Understanding and Applying Numerical Optimization Techniques

INTRODUCING NUMERICAL OPTIMIZATION

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Overview

Given a set of choices, optimization techniques help us make trade-offs

The optimization problem defines an objective and constraints

Any acceptable trade-off is called a feasible solution

The best trade-off is called the optimal solution

Powerful techniques exist to solve many types of optimization problems

Choices, Trade-offs and Optimization

"The best thing you can do is the right thing. The worst thing you can do is nothing"

Theodore Roosevelt

Decision:

How much of your personal savings should you invest in equities?

Choosing Is Complicated

Risk

Every investment carries risks

Return

No risk, no return

Decision:

You work as an actuary but love woodcarving. Can you quit your job and take it up professionally?

Choosing Is Complicated

Comfort

Stick to what you know and what you're good at

Excitement

Push yourself to learn and try something new

Decision:

Asleep in bed, you hear a loud sound. Should you wake up and investigate?

Choosing is Complicated

Think

Reflect on choices and seek more data and insights

Act

Do something (anything) and course-correct as needed

Choices, Trade-offs and Optimization

Choosing is complicated…

…Because resources are scarce

Optimization makes the right trade-offs

Resources Are Scarce

Risk

Every action carries risks

Return

No risk, no return

Capital is usually quite scarce and has a cost

Resources are Scarce

Comfort

Stick to what you know and what you're good at

Excitement

Push yourself to learn and try something new

Productive work years are finite, not using them well has a cost

Resources are Scarce

Think

Reflect on choices and seek more data and insights

Act

Do something (anything) and course-correct as needed

Reacting to emergencies requires speed, waiting too long often has a cost

Choices, Trade-offs and Optimization

Choosing is complicated…

…Because resources are scarce

Optimization makes the right trade-offs

Striking a balance is crucial

Balancing competing considerations calls for trade-offs

As the menu of options grows, these trade-offs become very hard

Optimization techniques help

Decision:

You work as an actuary but love woodcarving. Can you quit your job and take it up professionally?

Need job for income - but set aside family-time, then pursue passion on weekends

Optimization Helps Make Trade-offs

Comfort

Make sure you have enough time to spend with family and friends

Excitement

The rest of the time work on what truly excites you

Need job for income - but set aside family-time, then pursue passion on weekends

Decision:

How much of your personal savings should you invest in equities?

Set aside what you can't afford to lose, be aggressive with the rest

Optimization Helps Make Trade-offs

Risk

Decide how much you are willing to lose, never invest more than that

Return

Given that risk threshold, invest aggressively to maximize your return

Set aside what you can't afford to lose, be aggressive with the rest

The Optimization Problem: Objectives, Constraints and Decision Variables

Why Choosing Is Complicated

What do we really want to achieve?

What is slowing us down?

What do we really control?

Choosing involves answering complicated questions

Why Optimization Helps

What do we really want to achieve?

What is slowing us down?

What do we really control?

Optimization forces us to mathematically pin down answers to these questions

Framing the Optimization Problem

Objective Function

What we would like to achieve

Constraints

What slows us down

Decision Variables

What we really control

Collectively, these answers constitute the optimization problem

Decision:

How much of your personal savings should you invest in equities?

Personal Finance Choices

What do we really want to achieve?

What is slowing us down?

What do we really control?

Personal Finance Optimization

What do we really want to achieve?

Maximize returns on our investment

What is slowing us down?

Fear of losing money

What do we really control?

Investment amount, debt-equity split

Optimization Helps Make Trade-offs

Objective Function

Maximize return

Constraints

Upper bound on risk

Decision Variables

$-amount to invest?

Debt-equity split?

Decision:

You work as an actuary but love woodcarving. Can you quit your job and take it up professionally?

Career Choices

What do we really want to achieve?

What is slowing us down?

What do we really control?

Career Choice Optimization

What do we really want to achieve?

Minimize regrets

What is slowing us down?

Fear of not earning enough money, neglecting family

What do we really control?

Choice of job, leisure time

Optimization Helps Make Trade-offs

Objective Function

Minimize regrets

Constraints

Lower bound on income, family-time

Decision Variables

Quit job?

Work weekends?

Decision:

Asleep in bed, you hear a loud sound. Should you wake up and investigate?

Personal Safety Choices

What do we really want to achieve?

What is slowing us down?

What do we really control?

Personal Safety Optimization

What do we really want to achieve?

Ensure safety of yourself, others

What is slowing us down?

Fear of intruders, reluctance to leave a warm bed

What do we really control?

Call for help, act alone

Optimization Helps Make Trade-offs

Objective Function

Ensure safety of
yourself, others

Constraints

Lower and upper
bounds on seriousness
of threat

Decision Variables

Call 911?

Get up to investigate?

Correctly framing decisions as optimization problems is relatively easy, and very important

The Optimization Solution: Optimality and Feasibility

Framing the Optimization Problem

Objective Function

Constraints

Decision Variables

Decision Variables

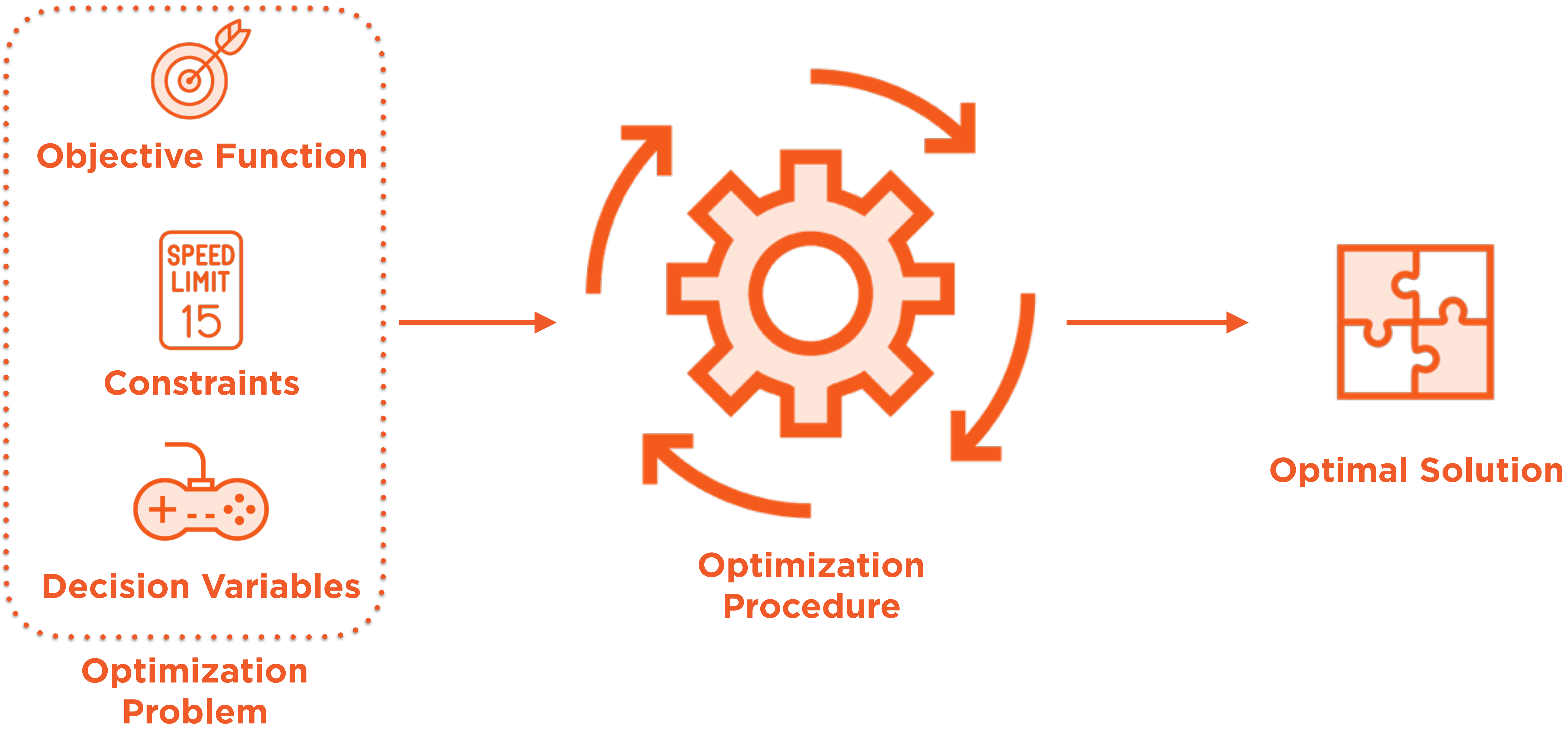
Any set of values for the decision variables is called a solution

This usage of the term solution can seem strange at first

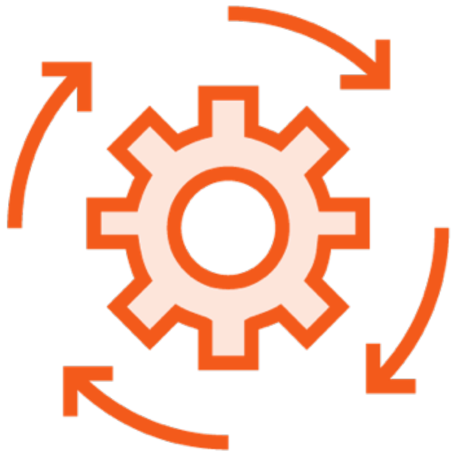
A solution might not even satisfy all of the constraints

Solutions to optimization problems are found via optimization procedures

Solving the Optimization Problem



Optimization Helps Make Trade-offs

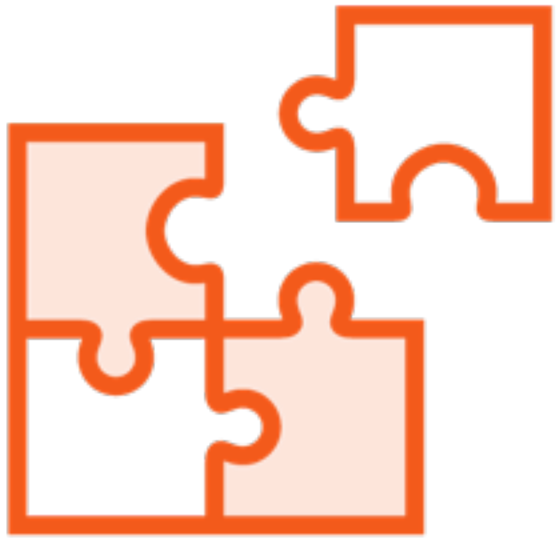


Optimization Procedure

Mathematical solution technique

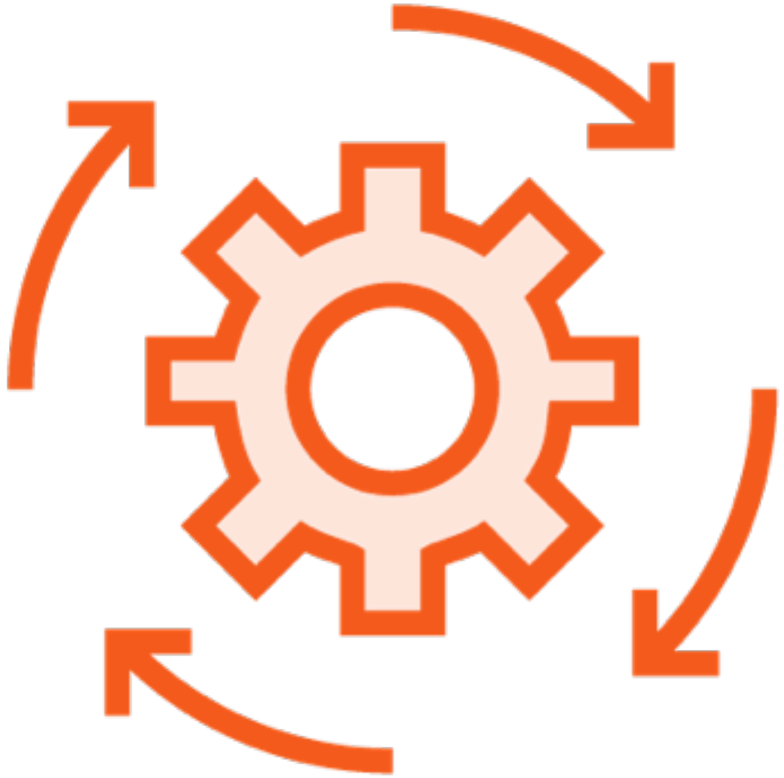
Optimal Solution

The “best” values of decision variables



Feasible Solution Set

Set of acceptable values of decision variables

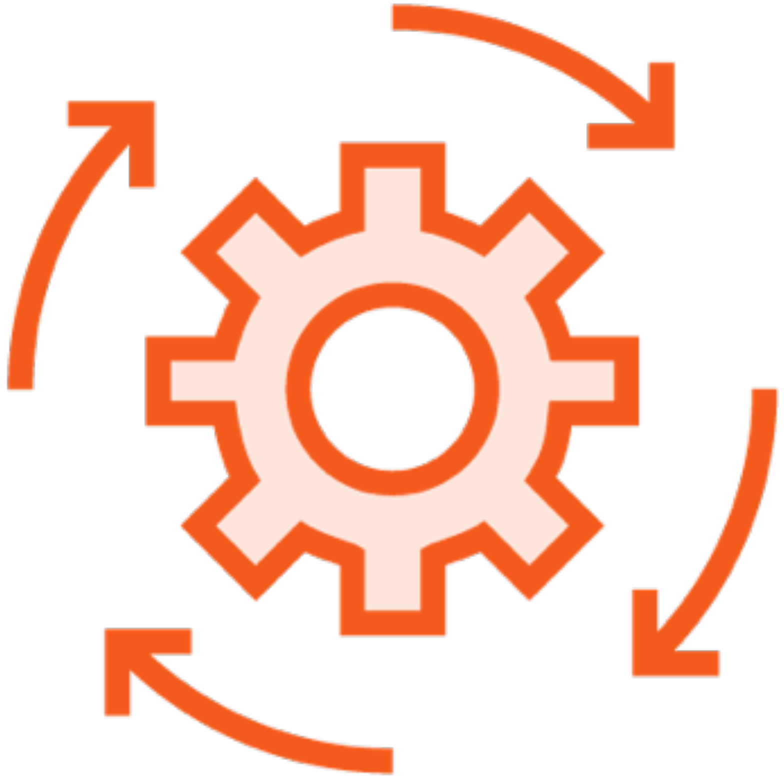


Optimization Procedure

The optimization procedure is usually very mathematical

Technologies like Excel, R and Python make this cookie-cutter

Using is easy, building is hard



Optimization Procedure

Optimization procedures vary widely based on type of problem

Linear programming

Integer programming

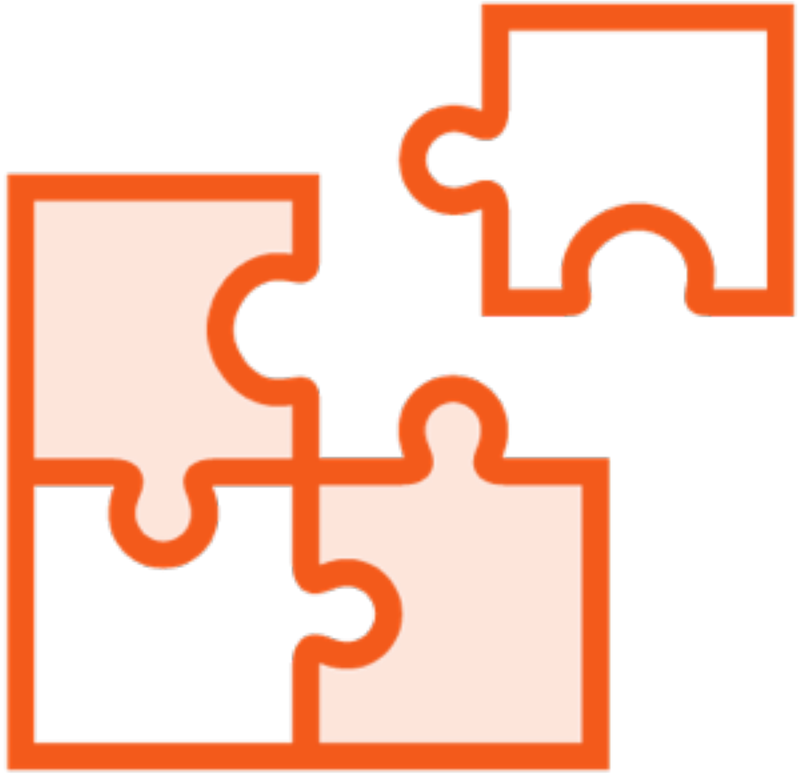
Second-order cone programming

Optimal Solution

The optimal solution represents the ‘best’ values of the decision variables

These ‘best’ values maximize/minimize the objective function…

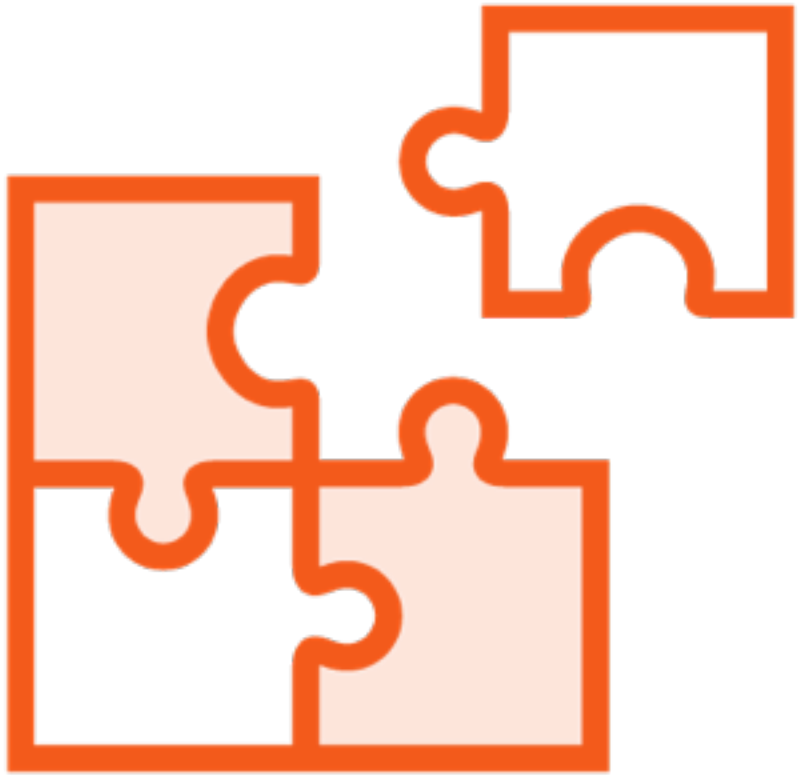
…while also satisfying all constraints



Feasible Solution Set

Optimal = Best

Feasible = Acceptable



Feasible Solution Set

The set of feasible solutions represents ‘acceptable’ values of the decision variables

These ‘acceptable’ values satisfy all constraints

The optimal solution is drawn from this set of feasible solutions

Framing the Optimization Problem

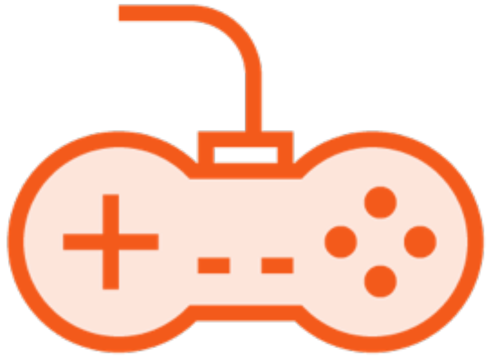
Objective Function

Earn $1M in a year

Constraints

Max loss = $10

Investment = $1000



Decision Variables

Debt-equity split

No feasible solution exists for this problem

Improperly framed optimization problems often have empty feasible solution sets (and no optimal solution)

Applications of Optimization

Optimization is Ubiquitous

Finance

Minimize risk subject to constraint on return

Mathematics

Find 'best' fit line through a set of points

Operations Research

Optimal working hours in a manufacturing facility

Economics

Maximise consumer utility within a given income

Medical Research

Find bonds between atoms that minimize potential

Portfolio as Sum of Random Variables

$$P = w_1 Y_E + w_2 Y_D + w_3 Y_G \ldots + w_k Y_A$$

P_i = % return of stock portfolio on day i

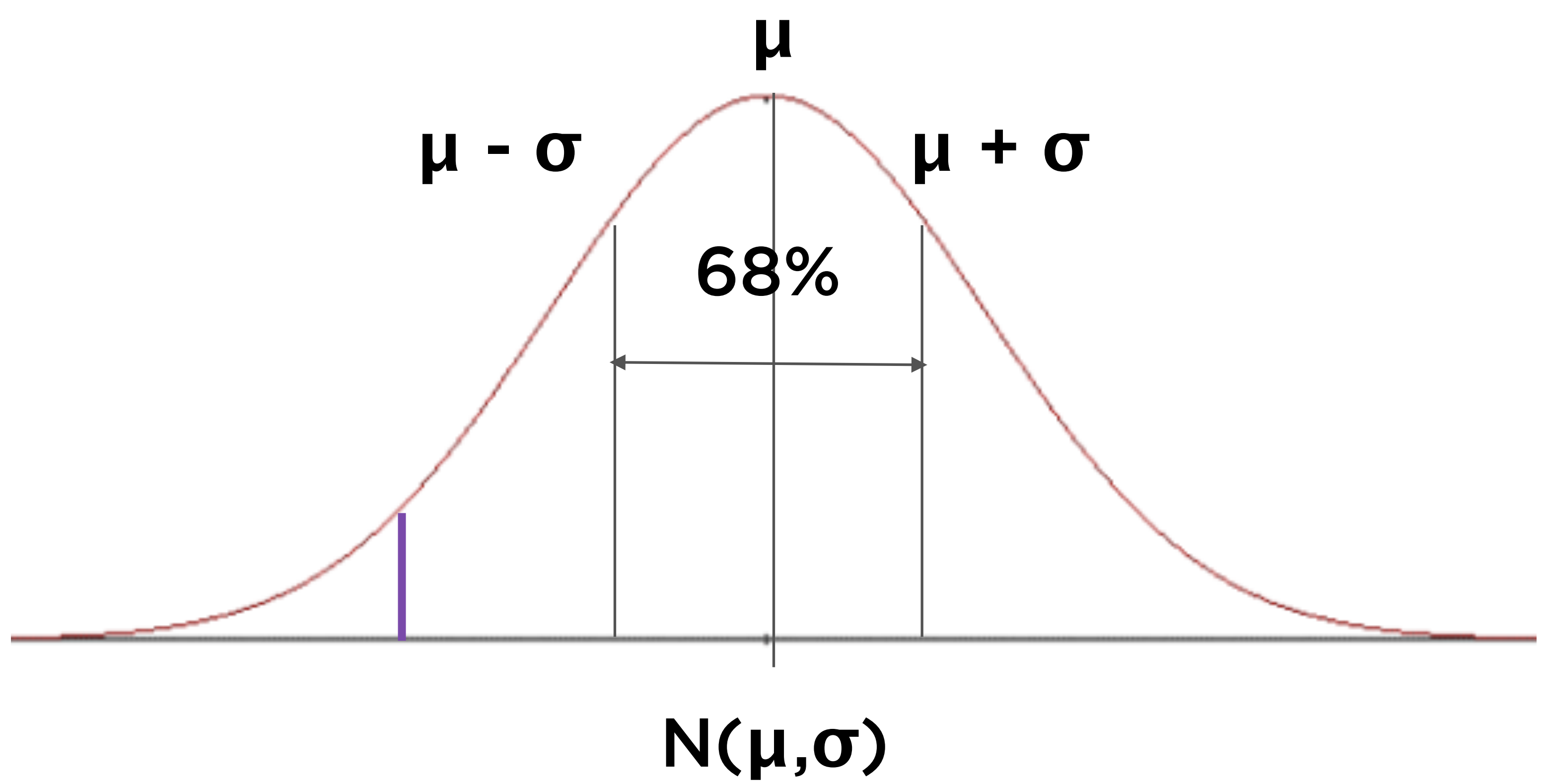
Portfolio P consists of stocks of value $\$w_1$ of Exxon, $\$w_2$ of the Dow, $\$w_3$ of Google and $\$w_k$ of Apple

Portfolio as Sum of Random Variables

$$P = w_1 Y_E + w_2 Y_D + w_3 Y_G \ldots + w_k Y_A$$

Modelling a portfolio as the sum of random variables is an extremely common use-case

Stock Returns



Movement of 1 stock over next 1 day is a random variable, usually modelled as a normal random variable with mean $\mu = 0$

Portfolio as Sum of Random Variables

$$P = w_1 Y_1 + w_2 Y_2 + w_3 Y_3 \ldots + w_k Y_k$$

Modelling a portfolio as the sum of random variables is an extremely common use-case

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

Mean(P) = ?

Variance(P) = ?

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$\left.\begin{aligned} \text{Mean}(P) = \; & w_1 \times \text{Mean}(Y_1) + \\ & w_2 \times \text{Mean}(Y_2) + \\ & w_3 \times \text{Mean}(Y_3) + \\ & \ldots \\ & w_k \times \text{Mean}(Y_k) \end{aligned}\right\} \text{k terms, all linear}$$

Mean of sum = sum of means

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$\begin{aligned} \text{Mean}(P) = \; & w_1\bar{Y}_1 + \\ & w_2\bar{Y}_2 + \\ & w_3\bar{Y}_3 + \\ & \ldots \\ & w_k\bar{Y}_k \end{aligned}$$

k terms, all linear

Mean of sum = sum of means

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

Mean(y)

Simple - mean of sum is sum of means

Variance(y) = ?

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

Mean(y)

Simple - mean of sum is sum of means

Variance(y)

Tricky - requires use of covariance matrix

Covariance Matrix

$$Y = Y_1 + Y_2 + Y_3 \ldots + Y_k$$

$$\begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) & \ldots & Cov(Y_1, Y_k) \\ Cov(Y_2, Y_1) & Var(Y_2) & \ldots & Cov(Y_2, Y_k) \\ Cov(Y_k, Y_1) & Cov(Y_k, Y_2) & \ldots & Var(Y_k) \end{bmatrix}$$

k rows

k columns

A kxk matrix - diagonal elements are variances, off-diagonal elements are covariances

Adding Random Variables

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$\text{Variance}(P) = \sum_{i=1}^{k} \sum_{j=1}^{k} w_i w_j \,\text{Covariance}(Y_i, Y_j)$$

k^2 terms,
quadratic

Variance of the portfolio can be found by multiplying the weight vector with the covariance matrix

Portfolio Variance

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$\underset{1 \times 1}{Var(P)} = \underset{1 \times k}{W} * \underset{k \times k}{Cov(Y)} * \underset{k \times 1}{W^T}$$

Variance of the portfolio can be found by multiplying the weight vector with the covariance matrix

Portfolio Variance

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$W = \begin{bmatrix} w_1 & w_2 & w_3 & \ldots & w_k \end{bmatrix}$$

1 row

1 x k

k columns

The weight vector simply contains the weights of different stocks in the portfolio

Portfolio Variance

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$W^T = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \cdots \\ w_k \end{bmatrix}$$

k x 1

k rows

1 column

Transposing a vector reverses its rows and columns

Portfolio Variance

$$P = w_1Y_1 + w_2Y_2 + w_3Y_3 \ldots + w_kY_k$$

$$\underset{1 \times 1}{Var(P)} = \underset{1 \times k}{W} * \underset{k \times k}{Cov(Y)} * \underset{k \times 1}{W^T}$$

Variance of the portfolio can be found by multiplying the weight vector with the covariance matrix

Portfolio Allocation as an Optimization Problem

Objective Function

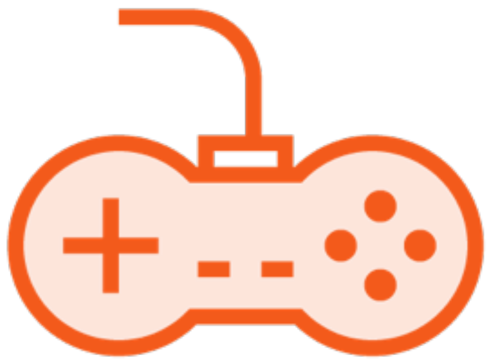
Minimize Var(P)

$$Var(P) = W * Cov(Y) * W^T$$

Constraints

$$\bar{P} >= R_{threshold}$$

$$\bar{P} = w_1\bar{Y}_1 + w_2\bar{Y}_2 + \ldots \; w_k\bar{Y}_k$$

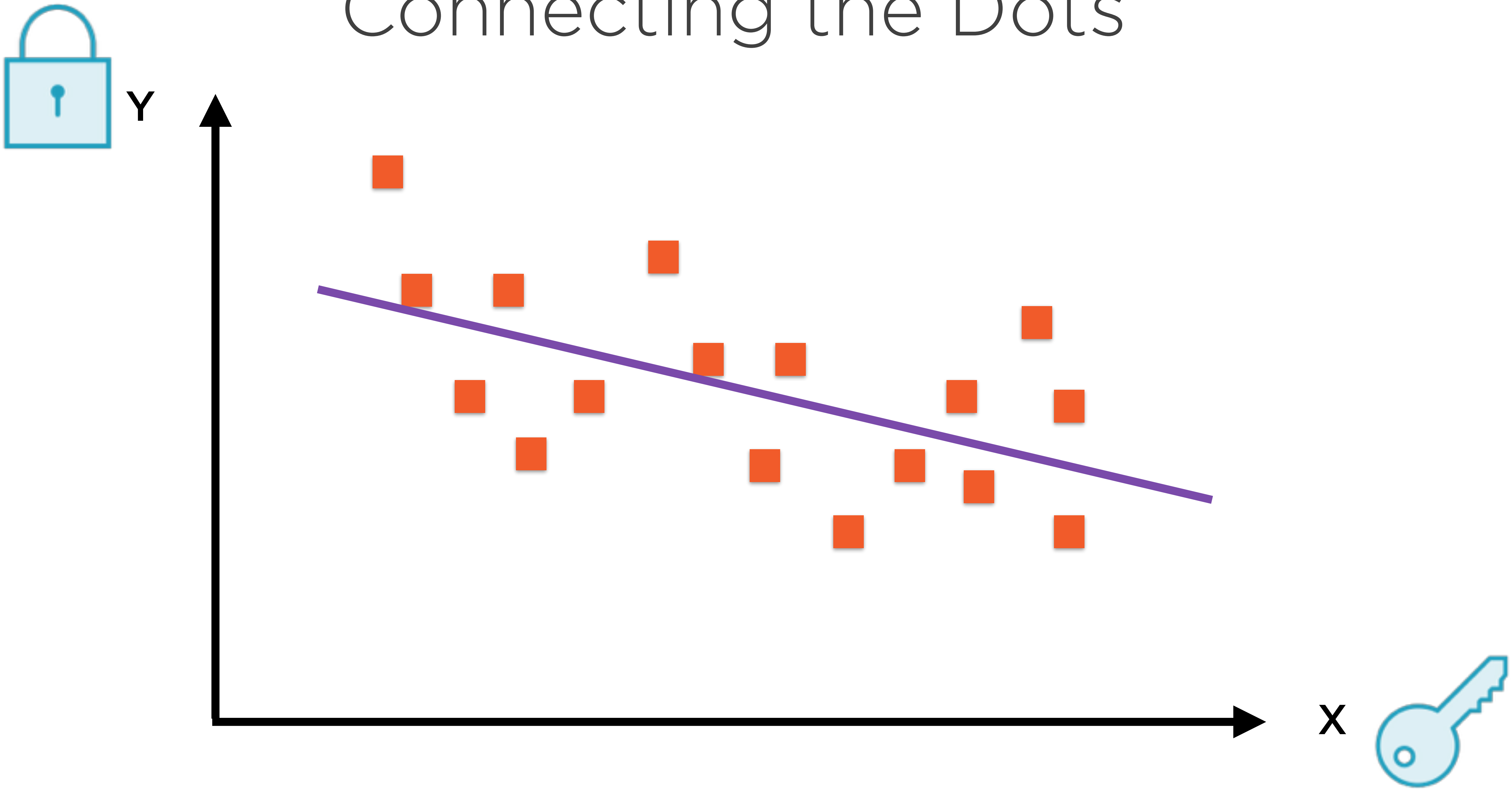


Decision Variables

W

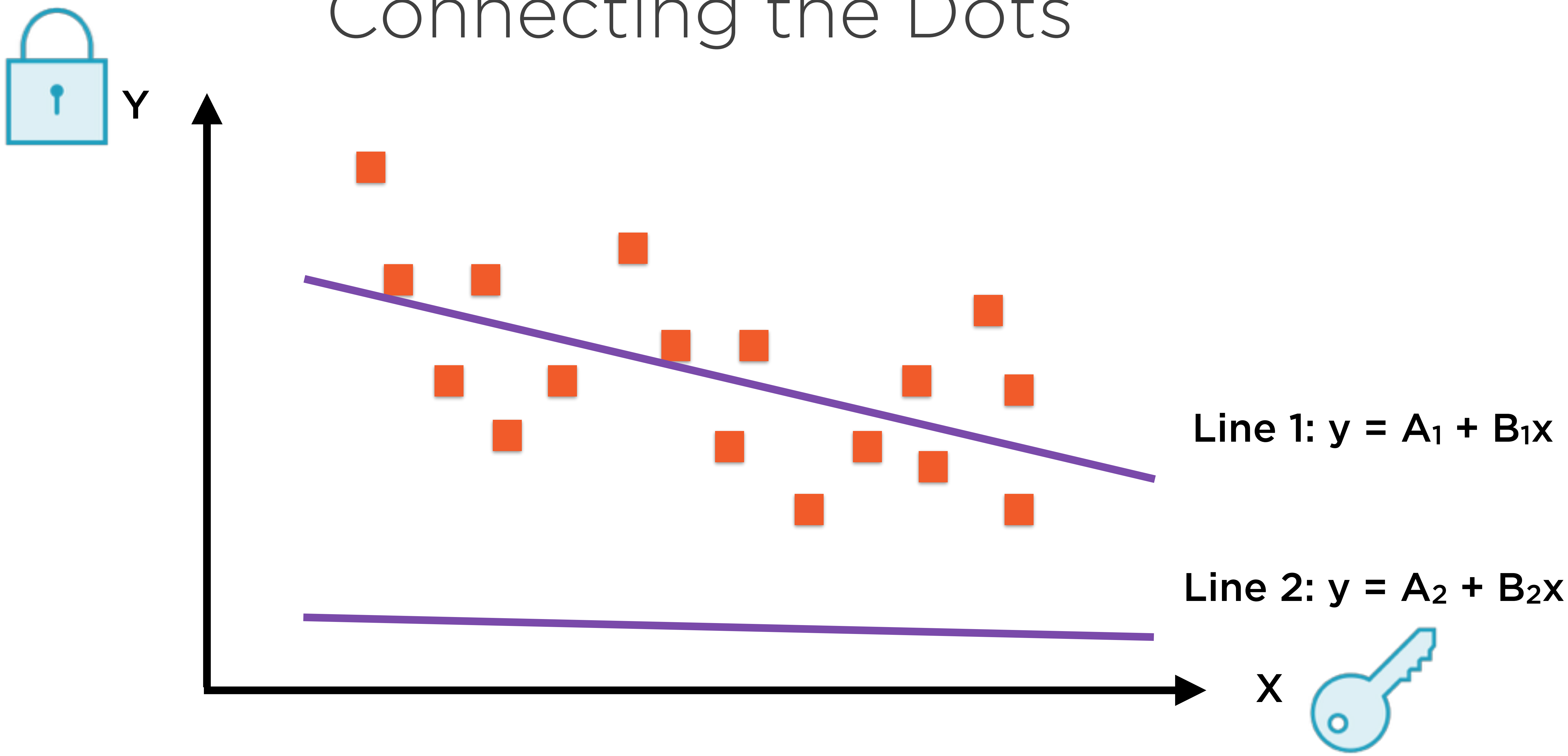
$$W = [\, w_1 \quad w_2 \quad w_3 \quad \ldots \quad w_k \,]$$

Connecting the Dots



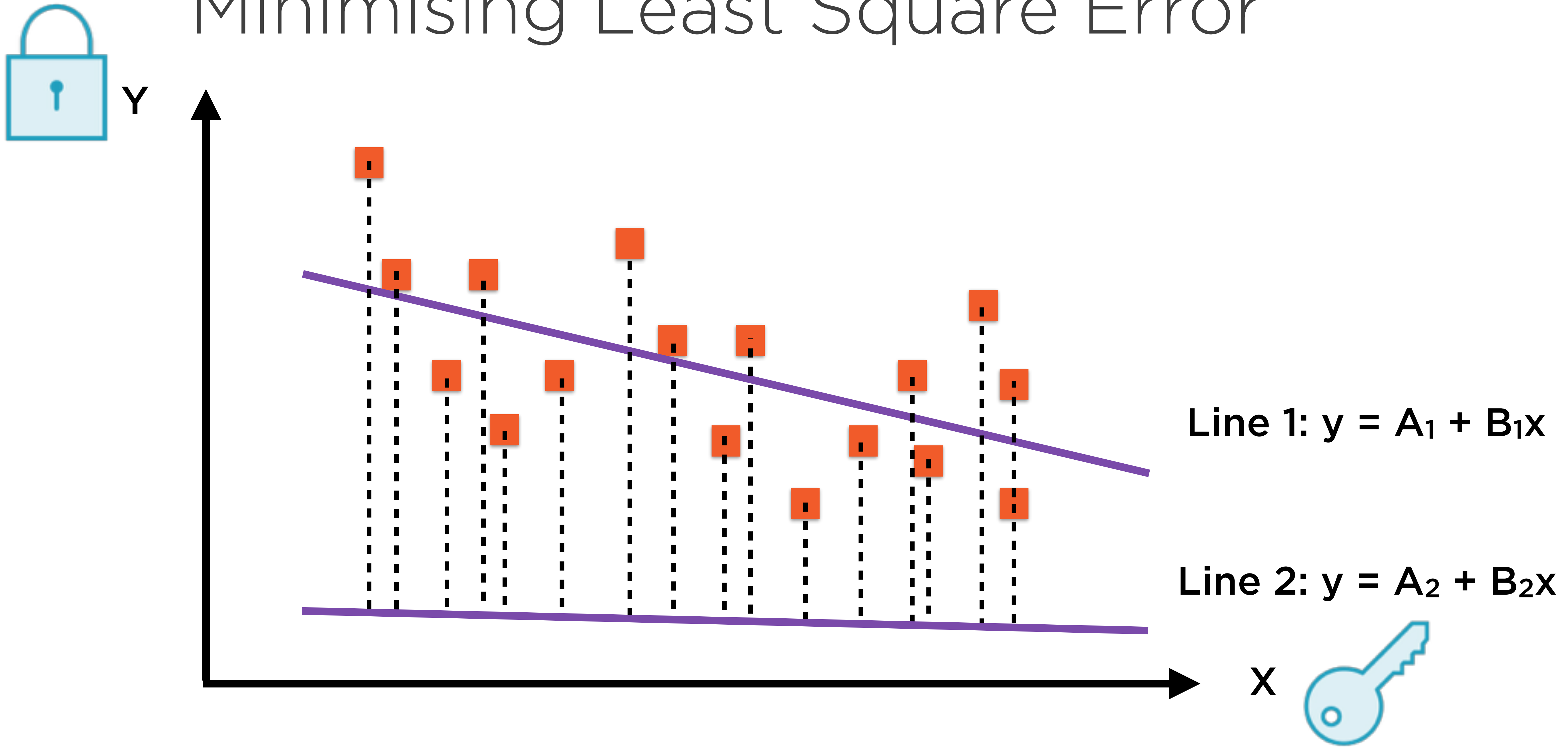
Linear Regression involves finding the "best fit" line

Connecting the Dots



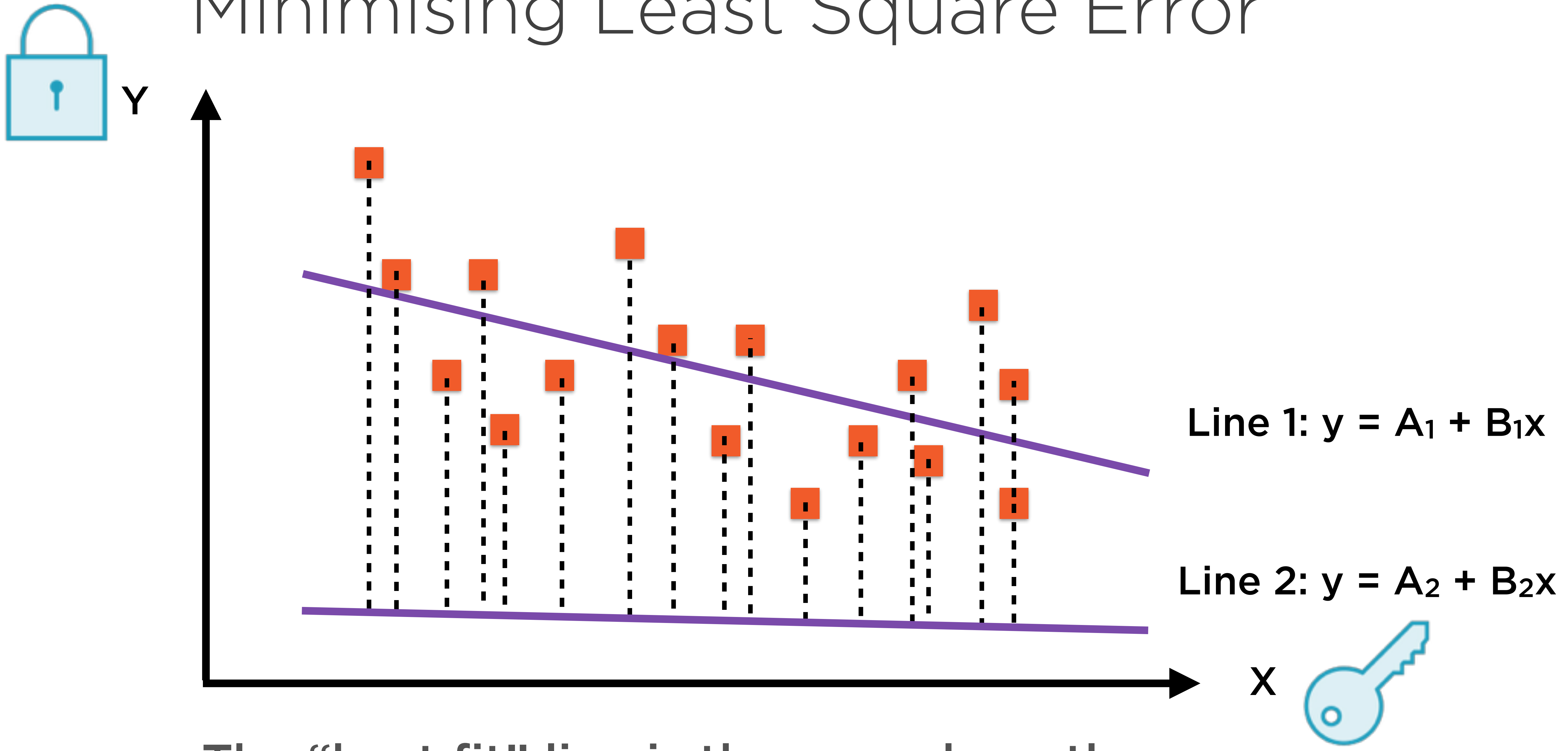
Let's compare two lines, Line 1 and Line 2

Minimising Least Square Error



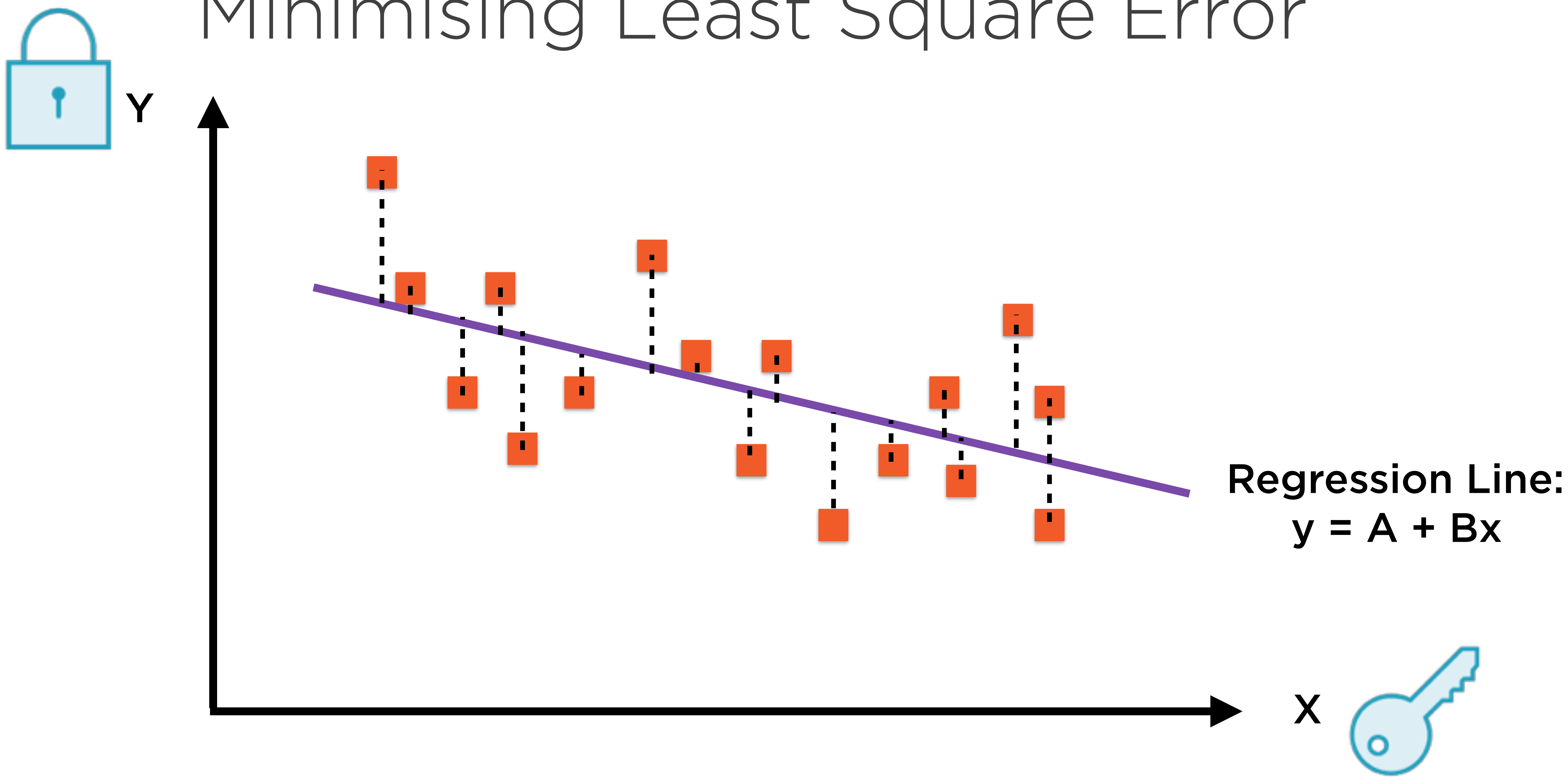
Drop vertical lines from each point to the lines

Minimising Least Square Error



The “best fit” line is the one where the sum of the squares of the lengths of these dotted lines is minimum

Minimising Least Square Error



The "best fit" line is called the regression line

Three Estimation Methods

Method of moments	Method of least squares	Maximum likelihood estimation

Cookie cutter techniques to determine the values of A and B (regression coefficients)

Linear Regression as an Optimization Problem

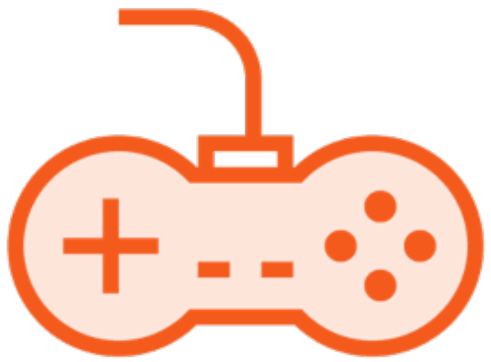
Objective Function

Minimize variance of the residuals

Constraints

Express relationship as a straight line

$y = Ax + B$

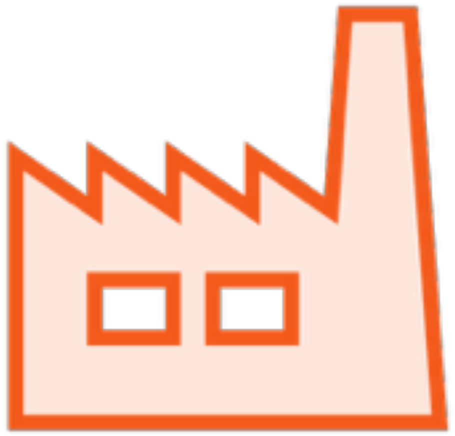


Decision Variables

Values of A and B

Coming Up Next

A Famous Case Study: Wyndor Glass



Three Factories

Different plants for wood, aluminium and glass

Two Products

Glass doors and glass windows

Cost and Profit

Profit and effort per unit product are known

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18

Profit per Batch	$3,000	$5,000

Tweak production to maximise profits

Manufacturing as an Optimization Problem

Objective Function

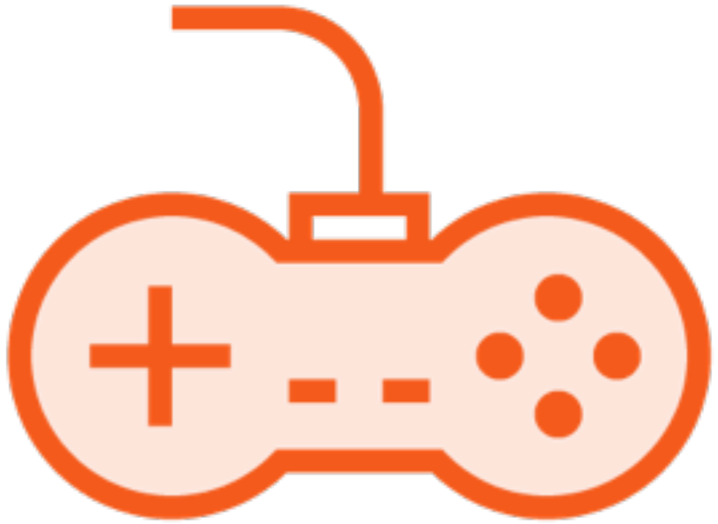
Maximize profits

Constraints

Plant capacity constraints

Decision Variables

How many batches of each product to produce



Decision Variables

x_1 = Number of batches of product 1 to produce

x_2 = Number of batches of product 2 to produce

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Batches of Product 1 = x_1

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Batches of Product 2 = x_2

Objective Function

Maximize profit Z

Z is total profit per week, in thousands of dollars

$Z = 3x_1 + 5x_2$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18

Profit per Batch	$3,000	$5,000

$$3x_1 \quad + \quad 5x_2$$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Profit $Z = 3x_1 + 5x_2$

Constraints

Infinite production is not possible

The production time available in the factories limits x_1 and x_2

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)			Production Time available per Week (hours)
	Product x_1	Product x_2		
Plant y_1	1 x_1 +	0 x_2	<=	4
Plant y_2	0	2		12
Plant y_3	3	2		18
Profit per Batch	$3,000	$5,000		

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 1: $x_1 <= 4$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)			Production Time available per Week (hours)
	Product x_1	Product x_2		
Plant y_1	1	0		4
Plant y_2	0 x_1 +	2 x_2	<=	12
Plant y_3	3	2		18
Profit per Batch	$3,000	$5,000		

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 2: $2x_2 <= 12$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3 x_1 +	2 x_2 $<=$	18
Profit per Batch	\$3,000	\$5,000	

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 3: $3x_1 + 2x_2 <= 18$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 4: $x_1 >= 0$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 5: $x_2 >= 0$

Linear Programming Problem Formulation

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$x_1 , x_2 >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

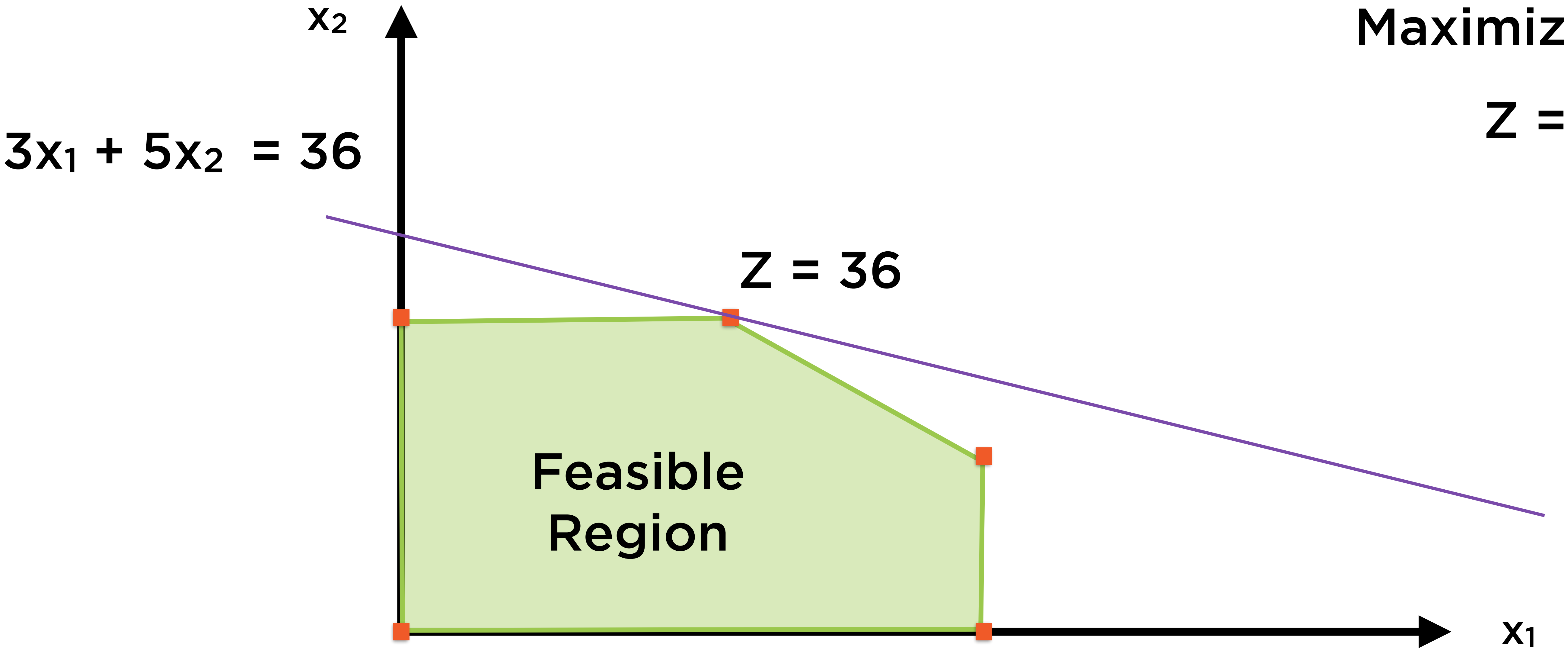
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n <= b_m$$

$x_1 , x_2 \ldots x_n >= 0$ **(Non-negativity constraints)**

Optimal Solution in Space



Maximize

$Z = 3x_1 + 5x_2$

Find the right-most such line that intersects the feasible region (for a maximization problem)

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution
- Test for optimality
- Not optimal, continue
- Optimal, stop
- Keep iterating until we run out of corner-points

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	$\boxed{36}$
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Right side of equation (0) gives the optimal value

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	$\boxed{x_2}$	0	0	1	0	1/2	0	$\boxed{6}$
(3)	$\boxed{x_1}$	0	1	0	0	-1/3	1/3	$\boxed{2}$

We can conclude that at optimal, $x_1 = 2$, $x_2 = 6$

Summary

Optimization techniques help us make smart trade-offs given a set of choices

Such techniques solve optimization problems set up in a particular form

- The objective function expresses what we would like to achieve
- The constraints are conditions that we must satisfy
- The decision variables are quantities we can control

The optimal solution is the ‘best’ set of values for those decision variables