Understanding Linear Programming

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Overview

Linear programming problems (LPPs) have a linear objective and constraints

They closely mirror an economic profit maximisation problem

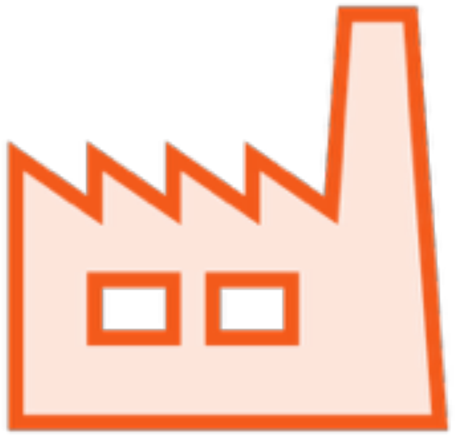
LPPs can be solved using the Simplex algorithm

Simplex is very powerful and widely used

A modified form of Simplex can be extended to quadratic programming

Linear Programming: Intuition

A Famous Case Study: Wyndor Glass



Three Factories

Different plants for wood, aluminium and glass

Two Products

Glass doors and glass windows

Cost and Profit

Profit and effort per unit product are known

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18

Profit per Batch	$3,000	$5,000

Tweak production to maximise profits

Manufacturing as an Optimization Problem

Objective Function

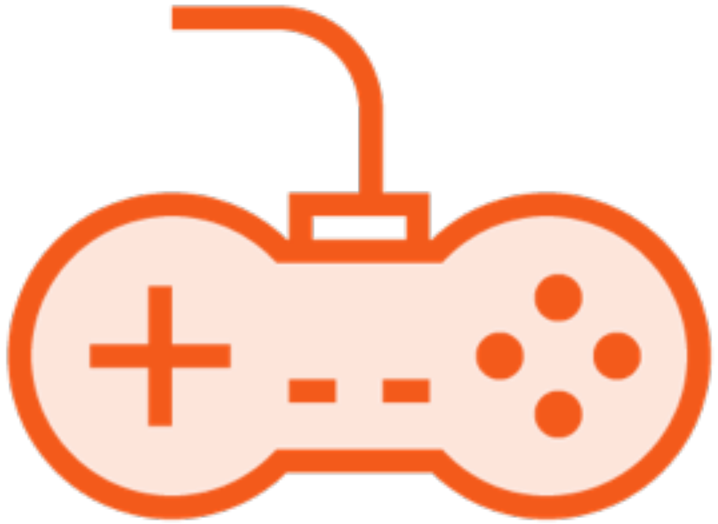
Maximize profits

Constraints

Plant capacity constraints

Decision Variables

How many batches of each product to produce



Decision Variables

x_1 = Number of batches of product 1 to produce

x_2 = Number of batches of product 2 to produce

A Famous Case Study: Wyndor Glass

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Batches of Product 1 = x_1

A Famous Case Study: Wyndor Glass

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Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Batches of Product 2 = x_2

Objective Function

Maximize profit Z

Z is total profit per week, in thousands of dollars

$Z = 3x_1 + 5x_2$

A Famous Case Study: Wyndor Glass

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Profit per Batch	\$3,000	\$5,000

$$3x_1 \quad + \quad 5x_2$$

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Profit per Batch	$3,000	$5,000	

$$\text{Profit } Z = 3x_1 + 5x_2$$

Constraints

Infinite production is not possible

The production time available in the factories limits x_1 and x_2

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)				Production Time available per Week (hours)
	Product x_1		Product x_2		
Plant y_1	1 x_1	+	0 x_2	<=	4
Plant y_2	0		2		12
Plant y_3	3		2		18

Profit per Batch	\$3,000	\$5,000

A Famous Case Study: Wyndor Glass

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Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 1: x_1 <= 4

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		
	Product x_1	Product x_2	Production Time available per Week (hours)
Plant y_1	1	0	4
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Profit per Batch	$3,000	$5,000	

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Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 2: $2x_2 <= 12$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3 x_1 +	2 x_2 <=	18
Profit per Batch	$3,000	$5,000	

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Plant y_1	1	0	4
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Plant y_3	3	2	18
Profit per Batch	\$3,000	\$5,000	

Constraint 3: $3x_1 + 2x_2 <= 18$

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	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
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Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 4: $x_1 >= 0$

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18
Profit per Batch	$3,000	$5,000	

Constraint 5: $x_2 >= 0$

Linear Programming Problem Formulation

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$x_1 , x_2 >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n <= b_m$$

$x_1 , x_2 \dots x_n >= 0$ **(Non-negativity constraints)**

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Objective function, interpret as profit

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

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$x_1 , x_2 \ldots x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

Maximize the profit function

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n <= b_1$$

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$x_1 , x_2 ... x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \dots + c_n\mathbf{x_n}$$

Decision variables: how much to produce of each product

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n <= b_2$$

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$x_1 , x_2 \dots x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \dots + c_n\mathbf{x_n}$$

Interpret each as an activity

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n <= b_m$$

$x_1 , x_2 \dots x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Increase in profit by increasing each activity by 1 unit

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

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$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n <= b_m$$

Amount of each resource that is available for use

$x_1 , x_2 \ldots x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

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Subject to constraints:

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Amount of resource allocated to each activity

$x_1 , x_2 \ldots x_n >= 0$ (Non-negativity constraints)

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Functional constraints

$x_1, x_2 \ldots x_n >= 0$ (Non-negativity constraints)

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Each functional constraint is a less-than inequality

$x_1 , x_2 \ldots x_n >= 0$ (Non-negativity constraints)

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Collectively referred to as the model parameters

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

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$x_1 , x_2 \ldots x_n >= 0$ (Non-negativity constraints)

Dual Problem

Minimize

$$W = b_1y_1 + b_2y_2 + \dots + b_my_m$$

Subject to constraints:

$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m >= c_1$$

$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m >= c_2$$

$$\vdots$$

$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m >= c_n$$

$y_1 , y_2 \dots y_m >= 0$ **(Non-negativity constraints)**

Primal and Dual Forms

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Minimize

$$W = b_1y_1 + b_2y_2 + \ldots + b_my_m$$

Primal and Dual Forms

Subject to constraints:

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Primal and Dual Forms

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Profit maximization

Subject to capacity constraints on production

Minimize

$$W = b_1y_1 + b_2y_2 + \dots + b_my_m$$

Cost minimization

Subject to minimal level of economic activity

The primal and dual problems have the same optimal solution

Linear Programming: Micro-economic Assumptions

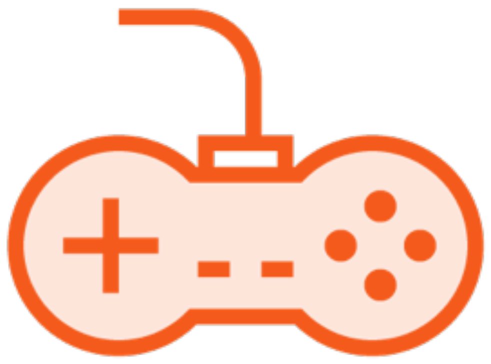
Manufacturing as an Optimization Problem

Objective Function

Maximize profits

Constraints

Plant capacity constraints



Decision Variables

How many batches of each product to produce

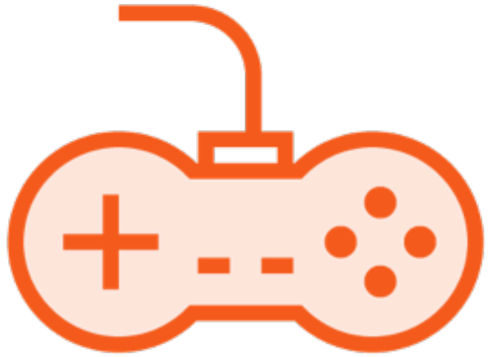
Micro-economic Assumptions

Proportionality Assumption

No start-up costs, constant returns to scale

Additivity Assumption

Products are neither complements nor substitutes



Divisibility Assumption

Fractional production is possible

Proportionality Assumption

Start-up costs would have caused profit function to have a constant term

Positive returns of scale would have profit function steepen as production increases

Negative returns of scale would have profit function flatten as production increases

Additivity Assumption

Complementary goods have positive synergies as scale increases

- reducing cost

or

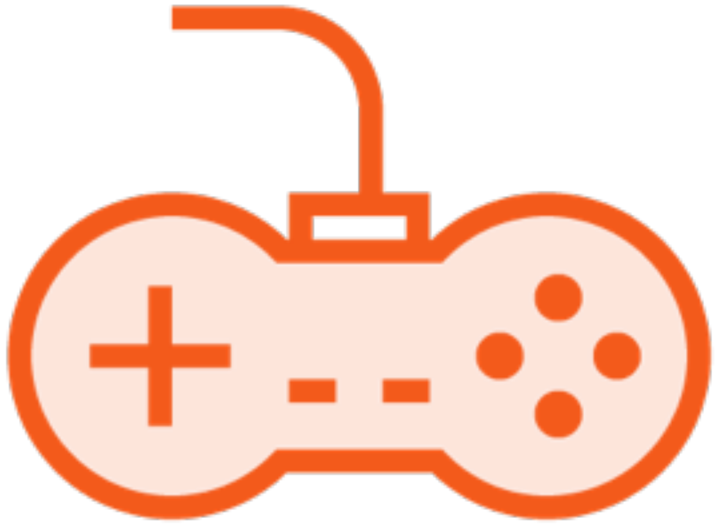
- increasing profit

Substitute goods have negative synergies as scale increases

- increasing cost

or

- reducing profit



Divisibility Assumptions

Fractional values of production are acceptable in optimal solution

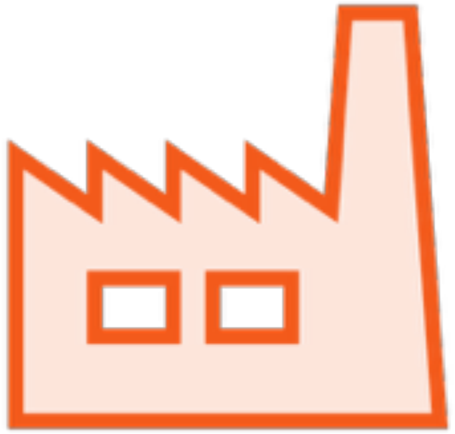
This assumption is more important than it seems

Requiring integer values makes the optimization much more difficult to solve

Integer programming techniques accomplish this

Linear Programming: Graphical Solutions

A Famous Case Study: Wyndor Glass



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Subject to constraints:

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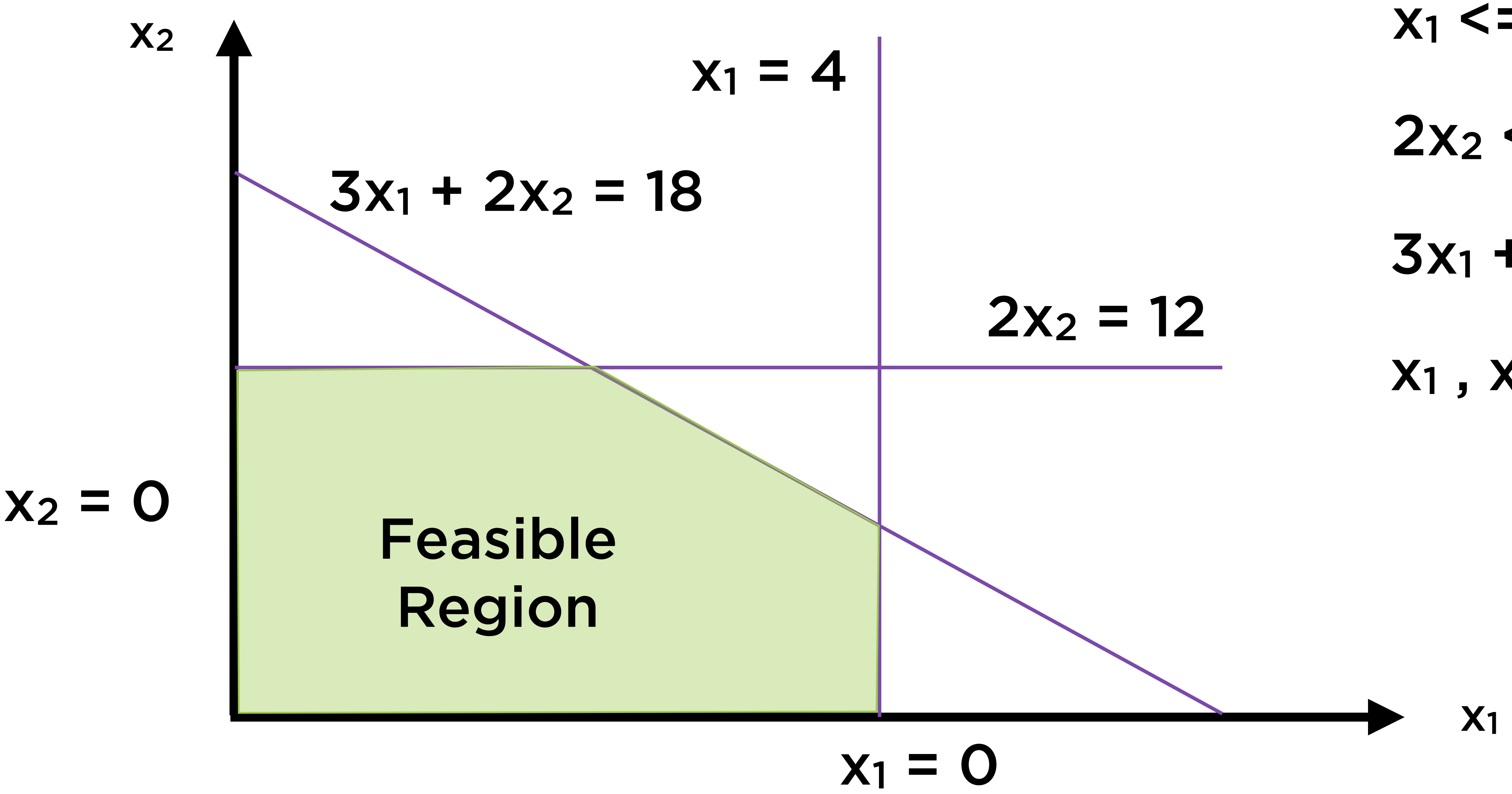
$x_1 , x_2 >= 0$ (Non-negativity constraints)

Decision Variables in Space

Two decision variables => two-dimensional space

Constraints in Space



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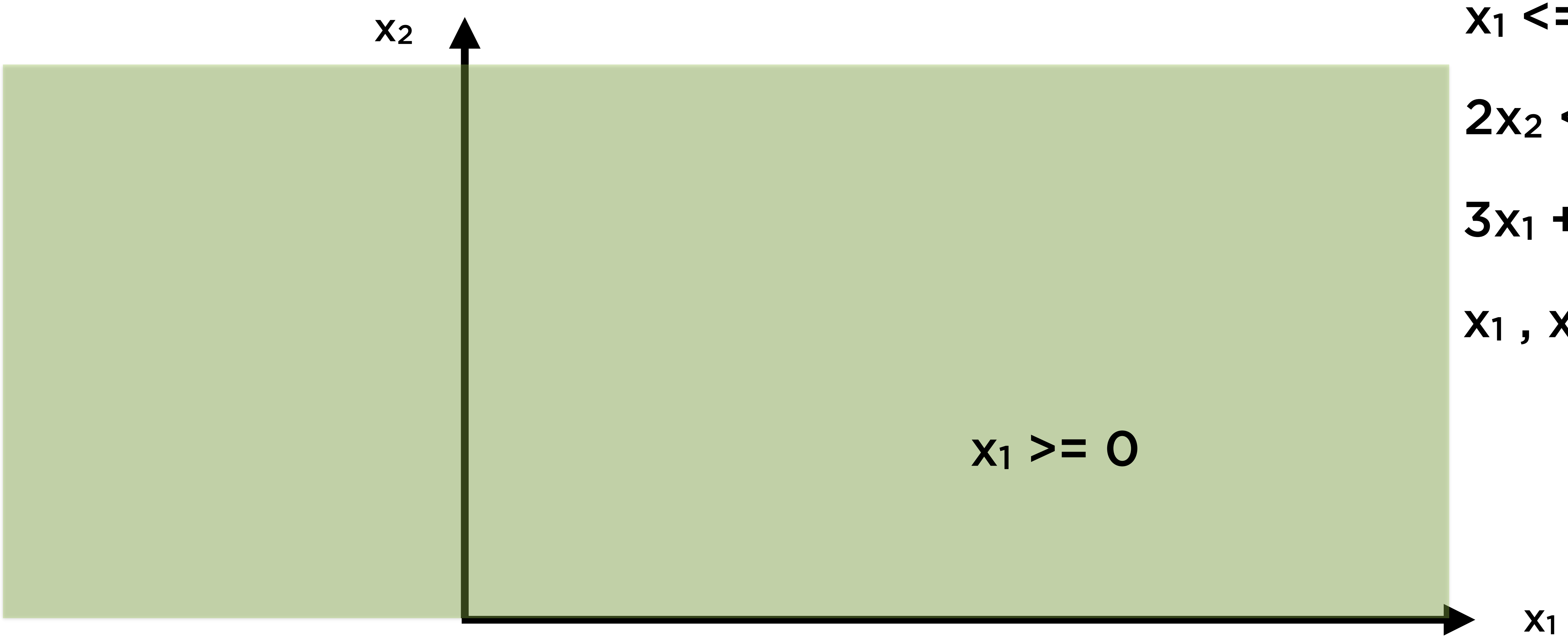
$x_1 , x_2 >= 0$

Each constraint bounds the feasible region

Constraints in Space



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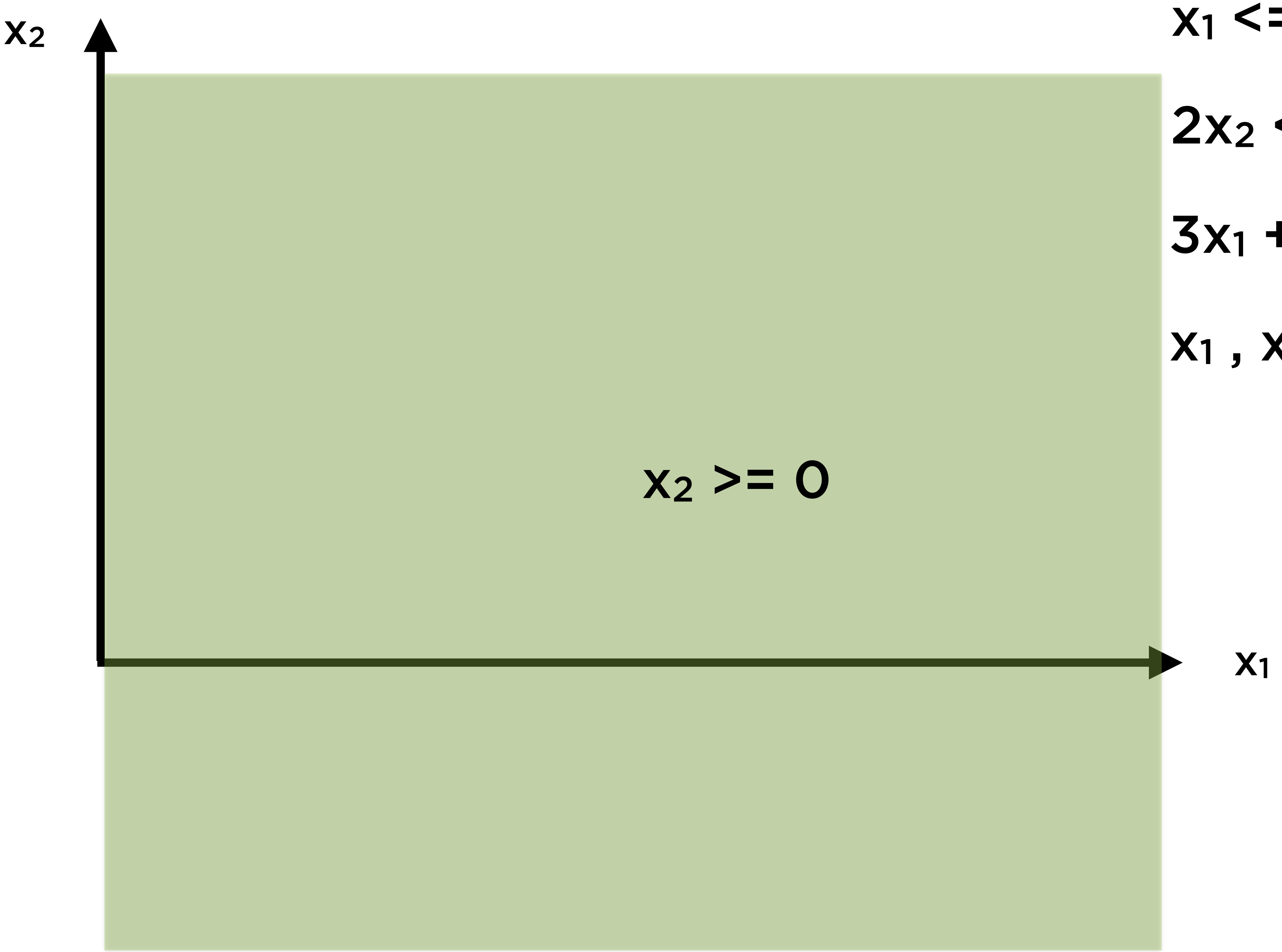

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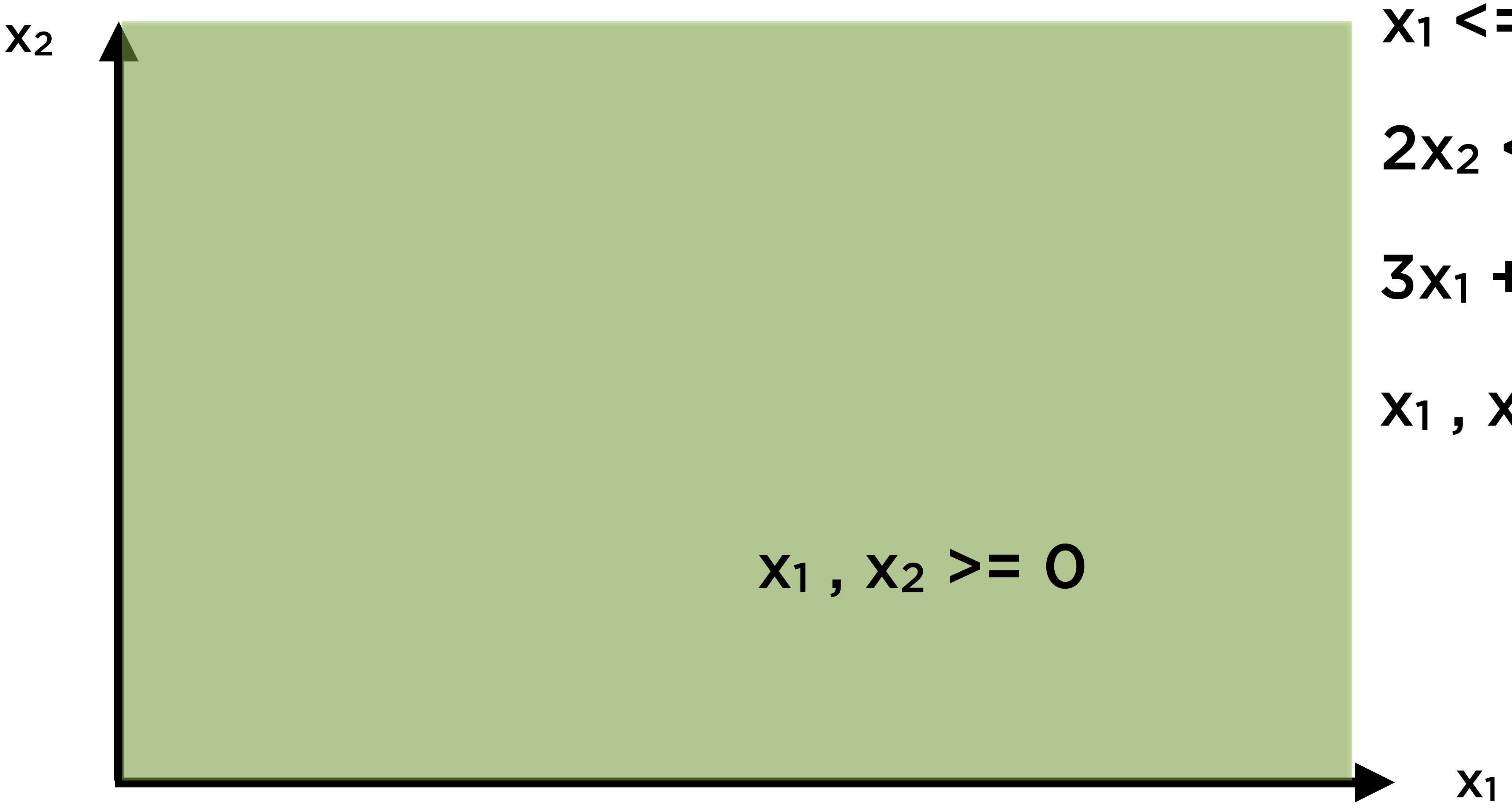

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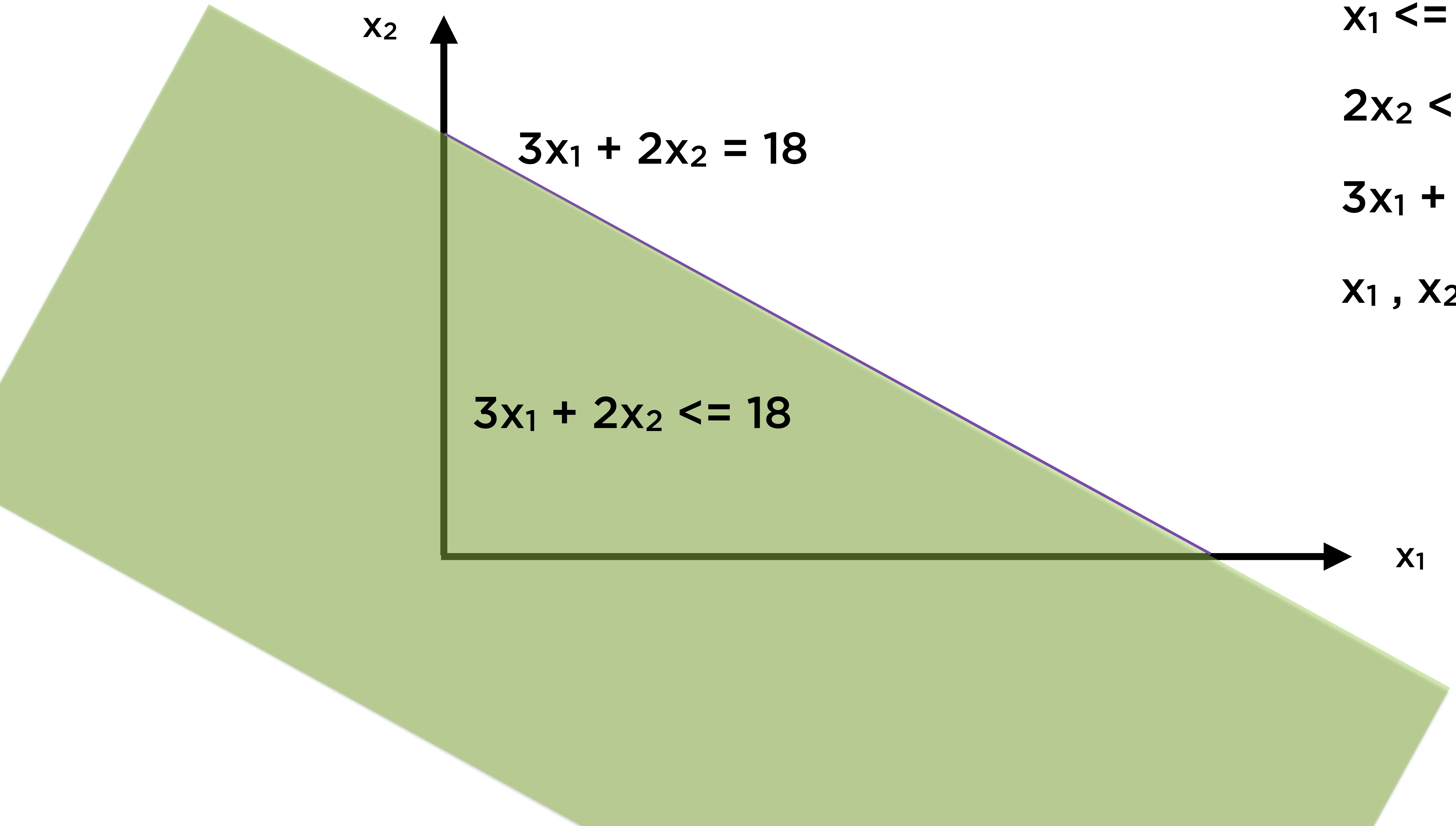
Constraints in Space



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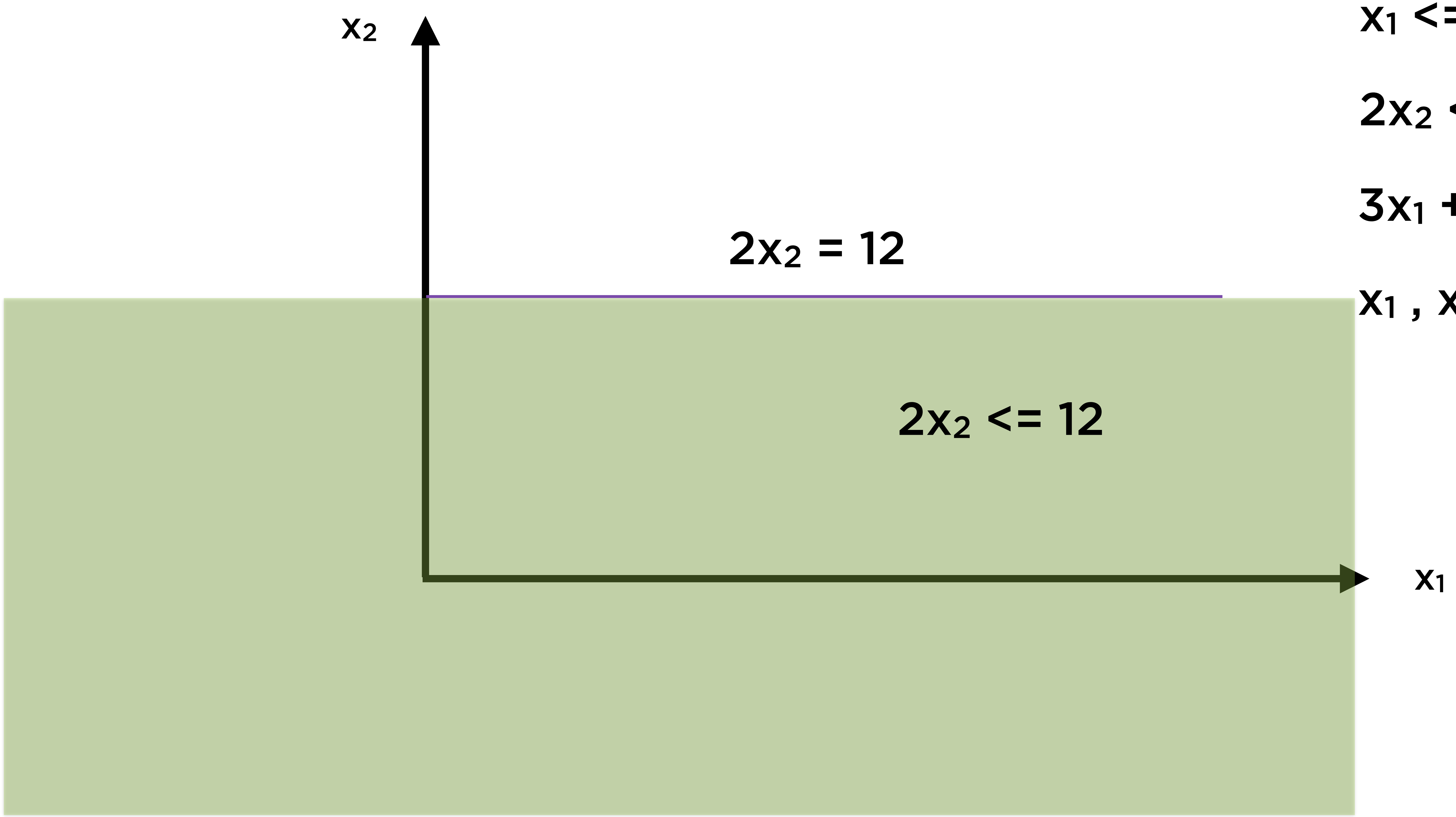
Constraints in Space



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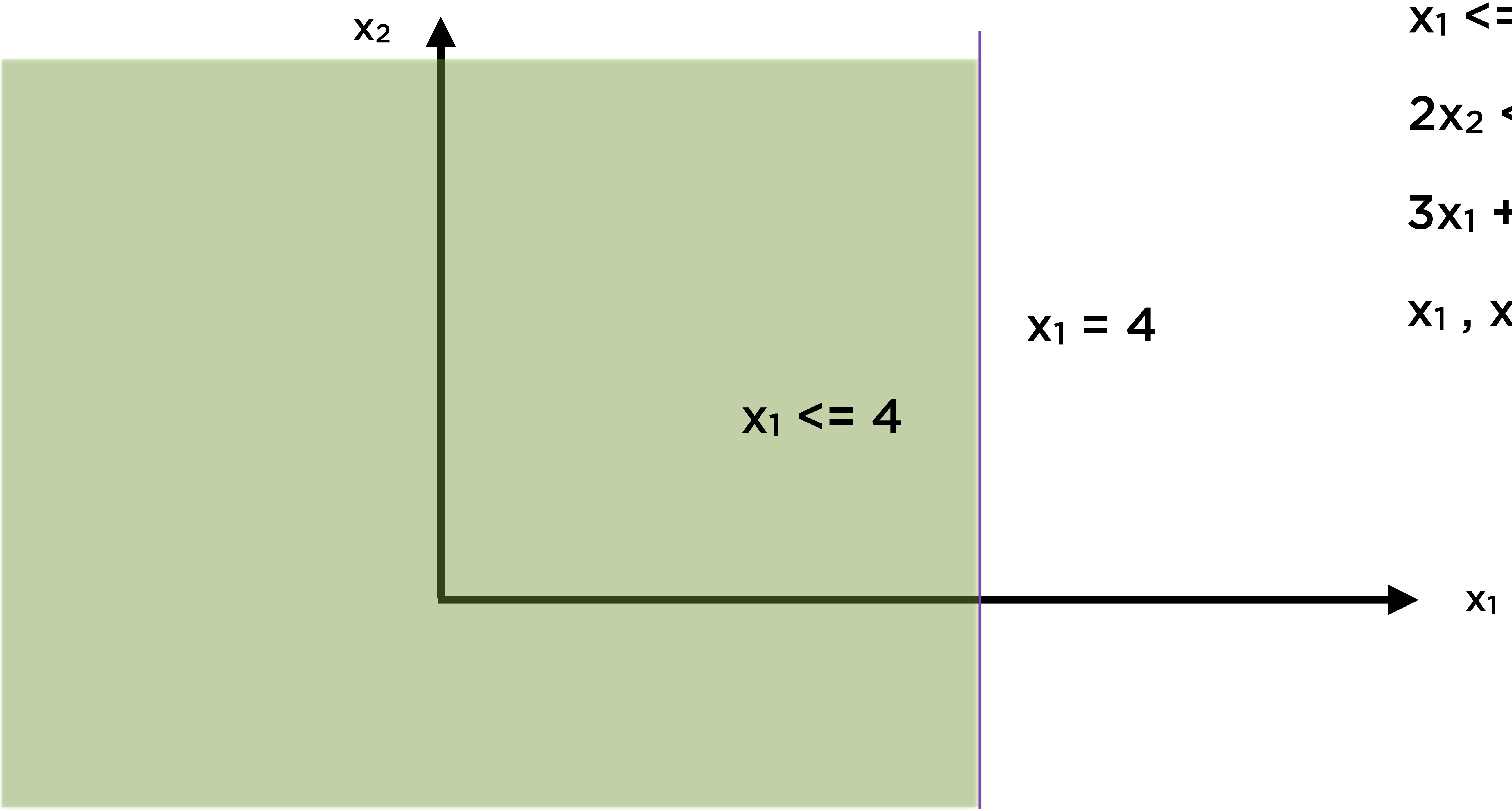
Constraints in Space



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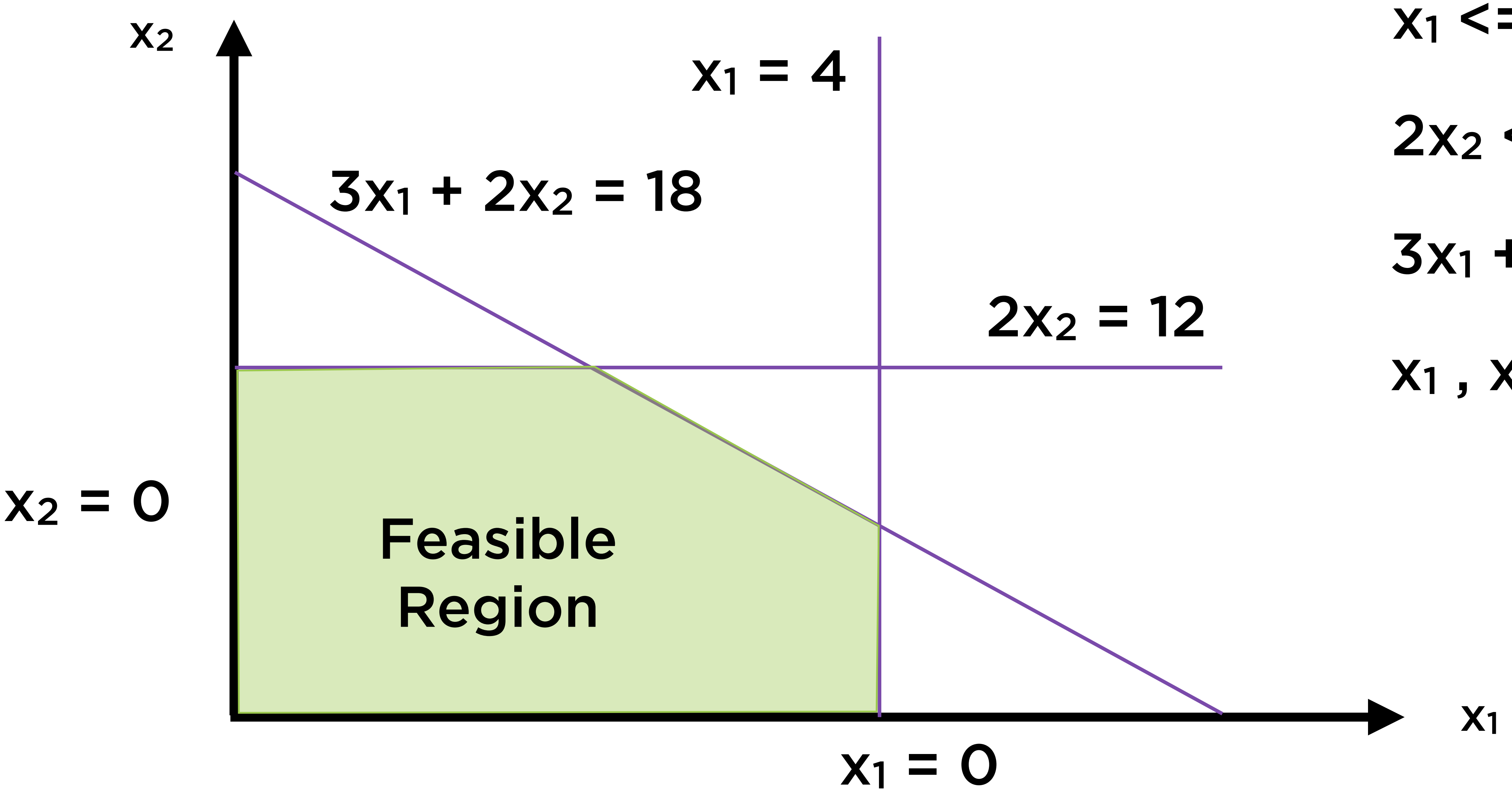
Constraints in Space



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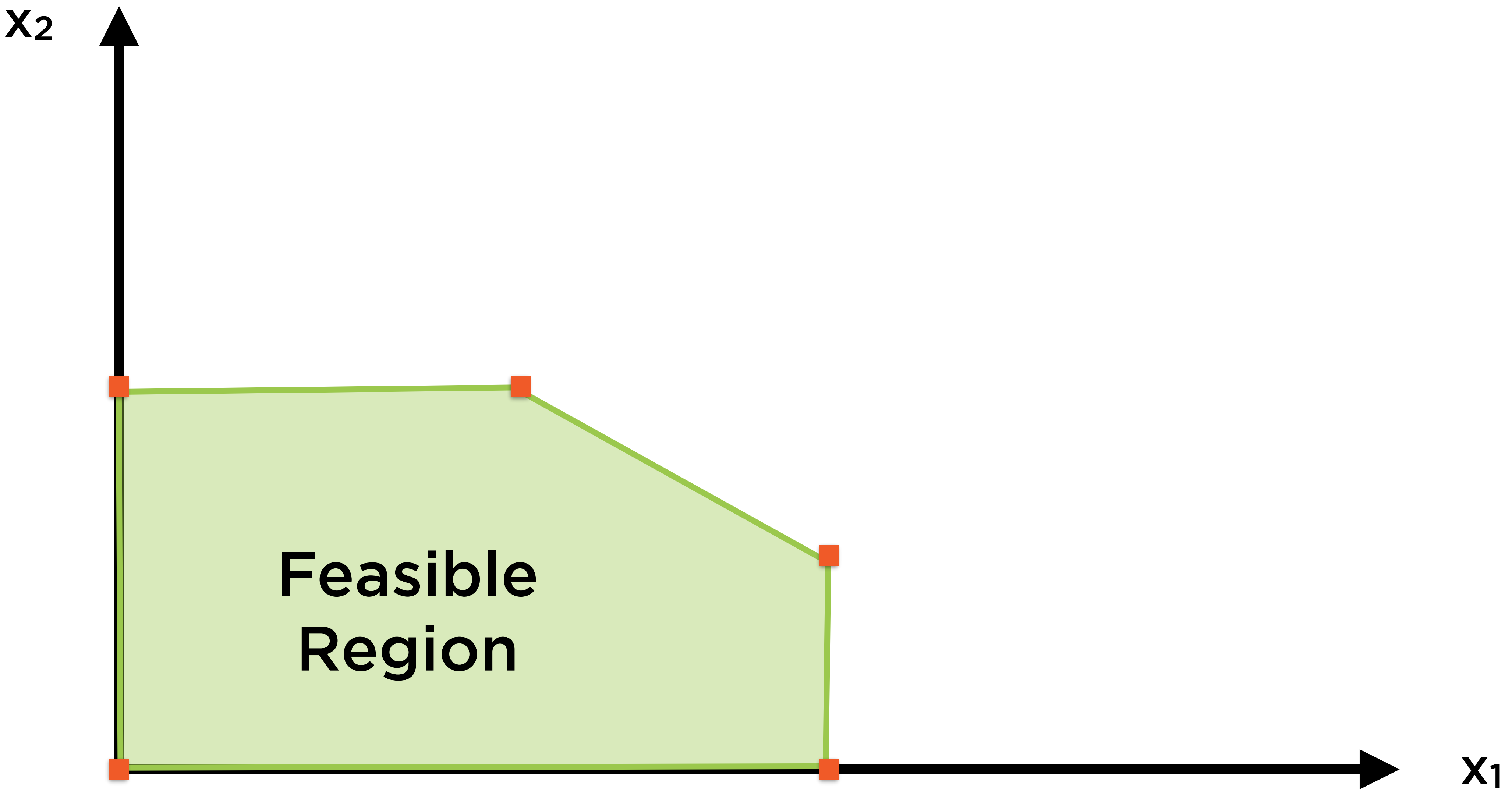
$x_1 , x_2 >= 0$

Each constraint bounds the feasible region

Constraints in Space



The optimal solution will always* be a corner point of this feasible region

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Linear Programming Problem Formulation

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

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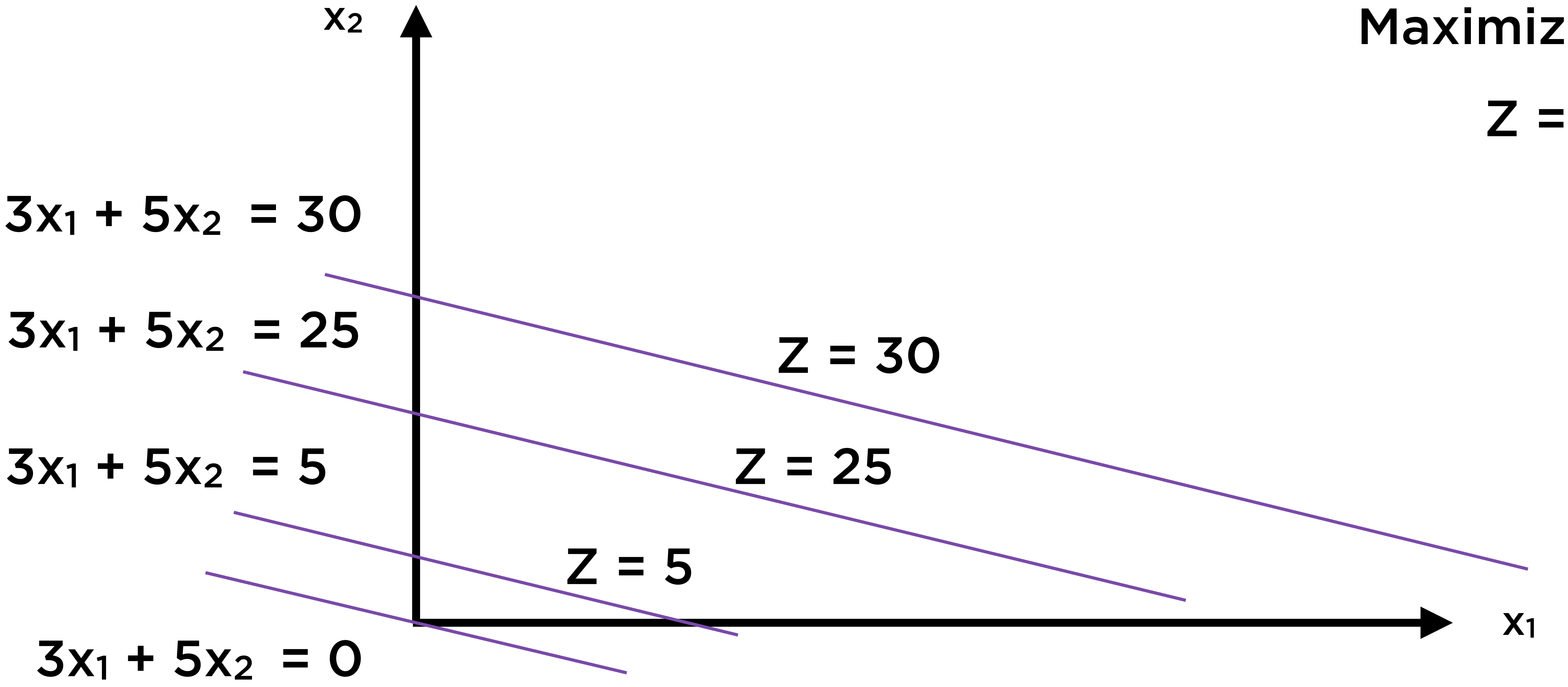
Objective Function in Space

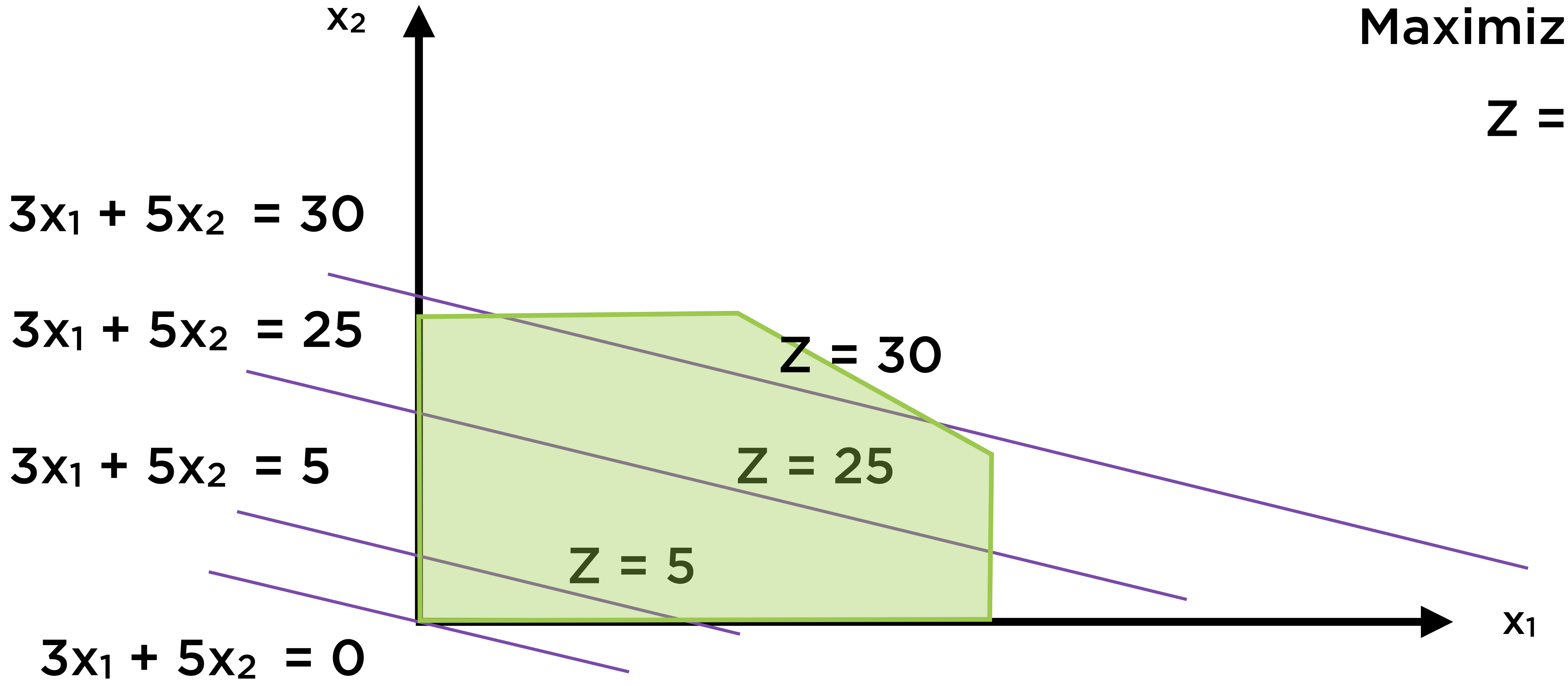
Maximize

$Z = 3x_1 + 5x_2$



An infinite number of lines exist, each with a different value of the objective function

Objective Function in Space

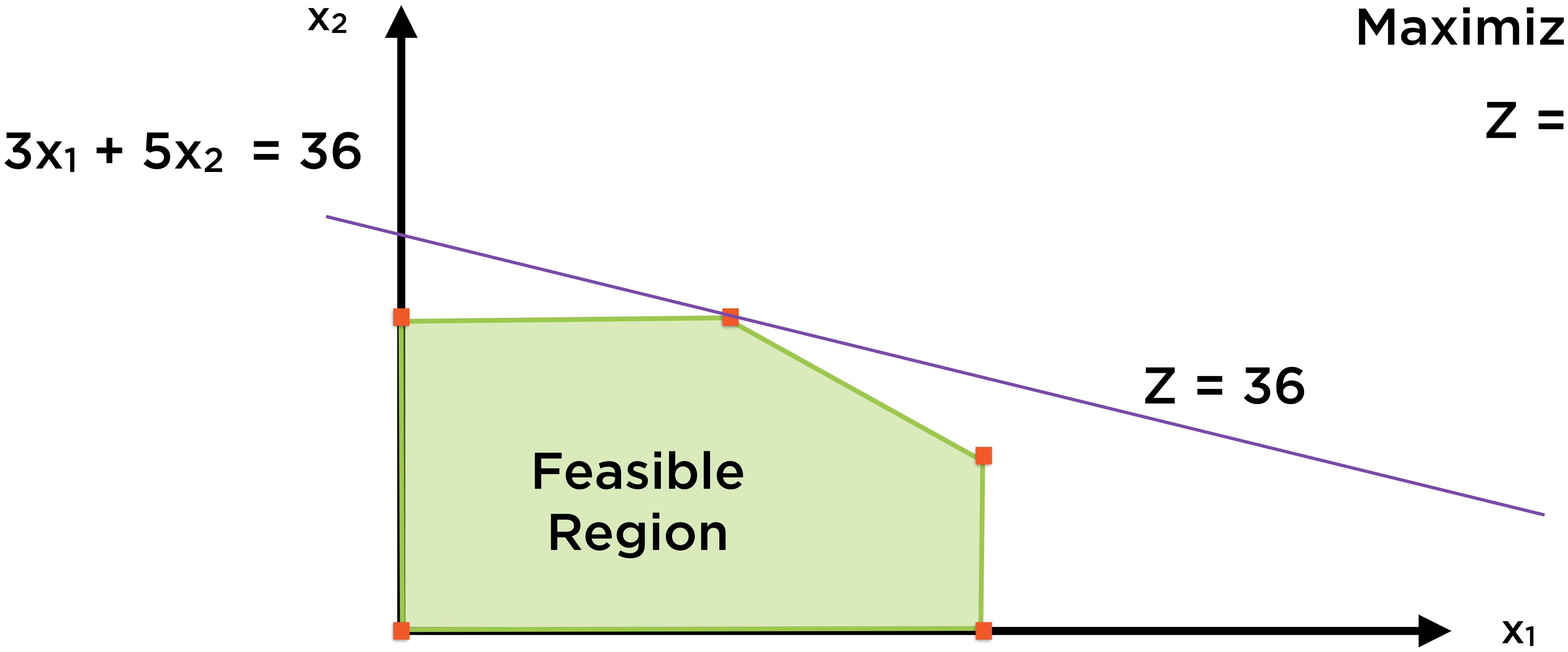


Maximize

$Z = 3x_1 + 5x_2$

Find the right-most such line that intersects the feasible region

Optimal Solution in Space

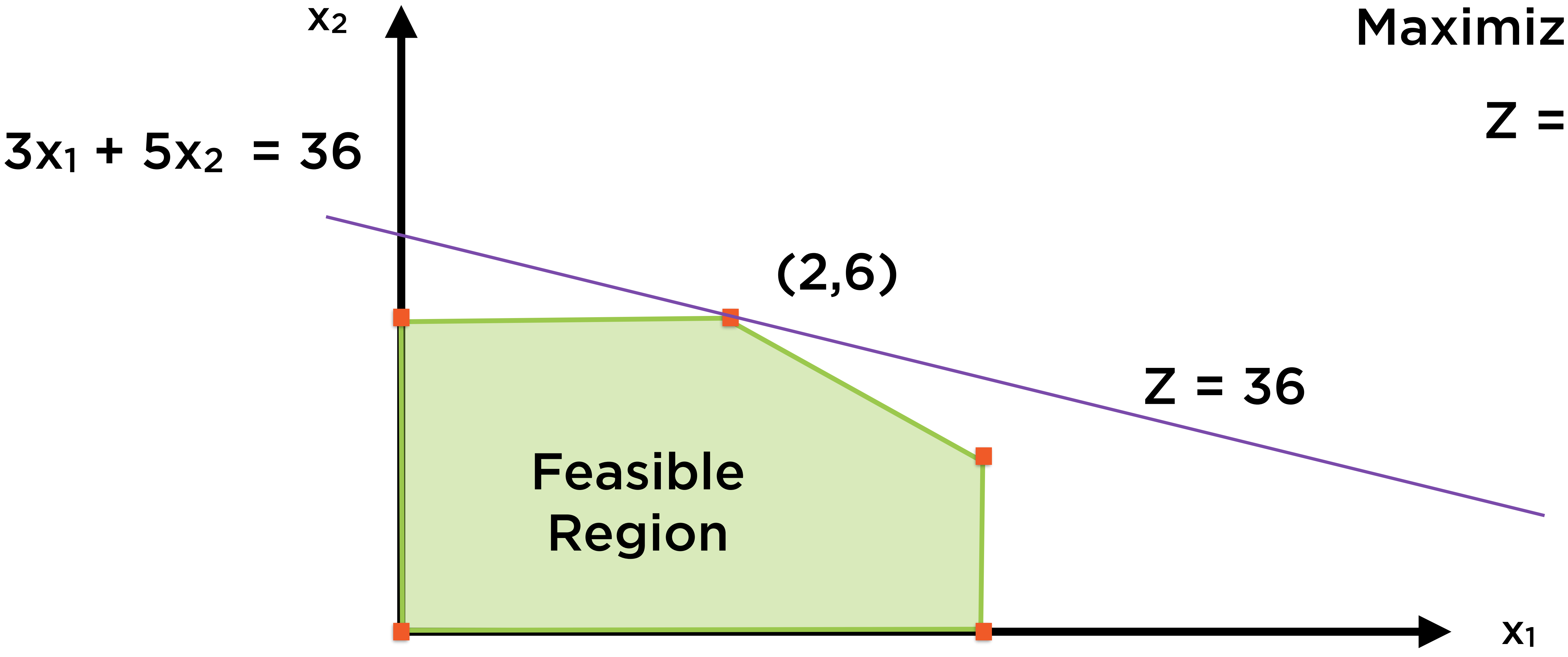


Maximize

$Z = 3x_1 + 5x_2$

Find the right-most such line that intersects the feasible region

Optimal Solution in Space

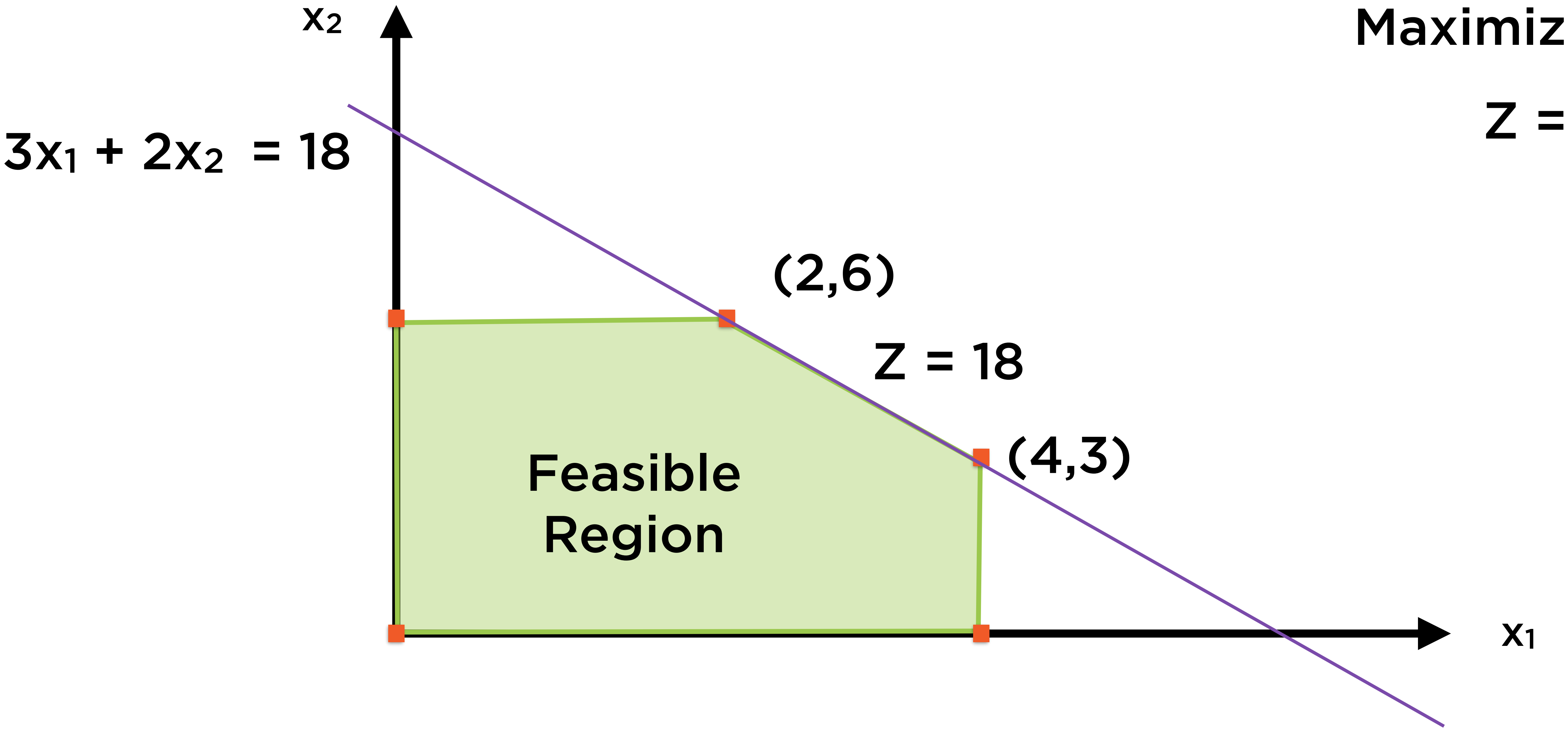


Maximize

$Z = 3x_1 + 5x_2$

The point of intersection represents the optimal solution

Multiple Optimum Solutions

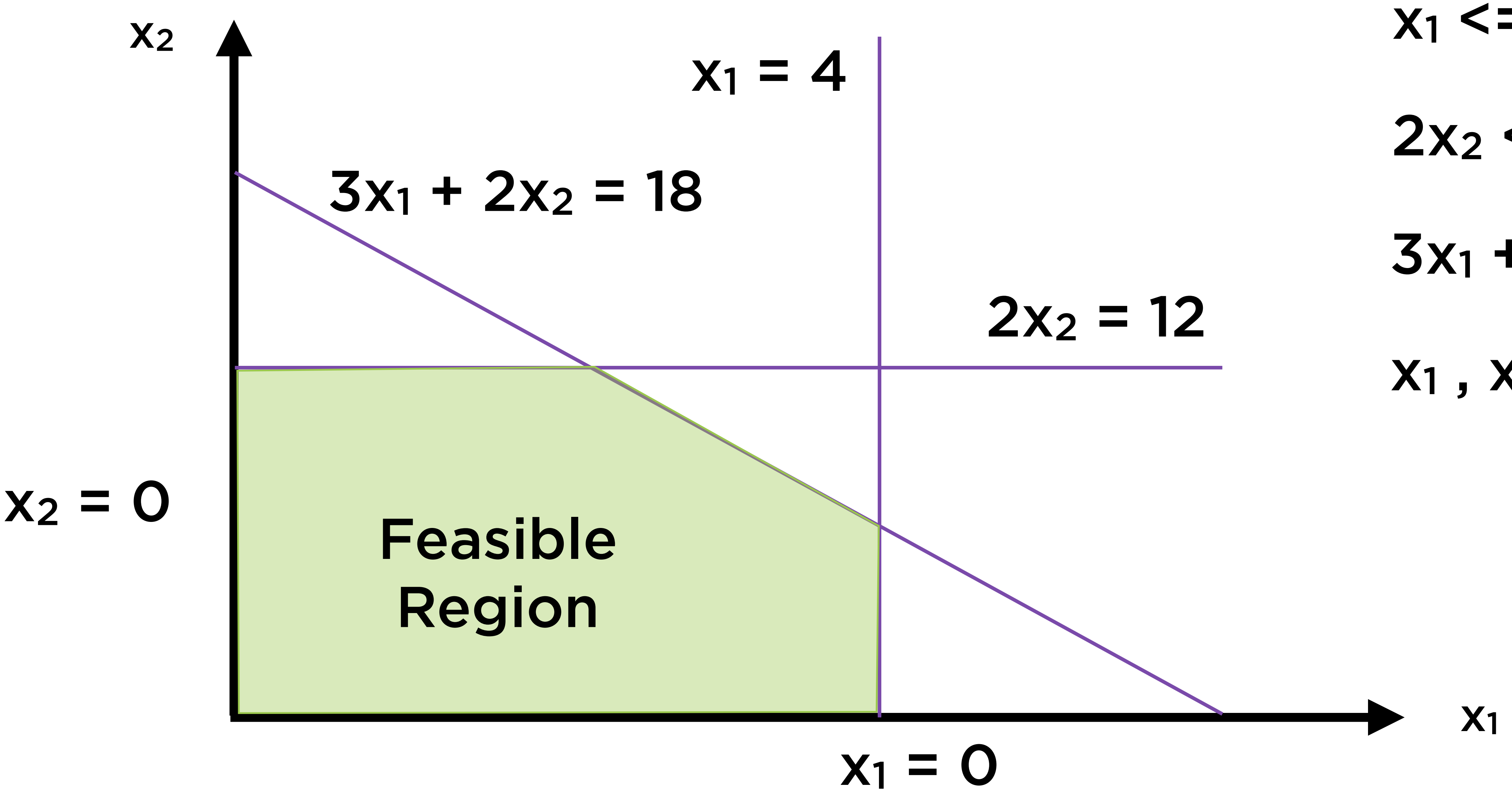


Maximize

$Z = 3x_1 + 2x_2$

Here the entire line segment contains optimal solutions

Constraints in Space



$x_1 <= 4$

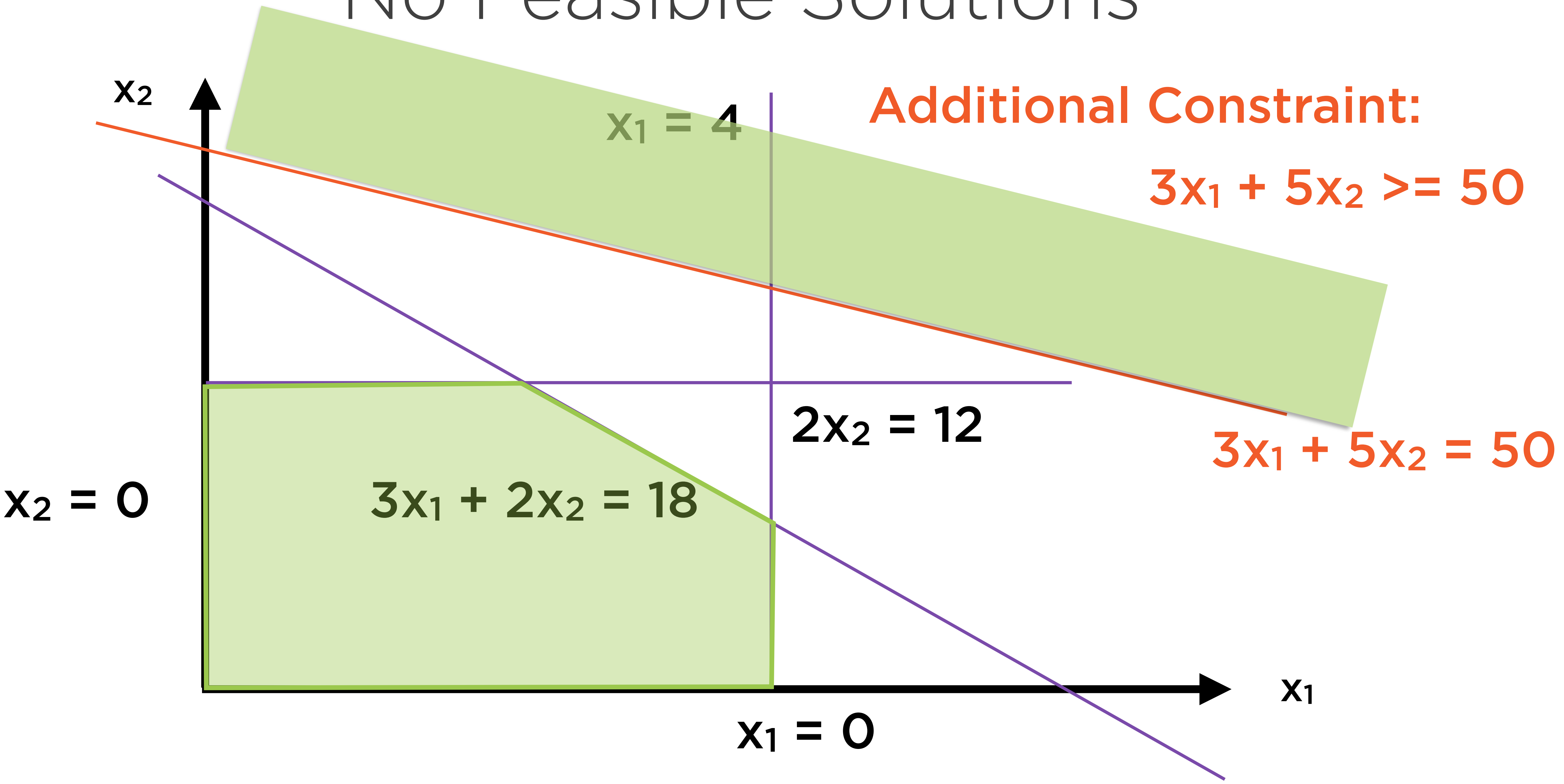
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

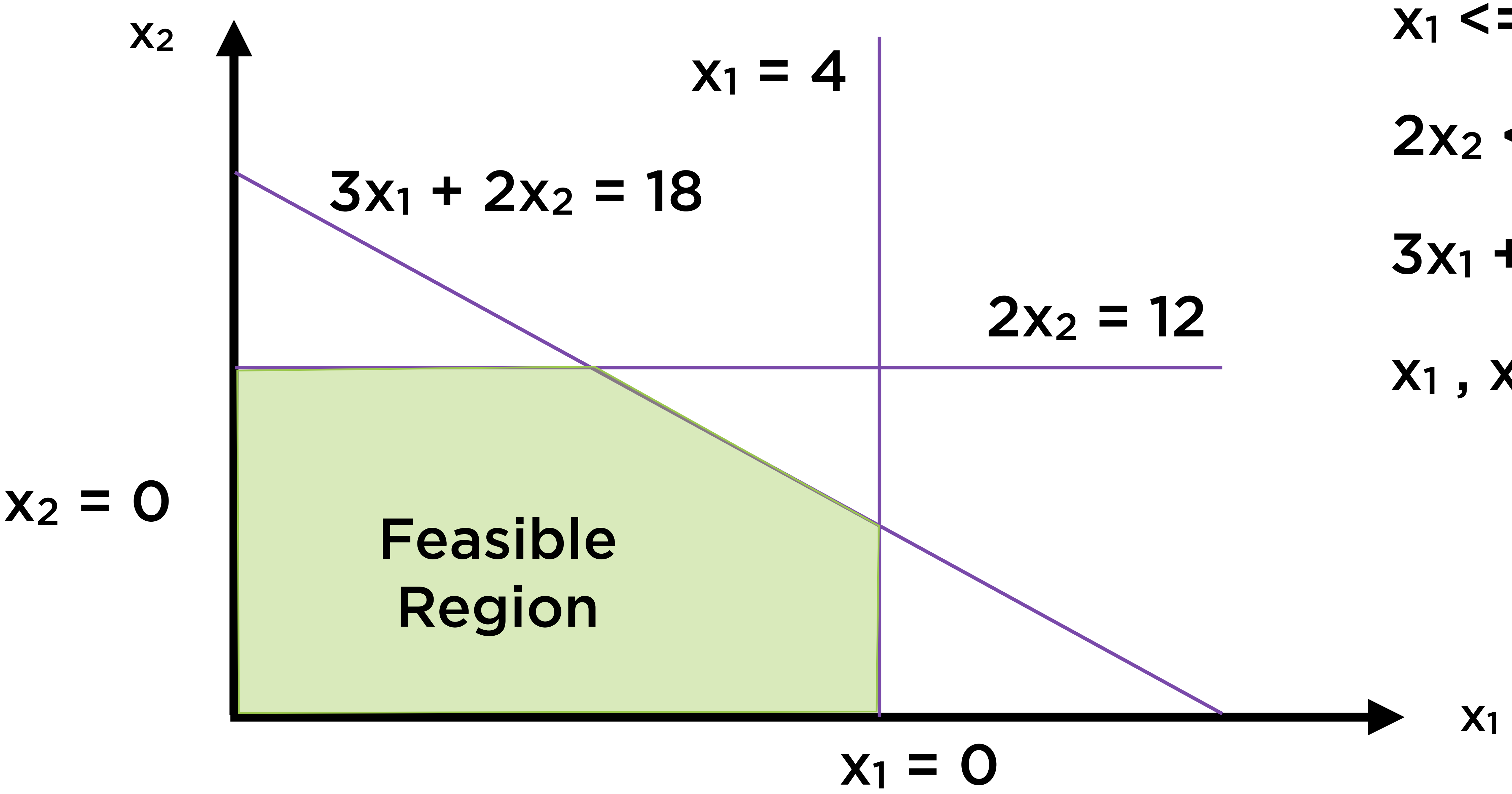
Each constraint bounds the feasible region

No Feasible Solutions



The intersection set of all feasible regions is the empty set

Constraints in Space



$x_1 <= 4$

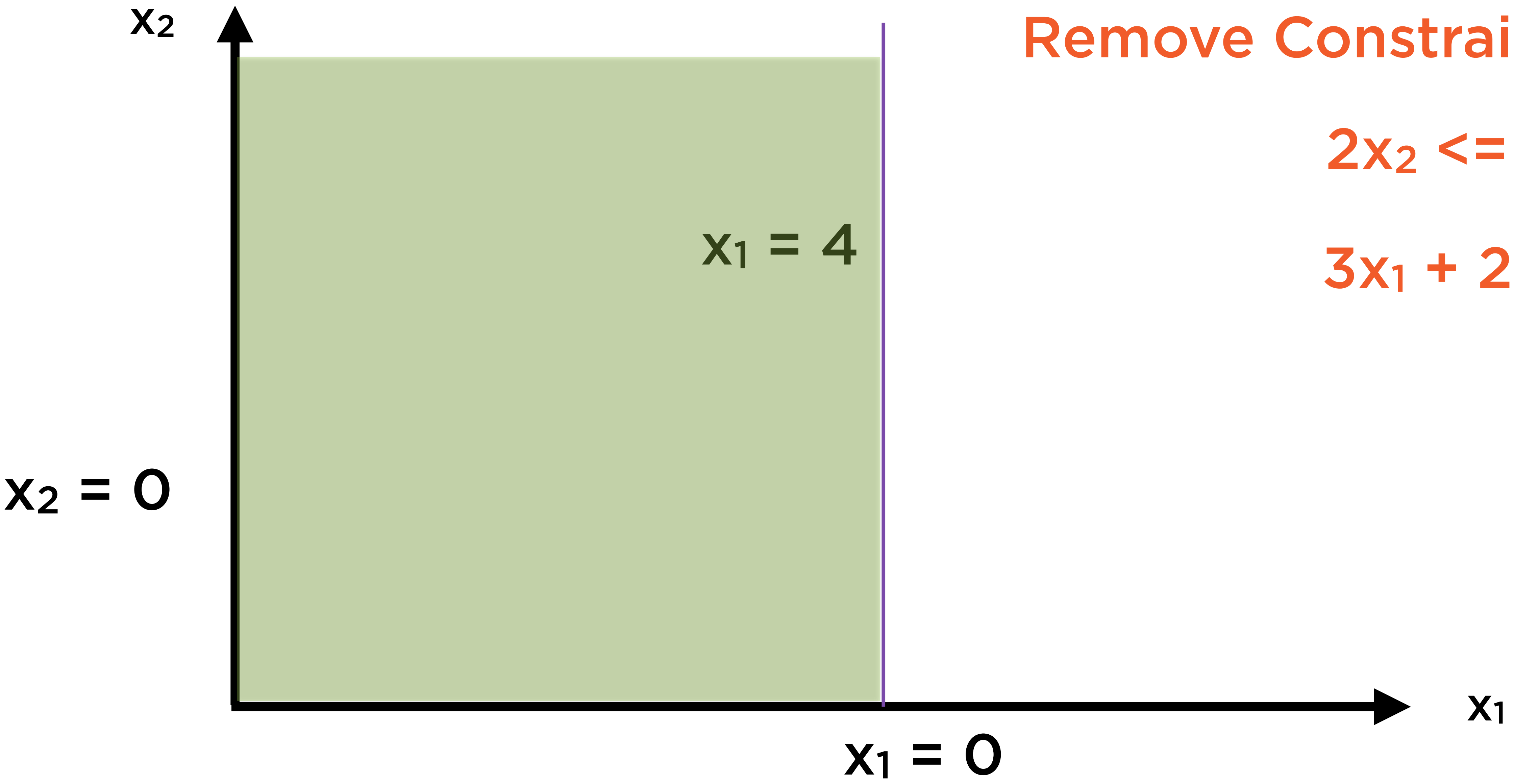
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Each constraint bounds the feasible region

No Optimal Solution



Remove Constraints

$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

Unbounded objective function (profits can increase to infinity)

Simplex Method: Intuition

The Simplex Method

Powerful

Easily extends to large numbers of variables, constraints

Versatile

Extends to sensitivity analysis and quadratic programming

Programmable

Easy to implement in software

Linear Programming Problem Formulation

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

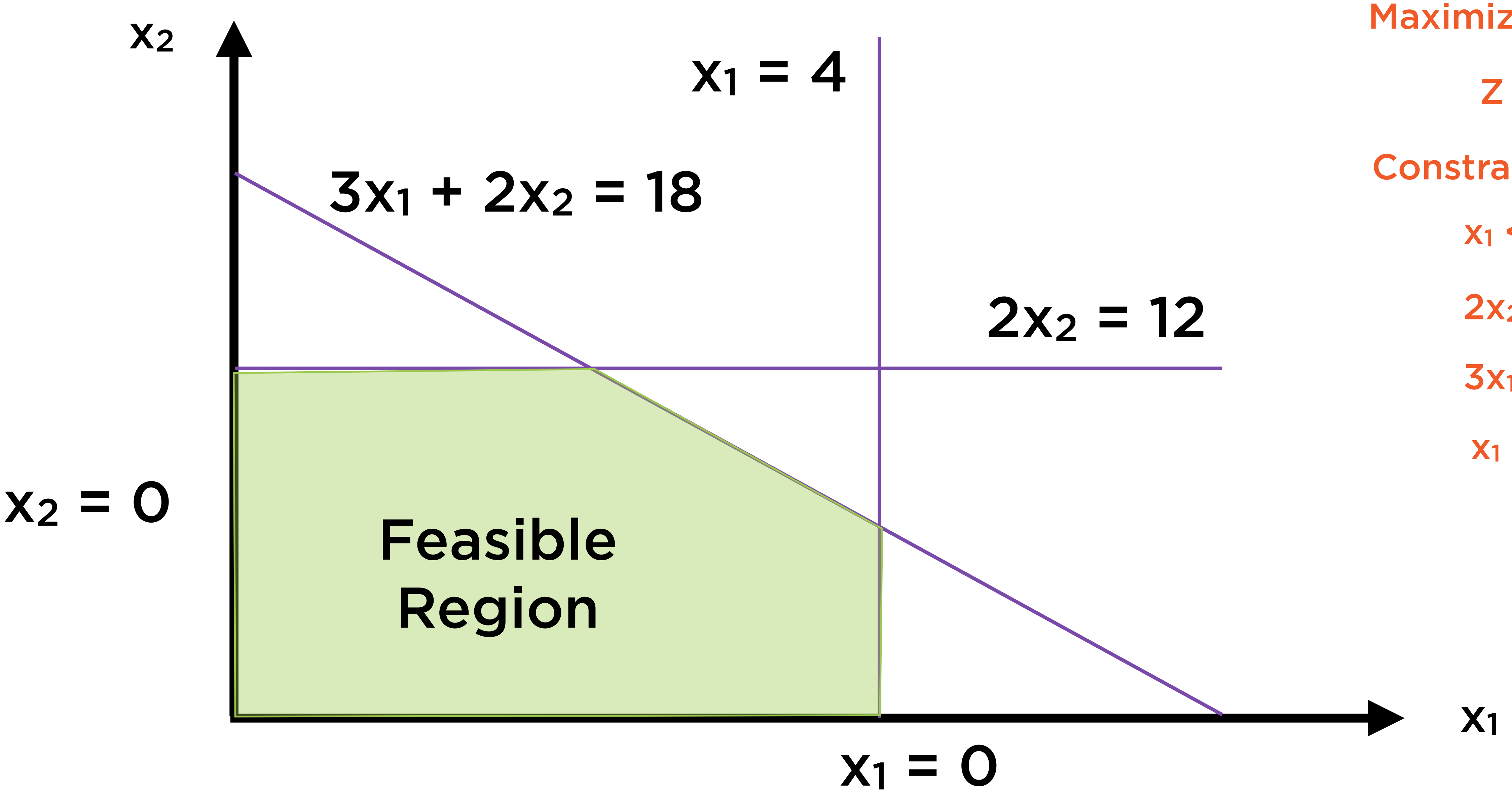
$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

$$x_1 , x_2 \ge 0$$ (Non-negativity constraints)

Constraints in Space



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

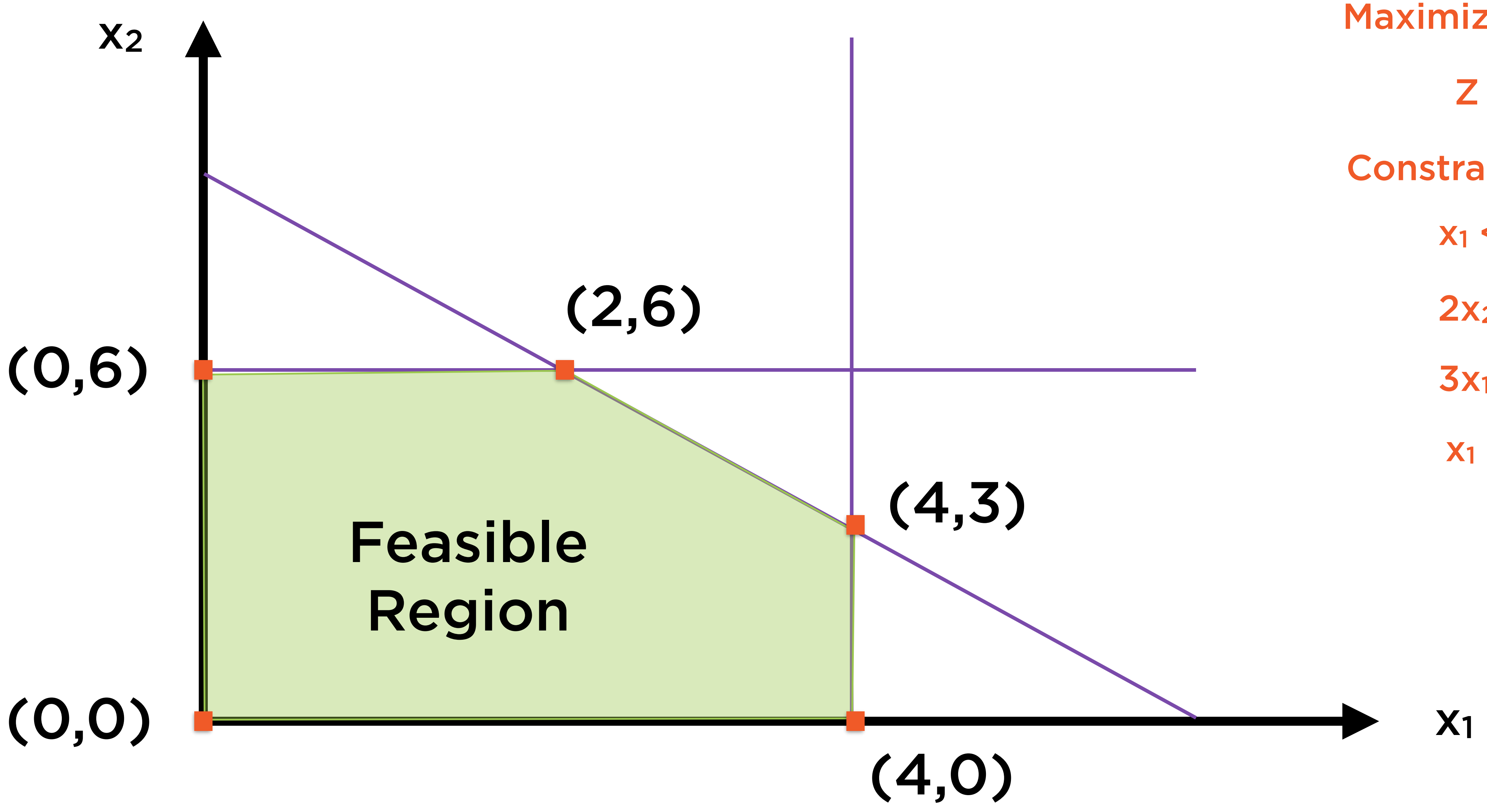
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Intersection points of the constraint lines are called corner points

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

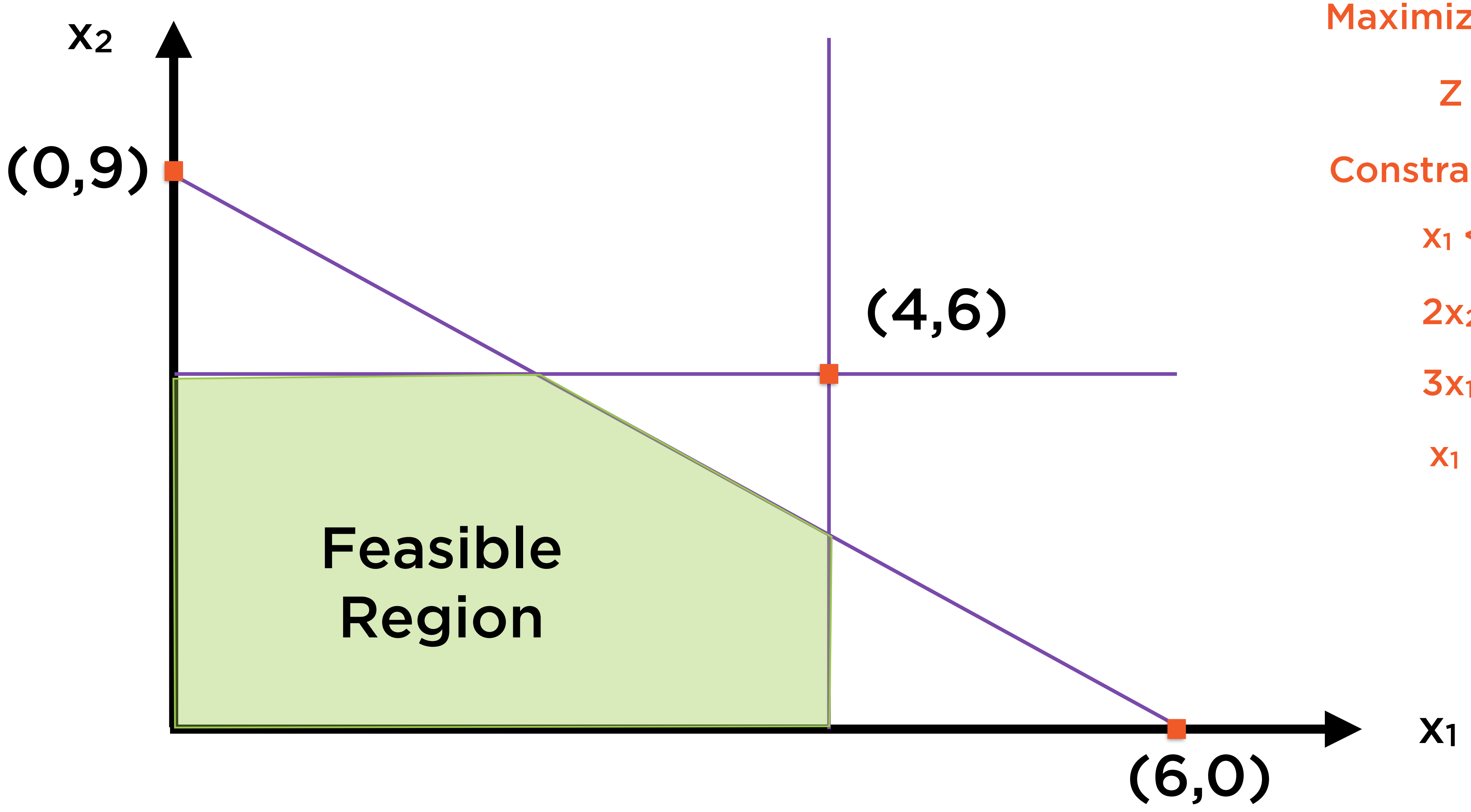
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Some corner points represent feasible solutions...

Corner-point Infeasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

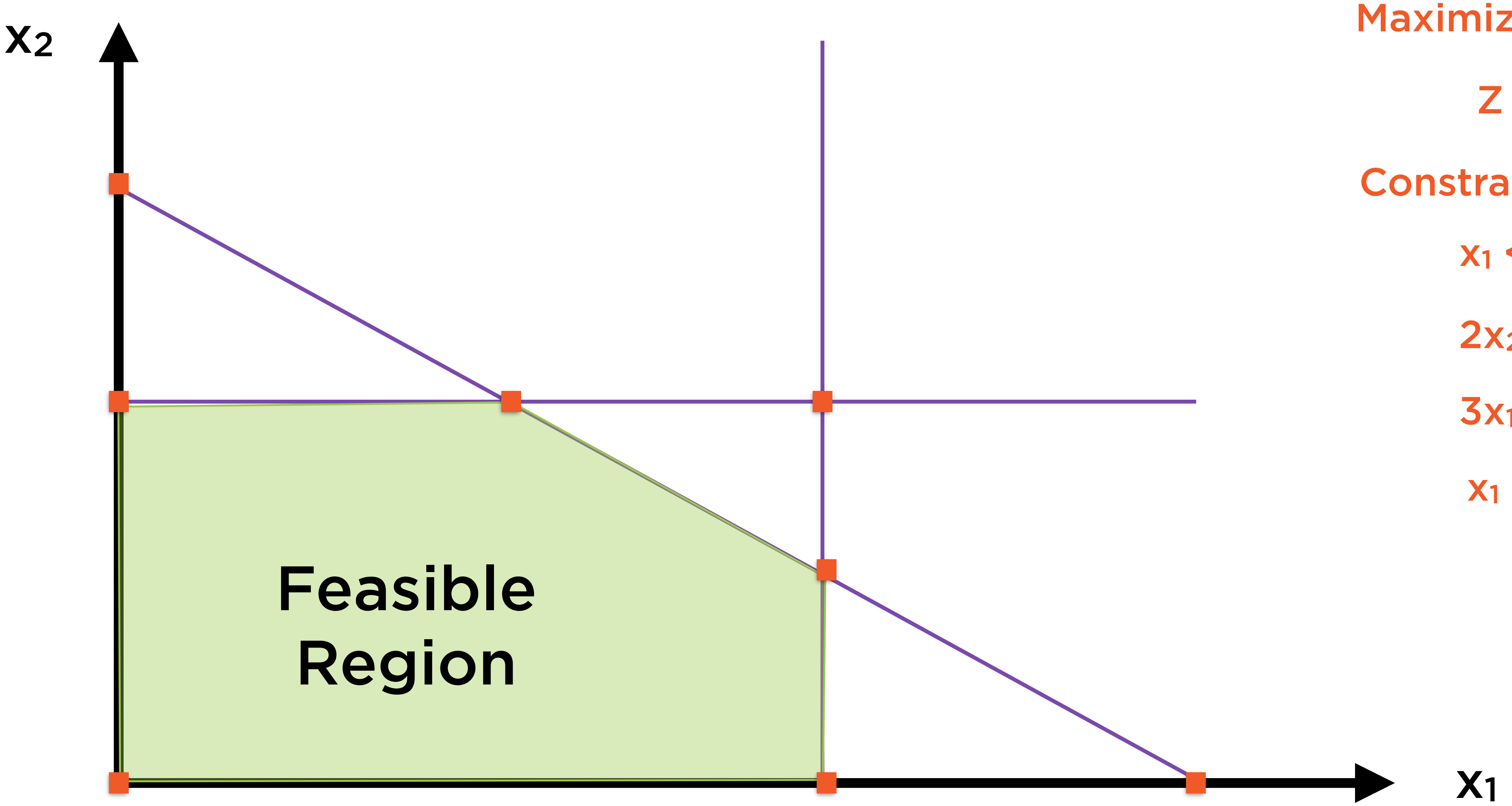
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

...While others do not.

All Corner-point Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

The optimal solution will always* be a corner point of the feasible region

The optimal solution will always* be a corner point of the feasible region

If a corner-point solution is “better” than any adjacent corner-point solution, it is the optimal solution*

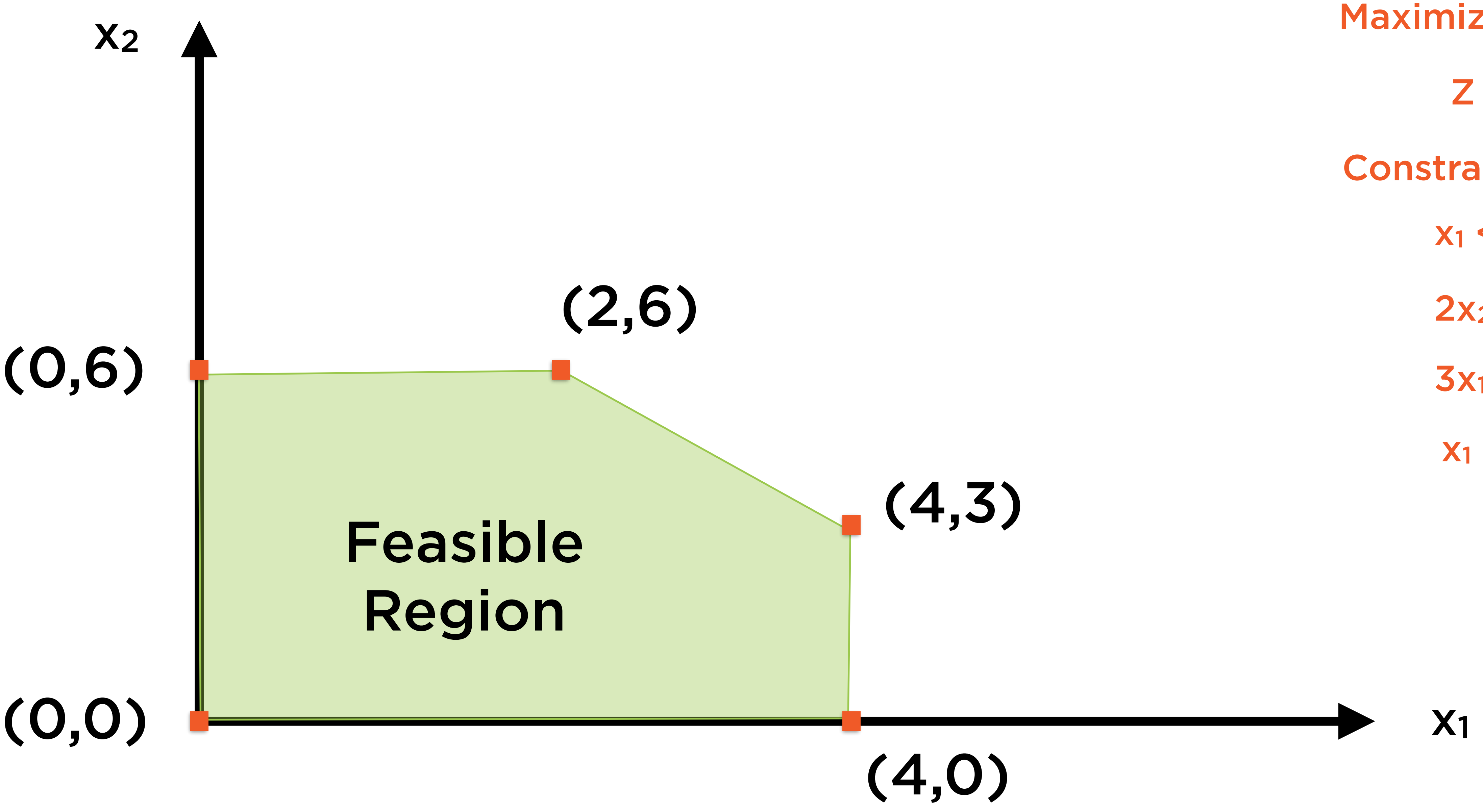
```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution
- Test for optimality
  - Not optimal, continue
  - Optimal, stop
- Keep iterating until we run out of corner-points

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

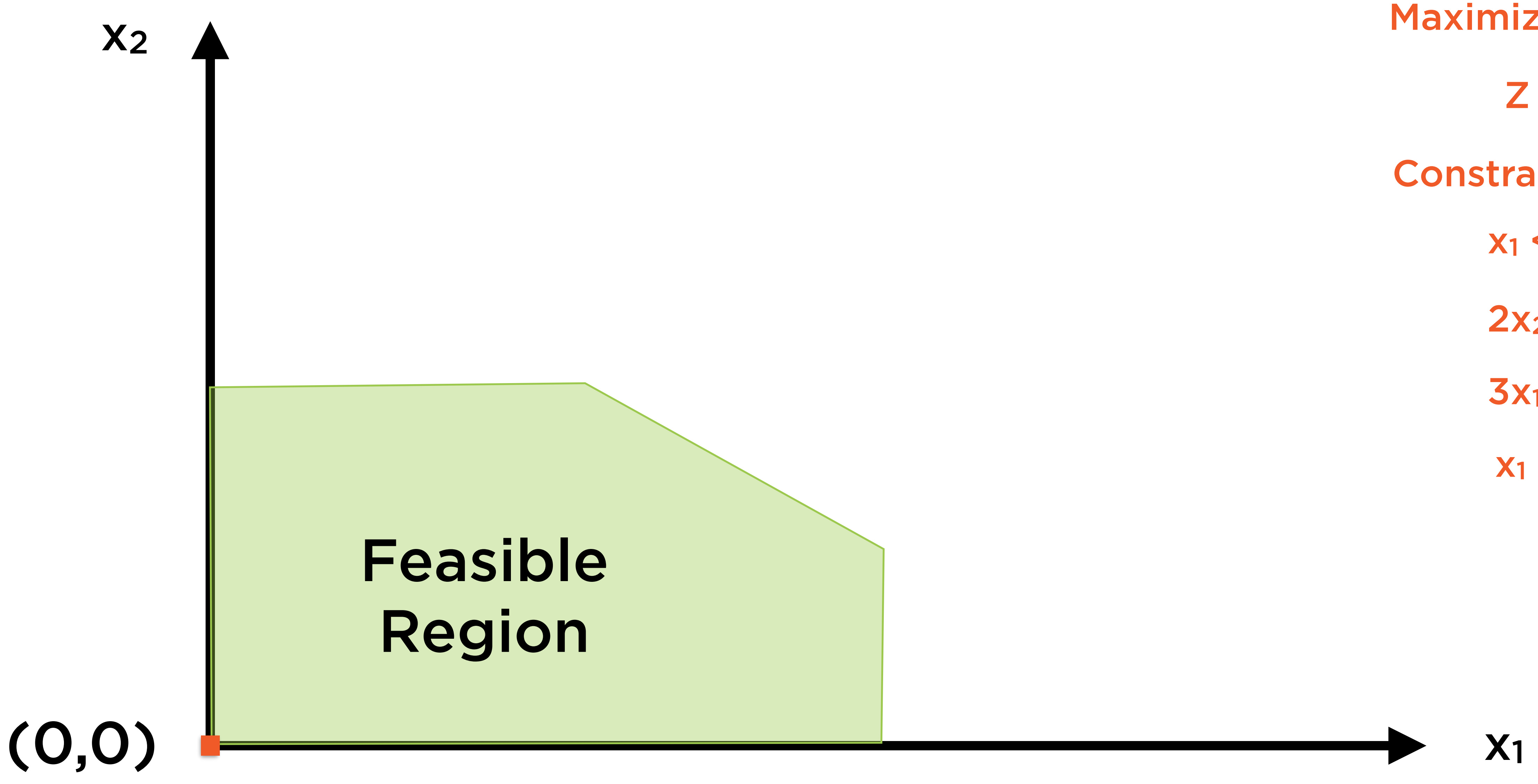
$x_1 <= 4$

$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$$x_1 <= 4$$

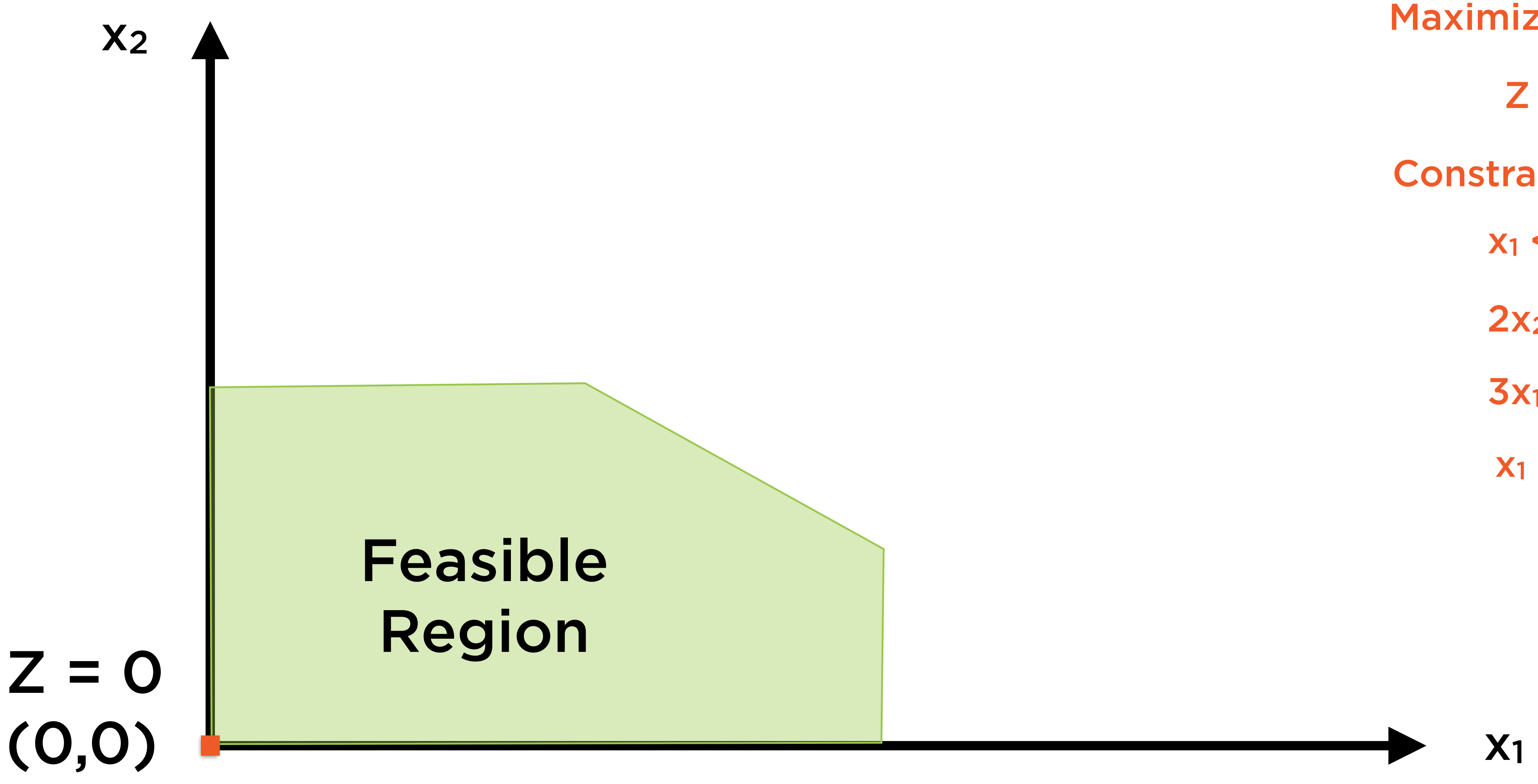
$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$$x_1 , x_2 >= 0$$

Initial guess = (0,0)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

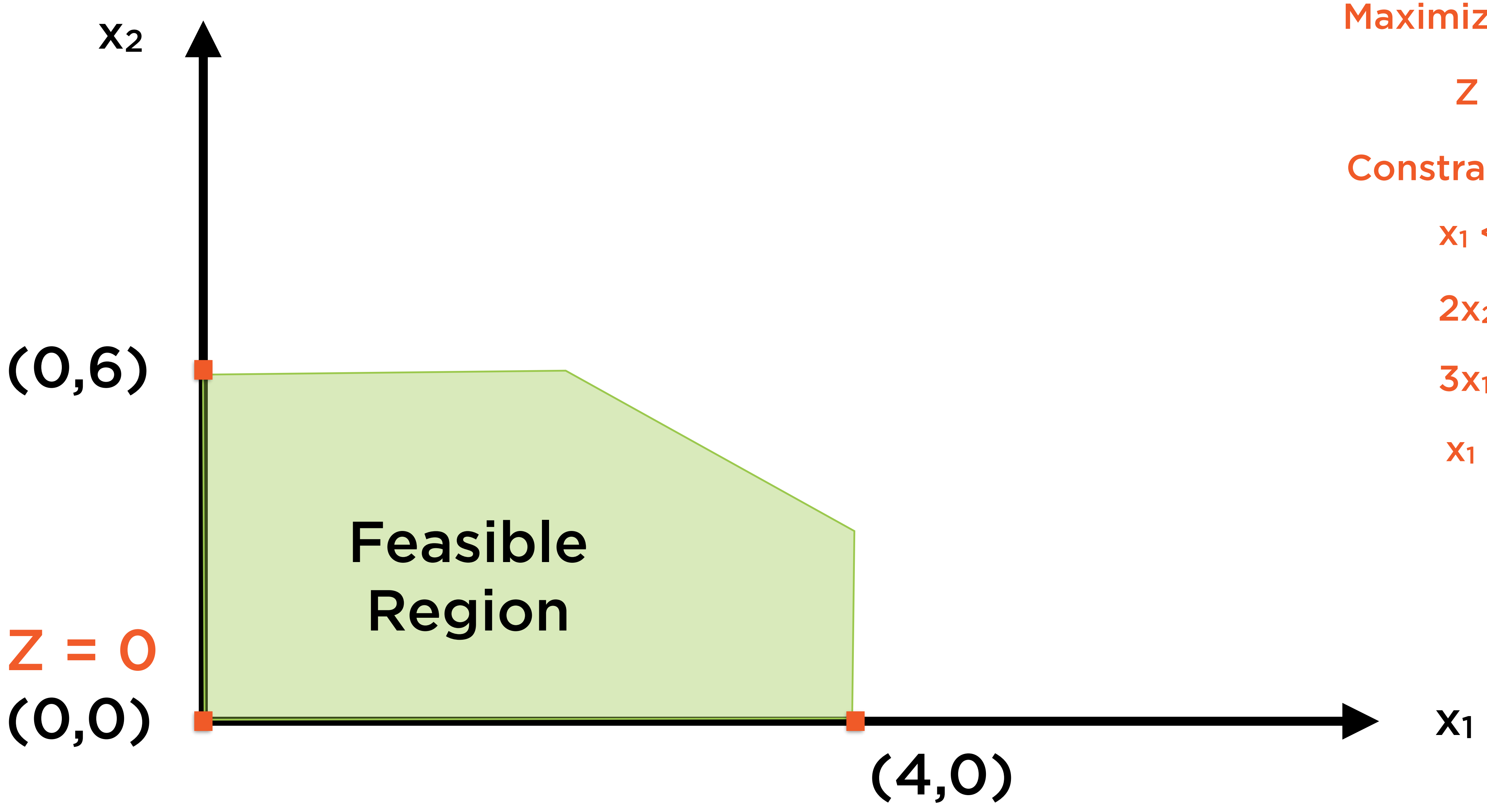
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Value of Z at (0,0) = 0

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

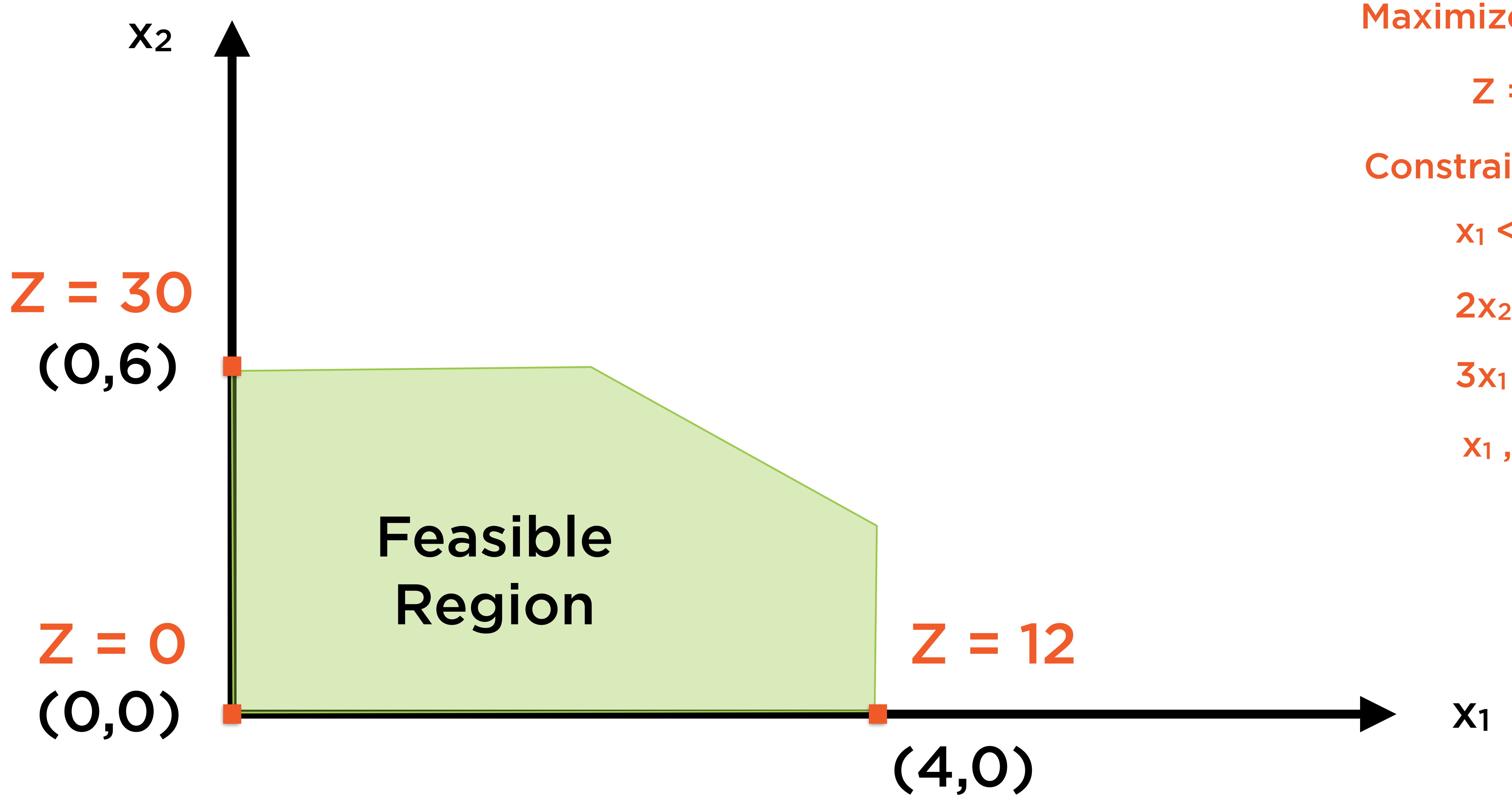
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,6) and (4,0)

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

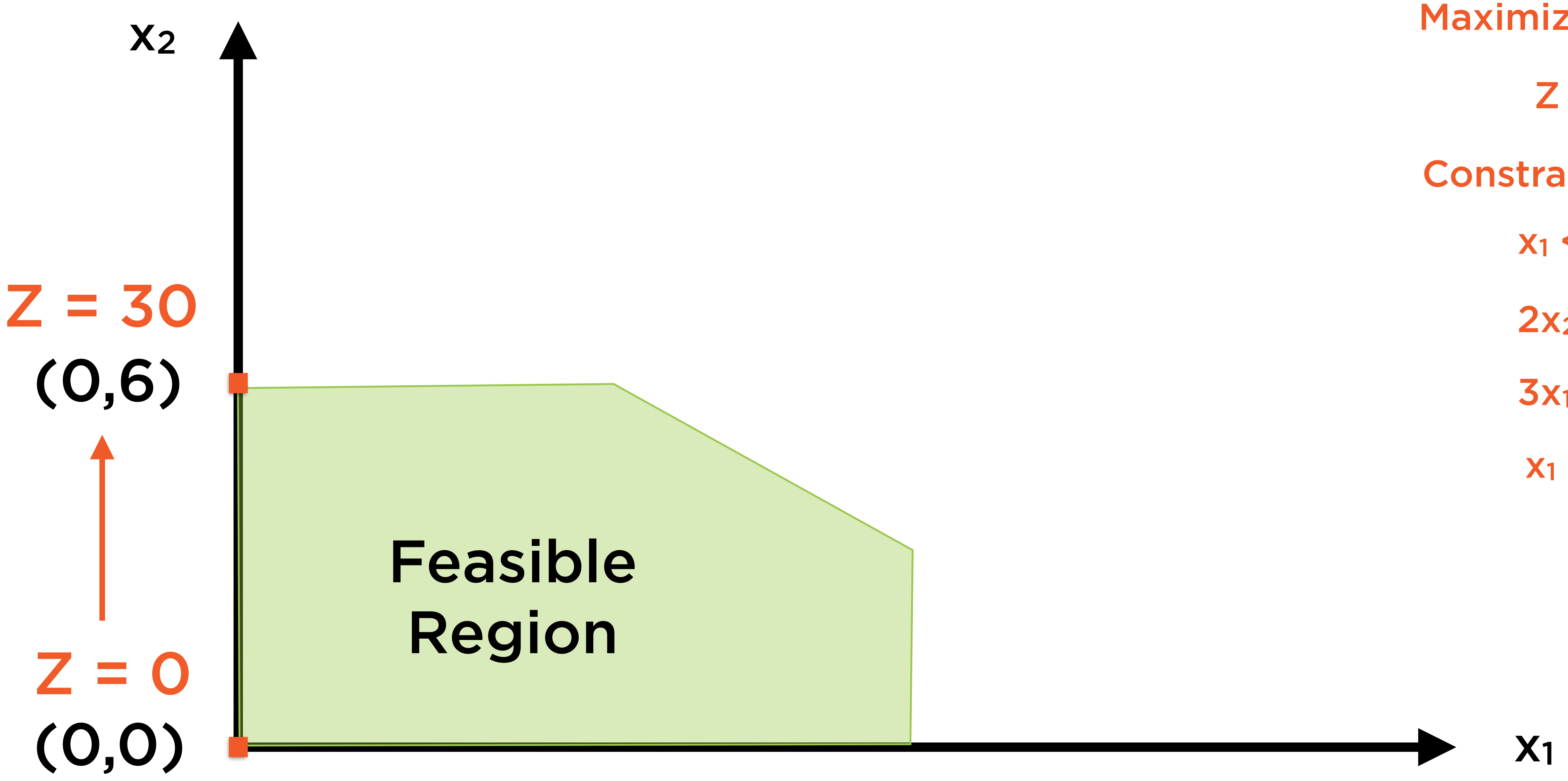
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

"Best" adjacent corner-point is (0,6)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

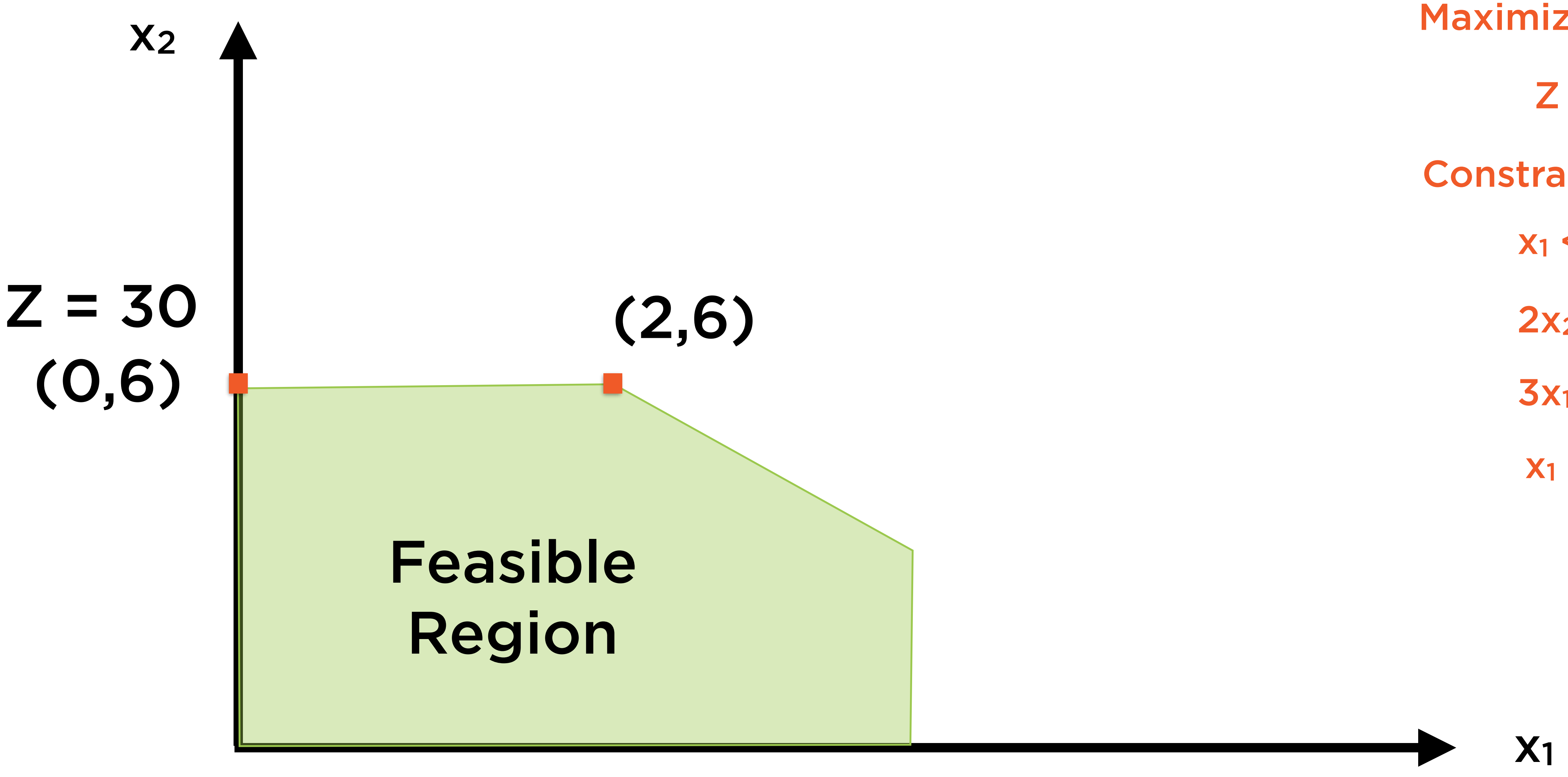
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Set (0,6) to be current solution

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

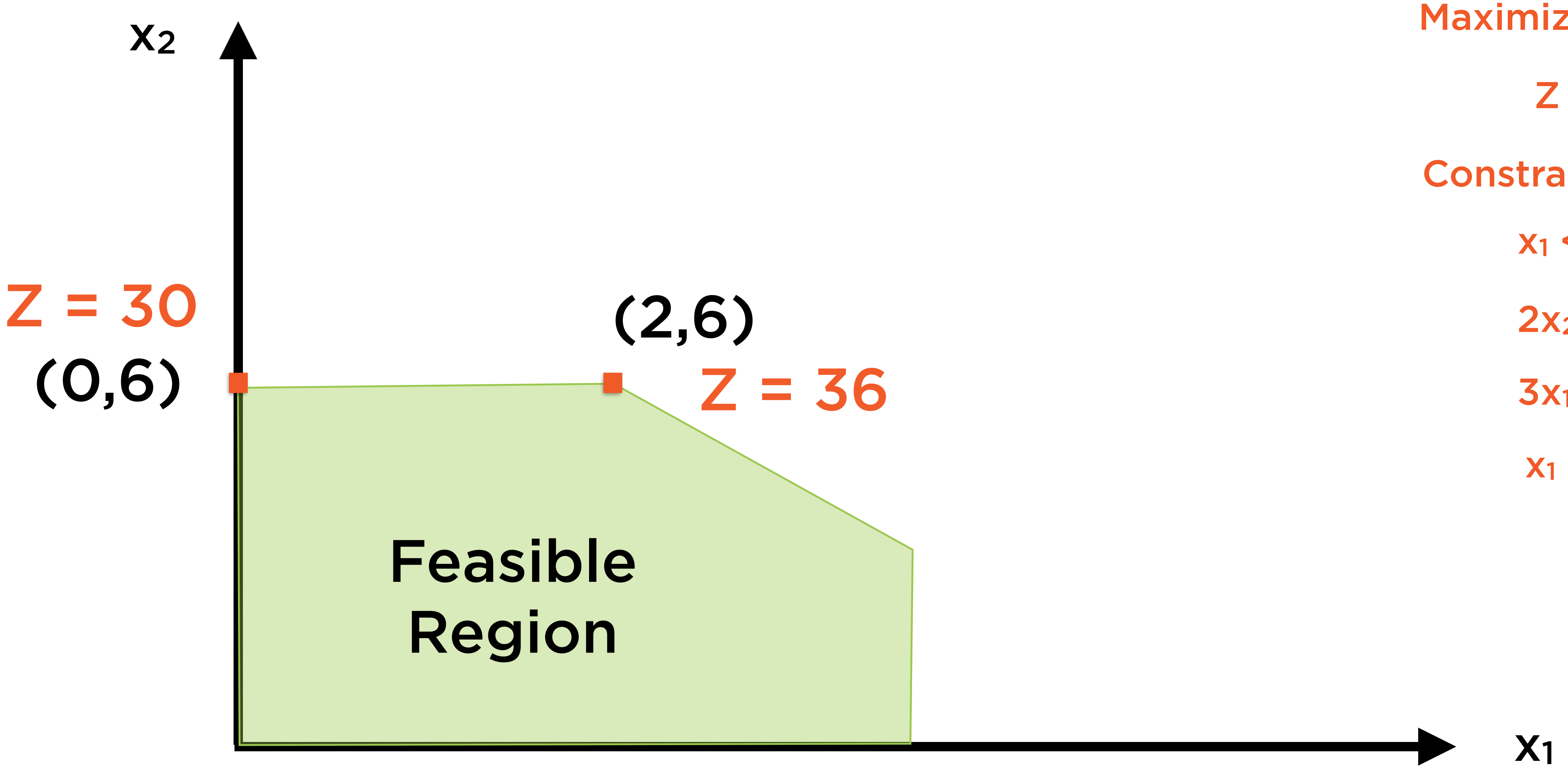
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,0) and (2,6), but we already know (0,0) is not better

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

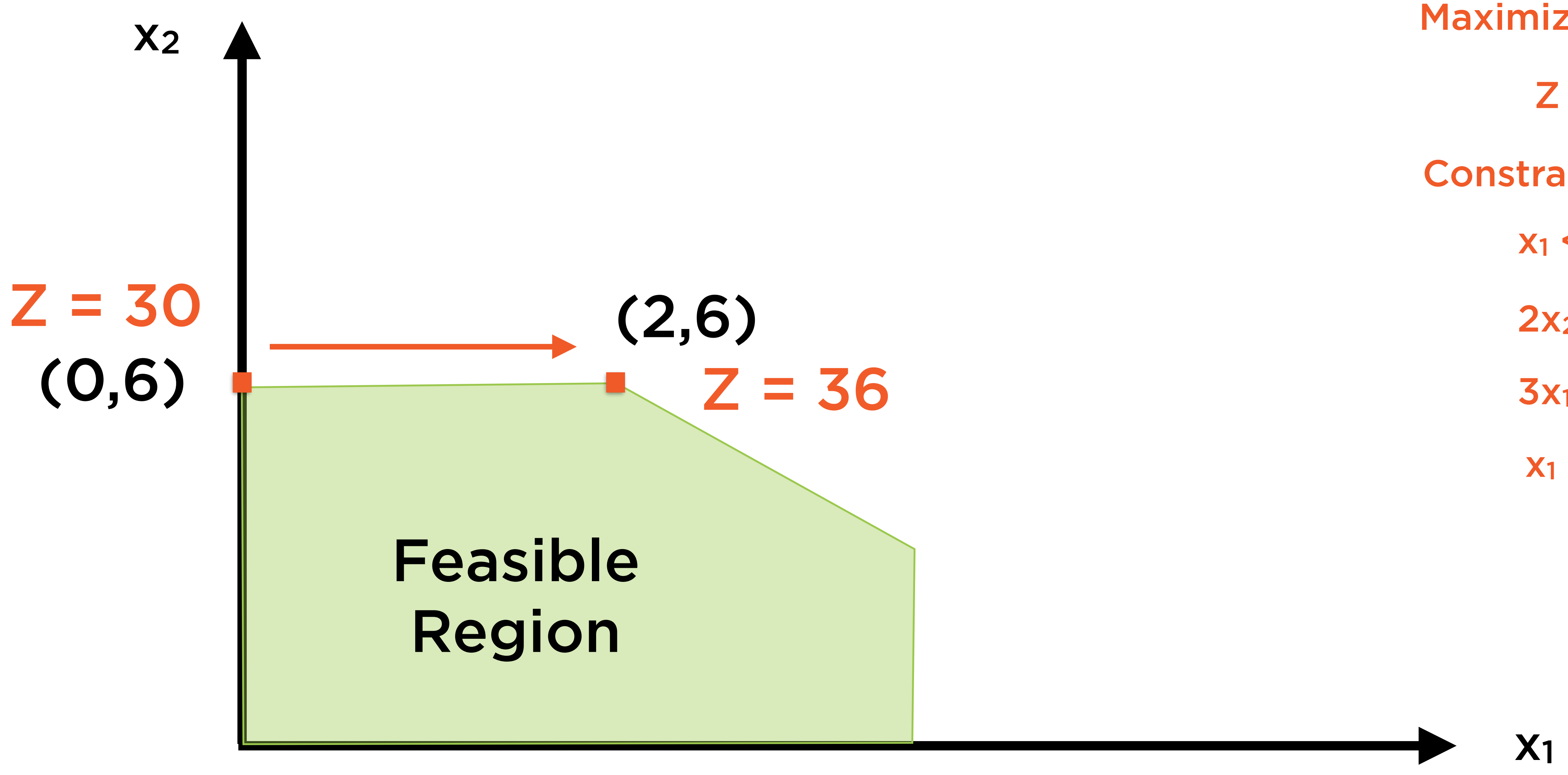
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

“Best” adjacent corner-point is (2,6)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

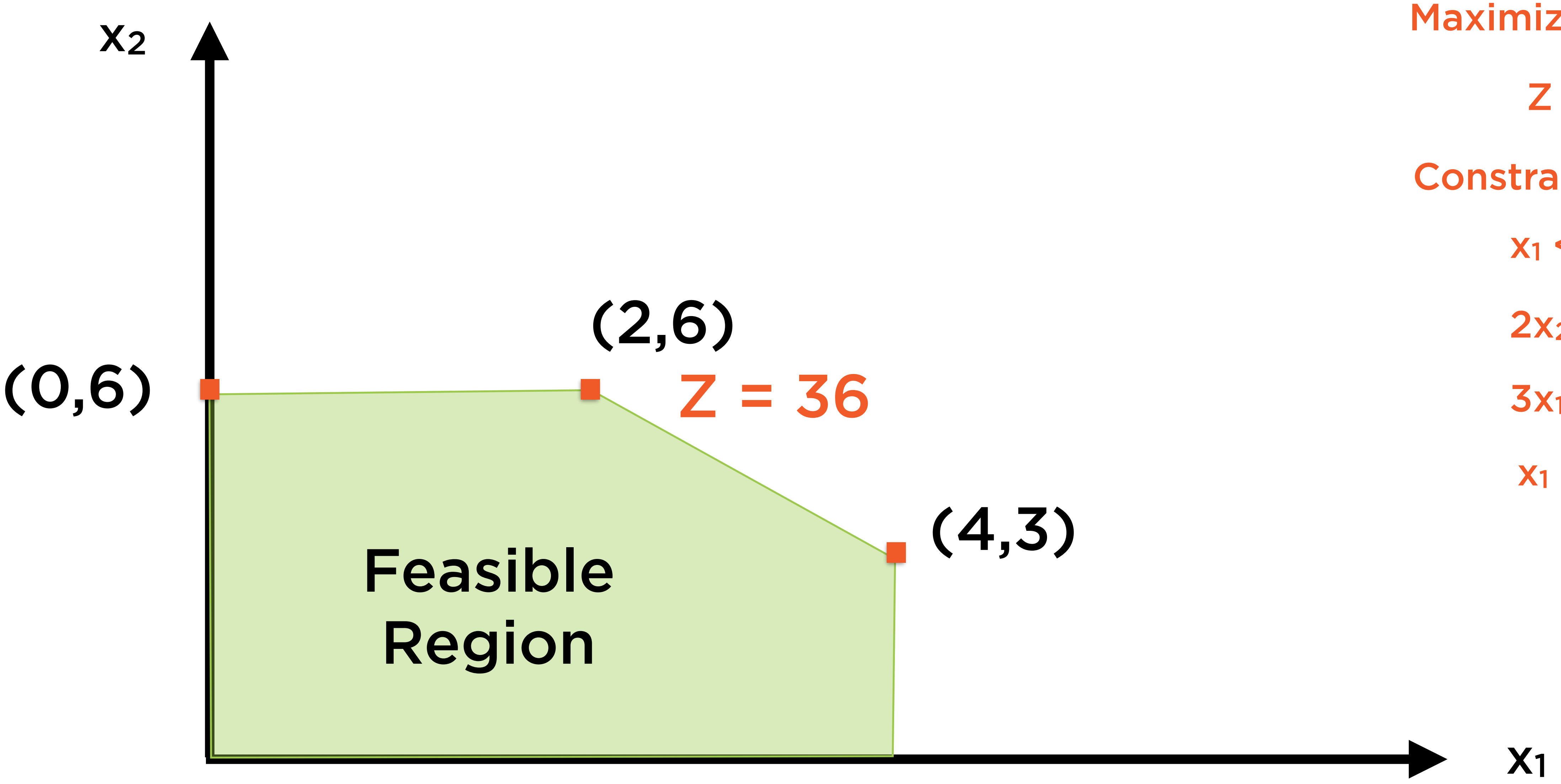
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Set (2,6) to be current solution

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

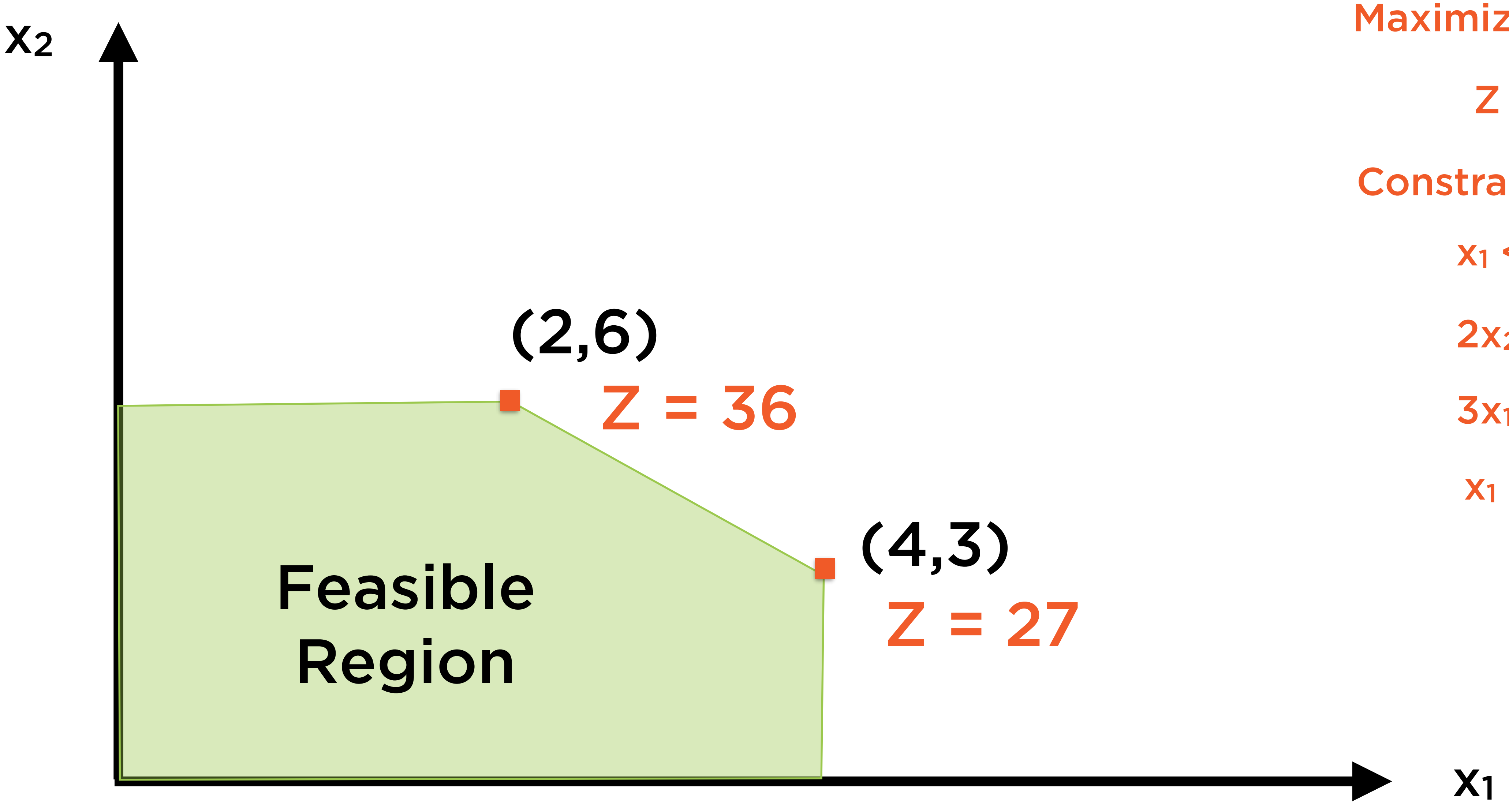
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,6) and (4,3), but we already know (0,6) is not better

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

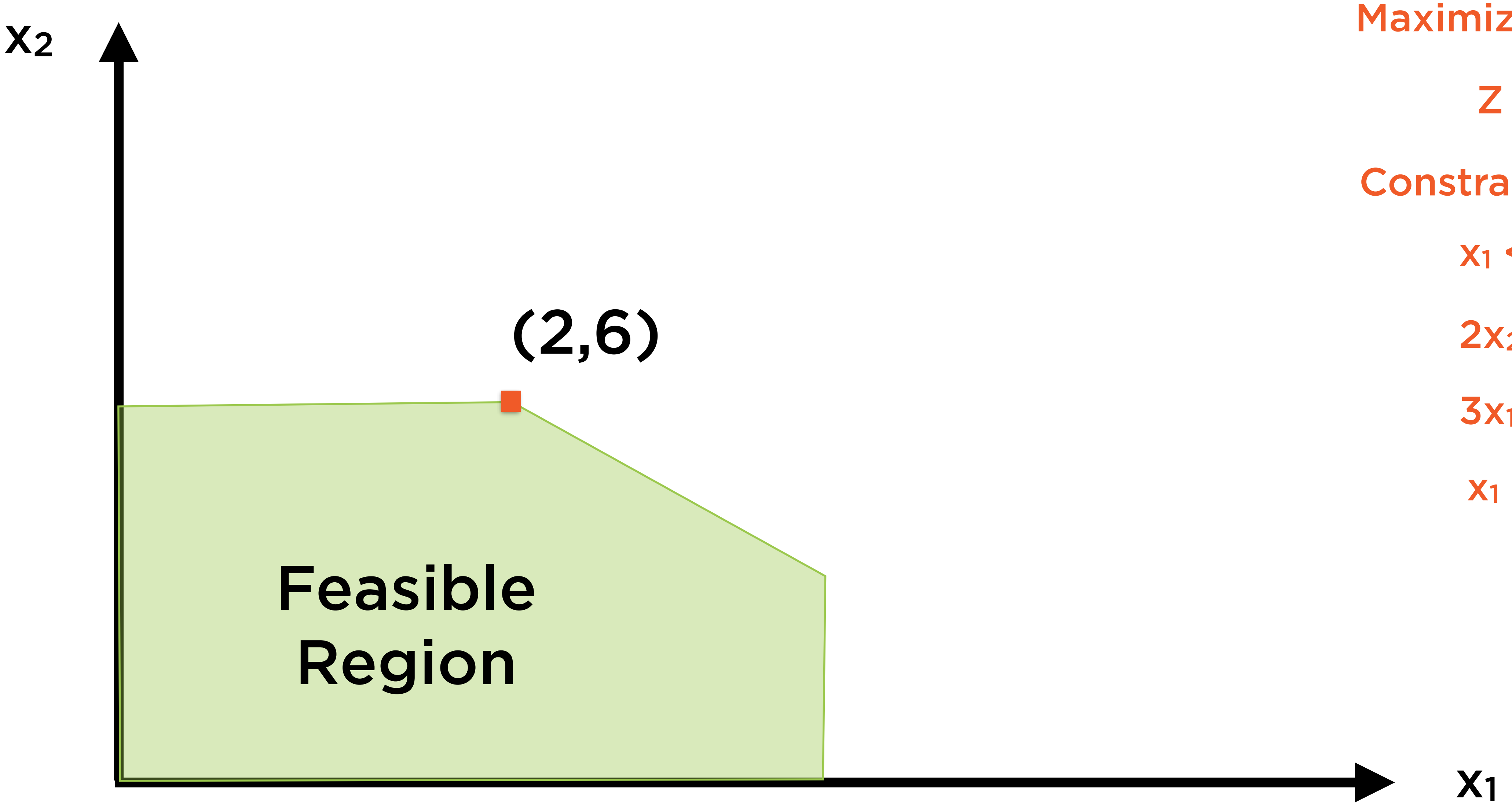
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

(4,3) is not better either - current solution
(2,6) is better than any adjacent corner-point

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

(2,6) is the optimal solution

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution
- Test for optimality
  - Not optimal, continue
  - Optimal, stop
- Keep iterating until we run out of corner-points

```
Pick an initial corner-point to
be the current solution

Is any adjacent corner-point
better than current solution?
    Yes: set that point to be
         the current solution
     No: stop, optimal point
         found

Have we run out of corner-
points?
    Yes: Sorry, no optimal
     No: Keep iterating
```

- The simplex method uses a trick to avoid actually recalculating Z at each adjacent corner-point

Pick an initial corner-point to be the current solution

Is any adjacent corner-point better than current solution?

Yes: set that point to be the current solution

No: stop, optimal point found

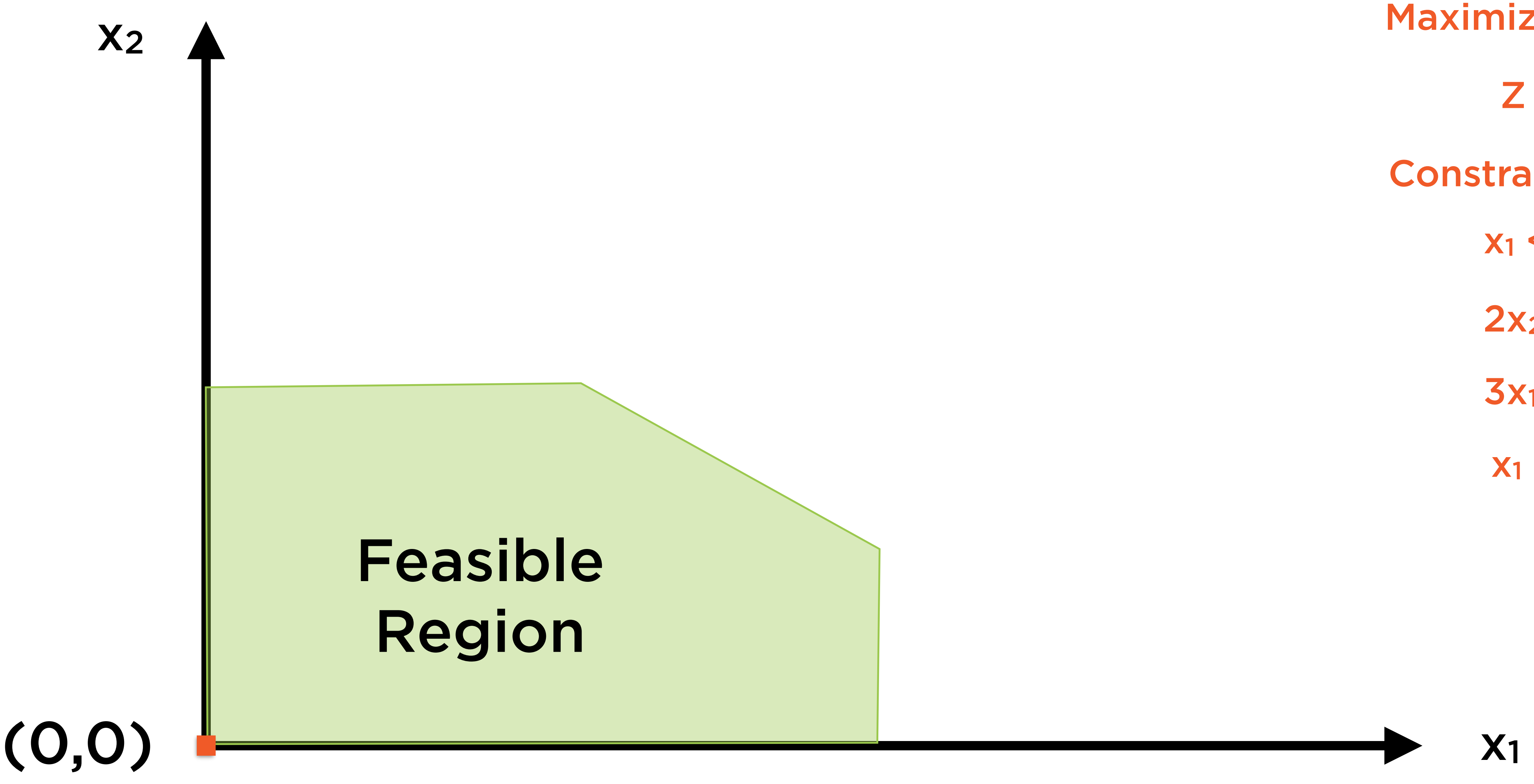
Have we run out of corner-points?

Yes: Sorry, no optimal

No: Keep iterating

The rate of improvement in each direction is calculated, using the coefficients of x_1 and x_2 in the objective function

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

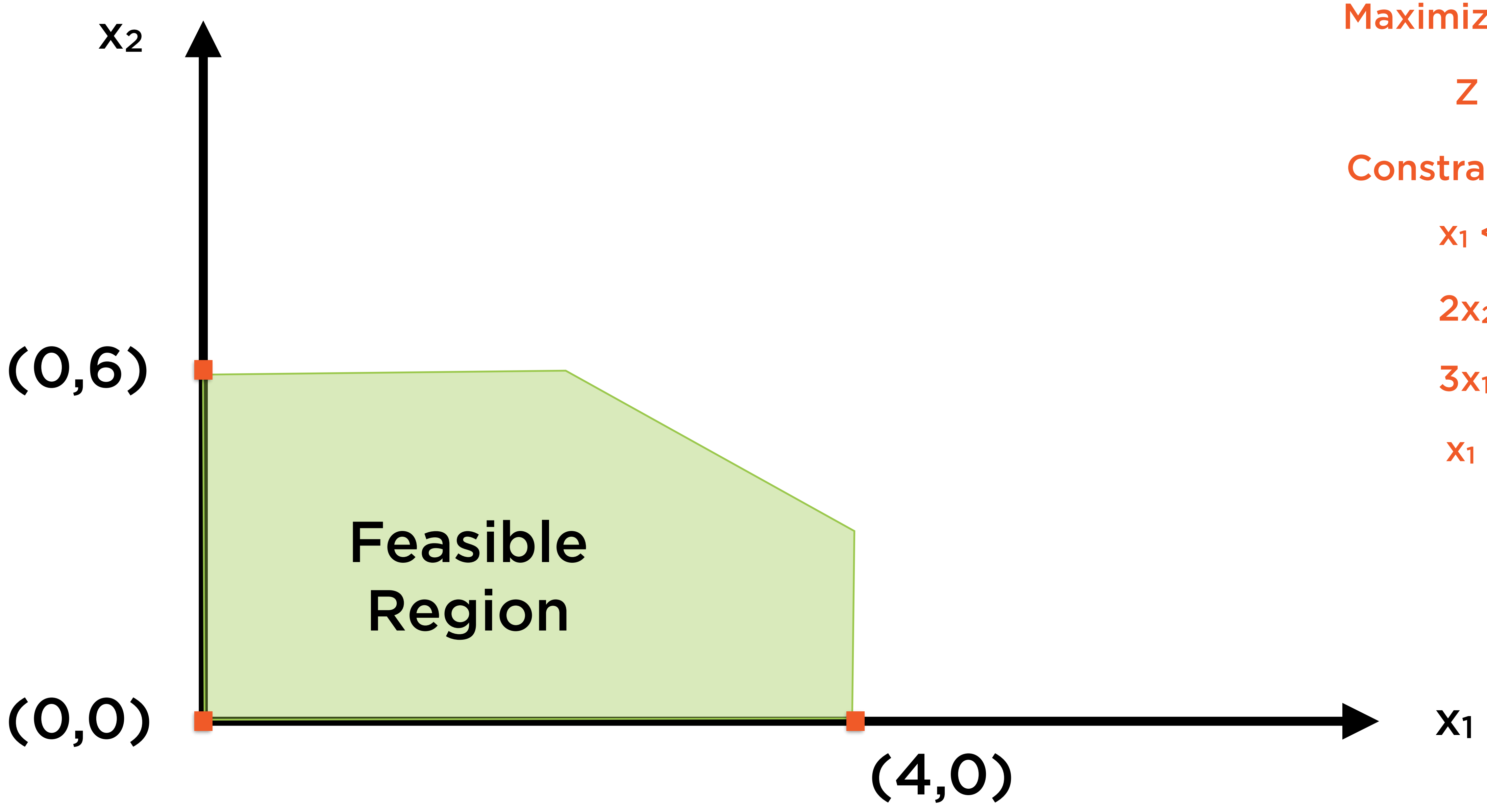
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Initial guess = (0,0)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

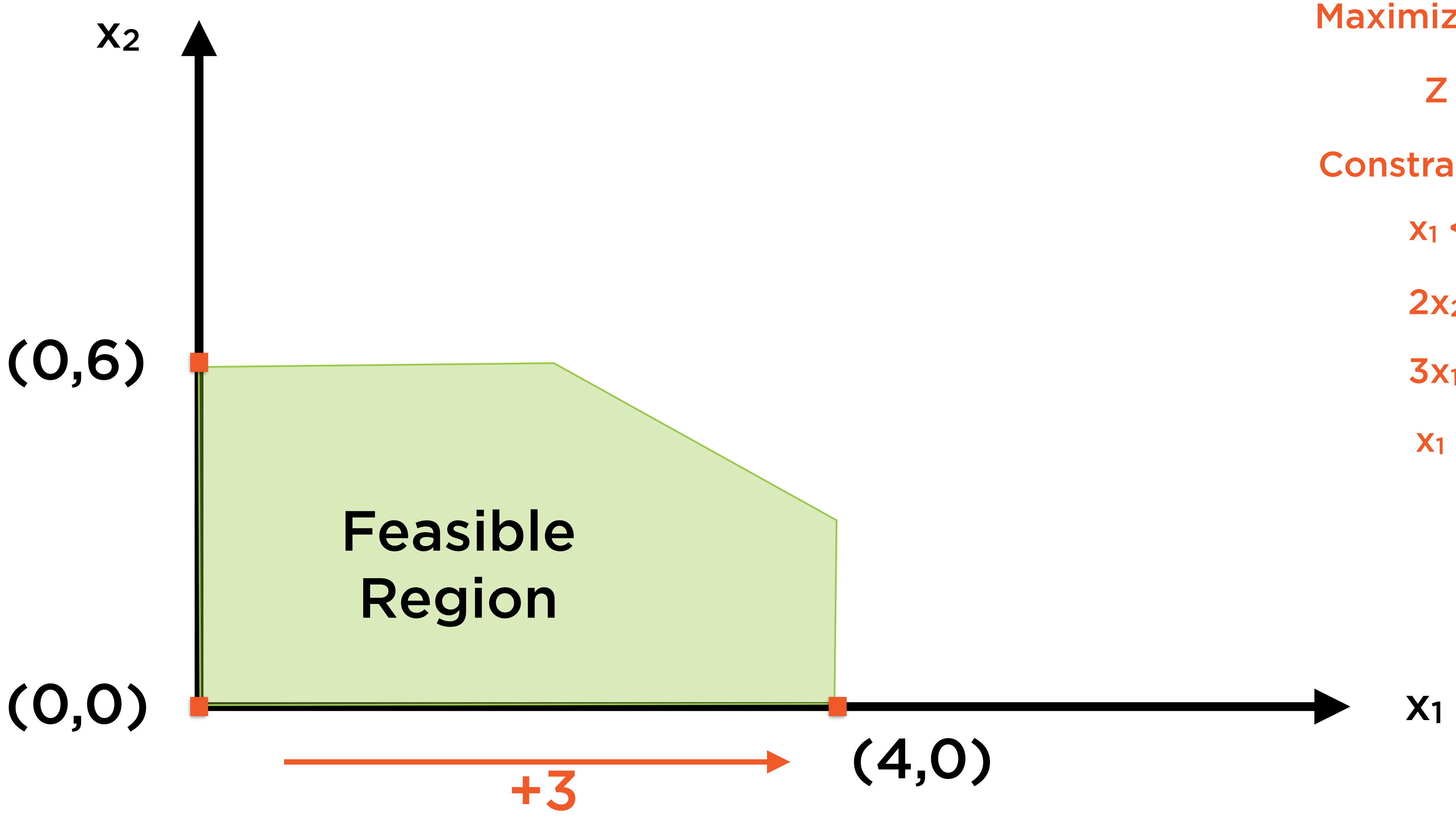
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,6) and (4,0)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

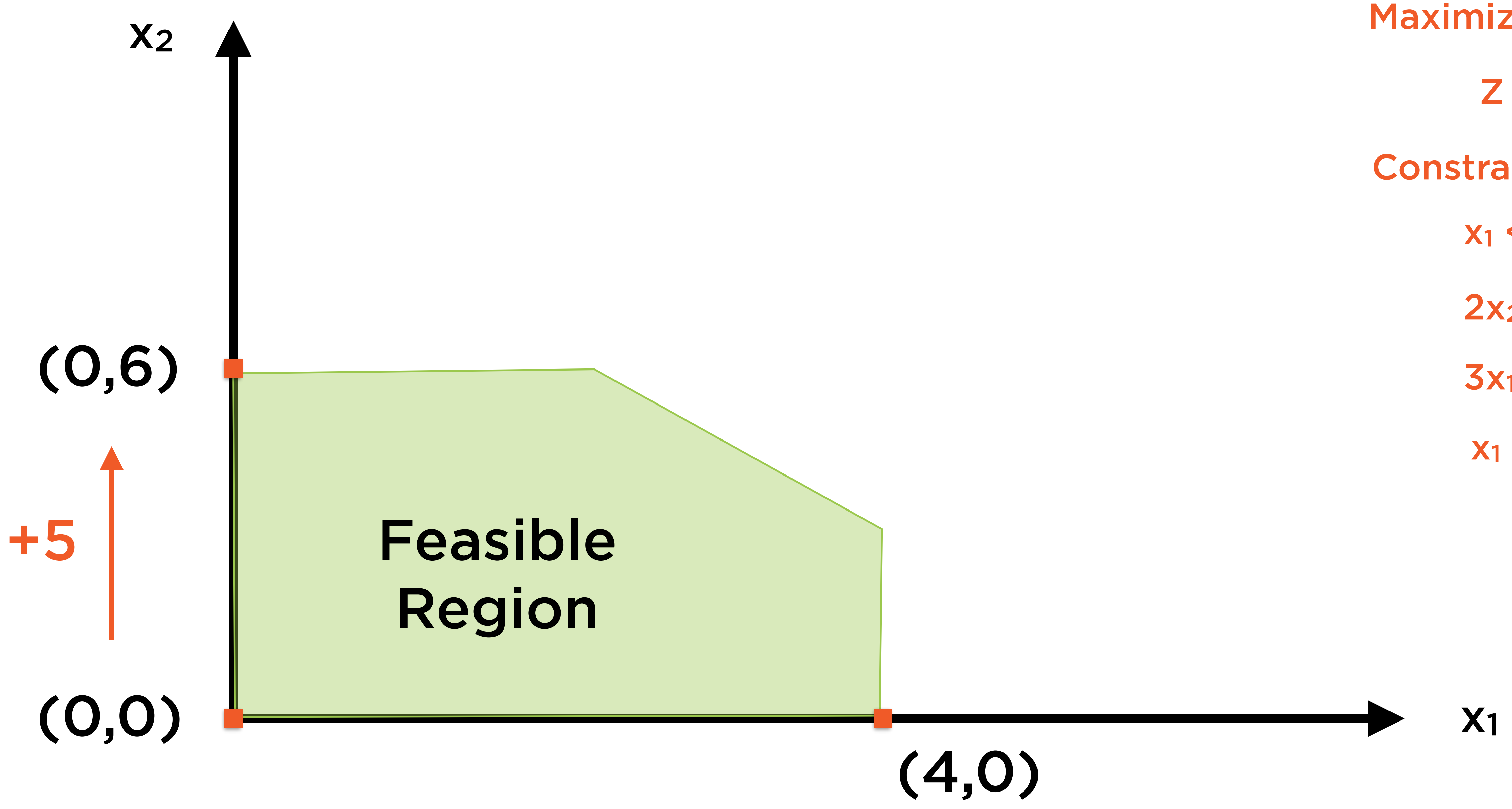
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving right increases x_1 by 1 unit, and so increases Z by 3 units (Since $Z = 3x_1 + 5x_2$)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

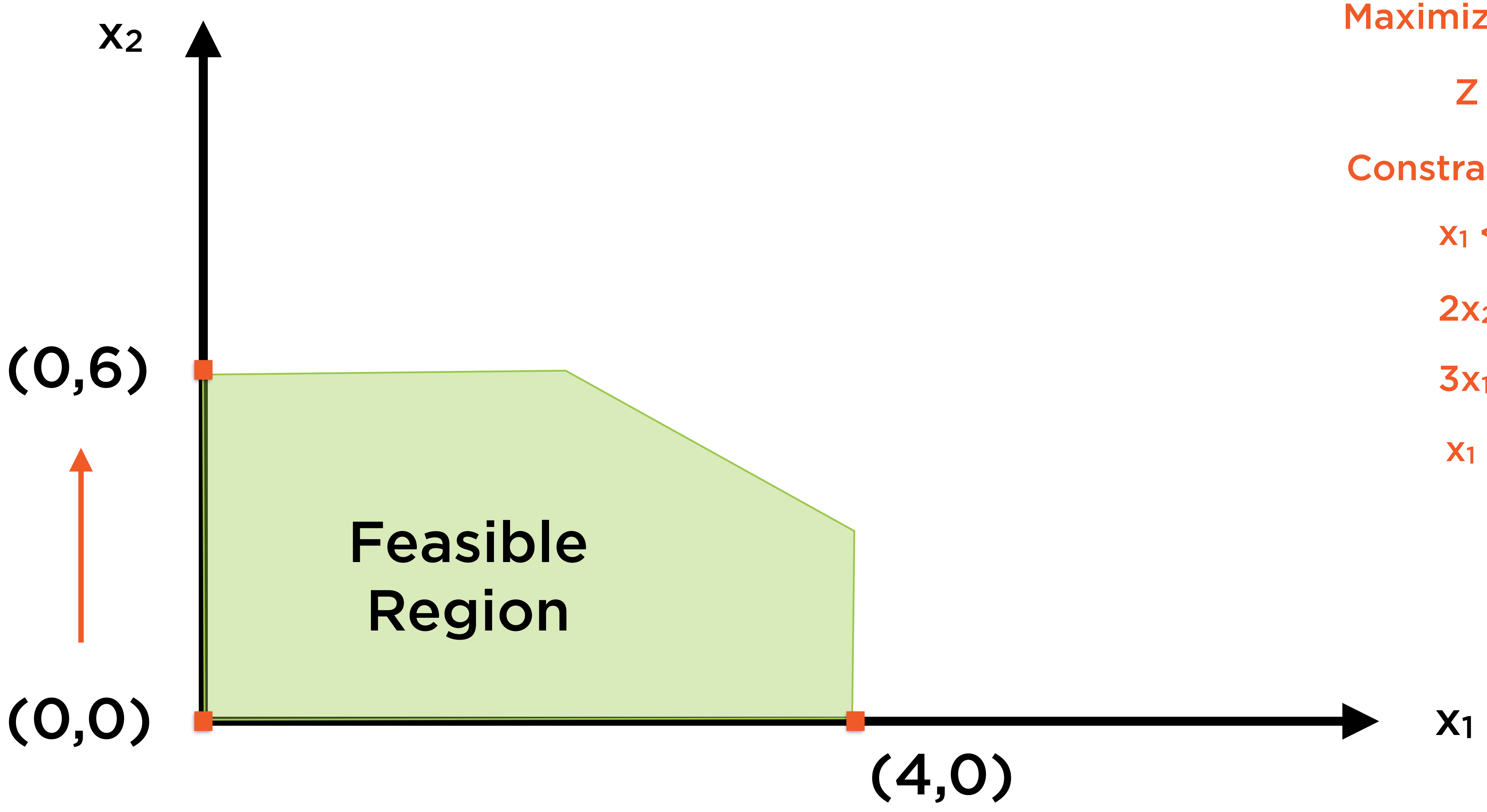
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving up increases x_2 by 1 unit, and so increases Z by 5 units (Since $Z = 3x_1 + 5x_2$)

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

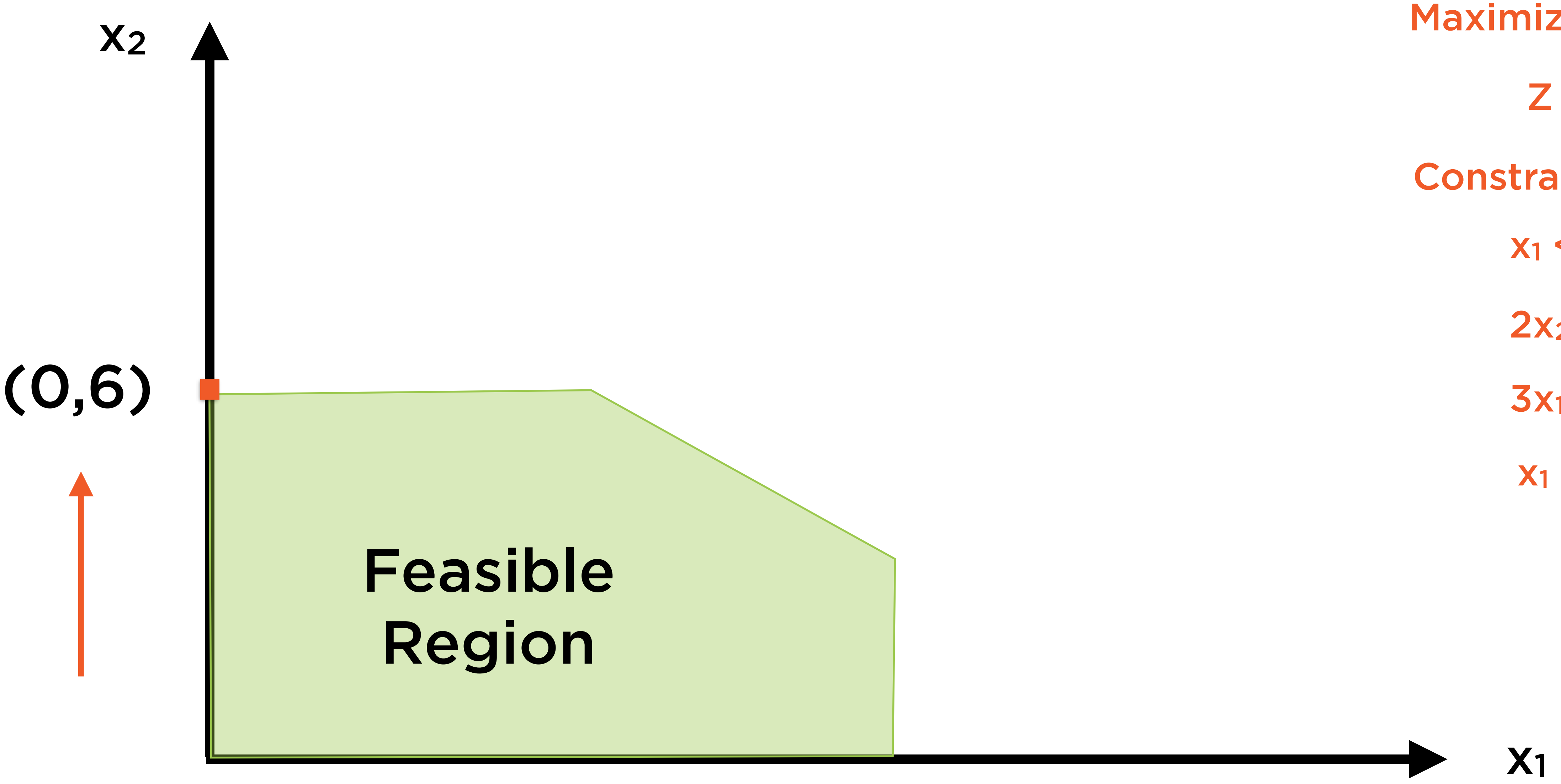
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving up increases Z faster than moving right, so move up in next iteration

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

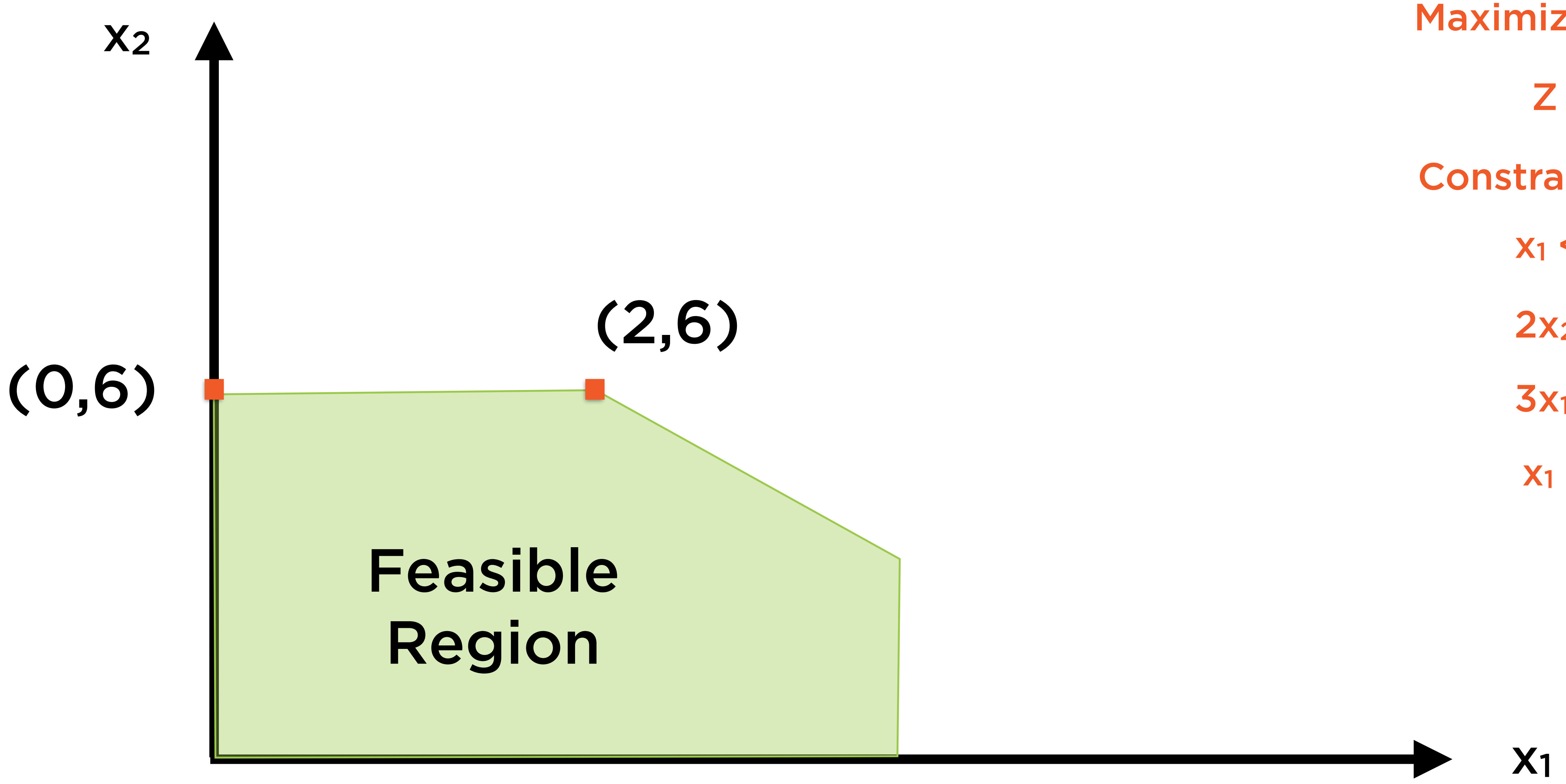
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Set (0,6) to be current solution

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

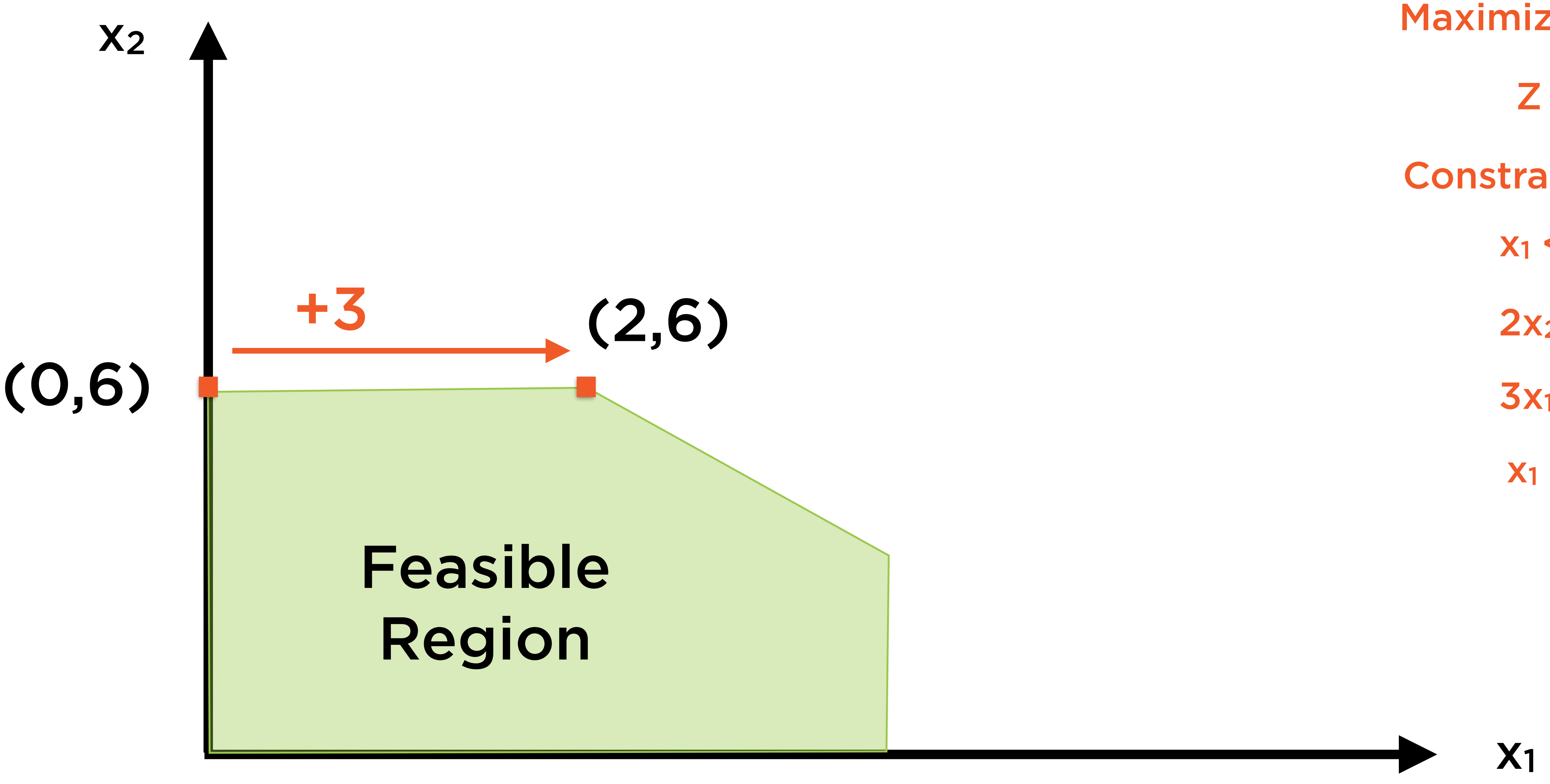
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,0) and (2,6), but we already know (0,0) is not better

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

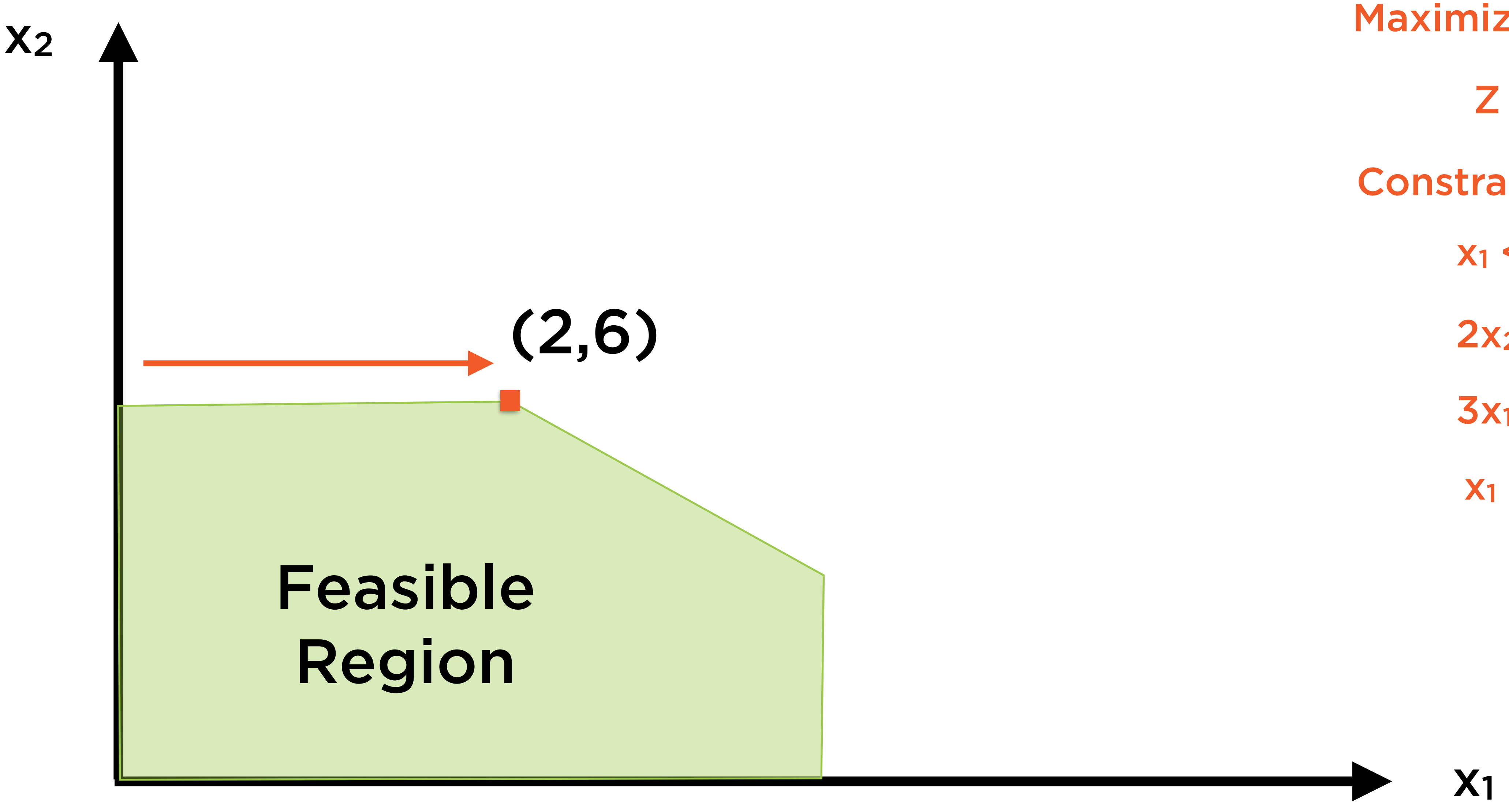
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving right increases x_1 by 1 unit, which increases Z by 3 units (Since $Z = 3x_1 + 5x_2$)

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

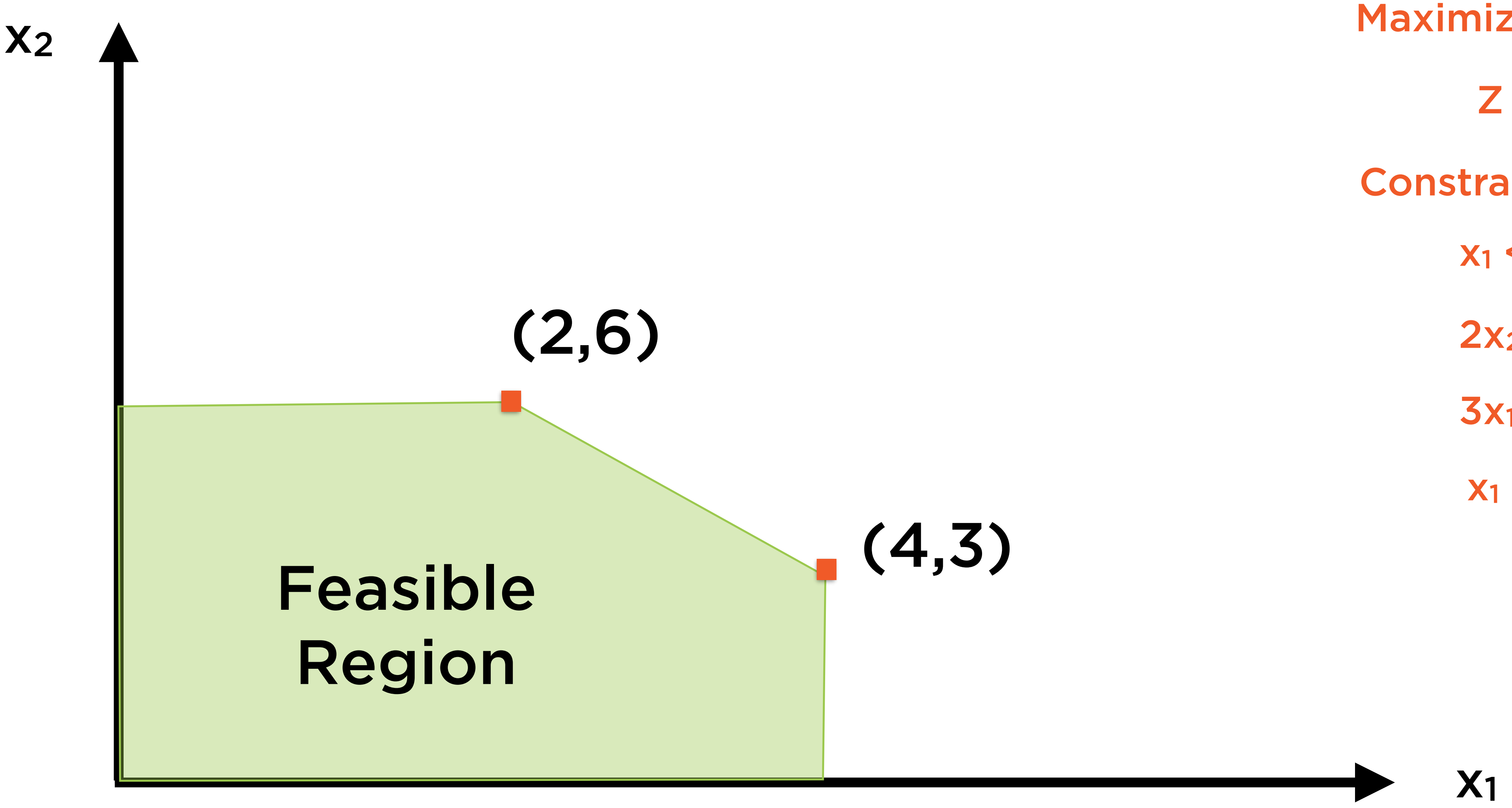
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Set (2,6) to be current solution

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

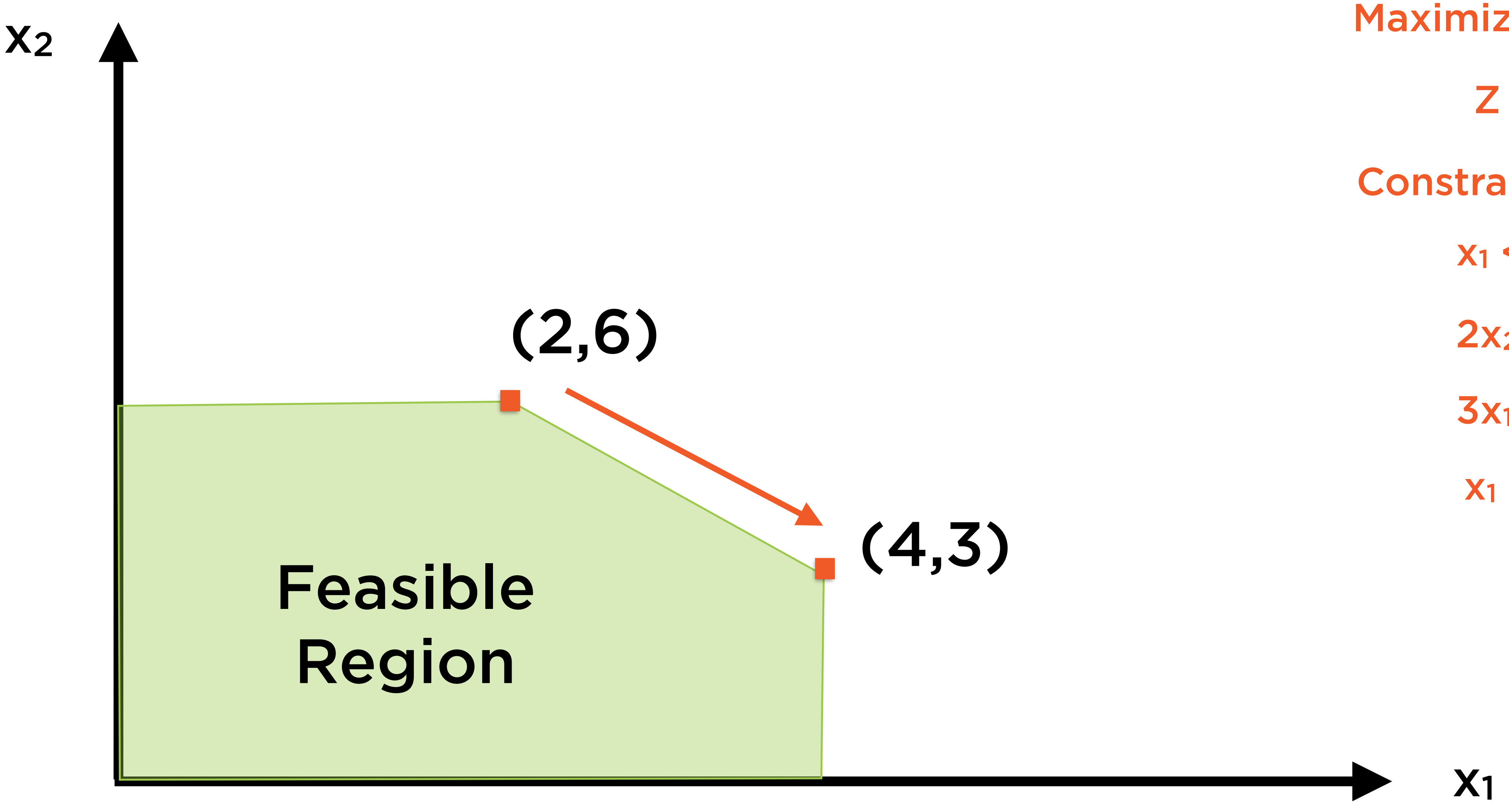
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Adjacent corner-points are (0,6) and (4,3), but we already know (0,6) is not better

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

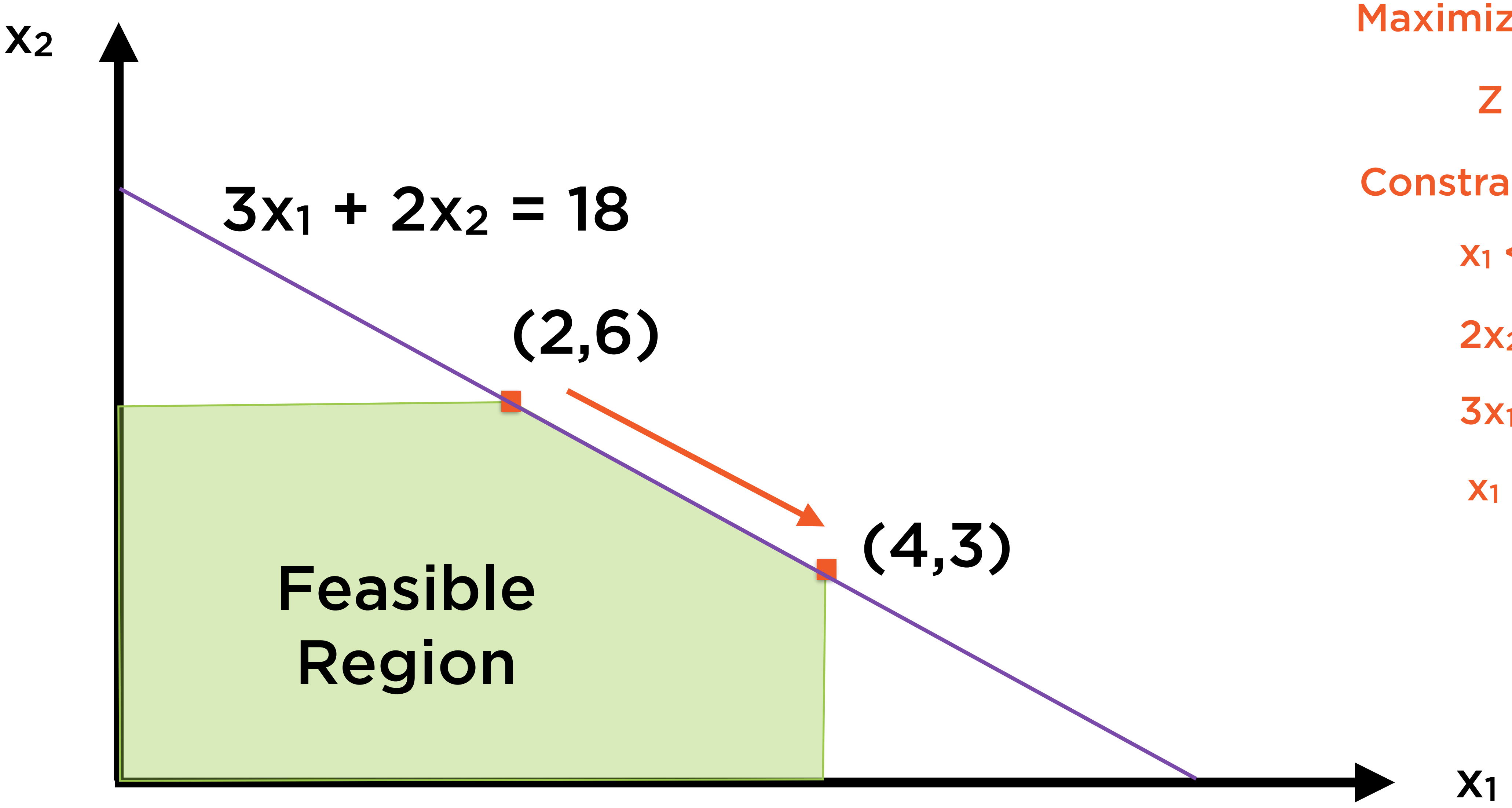
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving down and right decreases x_2 and increases x_1

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

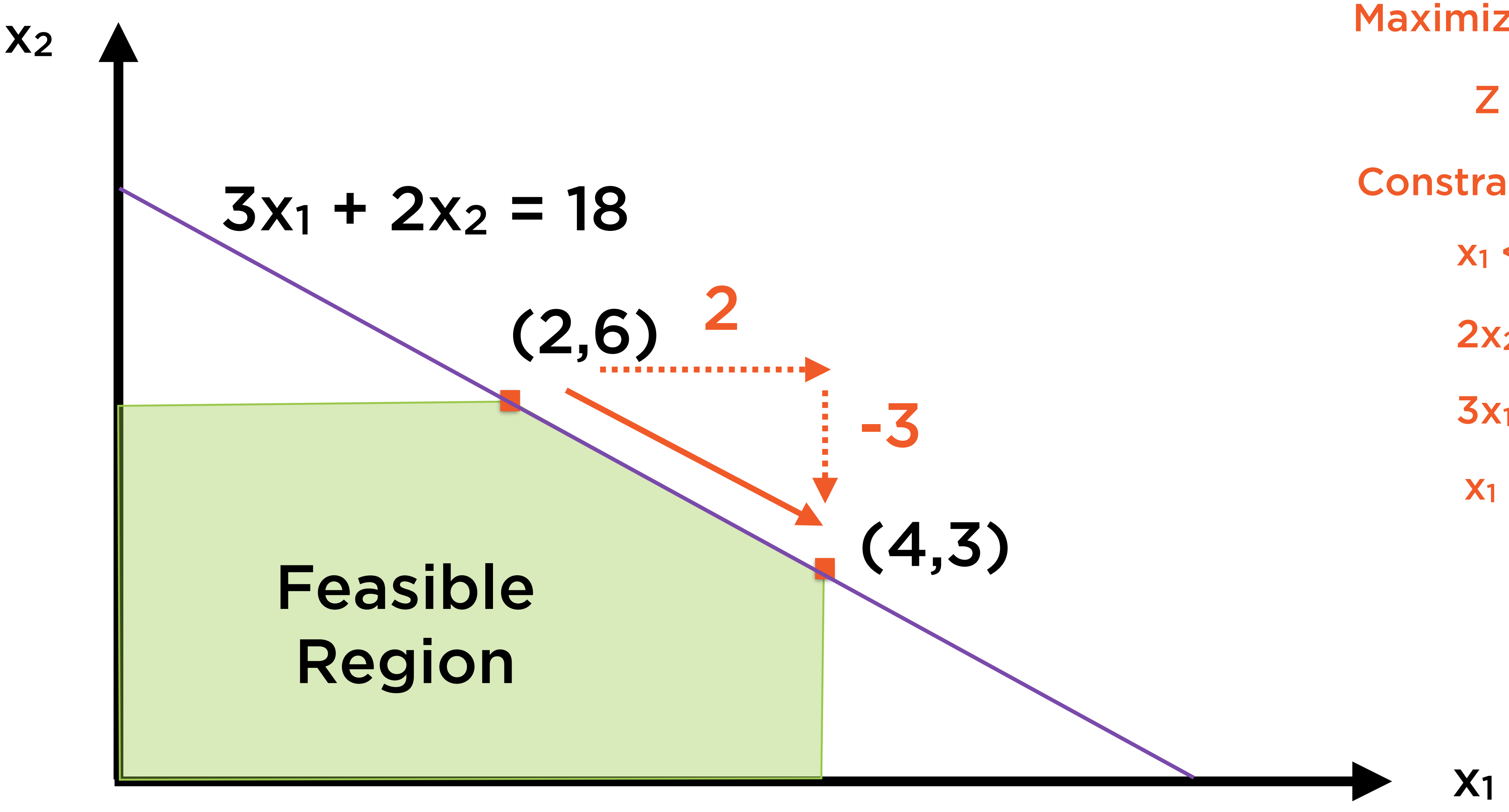
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Equation of that line: $3x_1 + 2x_2 = 18$

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

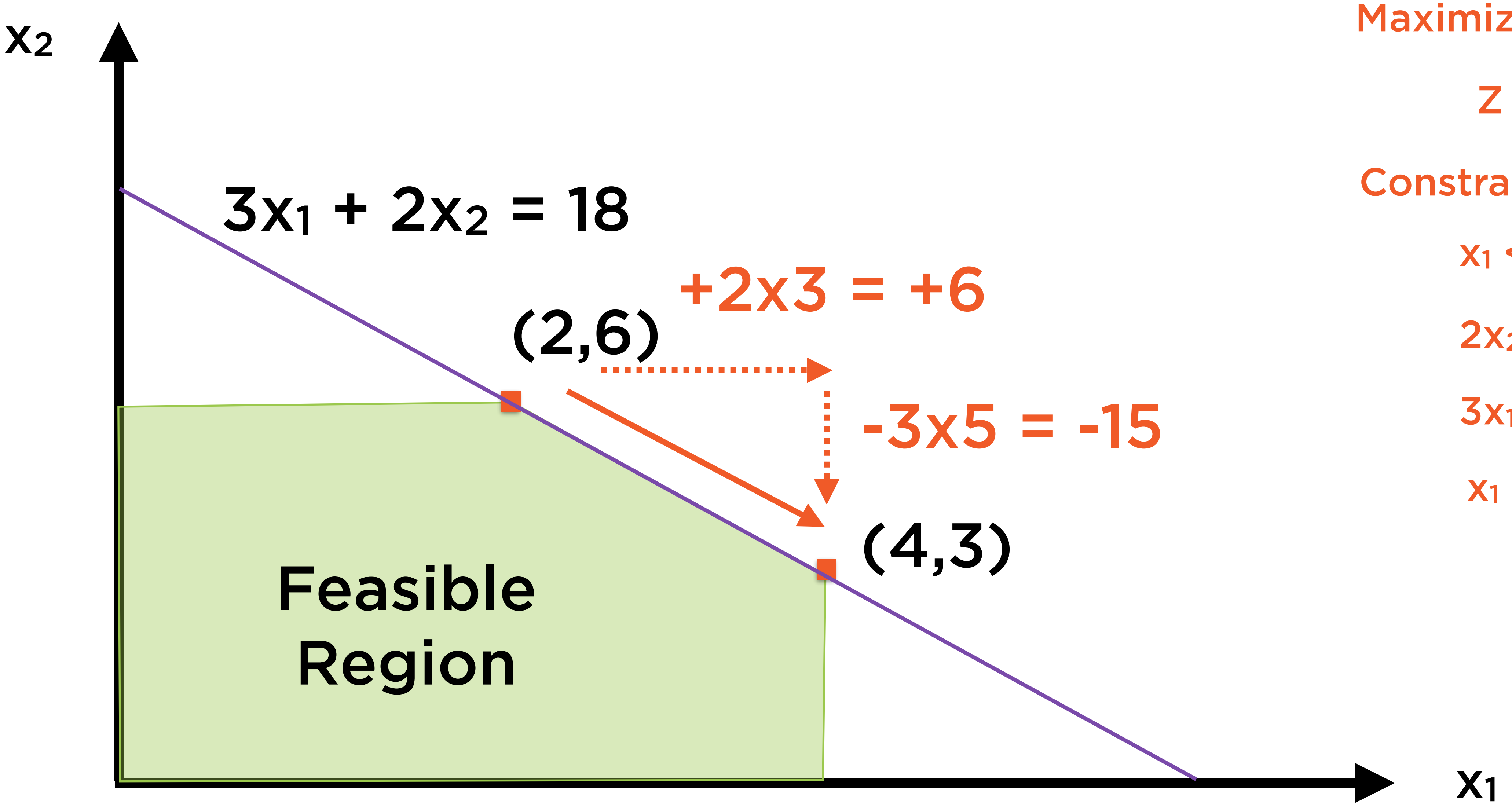
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

From slope of the line, we can conclude that moving 2 points right also moves us 3 points down

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

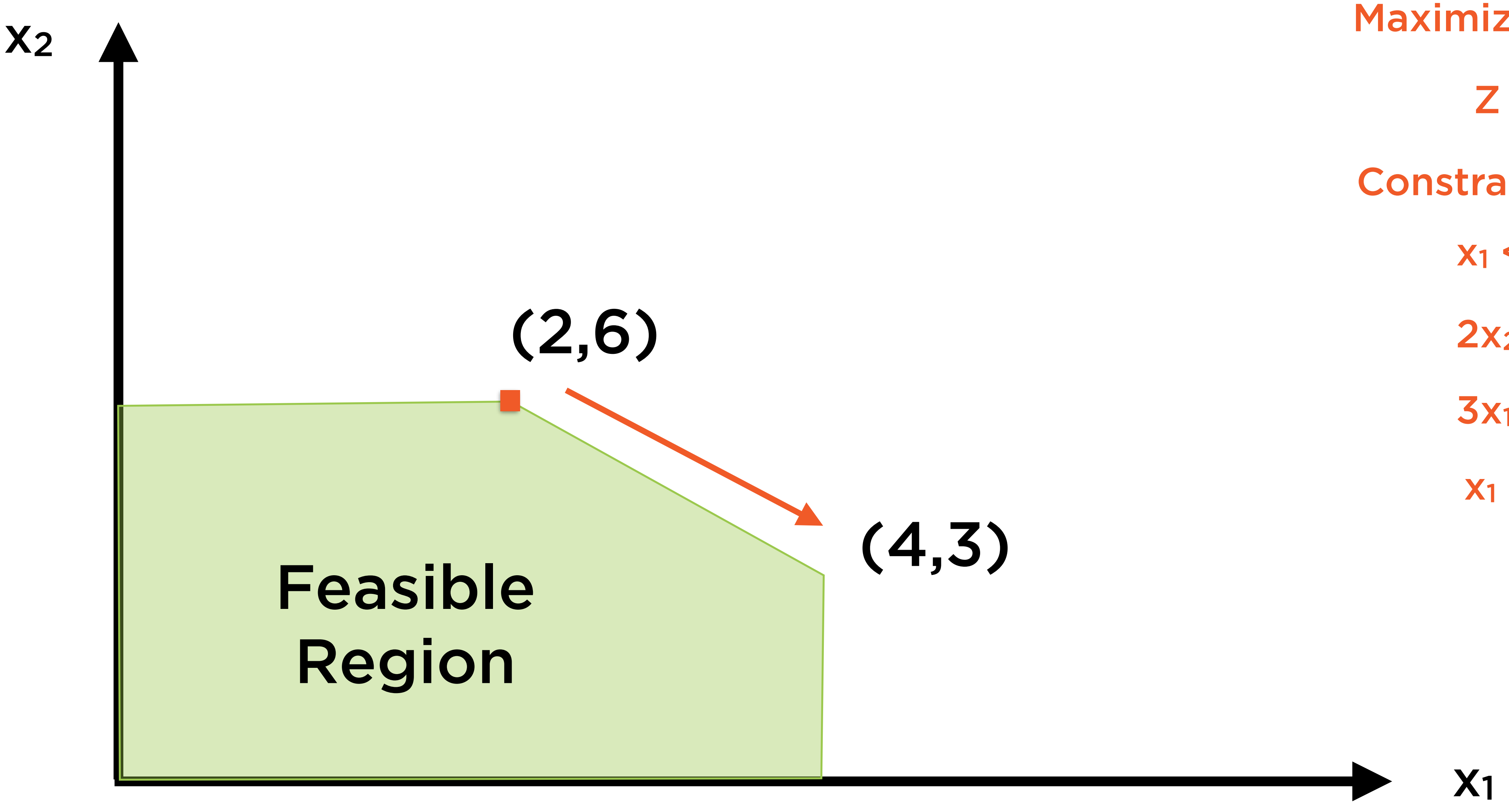
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1, x_2 >= 0$

3 unit decrease in x_2 causes Z to reduce by 15 units,
2 unit increase in x_1 causes Z to increase by only 6 units

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

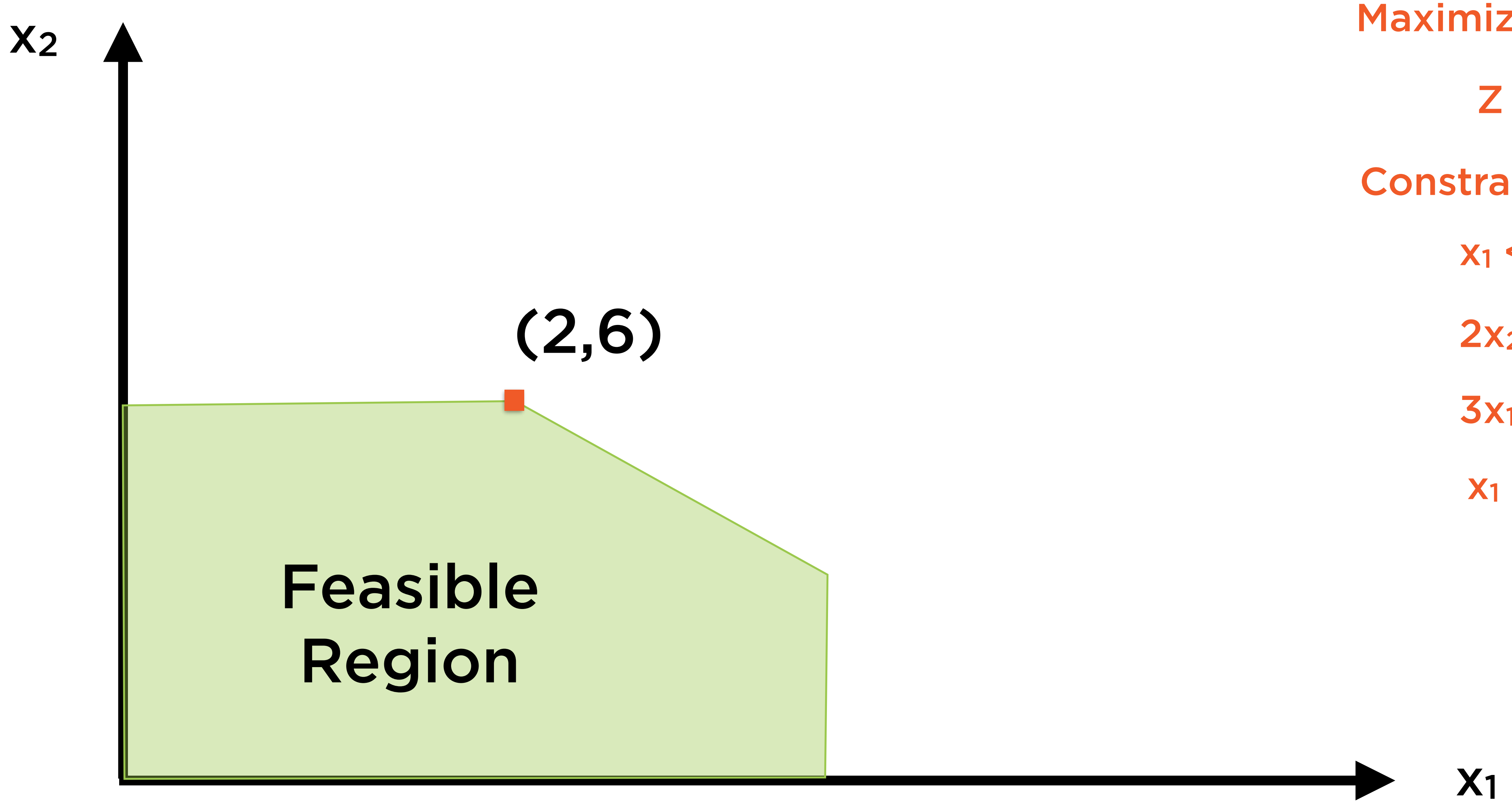
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Moving to (4,3) will cause value of objective function to decrease, i.e. (4,3) is worse than current solution (2,6)

Corner-point Feasible Solutions



Maximize

$Z = 3x_1 + 5x_2$

Constraints:

$x_1 <= 4$

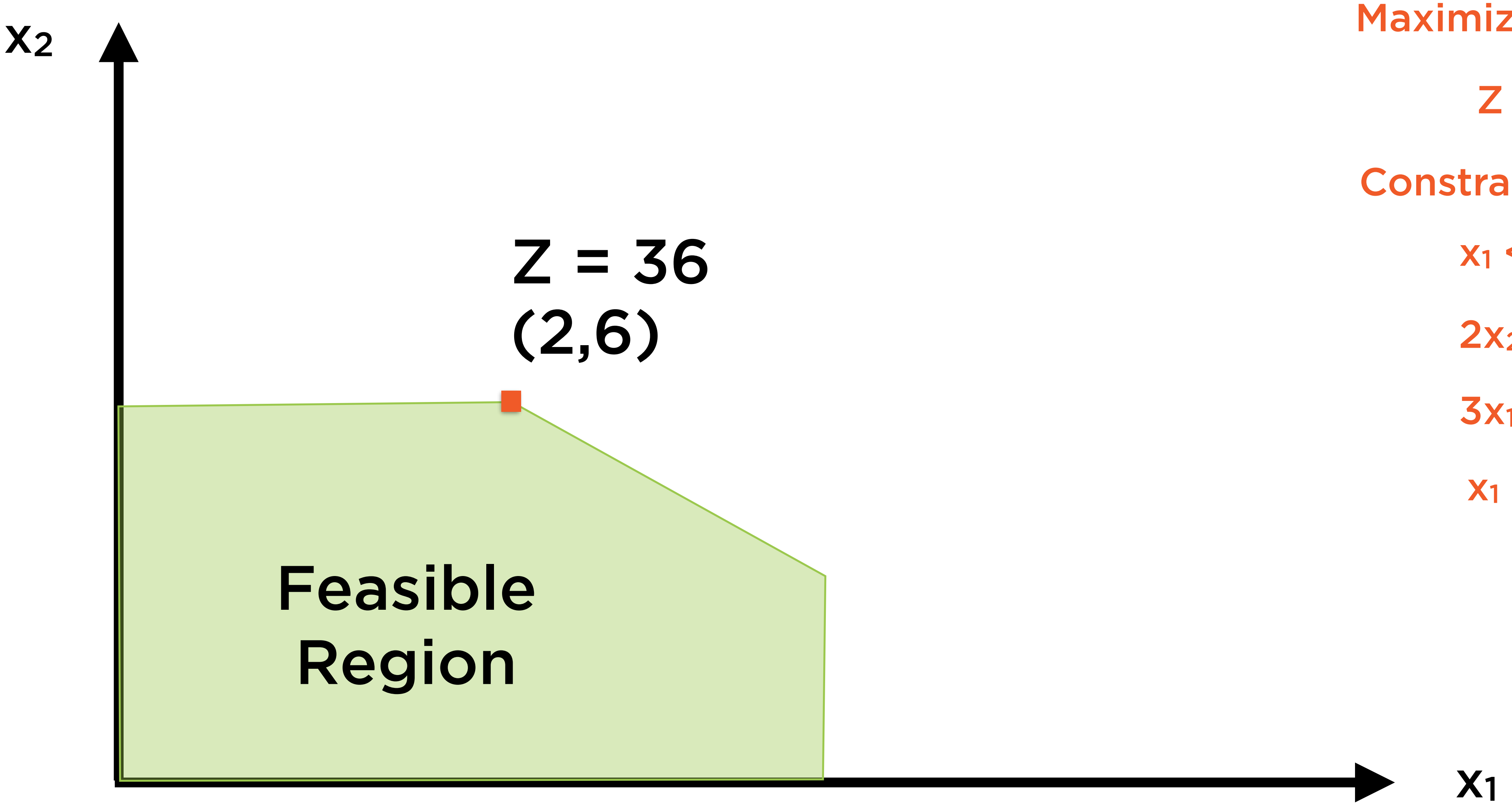
$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Current solution (2,6) is optimal - can stop now

Corner-point Feasible Solutions



Maximize

$$Z = 3x_1 + 5x_2$$

Constraints:

$x_1 <= 4$

$2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 , x_2 >= 0$

Evaluate objective function only at optimal

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution
- Test for optimality
  - Not optimal, continue
  - Optimal, stop
- Keep iterating until we run out of corner-points

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution (0,0)

- Iteratively evaluate
  - (0,0) in iteration 0
  - (0,6) in iteration 1
  - (2,6) in iteration 2

- (2,6) is optimal solution, Z = 36

The Simplex Method

Powerful

Easily extends to large numbers of variables, constraints

Versatile

Extends to sensitivity analysis and quadratic programming

Programmable

Easy to implement in software

Simplex Method: Mechanics and Interpretation

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution
- Test for optimality
  - Not optimal, continue
  - Optimal, stop
- Keep iterating until we run out of corner-points

Linear Programming Problem Formulation

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$x_1 , x_2 >= 0$ (Non-negativity constraints)

```
Pick an initial corner-point to
be the current solution


Is any adjacent corner-point
better than current solution?
   Yes: set that point to be
        the current solution
    No: stop, optimal point
        found

Have we run out of corner-
points?
   Yes: Sorry, no optimal
    No: Keep iterating
```

- Pick an initial solution (0,0)

- Iteratively evaluate
  - (0,0) in iteration 0
  - (0,6) in iteration 1
  - (2,6) in iteration 2

- (2,6) is optimal solution, Z = 36

Standard Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$$x_1 , x_2 >= 0$$ (Non-negativity constraints)

Augmented Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

x_3, x_4 and x_5 are called slack variables

Augmented Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

The slack variables convert inequality constraints into equalities

Augmented Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$Z - 3x_1 - 5x_2 = 0$$

Convert the objective function into a constraint too

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

Simplex Tableau Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$Z - 3x_1 - 5x_2 = 0 \qquad \text{Equation (0)}$$

$$x_1 + x_3 = 4 \qquad \text{Equation (1)}$$

$$2x_2 + x_4 = 12 \qquad \text{Equation (2)}$$

$$3x_1 + 2x_2 + x_5 = 18 \qquad \text{Equation (3)}$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12
(3)	x_5	0	3	2	0	0	1	18

This representation of a simplex problem is called a Simplex Tableau

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12
(3)	x_5	0	3	2	0	0	1	18

This tableau represents the initial state, with slack variables marked as 'basic variables'

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12
(3)	x_5	0	3	2	0	0	1	18

Find most negative coefficient in row (0)

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12
(3)	x_5	0	3	2	0	0	1	18

Corresponding column is called the pivot column

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12 / 2 = 6
(3)	x_5	0	3	2	0	0	1	18 / 2 = 9

Divide right hand side of each equation by pivot column (for all values in pivot column > 0)

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12 / 2 = 6
(3)	x_5	0	3	2	0	0	1	18 / 2 = 9

Highlight the smallest ratio, this identifies the pivot row

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12 / 2 = 6
(3)	x_5	0	3	2	0	0	1	18 / 2 = 9

Intersection of pivot row and pivot column yields the pivot number

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3	0	1	0	1	0	0	4
(2)	x_4	0	0	2	0	1	0	12 / 2 = 6
(3)	x_5	0	3	2	0	0	1	18 / 2 = 9

Divide pivot row by pivot number

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3							
(2)	x_4	0	0	1	0	1/2	0	6
(3)	x_5							

Divide pivot row by pivot number

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3							
(2)	x_4	0	0	1	0	1/2	0	6
(3)	x_5							

Pivot column represents the entering variable

Pivot row represents the leaving variable

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3							
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_5							

Replace the leaving variable with the entering variable in the column for basic variables

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	-3	-5	0	0	0	0
(1)	x_3							
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_5							

For other rows, do some complicated manipulation (not important exactly what)

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Rinse-and-repeat until all coefficients in row (0) are >=0

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

This final tableau represents the solution of the simplex algorithm

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	$\boxed{36}$
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Right side of equation (0) gives the optimal value

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Other values in that column give values of the decision variables (and some slack variables)

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

We can conclude that at optimal, $x_1 = 2$, $x_2 = 6$

Simplex Tableau Form

	Basic Variable	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right side of equation
(0)	Z	1	0	0	0	3/2	1	36
(1)	$\boxed{x_3}$	0	0	0	1	1/3	-1/3	$\boxed{2}$
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Any slack variable that is non-zero tells us that the corresponding constraint was non-binding at the optimal

Simplex Tableau Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$Z - 3x_1 - 5x_2 = 0 \quad \text{Equation (0)}$$

$$x_1 + x_3 = 4 \quad \text{Equation (1)}$$

$$2x_2 + x_4 = 12 \quad \text{Equation (2)}$$

$$3x_1 + 2x_2 + x_5 = 18 \quad \text{Equation (3)}$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

Simplex Tableau Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$Z - 3x_1 - 5x_2 = 0 \qquad \text{Equation (0)}$$

Any slack variable that is non-zero tells us that the corresponding constraint was non-binding at the optimal

$$x_1 + x_3 = 4 \qquad \text{Equation (1)}$$

$$2x_2 + x_4 = 12 \qquad \text{Equation (2)}$$

$$3x_1 + 2x_2 + x_5 = 18 \qquad \text{Equation (3)}$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

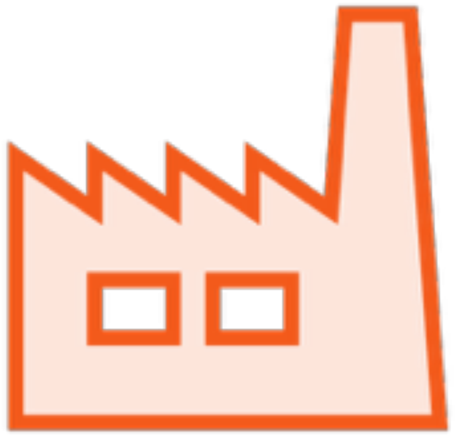
The values in row (0) for the slack variables are significant too

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

They represent the shadow prices of the resources (factory production time)

A Famous Case Study: Wyndor Glass



Three Factories

Different plants for wood, aluminium and glass

Two Products

Glass doors and glass windows

Cost and Profit

Profit and effort per unit product are known

A Famous Case Study: Wyndor Glass

	Production Time per Batch (Hours)		Production Time available per Week (hours)
	Product x_1	Product x_2	
Plant y_1	1	0	4
Plant y_2	0	2	12
Plant y_3	3	2	18

Profit per Batch	$3,000	$5,000

Tweak production to maximise profits

Simplex Tableau Form

	Basic Variable	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right side of equation	
(0)	Z	1	0	0	0	3/2	1	36	
(1)	x_3	0	0	0	1	1/3	-1/3	2	Plant y_1
(2)	x_2	0	0	1	0	1/2	0	6	Plant y_2
(3)	x_1	0	1	0	0	-1/3	1/3	2	Plant y_3

Shadow prices of the resources (factory production time)

Simplex Tableau Form

	Basic Variable	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right side of equation
(0)	Z	1	0	0	$\boxed{0}$	$\boxed{3/2}$	$\boxed{1}$	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Shadow prices of 1 hour of production time in each of the factories y_1, y_2 and y_3

Simplex Tableau Form

	Basic Variable	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right side of equation
(0)	Z	1	0	0	0	$\boxed{3/2}$	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Shadow price of 1 hour of production time in y_1 = 3/2

Unit increase in production time in y_1 will increase profit by $1,500 per week

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	$\boxed{1}$	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Shadow price of 1 hour of production time in y_2 = 1

Unit increase in production time in y_2 will increase profit by $1,000 per week

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	$\boxed{0}$	3/2	1	36
(1)	x_3	0	0	0	1	1/3	-1/3	2
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Shadow price of 1 hour of production time in y_3 = 0

Unit increase in production time in y_3 will not increase profit

Simplex Tableau Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$Z - 3x_1 - 5x_2 = 0$$ Equation (0)

Any slack variable that is non-zero tells us that the corresponding constraint was non-binding at the optimal

$$x_1 + x_3 = 4$$ Equation (1)

$$2x_2 + x_4 = 12$$ Equation (2)

$$3x_1 + 2x_2 + x_5 = 18$$ Equation (3)

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

Simplex Tableau Form

	Basic Variable	Coefficient of:						Right side of equation
		Z	x_1	x_2	x_3	x_4	x_5	
(0)	Z	1	0	0	0	3/2	1	36
(1)	$\boxed{x_3}$	0	0	0	1	1/3	-1/3	$\boxed{2}$
(2)	x_2	0	0	1	0	1/2	0	6
(3)	x_1	0	1	0	0	-1/3	1/3	2

Any slack variable that is non-zero tells us that the corresponding constraint was non-binding at the optimal

Simplex Method: Extensions

The Simplex Method

Powerful

Easily extends to large numbers of variables, constraints

Versatile

Extends to sensitivity analysis and quadratic programming

Programmable

Easy to implement in software

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n <= b_m$$

x_1 , x_2 … x_n >= 0 (Non-negativity constraints)

Equality Constraints

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

1 equality constraint is equivalent to 2 inequality constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n >= b_1$$

Equality Constraints: Big-M Method

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

Alternatively, use an artificial variable x' (Big-M method)

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n - Mx'$$

$M >> 0$ (penalty)

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x' = b_1$$

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
Minimization	Negative variables	Quadratic objectives

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
Minimization	Negative variables	Quadratic objectives

Negative Resources

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= \boxed{-10}$$

Multiply constraint by -1

$$-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n >= \boxed{10}$$

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
Minimization	Negative variables	Quadratic objectives

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Standard Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 <= 4$$

$$2x_2 <= 12$$

$$3x_1 + 2x_2 <= 18$$

$$x_1 , x_2 >= 0$$ (Non-negativity constraints)

Augmented Form

Maximize

$$Z = 3x_1 + 5x_2$$

Subject to constraints:

$$x_1 + x_3 = 4$$

$$2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5 >= 0$$

x_3, x_4 and x_5 are called slack variables

Greater-than Constraints

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n >= b_1$$

Need both a slack and a surplus variable in augmented form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - x_{surplus} + x_{slack} = b_1$$

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Minimization

Minimize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Multiply objective by -1 and maximize

Maximize

$$-Z = -c_1x_1 - c_2x_2 - \dots - c_nx_n$$

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
Minimization	Negative variables	Quadratic objectives

Extending Simplex

Equality constraints

Negative resources

Greater-than constraints

Minimization

Negative variables

Quadratic objectives

Bounded Negative Variables

Minimize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$x_1 >= -L$$

x_1 can be negative, but certain to be larger than -L (L is positive)

$$x'_1 = x_1 + L$$

Minimize

$$Z = c_1(x'_1 + L) + c_2x_2 + \dots + c_nx_n$$

$$x'_1 >= 0$$

Unbounded Negative Variables

Minimize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$x_1 <= 0$$

x_1 can be any negative value

$$x_1 = x_1^+ - x_1^-$$

Express x_1 as difference of two non-negative variables

Minimize

$$Z = c_1(x_1^+ - x_1^-) + c_2x_2 + \dots + c_nx_n$$

$$x_1^+, x_1^- >= 0$$

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
Minimization	Negative variables	Quadratic objectives

Extending Simplex

Equality constraints	Negative resources	Greater-than constraints
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Standard Form of Linear Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n <= b_m$$

$x_1 , x_2 \ldots x_n >= 0$ **(Non-negativity constraints)**

Quadratic Programming Problems

Maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$+ q_{11}x^2_1 + q_{12}x_1x_2 + \dots + q_{nn}x^2_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n <= b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n <= b_m$$

$$x_1 , x_2 \dots x_n >= 0$$

(Non-negativity constraints)

Quadratic Programming Problems

Maximize

$$Z = cx - \frac{1}{2} x^T Q x$$

Subject to constraints:

$$Ax <= B$$

$$x >= 0$$

Matrix form of quadratic programming problems

Quadratic Programming Problems

Maximize

$$Z = cx - \frac{1}{2}x^TQx$$

Subject to constraints:

$$Ax <= B$$

$$x >= 0$$

Can be solved using the Modified Simplex Method

Summary

Linear programming problems (LPPs) have a linear objective and constraints

They closely mirror an economic profit maximisation problem

LPPs can be solved using the Simplex algorithm

Simplex is very powerful and widely used

A modified form of Simplex can be extended to quadratic programming