allows us to combine the two sums, factor out x^m and apply Pascal's identity.

(c) We use this fact in the proof of the case when $a = 0$. We have $\binom{n}{n}0^{n-n}b^n = b^n$.

(d) The statement is true. (Consider applying the Binomial Theorem to $(2+a)^n$.)

Q2. Prove that for all $n \in \mathbb{Z}$, $4 \nmid (n^4 + 3)$ if and only if n is even.

Solution(s). Let $n \in \mathbb{Z}$. Since this is an if and only if proof we will prove it in both directions. We begin with the forward direction, which we will prove using the contrapositive: If n is odd, then $4 \mid (n^4+3)$. Assume that n is odd. Since n is odd, then $n = 2k+1$ for some integer k. Therefore,

$$\begin{aligned}
n^4 + 3 &= (2k+1)^4 + 3 \\
&= 16k^4 + 32k^3 + 24k^2 + 8k + 1 + 3 \\
&= 16k^4 + 32k^3 + 24k^2 + 8k + 4 \\
&= 4(4k^4 + 8k^3 + 6k^2 + 2k + 1)
\end{aligned}$$

Since $k \in \mathbb{Z}$, then $4k^4 + 8k^3 + 6k^2 + 2k + 1 \in \mathbb{Z}$. Therefore, $4 \mid (n^4 + 3)$.

Next we will prove the backward direction. We will prove it using a proof by contradiction. We begin by assuming n is even and $4 \mid (n^4 + 3)$. Since n is even, then $n = 2k$ for some integer k. Therefore, $n^4 = 16k^4$. Therefore, since $4 \mid 16$, applying *Transitivity of Divisibility (TD)*, gives $4 \mid 16k^4$. Therefore, $4 \mid n^4$. Applying *Divisibility of Integer Combinations (DIC)* gives $4 \mid (n^4 + 3 - n^4)$ which implies that $4 \mid 3$. This is a contradiction.

Q3. Prove that no natural number a exists such that $a^2 - 10$ is a perfect square.

Solution(s). Assume by way of contradiction, that there does exist a natural number a such that $a^2 - 10$ is a perfect square. Therefore, there exists an integer k such that $a^2 - 10 = k^2$. We can without loss of generality assume that $k \geq 0$. Therefore, $a^2 - k^2 = 10$ which implies that $(a-k)(a+k) = 10$. Since $a \in \mathbb{N}$ and $k \in \mathbb{Z}$, then $a-k, a+k \in \mathbb{Z}$. Since $a > 0$ and $k \geq 0$, then $a + k > 0$ and since $10 > 0$, then $a - k > 0$. Also, we obtain that $a + k \geq a - k$.

The only positive factor pairs of 10 are 10 and 1, and 5 and 2. Therefore, we have two possibilities: (i) $a + k = 10$ and $a - k = 1$ or (ii) $a + k = 5$ and $a - k = 2$.

In case (i), we add the equations to obtain that $2a = 11$ which implies that $a = \frac{11}{2}$. This is a contradiction, since $a \in \mathbb{N}$.

In case (ii), we add the equations to obtain that $2a = 7$ which implies that $a = \frac{7}{2}$. This is a contradiction, since $a \in \mathbb{N}$.

Q4. For all integers w, x, y and z with $w \neq y$ and $wz - xy \neq 0$, prove that there exists a unique rational number r such that $\dfrac{wr + x}{yr + z} = 1$.

Solution(s). We begin by first showing that the rational number r exists.

Assume that $w, x, y, z \in \mathbb{Z}$ such that $w \neq y$ and $wz - xy \neq 0$. Consider $r = \dfrac{z - x}{w - y}$. Since $w, x, y, z \in \mathbb{Z}$ and $w \neq y$, then $r \in \mathbb{Q}$.

Therefore,

$$\begin{aligned}
\frac{wr + x}{yr + z} &= \frac{w\left(\dfrac{z - x}{w - y}\right) + x}{y\left(\dfrac{z - x}{w - y}\right) + z} \\
&= \frac{w(z - x) + x(w - y)}{y(z - x) + z(w - y)} \qquad \left(\text{Since } w \neq y, \text{ we multiply by } \frac{w - y}{w - y}.\right) \\
&= \frac{wz - wx + xw - xy}{yz - yx + zw - zy} \\
&= \frac{wz - xy}{zw - yx} \\
&= 1
\end{aligned}$$

as desired. We note that the denominators of the expressions above are non-zero and that the final ratio is 1 since $wz - xy \neq 0$.

Next we will show that the rational number r is unique. Assume that there are two rational number r_1 and r_2 such that $\dfrac{wr_1 + x}{yr_1 + z} = 1$ and $\dfrac{wr_2 + x}{yr_2 + z} = 1$.

Therefore,

$$\begin{aligned}
\frac{wr_1 + x}{yr_1 + z} &= \frac{wr_2 + x}{yr_2 + z} \\
wyr_1r_2 + wzr_1 + xyr_2 + xz &= wyr_1r_2 + xyr_1 + wzr_2 + xz \\
wzr_1 + xyr_2 &= xyr_1 + wzr_2 \\
wzr_1 - xyr_1 &= wzr_2 - xyr_2 \\
r_1(wz - xy) &= r_2(wz - xy) \\
r_1 &= r_2 \qquad (\text{Since } wz - xy \neq 0)
\end{aligned}$$

Therefore, the rational number r is unique.

Q5. Prove by induction that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(-1)^i\, i^2 = \frac{(-1)^n n(n+1)}{2}.$$

Solution(s). The proof is by induction on n, where $P(n)$ is the open sentence

$$\sum_{i=1}^{n}(-1)^i\, i^2 = \frac{(-1)^n n(n+1)}{2}.$$

Base Case: The statement $P(1)$ is given by

$$\sum_{i=1}^{1}(-1)^i\, i^2 = \frac{(-1)^1 1(1+1)}{2}.$$

The expression on the left hand side of this equation evaluates to

$$\sum_{i=1}^{1}(-1)^i\, i^2 = (-1)^1 1^2 = -1,$$

and the expression on the right hand side evaluates to

$$\frac{(-1)^1 1(1+1)}{2} = \frac{-1(2)}{2} = -1.$$

Since both sides are equal to each other, $P(1)$ is true.

Inductive Step: Let k be an arbitrary natural number.

Assume the inductive hypothesis $P(k)$. That is, assume that

$$\sum_{i=1}^{k}(-1)^i\, i^2 = \frac{(-1)^k k(k+1)}{2}.$$

We wish to prove $P(k+1)$, which stands for

$$\sum_{i=1}^{k+1}(-1)^i\, i^2 = \frac{(-1)^{k+1}(k+1)(k+2)}{2}.$$

Then, starting with the summation on the left hand side of $P(k+1)$, we obtain

$$\begin{aligned}
\sum_{i=1}^{k+1}(-1)^i\, i^2 &= \left(\sum_{i=1}^{k}(-1)^i\, i^2\right) + (-1)^{k+1}(k+1)^2 && \text{by properties of summation notation}\\
&= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2 && \text{by the inductive hypothesis}\\
&= \frac{(-1)^k k(k+1) + 2(-1)^{k+1}(k+1)^2}{2}\\
&= \frac{(-1)^k (k+1)[k+2(-1)(k+1)]}{2}\\
&= \frac{(-1)^k (k+1)(-k-2)}{2}\\
&= \frac{(-1)^{k+1} (k+1)(k+2)}{2}
\end{aligned}$$

The result is true for $n = k+1$, and hence holds for all $n \geq 1$ by the *Principle of Mathematical Induction (POMI)*.

Q6. Let x and y be real numbers with $0 < y < x$. Prove by induction that for all $n \in \mathbb{N}$,

$$x^n - y^n \leq nx^{n-1}(x-y).$$

Solution(s). Let x and y be real numbers with $0 < y < x$.

Base Case: When $n = 1$, the left-hand side is $x^n - y^n = x - y$. The right-hand side is $nx^{n-1}(x-y) = x^0(x-y) = x - y$. Since $x - y \leq x - y$, then the inequality holds for $n = 1$.

Inductive step: Let k be an arbitrary natural number. Assume the inductive hypothesis, that is, assume $x^k - y^k \leq kx^{k-1}(x-y)$.

We wish to show that $x^{k+1} - y^{k+1} \leq (k+1)x^k(x-y)$.

Starting with the right-hand side,

$$\begin{aligned}
(k+1)x^k(x-y) &= kx^k(x-y) + x^k(x-y) \\
&= kxx^{k-1}(x-y) + x^k(x-y) \\
&\geq x(x^k - y^k) + x^{k+1} - x^k y && \text{(by IH and the fact that } x > 0\text{)} \\
&= x^{k+1} - xy^k + x^{k+1} - x^k y \\
&= x^{k+1} - xy^k - x^k y + y^{k+1} + x^{k+1} - y^{k+1} \\
&= (x-y)(x^k - y^k) + x^{k+1} - y^{k+1}.
\end{aligned}$$

Since $0 < y < x$, then $y^k < x^k$. Therefore, $x - y > 0$ and $x^k - y^k > 0$. Therefore, $(x-y)(x^k - y^k) > 0$.

Using this final inequality and the inequality $(k+1)x^k(x-y) \geq (x-y)(x^k-y^k)+x^{k+1}-y^{k+1}$, which we derived above, we can conclude that $(k+1)x^k(x-y) \geq x^{k+1} - y^{k+1}$. Therefore, the inequality holds for $n = k+1$.

Therefore, by the *Principle of Mathematical Induction (POMI)*, $x^n - y^n \leq nx^{n-1}(x-y)$ is true for all natural numbers n.

Alternate Proof of the Inductive Step

Let k be an arbitrary natural number. Assume the inductive hypothesis, that is, assume $x^k - y^k \leq kx^{k-1}(x-y)$.

We wish to show that $x^{k+1} - y^{k+1} \leq (k+1)x^k(x-y)$.

We begin with the inductive hypothesis which we have assumed to be true, that is, we begin with

$$x^k - y^k \leq kx^{k-1}(x-y).$$

Since x and y are positive real numbers with $y < x$ we can multiply the left-hand side of the inequality by y and the right-hand side of inequality to obtain

$$x^k y - y^{k+1} \leq kx^k(x-y).$$

We then add $x^k(x-y)$ to both sides to obtain

$$x^k y - y^{k+1} + x^k(x-y) \leq (k+1)x^k(x-y).$$

The left-hand side simplifies to $x^{k+1} - y^{k+1}$ and so we have $x^{k+1} - y^{k+1} \leq (k+1)x^k(x-y)$, as required.