11

Hexahedron Elements

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§11.1. Introduction

Triangles in two dimensions generalize to tetrahedra in three. The corresponding generalization of a quadrilateral is a *hexahedron*, also known in the finite element literature as *brick*. A hexahedron is topologically equivalent to a cube. It has eight corners, twelve edges or sides, and six faces.

Finite elements with this geometry are extensively used in modeling three-dimensional solids. Hexahedra also have been the motivating factor for the development of "Ahmad-Pawsey" shell elements through the use of the "degenerated solid" concept.

The construction of hexahedra shape functions and the computation of the stiffness matrix was greatly facilitated by three advances in finite element technology: natural coordinates, isoparametric mapping and numerical integration. Together these revolutionized FEM in the mid-1960's, making possible the construction of finite element *families*.

§11.2. Hexahedron Natural Coordinates

Before presenting examples of hexahedron elements, we have to introduce the appropriate *natural coordinate system* for that geometry. The natural coordinates for this geometry are called ξ, η and μ, and are called *isoparametric hexahedral coordinates* or simply *natural coordinates*.

These coordinates are illustrated in Figure **?**. As can be seen they are very similar to the quadrilateral coordinates ξ and η used in IFEM. They vary from -1 on one face to $+1$ on the opposite face, taking the value zero on the "median" face. As in the case of quadrilaterals, this particular choice of limits was made to facilitate the use of the standard Gauss integration formulas.

§11.2.1. Corner Numbering Rules

The eight corners of a hexahedron element are locally numbered $1, 2 \ldots 8$. The corner numbering rule is similar to that given for the 4-node tetrahedron in Chapter 14. Again the purpose is to guarantee a positive volume (or, more precisely, a positive Jacobian determinant at every point). The transcription of those rules to the hexahedron element is as follows:

1. Chose one starting corner, which is given number 1, and one initial face pertaining to that corner (given a starting corner, there are three possible faces meeting at that corner that may be selected).

2. Number the other 3 corners as 2,3,4 traversing the initial face counterclockwise[1] while one looks at the initial face from the opposite one.

3. Number the corners of the opposite face directly opposite 1,2,3,4 as 5,6,7,8, respectively.

[1] "Anticlockwise" in British.

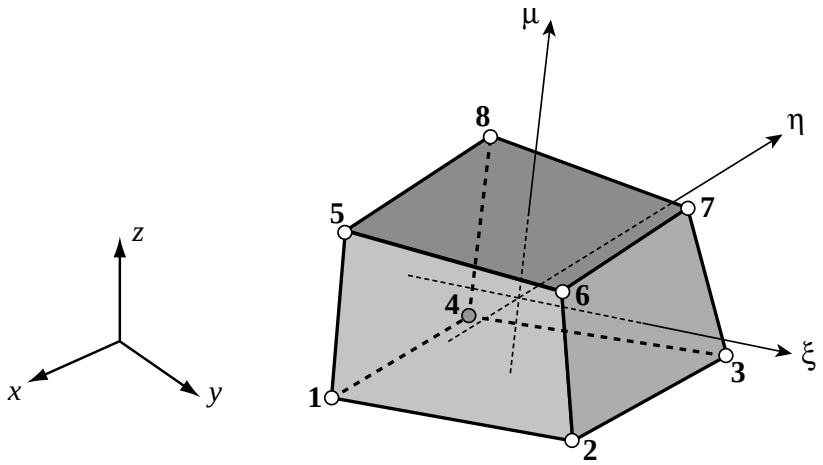

FIGURE 11.1. The 8-node hexahedron and the natural coordinates ξ, η, μ. The definitio of these coordinates is the same for higher order models.

The definition of ξ, η and μ can be now be made more precise:

ξ goes from -1 from (center of) face 1485 to $+1$ on face 2376

η goes from -1 from (center of) on face 1265 to $+1$ on face 3487

μ goes from -1 from (center of) on face 1234 to $+1$ on face 5678

The center of a face is the intersection of the two medians.

§11.3. The Eight Node (Trilinear) Hexahedron

The eight-node hexahedron shown in Figure **?** is the simplest member of the hexahedron family. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ v_{x1} & v_{x2} & v_{x3} & v_{x4} & v_{x5} & v_{x6} & v_{x7} & v_{x8} \\ v_{y1} & v_{y2} & v_{y3} & v_{y4} & v_{y5} & v_{y6} & v_{y7} & v_{y8} \\ v_{z1} & v_{z2} & v_{z3} & v_{z4} & v_{z5} & v_{z6} & v_{z7} & v_{z8} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \\ \vdots \\ \\ N_8^{(e)} \end{bmatrix} \tag{11.1}$$

The hexahedron coordinates of the corners are (see Figure **?**)

Figure **?**. The 20-node hexahedron element — note node numbering conventions.

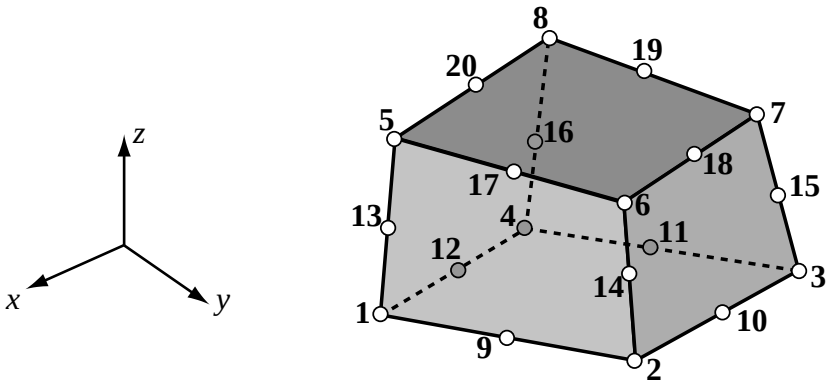

FIGURE 11.2. The 20-node hexahedron element — note node numbering conventions.

node	ξ	η	μ
1	-1	-1	-1
2	$+1$	-1	-1
3	$+1$	$+1$	-1
4	-1	$+1$	-1
5	-1	-1	$+1$
6	$+1$	-1	$+1$
7	$+1$	$+1$	$+1$
8	-1	$+1$	$+1$

The shape functions are

$$\begin{aligned} N_1^{(e)} &= \tfrac{1}{8}(1-\xi)(1-\eta)(1-\mu), & N_2^{(e)} &= \tfrac{1}{8}(1+\xi)(1-\eta)(1-\mu) \\ N_3^{(e)} &= \tfrac{1}{8}(1+\xi)(1+\eta)(1-\mu), & N_4^{(e)} &= \tfrac{1}{8}(1-\xi)(1+\eta)(1-\mu) \\ N_5^{(e)} &= \tfrac{1}{8}(1-\xi)(1-\eta)(1+\mu), & N_6^{(e)} &= \tfrac{1}{8}(1+\xi)(1-\eta)(1+\mu) \\ N_7^{(e)} &= \tfrac{1}{8}(1+\xi)(1+\eta)(1+\mu), & N_8^{(e)} &= \tfrac{1}{8}(1-\xi)(1+\eta)(1+\mu) \end{aligned} \tag{11.2}$$

These eight formulas can be summarized in a single expression:

$$N_1^{(e)} = \tfrac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\mu\mu_i) \tag{11.3}$$

where ξ_i, η_i and μ_i denote the coordinates of the i^{th} node.

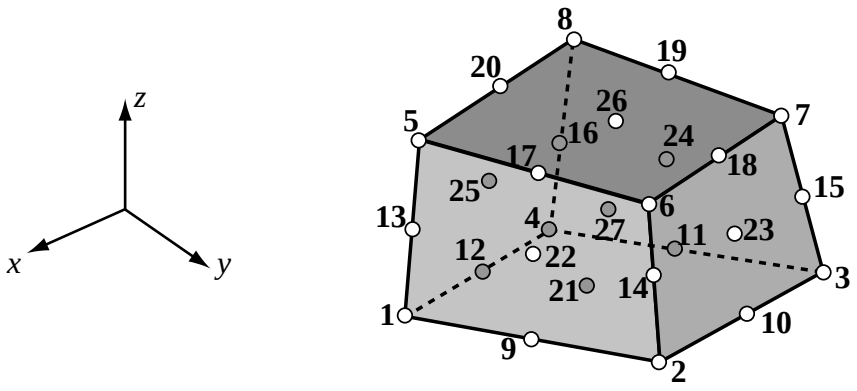

FIGURE 11.3. The 27-node hexahedron element — note node numbering conventions.

§11.4. The 20-Node (Serendipity) Hexahedron

The 20-node hexahedron is the analog of the 8-node "serendipity" quadrilateral. The 8 corner nodes are augmented with 12 side nodes which are usually located at the midpoints of the sides. The numbering scheme is illustrated in Figure **?**. For elasticity applications this element have $20 \times 3 = 60$ degrees of freedom.

The 8-node quadrilateral studied in IFEM cannot represent a complete biquadratic expansion in the quadrilateral coordinates ξ and η, that is, the nine terms $1, \xi, \eta, \xi^2, \ldots, \xi^2\eta^2$. One has to go to the 9-node (biquadratic) quadrilateral to achieve that.

Likewise, the 20 node hexahedron is incapable of accomodating a full triquadratic expansion in ξ, η and μ; that is $1, \xi, \eta, \mu, \eta^2, \ldots, \xi^2\eta^2\mu^2$. A 27-node hexahedron is required for that. That element is described in the next section.

The shape functions of the 20-node hexahedron can be grouped as follows. For the corner nodes $i = 1, 2, \ldots, 8$:

$$N_i^{(e)} = \tfrac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \mu\mu_i)(\xi\xi_i + \eta\eta_i + \mu\mu_i - 2). \tag{11.4}$$

For the midside nodes $i = 9, 11, 17, 19$:

$$N_i^{(e)} = \tfrac{1}{4}(1 - \xi^2)(1 + \eta\eta_i)(1 + \mu\mu_i). \tag{11.5}$$

For the midside nodes $i = 10, 12, 18, 20$:

$$N_i^{(e)} = \tfrac{1}{4}(1 - \eta^2)(1 + \xi\xi_i)(1 + \mu\mu_i). \tag{11.6}$$

For the midside nodes $i = 13, 14, 15, 16$:

$$N_i^{(e)} = \tfrac{1}{4}(1 - \mu^2)(1 + \xi\xi_i)(1 + \eta\eta_i). \tag{11.7}$$

§11.5. The 27-Node (Triquadratic) Hexahedron

A 27-node hexahedron can indeed be constructed by adding 7 more nodes: 6 on each face center, and 1 interior node at the hexahedron center. See Figure **?**. In elasticity application such an element has $27 \times 3 = 81$ degrees of freedom.

(To be completed).

§11.6. Partial Derivatives

The calculation of partial derivatives of hexahedron shape functions with respect to Cartesian coordinates follows techniques similar to that discussed for two-dimensional quadrilateral elements in IFEM. Only the size of the matrices changes because of the appearance of the third dimension.

§11.6.1. The Jacobian Matrix

The derivatives of the shape functions are given by the usual chain rule formulas:

$$\begin{aligned}
\frac{\partial N_i^{(e)}}{\partial x} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial x},\\
\frac{\partial N_i^{(e)}}{\partial y} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial y},\\
\frac{\partial N_i^{(e)}}{\partial z} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial z}.
\end{aligned} \tag{11.8}$$

In matrix form

$$\begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} \\ \frac{\partial N_i^{(e)}}{\partial y} \\ \frac{\partial N_i^{(e)}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \mu}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \mu}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \mu}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial \xi} \\ \frac{\partial N_i^{(e)}}{\partial \eta} \\ \frac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}. \tag{11.9}$$

The 3×3 matrix that appears in (11.9) is $\mathbf{J}^{-1}$, the inverse of:

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \mu} & \frac{\partial y}{\partial \mu} & \frac{\partial z}{\partial \mu} \end{bmatrix}. \tag{11.10}$$

Matrix $\mathbf{J}$ is called the *Jacobian matrix* of (x, y, z) with respect to (ξ, η, μ). In the finite element literature, matrices $\mathbf{J}$ and $\mathbf{J}^{-1}$ are called simply the *Jacobian* and *inverse Jacobian*, respectively, although such a short name is sometimes ambiguous. The notation

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)}, \qquad \mathbf{J}^{-1} = \frac{\partial(\xi, \eta, \mu)}{\partial(x, y, z)}. \tag{11.11}$$

is standard in multivariable calculus and suggests that the Jacobian may be viewed as a generalization of the ordinary derivative, to which it reduces for a scalar function $\mathbf{x} = x(\xi)$.

§11.6.2. Computing the Jacobian Matrix

The isoparametric definition of hexahedron element geometry is

$$x = x_i\, N_i^{(e)}, \qquad y = y_i\, N_i^{(e)}, \qquad z = z_i\, N_i^{(e)}, \tag{11.12}$$

where the summation convention is understood to apply over $i = 1, 2, ...n$, in which n denotes the number of element nodes.

Differentiating these relations with respect to the hexahedron coordinates we construct the matrix $\mathbf{J}$ as follows:

$$\mathbf{J} = \begin{bmatrix} x_i \dfrac{\partial N_i^{(e)}}{\partial \xi} & y_i \dfrac{\partial N_i^{(e)}}{\partial \xi} & z_i \dfrac{\partial N_i^{(e)}}{\partial \xi} \\ x_i \dfrac{\partial N_i^{(e)}}{\partial \eta} & y_i \dfrac{\partial N_i^{(e)}}{\partial \eta} & z_i \dfrac{\partial N_i^{(e)}}{\partial \eta} \\ x_i \dfrac{\partial N_i^{(e)}}{\partial \mu} & y_i \dfrac{\partial N_i^{(e)}}{\partial \mu} & z_i \dfrac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}. \tag{11.13}$$

Given a point of hexahedron coordinates (ξ, η, μ) the Jacobian $\mathbf{J}$ can be easily formed using the above formula, and numerically inverted to form $\mathbf{J}^{-1}$.

Remark 11.1. The inversion formula for a matrix of order 3 is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{11.14}$$

where

$$\begin{aligned} A_{11} &= a_{22}a_{33} - a_{23}a_{32}, & A_{22} &= a_{33}a_{11} - a_{31}a_{13}, \\ A_{33} &= a_{11}a_{22} - a_{12}a_{21}, & A_{12} &= a_{23}a_{31} - a_{21}a_{33}, \\ A_{23} &= a_{31}a_{12} - a_{32}a_{11}, & A_{31} &= a_{12}a_{23} - a_{13}a_{22}, \\ A_{21} &= a_{32}a_{13} - a_{12}a_{33}, & A_{32} &= a_{13}a_{21} - a_{23}a_{11}, \\ A_{13} &= a_{21}a_{22} - a_{31}a_{22}, & |\mathbf{A}| &= a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}. \end{aligned} \tag{11.15}$$

(The determinant can in fact be computed in 9 different ways.)

§11.7. The Strain Displacement Matrix

Having obtained the shape function derivatives, the matrix $\mathbf{B}$ for a hexahedron element displays the usual structure for 3D elements:

$$\mathbf{B} = \mathbf{D}\mathbf{\Phi} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} \mathbf{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q}_z \\ \mathbf{q}_y & \mathbf{q}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_z & \mathbf{q}_y \\ \mathbf{q}_z & \mathbf{0} & \mathbf{q}_x \end{bmatrix} \tag{11.16}$$

where

$$\mathbf{q} = [\, N_1^{(e)} \quad \cdots \quad N_n^{(e)} \,]$$
$$\mathbf{q}_x = \left[\, \frac{\partial N_1^{(e)}}{\partial x} \quad \cdots \quad \frac{\partial N_n^{(e)}}{\partial x} \,\right]$$
$$\mathbf{q}_y = \left[\, \frac{\partial N_1^{(e)}}{\partial y} \quad \cdots \quad \frac{\partial N_n^{(e)}}{\partial y} \,\right]$$
$$\mathbf{q}_z = \left[\, \frac{\partial N_1^{(e)}}{\partial z} \quad \cdots \quad \frac{\partial N_n^{(e)}}{\partial z} \,\right]$$

are row vectors of length n, n being the number of nodes in the element.

§11.8. Stiffness Matrix Evaluation

The element stiffness matrix is given by

$$\mathbf{K}^{(e)} = \int_{V^{(e)}} \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV^{(e)}. \tag{11.17}$$

As in the two-dimensional case, this is replaced by a numerical integration formula which now involves a triple loop over conventional Gauss quadrature rules. Assuming that the stress-strain matrix $\mathbf{E}$ is constant over the element,

$$\mathbf{K}^{(e)} = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \sum_{k=1}^{p_3} w_i w_j w_k \mathbf{B}_{ijk}^T \mathbf{E} \mathbf{B}_{ijk}\, J_{ik\ell}. \tag{11.18}$$

Here p_1, p_2 and p_3 are the number of Gauss points in the ξ, η and μ direction, respectively, while $\mathbf{B}_{ijk}$ and $\mathbf{J}_{ij\ell}$ are abbreviations for

$$\mathbf{B}_{ijk} \equiv \mathbf{B}(\xi_i, \eta_j, \mu_k), \qquad J_{ik\ell} \equiv \det \mathbf{J}(\xi_i, \eta_j, \mu_k). \tag{11.19}$$

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$. The total number of Gauss points is thus p^3. Each point adds at most 6 to the stiffness matrix rank. The minimum rank-sufficient rules for the 8-node and 20-node hexahedra are $p = 2$ and $p = 3$, respectively.

Remark 11.2. The computation of consistent node forces corresponding to body forces is straightforward. The treatment of prescribed surface tractions such as pressure, presents, however, some computational difficulties because hexahedron faces are not generally plane.

§11.9. Numerical Integration Over Hexahedra

Numerical integration is essential for evaluating integrals over isoparametric hexahedral elements. As in the two-dimensional case discussed in [247, Ch 17], standard practice has been to use *Gauss integration* because such rules use a *minimal number of sample points to achieve a desired level of accuracy*. This economy is important for efficient stiffness matrix calculations, since a *matrix product*, namely $\mathbf{B}^T \mathbf{E} \mathbf{B}$, is evaluated at each sample point. The fact that the location of the sample points is usually given by non-rational numbers is of no concern in digital computation.

Gauss integration rules used in hexahedra are *tensor products* of one-dimensional (1D) rules. For completeness the lowest order 1D rules are summarized in the next subsection, which is taken verbatim from Chapter 17 of the IFEM Notes [247].

Table 11.1 - One-Dimensional Gauss Rules with 1 through 5 Sample Points

Points	Rule
1	$\int_{-1}^{1} F(\xi)\,d\xi \approx 2F(0)$
2	$\int_{-1}^{1} F(\xi)\,d\xi \approx F(-1/\sqrt{3}) + F(1/\sqrt{3})$
3	$\int_{-1}^{1} F(\xi)\,d\xi \approx \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$
4	$\int_{-1}^{1} F(\xi)\,d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$
5	$\int_{-1}^{1} F(\xi)\,d\xi \approx w_{15}F(\xi_{15}) + w_{25}F(\xi_{25}) + w_{35}F(\xi_{35}) + w_{45}F(\xi_{45}) + w_{55}F(\xi_{55})$

For the 4-point rule, $\xi_{34} = -\xi_{24} = \sqrt{(3-2\sqrt{6/5})/7}$, $\xi_{44} = -\xi_{14} = \sqrt{(3+2\sqrt{6/5})/7}$, $w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{5/6}$, and $w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{5/6}$.

For the 5-point rule, $\xi_{55} = -\xi_{15} = \frac{1}{3}\sqrt{5+2\sqrt{10/7}}$, $\xi_{45} = -\xi_{35} = \frac{1}{3}\sqrt{5-2\sqrt{10/7}}$, $\xi_{35} = 0$, $w_{15} = w_{55} = (322 - 13\sqrt{70})/900$, $w_{25} = w_{45} = (322 + 13\sqrt{70})/900$ and $w_{35} = 512/900$.

§11.9.1. One Dimensional Gauss Rules

The classical Gauss integration rules are defined by

$$\int_{-1}^{1} F(\xi)\,d\xi \approx \sum_{i=1}^{p} w_i F(\xi_i). \tag{11.20}$$

Here $p \geq 1$ is the number of Gauss integration points (also known as sample points), w_i are the integration weights, and ξ_i are sample-point abcissae in the interval $[-1,1]$. The use of the canonical interval $[-1,1]$ is no restriction, because an integral over another range, say from a to b, can be transformed to $[-1,+1]$ via a simple linear transformation of the independent variable, as shown in the Remark below.

The first five unidimensional Gauss rules, illustrated in Figure 11.4, are listed in Table 11.1. These integrate exactly polynomials in ξ of orders up to 1, 3, 5, 7 and 9, respectively. In general a 1D Gauss rule with p points integrates exactly polynomials of order up to $2p-1$. This is called the *degree* of the formula.

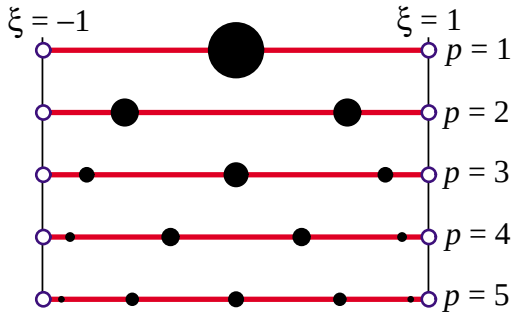

FIGURE 11.4. The first five one-dimensional Gauss rules $p = 1, 2, 3, 4, 5$ depicted over the line segment $\xi \in [-1,+1]$. Sample point locations are marked with black circles. The radii of those circles are proportional to the integration weights.

```
LineGaussRuleInfo[{rule_,numer_},point_]:= Module[
  {g2={-1,1}/Sqrt[3],w3={5/9,8/9,5/9},
   g3={-Sqrt[3/5],0,Sqrt[3/5]},
   w4={(1/2)-Sqrt[5/6]/6, (1/2)+Sqrt[5/6]/6,
       (1/2)+Sqrt[5/6]/6, (1/2)-Sqrt[5/6]/6},
   g4={-Sqrt[(3+2*Sqrt[6/5])/7],-Sqrt[(3-2*Sqrt[6/5])/7],
        Sqrt[(3-2*Sqrt[6/5])/7], Sqrt[(3+2*Sqrt[6/5])/7]},
   g5={-Sqrt[5+2*Sqrt[10/7]],-Sqrt[5-2*Sqrt[10/7]],0,
        Sqrt[5-2*Sqrt[10/7]], Sqrt[5+2*Sqrt[10/7]]}/3,
   w5={322-13*Sqrt[70],322+13*Sqrt[70],512,
       322+13*Sqrt[70],322-13*Sqrt[70]}/900,
   i=point,p=rule,info={{Null,Null},0}},
  If [p==1, info={0,2}];
  If [p==2, info={g2[[i]],1}];
  If [p==3, info={g3[[i]],w3[[i]]}];
  If [p==4, info={g4[[i]],w4[[i]]}];
  If [p==5, info={g5[[i]],w5[[i]]}];
  If [numer, Return[N[info]], Return[Simplify[info]]];
];
```

FIGURE 11.5. A *Mathematica* module that returns the first five one-dimensional Gauss rules.

Remark 11.3. A more general integral, such as $F(x)$ over $[a, b]$ in which $\ell = b - a > 0$, is transformed to the canonical interval $[-1, 1]$ through the mapping $x = \frac{1}{2}a(1-\xi) + \frac{1}{2}b(1+\xi) = \frac{1}{2}(a+b) + \frac{1}{2}\ell\xi$, or $\xi = (2/\ell)(x - \frac{1}{2}(a+b))$. The Jacobian of this mapping is $J = dx/d\xi = 1/2\ell$. Thus

$$\int_a^b F(x)\,dx = \int_{-1}^{1} F(\xi)\,J\,d\xi = \int_{-1}^{1} F(\xi)\,\tfrac{1}{2}\ell\,d\xi. \tag{11.21}$$

Remark 11.4. Higher order Gauss rules are tabulated in standard manuals for numerical computation. For example, the widely used Handbook of Mathematical Functions [2] lists (in Table 25.4) rules with up to 96 points. For $p > 6$ the abscissas and weights of sample points are not expressible as rational numbers or radicals, and can only be given as floating-point numbers.

§11.9.2. Implementation of 1D Rules

The *Mathematica* module shown in Figure 11.5 returns either exact or floating-point information for the first five 1D Gauss rules. To get information for the i^{th} point of the p^{th} rule, in which $1 \le i \le p$ and $p = 1, 2, 3, 4, 5$, call the module as

```
{xii,wi}=LineGaussRuleInfo[{p,numer},i]
```  (11.22)

Logical flag `numer` is `True` to get numerical (floating-point) information, or `False` to get exact information. The module returns the sample point abcissa ξ_i in `xii` and the weight w_i in `wi`. If `p` is not in the implemented range 1 through 5, the module returns `{Null,0}`.

Example 11.1. `{xi,w}=LineGaussRuleInfo[{3,False},2]` returns `xi=0` and `w=8/9`, whereas `{xi,w}=LineGaussRuleInfo[{3,True},2]` returns (to 16 places) `xi=0.` and `w=0.888888888888889`.

§11.9.3. Three Dimensional Gauss Rules

The simplest three-dimensional Gauss rules are called *product rules*. They are obtained by applying the one-dimensional rules described in the previous subsection to each natural coordinate in turn. To do that we must first reduce the integrand, say F, to the *canonical form* in natural coordinates

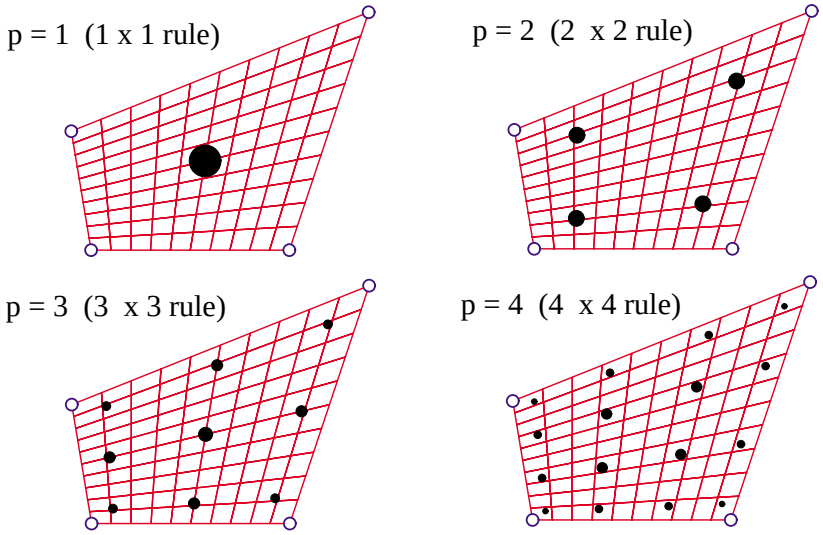

FIGURE 11.6. The first four two-dimensional Gauss product rules $p = 1, 2, 3, 4$ depicted over a straight-sided quadrilateral region. Sample points are marked with black circles. The areas of these circles are proportional to the integration weights. (For now this is a placeholder figure, to be replaced by a hexahedron picture once hex plot module works.)

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta,\mu)\,d\xi\,d\eta\,d\mu = \int_{-1}^{1} d\mu \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta,\mu)\,d\xi. \tag{11.23}$$

Once this is done we can process numerically each integral in turn:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta,\ \mu)\,d\xi\,d\eta\,d\mu \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{k=1}^{p_3} w_i\,w_j\,w_k\,F(\xi_i,\eta_j,\mu_k). \tag{11.24}$$

Here p_1, p_2, and p_3 are the number of Gauss points in the ξ, η and μ directions, respectively. Usually the same number $p = p_1 = p_2 = p_3$ is chosen if the shape functions are taken to be the same in the ξ and η directions. This is in fact the case for all hexahedral elements presented here. The first four two-dimensional Gauss product rules with $p = p_1 = p_2 = p_3$ are illustrated in Figure 11.6.

§11.9.4. Implementation of 3D Gauss Rules

The *Mathematica* module listed in Figure 11.8 implements three-dimensional product Gauss rules having 1 through 5 points in each natural coordinate direction. The number of points along those directions may be the same or different. If the rule has the same number of points p in all three directions the module is called in either of two ways:

```
{{ξi,ηj,μk},wijk}=HexaGaussRuleInfo[{p,numer},{i,j,k}]
{{ξi,ηj,μk},wijk}=HexaGaussRuleInfo[{p,numer},m]
```
(11.25)

The first form is used to get information for point $\{i,\ j,\ k\}$ of the $p \times p \times p$ rule, in which $1 \le i \le p$, $1 \le j \le p$, and $1 \le k \le p$. Indices i, j and k index the Gauss points along the ξ, η and μ directions, respectively.

```
HexaGaussRuleInfo[{rule_,numer_},point_]:= Module[
 {ξ,η,μ,p1,p2,p3,p12,i,j,jj,k,m,w1,w2,w3,
  info={{Null,Null,Null},0}}, i=j=k=0;
  If [Length[rule]==3,  {p1,p2,p3}=rule, p1=p2=p3=rule];
  If [Length[point]==3, {i,j,k}=point,   m=point;
     p12=p1*p2; k=Floor[(m-1)/p12]+1; jj=m-p12*(k-1);
     j=Floor[(jj-1)/p1]+1;  i=jj-p1*(j-1)];
  If [i<1||i>5||j<1||j>5||k<1||k>5, Return[info]];
  {ξ,w1}=  LineGaussRuleInfo[{p1,numer},i];
  {η,w2}=  LineGaussRuleInfo[{p2,numer},j];
  {μ,w3}=  LineGaussRuleInfo[{p3,numer},k];
  info={{ξ,η,μ},w1*w2*w3};
  If [numer, Return[N[info]], Return[Simplify[info]]]];
```

FIGURE 11.7. A *Mathematica* module that returns three-dimensional product Gauss rules having1 through 5 points along each natural coordinate direction.

The second form specifies that point by a "visiting counter" m that runs from 1 through p^3; if so $\{i,\, j,\, k\}$ are internally extracted from m by the statements shown in the listing of Figure 11.8.

If the integration rule has p_1, p_2 and p_3 points in the ξ, η and μ directions, respectively, the module may be also called in two ways:

```
{{ξi,ηj,μk},wijk}=HexaGaussRuleInfo[{p1,p2,p3},numer},{i,j,k}]
{{ξi,ηj,μk},wijk}=HexaGaussRuleInfo[{p1,p2,p3},numer},m]
```
(11.26)

The first form is used to explicitly specify the Gauss position through the $\{i,\, j,\, k\}$ index triplet. In this case $1 \le i \le p_1$, $1 \le j \le p_2$, and $1 \le k \le p_3$. In the second form m is a "visiting index" that runs from 1 through $p_1\, p_2\, p_3$; if so i, j and k are internally extracted from m by the statements shown in the listing.

In all four invocation forms, logical flag `numer` is set to `True` if numerical (floating-point to double precision accuracy) information is desired and to `False` to get exact information.

The module returns the Gauss point abcissas in the one-dimensional list {ξi,ηj,μk}, and the weight product $w = w_i w_j w_k$ in `wijk`. If inputs are incorrect (for instance, the number of points in one direction is outside the implemented range), the module returns {{Null,Null,Null},0}.

Example 11.2. The call {{xi,eta,mu},w}=HexaGaussRuleInfo[{3,False},{2,3,2}] returns xi=0, eta=Sqrt[3/5], mu=0, and w=$(5/9)\times(8/9)\times(5/9) = 200/727$.

Example 11.3. The variant call {{xi,eta,mu},w}=HexaGaussRuleInfo[{3,True},{2,3,2}] returns (to 16-place precision) xi=0., eta=0.7745966692414834, mu=0., and w=0.2751031636863824.

Remark 11.5. Different number of integration points in each direction are used in certain hexahedral elements called Ahmad-Pawsey elements or, coloquially, "degenerated brick" elements. These are intended to model thick shell structures by making special behavioral assumptions along the μ "thickness" direction.

§11.9.5. Selecting the Integration Rule

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$, except in special circumstances such as those noted in Remark 11.5. The total number of Gauss

```
HexaGaussRuleInfo[{rule_,numer_},point_]:= Module[
 {ξ,η,μ,p1,p2,p3,p12,i,j,jj,k,m,w1,w2,w3,
  info={{Null,Null,Null},0}}, i=j=k=0;
  If [Length[rule]==3,  {p1,p2,p3}=rule, p1=p2=p3=rule];
  If [Length[point]==3, {i,j,k}=point,   m=point;
     p12=p1*p2; k=Floor[(m-1)/p12]+1; jj=m-p12*(k-1);
     j=Floor[(jj-1)/p1]+1;  i=jj-p1*(j-1)];
  If [i<1||i>5||j<1||j>5||k<1||k>5, Return[info]];
  {ξ,w1}=  LineGaussRuleInfo[{p1,numer},i];
  {η,w2}=  LineGaussRuleInfo[{p2,numer},j];
  {μ,w3}=  LineGaussRuleInfo[{p3,numer},k];
  info={{ξ,η,μ},w1*w2*w3};
  If [numer, Return[N[info]], Return[Simplify[info]]]];
```

FIGURE 11.8. A *Mathematica* module that returns three-dimensional product Gauss rules having1 through 5 points along each natural coordinate direction.

points is then p^3. Each point adds at most $n_E = 6$ to the stiffness matrix rank, in which n_E denotes the rank of the elasticity matrix $\mathbf{E}$. For the 8-node hexahedron this rule gives $p \geq 2$ because $2^3 \times 6 = 48 > 24 - 6 = 18$ whereas $p = 1$ would incur a rank deficiency of $18 - 6 = 12$. For other configurations see Exercise 11.3.

Homework Exercises for Chapter 11

Hexahedron Elements

EXERCISE 11.1 [A:20] Find the shape functions associated with the 16-node hexahedron depicted in Figure E11.1(a) for all nodes. (This kind of element is historically important as pitstop on the way to the "degenerated solid" thick-plate and thick-shell elements developed in the late 1960s; those are called Ahmad-Pawsey elements in the FEM literature, and are bread-and-butter in nonlinear commercial codes such as ABAQUS.) Verify that your shape functions satisfy two important conditions:

(1) Interelement compatibility over a typical 6-node face, say 1-2-6-5-9-13. (If used as a thick-plate or solid-shell element, those will be the faces connected to neighboring elements; μ is conventionally the plate or shell "thickness" direction.)

(2) Completeness in the sense that the sum of all shape functions must be identically one. (This must be verified algebraically, not numerically).

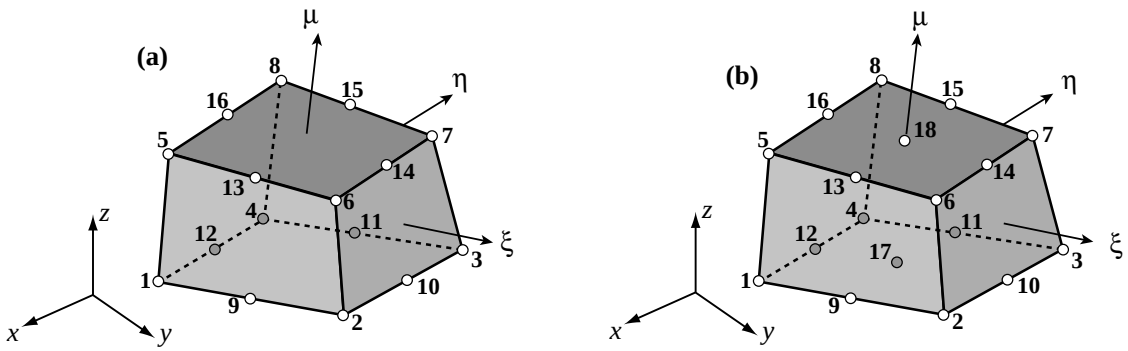

FIGURE E11.1. (a): 16-node hexahedron for Exercise 11.1; (b): 18-node hexahedron for Exercise 11.2.

EXERCISE 11.2 [A:20] Find all shape functions associated with the 18-node hexahedron depicted in Figure E11.1(b). (As in the previous case, this configuration is used for some thick-plate and solid-shell elements discussed in the last part of the course.) Verify that your shape functions satisfy two important conditions:

(1) Interelement compatibility over a typical 6-node face, say 1-2-6-5-9-13.(If used as a thick-plate or solid-shell element, those will be the faces connected to neighboring elements; μ is conventionally the plate or shell "thickness" direction.)

(2) Completeness in the sense that the sum of all shape functions must be identically one. (This must be verified algebraically, not numerically).

EXERCISE 11.3 [A:15] Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for (a) the 20-node hexahedron, (b) the 27-node hexahedron, (c) the the 16-node hexahedron of Exercise 11.1 and (d) the 18-node hexahedron of Exercise 11.2. For the last two, would a formula containing less Gauss sample points in the μ direction (for example: $3 \times 3 \times 1$, work, at least on paper?