Hat Problems on Bipartite Graphs and the Generalized Line of Sages Problem

William Bass, Tyler Huang, Reynald Oliveria, Kevin A. Zhou

Abstract

Hat problems are a staple in recreational mathematics. Usually these problems involve people with hats and they can see everyone else's hats but theirs, or person 1 only seeing person 2's hat, person 2 seeing only person 3's hat and so on, or people stand in a circle and can only see the hats of the people standing immediately next to them. These three situations can be represented by a graph that we call a sight graph. For example, the first case above would be a complete graph. Many papers prior have explored hat problems with differing sight graphs and this continues such tradition. We prove that for a sight graph of $K_{m,n}$ and three hat colors, then the number of guaranteed answers is $\lfloor \frac{\min(m,n)}{2} \rfloor$. A similar problem is the Line of Sages problem, presented by Tanya Khovanova [1]. We explore the generalized version of varying number of colors and people and we have shown that for any number of colors and three people, there is a strategy to guarantee two correct answers.

Contents

1 Introduction

In general, hat problems involve people who are wearing hats of different colors but are not able to see their own hat. In one way or another, each person will try to guess their own hat color based on some information given. The variety comes in the goals and knowledge of each person. There are hat problems where the people wearing hats attempt to maximize the correct answers *most* of the time. Others require that no one person will answer incorrectly given that they are able to pass. By nature, some hat problems require probabilistic strategies, whereas some require deterministic strategies. The website, https://www.cs.umd.edu/ gasarch/TOPICS/hats/hats.html, is a compilation of various papers on hat problems.

There are two types of hat problems that this paper focuses on. First are a set of hat problems where the people who are wearing hats can see other people's hats but cannot see their own. Which person can see who can vary. The sight of each person can be represented by a graph whose vertices represent each person and an edge shows that the two people can see each other. This is called a sight graph. There has been some work on hat problems with a variety of sight graphs. Our work continues this tradition. Finally, we explore variation of the Line of Sages problem, propose by Tanya Khovanova [1].

1.1 2 People, 2 Color

1.1.1 Problem Statement

There are 2 people facing each other. Each person is wearing a red hat or a blue hat. Each person can see the other person's hat but not their own. Each person guesses their hat color simultaneously. Develop a strategy that guarantees the most correct guesses. The strategy must be deterministic; in other words, it cannot involve randomness.

1.1.2 Solution

The solution to this problem is that one person, person 0, will guess the hat color of the other person, person 1. Person 1 will guess the opposite color of person 0's hat. Either they are wearing the same color, in which case person 0 is correct, or they are wearing different colors, in which case person 1 is correct. Therefore, this strategy guarantees that one person will always be correct.

Theorem 1.1
The best strategy for the 2 people facing each other and 2 color problem guarantees 1 correct answer.

Proof. We will prove that the best strategy guarantees at least 1 correct answer, and that it cannot get more than 1 correct answer.

Lower bound: The strategy described above guarantees 1 correct answer, so the best strategy guarantees at least 1 correct answer.

Upper bound: Consider an arbitrary strategy. Fix the hat color of person 0. By doing so, the guess of person 1 is fixed as well since their strategy is deterministic. Person 1 can be wearing either a red hat or a blue hat, but person 1 guesses the same color in both scenarios. So, in one of those configurations, person 1 will guess incorrectly. Therefore, no strategy can guarantee more than 1 correct answer. □

1.2 Definitions and Notation

1.2.1 Sight Graphs

The hat problems explored in this paper have a corresponding **sight graph**. A sight graph represents the information given to each person. Each vertex in a sight graph represents a person in the hat problem. Adjacent vertices represent which hats the person can see.

Definition 1.2 (Sight Graph)
Formally, given a sight graph $G = (V, E)$, where G is undirected, if $\{x, y\} \in E$, then person x and person y can see each other's hats.

Example 1.3 (Sight Graph for 2 people, 2 color)
The sight graph for the 2 people, 2 color problem is K_2. Each person may see each other's hat, but not their own.

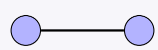


Figure 1: K_2

1.2.2 HAT notation

A hat problem consists of a sight graph, the number of colors, and the order in which people answer. When everyone answers simultaneously, a person's answer is not influenced by the other people's answers. This type of hat problems will be denoted by $\mathrm{HAT_S}(G, k)$, where G is the sight graph and k is the number of colors. For example, the 2 people, 2 color problem would be denoted by $\mathrm{HAT_S}(K_2, 2)$. Note that $\mathrm{HAT_S}(G, k)$ is also a function which outputs the maximum amount of guaranteed correct answers for some specific hat problem. To formalize, we shall define other functions.

Notation 1.4 ($h(i)$ and $g(i)$)
$h(i)$, where i is a vertex in the sight graph (equivalently a person in the problem), is the hat color of vertex i.

$g(i)$, where i is a vertex in the sight graph, is the answer of vertex i.

$h(i)$ and $g(i)$ return nonnegative integers, as each color can be matched with a unique number. For example, if there are k colors, then the colors are $0, 1, \cdots, k-1$. Also, the equation $g(i) = h(i)$ means that vertex i guesses correctly.

Definition 1.5 (Strategy)
A **strategy**, s, is an algorithm to determine the value of $g(i)$. In a strategy, the value of $g(i)$ can only change if $h(a)$ also changes, for some vertex a adjacent to i. In other words, there is no random guessing allowed, as a person will always answer the same color if that person sees the same thing. $s(c)$, where c is a configuration of hat colors, is the number of correct answers produced by s.

We will use the 2 person, 2 color problem to give an example of this notation. Let the person who guesses the same color as the other person's hat be a_0, and let the other person be a_1. Also, let the color red be 0, and let blue be 1. The strategy, s, for that problem is $g(a_0) = h(a_1)$ and $g(a_1) \equiv h(a_0) + 1 \pmod 2$. This guarantees that one person is always correct. So, $s(c) = 1$ for all configurations c. Additionally, no strategy can guarantee more than 1 correct answer. Therefore, $\mathrm{HAT_S}(K_2, 2) = 1$.

Notation 1.6 ($\mathrm{HAT_S}(G, k)$)
Let Σ be the set of all strategies applicable to $\mathrm{HAT_S}(G, k)$, and let χ be the set of all hat configurations applicable to $\mathrm{HAT_S}(G, k)$.

Formally,
$$\mathrm{HAT_S}(G, k) = \max_{s\in\Sigma}(\min_{c\in\chi}(s(c))).$$

2 Background

2.1 Hat Problems on Complete Graphs, Any Number of Colors

2.1.1 n people, 2 colors

A generalization of the first problem, $\mathrm{HAT_S}(K_2, 2)$, is $\mathrm{HAT_S}(K_n, 2)$. In a K_n graph, each person can see every other person. For each pair of people, the strategy for $\mathrm{HAT_S}(K_2, 2)$ can be applied. This guarantees 1 correct answer for each every pair of people. Since there are exactly $\lfloor\frac{n}{2}\rfloor$ pairs, this strategy guarantees $\lfloor\frac{n}{2}\rfloor$ correct answers.

Theorem 2.1 ($\mathrm{HAT_S}(K_n, 2)$)
$$\mathrm{HAT_S}(K_n, 2) = \lfloor\frac{n}{2}\rfloor$$

Proof. **Lower bound:** The strategy of pairing people yields $\lfloor\frac{n}{2}\rfloor$ correct answers. Therefore, $\mathrm{HAT_S}(K_n, 2) \geq \lfloor\frac{n}{2}\rfloor$.

Upper bound: Given an arbitrary strategy, consider an arbitrary person, Ada. Fix all the hat colors apart from Ada's. By doing so, Ada's answer is fixed. Therefore, Ada is correct in exactly one of the two possible cases for her hat color. Therefore, across all 2^n configuration of hat colors, Ada is correct in $\frac{1}{2}$ of them, or 2^{n-1} times. Thus, Ada is incorrect the remaining

2^{n-1} of the configurations. Since there are n people, the total number of incorrect answers for any given strategy is $n(2^{n-1})$.

By the pigeonhole principle, for any strategy, there exists at least one configuration in which there are $\lceil \frac{n(2^{n-1})}{2^n} \rceil = \lceil \frac{n}{2} \rceil$ incorrect answers. Since $n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$, a strategy can guarantee at most $\lfloor \frac{n}{2} \rfloor$ correct answers. Therefore, $\mathrm{HAT_S}(K_n, 2) \leq \lfloor \frac{n}{2} \rfloor$.

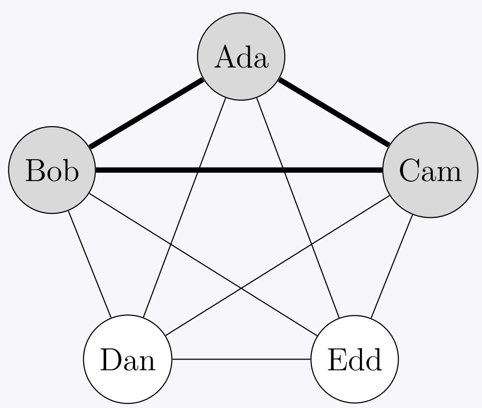
An alternate proof uses expected value. Since each person is correct in exactly $\frac{1}{2}$ of the configurations, the expected value of the amount of correct answers for any strategy is $\frac{n}{2}$, by the linearity of expectations. Therefore, if there exists a configuration in which a strategy produces more than $\frac{n}{2}$ correct answers, there must be a configuration that produces less than $\frac{n}{2}$ correct answers. This proves that $\mathrm{HAT_S}(K_n, 2) \leq \lfloor \frac{n}{2} \rfloor$. □

2.1.2 Any number of colors, any number of people

A further generalization of problems is when the number of colors is arbitrary and the number of people are also arbitrary. These are problems in the form of $\mathrm{HAT_S}(K_n, k)$. A strategy for this problem is to split the n people into cliques of k people. Then, each person in a clique would assume that the sum of hat colors in that clique is a different number $\pmod{k}$. Since the cliques are k in size, and there are only k colors, this covers all possible sums $\pmod{k}$. Therefore, each clique guarantees 1 correct answer. Since there are $\lfloor \frac{n}{k} \rfloor$ cliques, this strategy guarantees $\lfloor \frac{n}{k} \rfloor$ correct answers.

Example 2.2

Consider $\mathrm{HAT_S}(K_5, 3)$. Let the 5 people be Ada, Bob, Cam, Dan, and Edd. Let the colors be $\{0, 1, 2\}$. Since there are 5 people, there can only be 1 clique of size 3. Let that clique be Ada, Bob, and Cam.



Let $s = h(\mathrm{Ada}) + h(\mathrm{Bob}) + h(\mathrm{Cam})$. Ada would assume that $s \equiv 0 \pmod{3}$, Bob assumes $s \equiv 1 \pmod{k}$, and Cam assumes $s \equiv 2 \pmod{k}$. Dan and Edd's answers do not matter. Since the clique covers all possible values of $s \pmod{3}$, one of them must be correct in any configuration of their hats. So, the strategy yields 1 correct answer, which is equal to $\lfloor \frac{5}{3} \rfloor$.

To see how this strategy works, suppose that all 5 people are wearing hat color 1. The following table shows their strategy. Ada guesses her hat color correctly.

	Ada	Bob	Cam
Assumed sum	0	1	2
Hat color	1	1	1
Other hats	$h(\text{Bob}) + h(\text{Cam}) = 2$	$h(\text{Ada}) + h(\text{Cam}) = 2$	$h(\text{Ada}) + h(\text{Bob}) = 2$
Guess	1	2	0

Theorem 2.3 ($\text{HAT}_\text{S}(K_n, k)$)

$$\text{HAT}_\text{S}(K_n, k) = \lfloor \frac{n}{k} \rfloor$$

Proof. **Lower bound:** Since the strategy of separating persons into k-sized cliques results in $\lfloor \frac{n}{k} \rfloor$ correct answers, $\text{HAT}_\text{S}(K_n, k) \geq \lfloor \frac{n}{k} \rfloor$.

Upper bound: Given an arbitrary strategy, consider an arbitrary person, Ada. Fix the hats of everyone but Ada, which thus fixes her answer. There are now k possibilities for Ada's hat color. In only 1 of those k possibilities would Ada answer correctly. Therefore, Ada is correct in $\frac{1}{k}$ of all configurations. Since Ada was an arbitrary person, this can be generalized to all n people; each of them is correct in $\frac{1}{k}$ of all configurations. By the linearity of expectation, the expected value for the number of correct answers is $\frac{n}{k}$. If the strategy produces more than $\frac{n}{k}$ correct answers for one configuration, then it must produce fewer than $\frac{n}{k}$ correct answers for another configuration. Therefore, $\text{HAT}_\text{S}(K_n, k) \leq \frac{n}{k}$, and since $\text{HAT}_\text{S}(K_n, k)$ is an integer, $\text{HAT}_\text{S}(K_n, k) \leq \lfloor \frac{n}{k} \rfloor$. □

2.2 2 Colors, Tree Graphs, Cycle Graphs

2.2.1 Any graph, 2 colors

A further generalization of Theorem 2.1 ($\text{HAT}_\text{S}(K_n, 2)$) is $\text{HAT}_\text{S}(G, 2)$, where G is an arbitrary graph. A solution to this is to make as many pairs of vertices as possible, which is the maximal pairing of G. This is formally the number of edges in the largest set of independent edges.

Theorem 2.4 ($\text{HAT}_\text{S}(G, 2)$)
$\text{HAT}_\text{S}(G, 2)$ is the maximal pairing of G, proven in [2].

2.2.2 Tree graphs, 3+ colors

Theorem 2.5 ($\mathrm{HAT}_{\mathrm{S}}(T,3)$)
$\mathrm{HAT}_{\mathrm{S}}(T,3) = 0$, where T is a tree graph.

Proof. Consider $\mathrm{HAT}_{\mathrm{S}}(T,3)$. Consider colors c_1 and c_2. Suppose that T has one vertex, it is obvious that $\mathrm{HAT}_{\mathrm{S}}(T,3) = 0$ and that the lone vertex can be wearing either hat color c_1 or c_2, in at least one case it would still be wrong. Otherwise select a vertex v in T, and delete v. This creates some number, j, subtrees. Call these subtrees $T_1, T_2, ..., T_j$. Let $r(T_i)$ be the vertex in T_i solely adjacent to v. Let $\Gamma_1(T_i)$ and $\Gamma_2(T_i)$ be the strategy for the persons in T_i if v is wearing c_1 and c_2, respectively. Let $B_1(T_i)$ be the set of configurations causing all vertices in T_i to be incorrect using $\Gamma_1(T_i)$, and let $C_1(T_i)$ be the set of colors $B_1(T_i)$ assigns to $r(T_i)$. Similarly define $B_2(T_i)$ and $C_2(T_i)$, but with $\Gamma_2(T_i)$.

Example 2.6 (Notation)
Consider the tree in Figure 2. Let v be the gray vertex.

- The T_i would be the green subtree, the blue subtree, and the red subtree in whichever order. But for the sake of the example, let the green subtree be T_1, the blue subtree be T_2, and the red subtree be T_3.
- The $r(T_i)$ would be the vertex that is outlined black in T_i. So the blue vertex outlined black is $r(T_2)$.
- If v were to be wearing hat color c_1, then
  - The green vertices (T_1) would use the strategy $\Gamma_1(T_1)$.
  - If a configuration of hat colors were to cause $\Gamma_1(T_1)$ to fail to produce at least one correct answer:
    * The configuration would be in $B_1(T_1)$.
    * The hat color $r(T_1)$ would be wearing in that configuration would be in $C_1(T_1)$.
- If v were to wear hat color c_2, then
  - The red vertices (T_3) would use the strategy $\Gamma_2(T_3)$.
  - If a configuration of hat colors were to cause $\Gamma_2(T_3)$ to fail to produce at least one correct answer:
    * The configuration would be in $B_2(T_3)$.
    * The hat color $r(T_3)$ would be wearing in that configuration would be in $C_2(T_3)$.

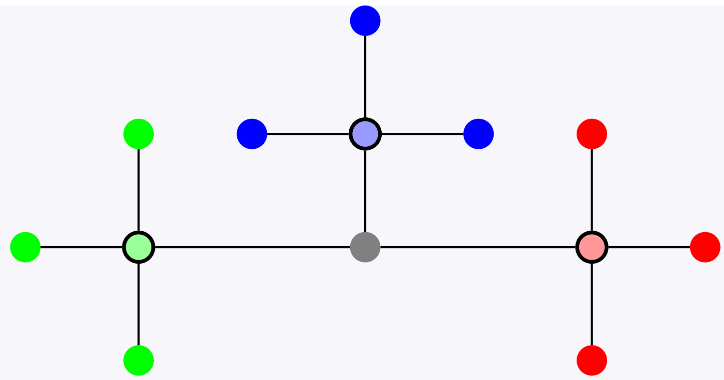


Figure 2: A tree

Inductive Hypothesis: Given strategies $\Gamma_1(T_i)$ and $\Gamma_2(T_i)$, and a vertex v_i in T_i, then there exists a configuration of hats such that all vertices in T_i are incorrect ($\text{HAT}_\text{S}(T_i, 3) = 0$), and that v_i is either color a or color b.

Fact

Both $|C_1(T_i)|$ and $|C_2(T_i)|$ are at least 2

Proof. For the sake of contradiction, suppose that $|C_1(T_i)| < 2$. Since there are only 3 colors, the complement of $C_1(T_i)$ has at least two colors, call these colors c_3 and c_4. By the inductive hypothesis, given $\Gamma_1(T_i)$, $r(T_i)$, and two colors c_3 and c_4, there exists a configuration that causes all vertices in T_i to be incorrect. Therefore $B_1(T_i)$ is non-empty. Since $B_1(T_i)$ is non-empty, and by the inductive hypothesis, $r(T_i)$ is wearing either c_3 or c_4 in some hat configuration in $B_1(T_i)$. Therefore, either c_3 or c_4 is in $C_1(T_i)$. This contradicts the earlier conclusion that both c_3 and c_4 are not in $C_1(T_i)$. Therefore, $|C_1(T_i)| \geq 2$. Similarly it can be proven that $|C_2(T_i)| \geq 2$. $\square$

Since there are only 3 colors and both $|C_1(T_i)|$ and $|C_2(T_i)|$ are at least 2, $C_1(T_i) \cap C_2(T_i)$ must have at least one element, k_i. This means that regardless of which strategy ($\Gamma_1(T_i)$ or $\Gamma_2(T_i)$) is applied, then there exists a configuration in which $r(T_i)$ is wearing a k_i-colored hat, and everyone in the T_i is incorrect.

Let the color, for each i, of $r(T_i)$ be k_i. Insodoing there exists a configuration in which all the vertices in the T_i is incorrect given that v is wearing either c_1 or c_2. However, by also setting the hat colors of all of the $r(T_i)$, the answer of v is set. This answer cannot be both c_1 and c_2. And so, place the hat color that is not the answer of v. Therefore, there exists a configuration that causes all vertices in T to be incorrect.

The assumption, that the subtrees (T_i) formed by removing v cannot guarantee a correct answer and some vertex (v_i) in T_i is wearing hat color either a or b, implies that T cannot guarantee a correct answer, and some vertex, v, in T is wearing hat color either c_1 or c_2. Also if T has one vertex, v, T cannot guarantee a correct answer and v is wearing either hat color c_1 or c_2. Therefore, by induction, for any tree T, no strategy can guarantee at least 1 correct answer. Hence $\text{HAT}_\text{S}(T, 3) = 0$. A similar proof can be found in [2]. $\square$

2.2.3 Cycle graph, 3 colors

We define that a **viable cycle** is a cycle graph of size n where $3|n$ or $n = 4$.

Theorem 2.7 ($\mathrm{HAT_S}(C_n, 3)$)
Proven in [3], $\mathrm{HAT_S}(C_n, 3) = 1$ if C_n is a viable cycle, and $\mathrm{HAT_S}(C_n, 3) = 0$ if C_n is not a viable cycle.

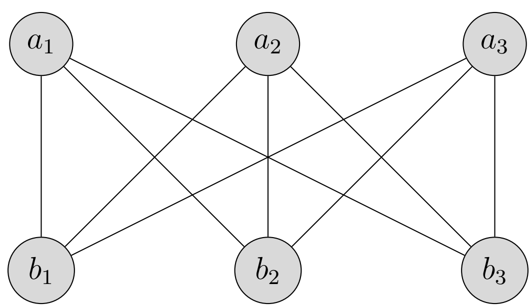
Additionally, combined with Theorem 2.5 ($\mathrm{HAT_S}(T, 3)$), we know that $\mathrm{HAT_S}(G, 3) \geq 1$ if and only if there exists a viable cycle.

3 Results

3.1 Hat Problems on Bipartite Graphs, 3 Colors

Consider the graph $K_{m,n}$, the complete bipartite graph with independent sets of sizes m and n. Each vertex represents a person. An edge between two people indicates that the two connected people can see each other's hat. There are 3 different colors of hats. All of the people guess their hat color at once. Determine the strategy that maximizes the number of correct guesses.

3.1.1 $\mathrm{HAT_S}(K_{3,3}, 3)$



Theorem 3.1 ($\mathrm{HAT_S}(K_{3,3}, 3)$)
$\mathrm{HAT_S}(K_{3,3}, 3) = 1$.

Proof. **Lower bound:** Since a_1, b_1, a_2, b_2 form a cycle of length 4, they can guarantee 1 correct answer. Thus, $\mathrm{HAT_S}(K_{3,3}, 3) \geq 1$.

Upper bound: We may use an expected value argument. There are 3^6 hat configurations total. Consider some arbitrary person Ada. There are 3^5 hat configurations if we ignore Ada. In each of these configurations, Ada's guess is independent of his own hat color, and therefore he will be correct in 1 out of 3 cases. Therefore, Ada will answer correctly in 3^5 hat configurations. Since there are 6 people total, the total amount of correct answers is $6 * 3^5$.

The expected amount of correct guesses is $\frac{6*3^5}{3^6} = 2$. Let us fix the hat colors of b_1, b_2, b_3. Suppose that a_i guesses $h(a_i)$. If the adversary places $h(a_i)$ on each of a_i's heads, then there will be 3 correct answers. But since the expected amount of correct guesses is 2, then there must exist a configuration where there are less than 2 correct answers. Thus, $\mathrm{HAT_S}(K_{3,3}, 3) \leq 1$, and our proof is complete. □

3.1.2 HAT$_\mathrm{S}(K_{m,n}, 3)$

Theorem 3.2 ($\mathrm{HAT_S}(K_{m,n}, 3)$)
$\mathrm{HAT_S}(K_{m,n}, 3) = \lfloor \frac{\min(m,n)}{2} \rfloor$.

Proof. **Lower Bound:** We may divide the people into independent cycles of length 4, each of which guarantees 1 correct answer. Each cycle is formed by taking 2 people from each group. Whichever group runs out of people first will determine the amount of correct answers. Therefore, using this strategy, we find $\mathrm{HAT_S}(K_{m,n}, 3) \geq \lfloor \frac{\min(m,n)}{2} \rfloor$.

Upper Bound: Without loss of generality, we will let $m \leq n$. So, our new goal is to show that $\mathrm{HAT_S}(K_{m,n}, 3) \leq \lfloor \frac{m}{2} \rfloor$.

Consider when all people in the m group are wearing a red hat (the "Red Case"). Now, the guesses of the n group are determined. Let's also consider when all people in the m group are wearing a blue hat (the "Blue Case"). Again, the guesses of the n group are determined.

Consider some arbitrary person in the n group, Ada. Suppose Ada guesses the color r_{Ada} in the Red Case and s_{Ada} in the Blue Case. Even though r_{Ada} and s_{Ada} are not necessarily distinct, note that there exists a color t_{Ada} that Ada does not guess in the Red Case or Blue Case. We can apply this logic to every person in the n group. Let us place all of these t's on the n group's heads, so that they will all be wrong in both the Red Case and the Blue Case.

Therefore, in the Red Case and the Blue Case, the only people who are correctly guessing their hats are in the m group. Also, note that they must guess the same thing in both cases, since the same hats are on the n group's heads. So, by the pigeonhole principle, the amount of correct answers $\leq \lfloor \frac{m}{2} \rfloor$ in either the Red Case or the Blue Case. The adversary will choose the appropriate case, and thus $\mathrm{HAT_S}(K_{m,n}, 3) \leq \lfloor \frac{m}{2} \rfloor$, as we wanted to show. □

Example 3.3 ($K_{4,5}$)
We'll walk through the proof for specific m, n. To create the Red Case, we put red hats on all people in the smaller group.

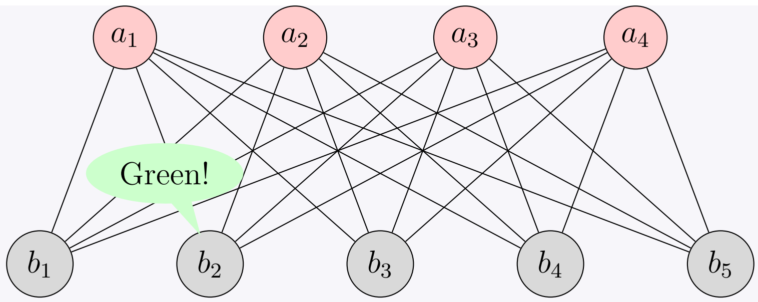


Figure 1. This shows the Red Case. Let's suppose b_2 guesses green in this situation.

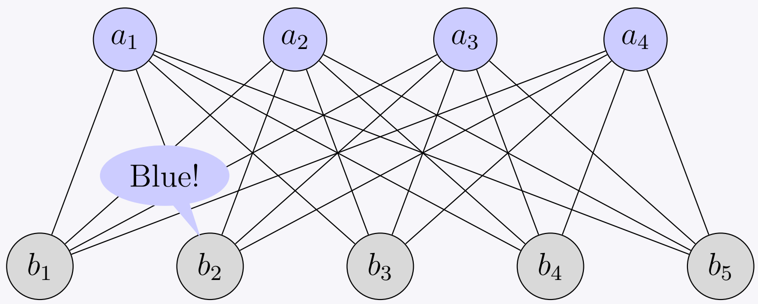


Figure 2. This shows the Blue Case. Let's suppose b_2 guesses blue in this situation.

In both the Red Case and the Blue Case, b_2 will not guess red. Therefore, the adversary places a red hat on his head. The same can be done for all other b_i. Therefore, none of b_i will guess correctly, no matter it's the Red Case or the Blue Case. Additionally, each a_i must guess the same thing in the Red Case as in the Blue Case, because they see the same hat colors on the b people.

The best case scenario for the a_i's (and all of the people) is to have 2 guess red and the other 2 guess blue, so that they will have 2 correct. If 1 guesses red and 3 guess blue, then the hatter will simply pick the Red Case, and there will only be 1 correct answer. So, this is not their optimal strategy. Therefore, the maximum amount of correct answers is $\lfloor \frac{4}{2} \rfloor = 2$, as desired.

The people also have a strategy to guarantee 2 correct answers, because there are 2 cycles of length 4: $\{a_1, b_1, a_2, b_2\}, \{a_3, b_3, a_4, b_4\}$.

Thus, $\mathrm{HAT}_{\mathrm{S}}(K_{4,5}, 3) = 2$.

3.2 A Line of Sages Generalized

3.2.1 Problem Statement

Suppose there are n people standing in a line, where person i can see the hats of persons $i+1, i+2, \cdots, n$. The hat colors are $0, 1, \cdots, k$; each color may be used either once or not at all. The people can guess in any order they want. No person can repeat a previous person's guess. Develop a strategy that maximizes the amount of correct guesses.

3.2.2 Notation

First, we introduce HAT notation for the Line of Sages. This type of problem will be denoted by $\text{HAT}_{\text{LS}}(n, k)$, where n is the number of people, and k is the number of colors. A paper by Tanya Khovanova [1] shows that $HAT_{LS}(n, n+1) = n - 1$. We conjecture that $HAT_{LS}(n, k) = n - 1$, but we have only proven this for $n = 3$. (The proof for $n = 2$ is simple– the first person just says the hat color 1 before the next person, and the next person now knows their own hat color.)

3.2.3 Example: 3 people, 5 colors

We begin by creating person 1's strategy. Consider the 20 possibilities that person 1 may see. The first number in the ordered pair represents person 2's hat color; the second number represents person 3's hat color.

(1,0)	(2,0)	(3,0)	(4,0)
(0,1)	(2,1)	(3,1)	(4,1)
(0,2)	(1,2)	(3,2)	(4,2)
(0,3)	(1,3)	(2,3)	(4,3)
(0,4)	(1,4)	(2,4)	(3,4)

We want to guarantee that persons 2 and 3 can always call their hat color correctly. So, person 1 cannot be ambiguous. For example, suppose if person 1 sees (1,0), then person 1 calls "4"; also, if person 1 sees (2,0), then person 1 calls "4" also. Now, when person 2 sees that person 3 has hat color 0 and hears "4", there are still two possibilities for what his own hat color is, which is bad. We want to avoid this from happening, and thankfully, it is possible. Here is one of many solutions.

(0,1,2)	(0,2,1)	(0,3,4)	(0,4,3)
(1,0,3)	(1,2,4)	(1,3,0)	(1,4,2)
(2,0,4)	(2,1,3)	(2,3,1)	(2,4,0)
(3,0,2)	(3,1,4)	(3,2,0)	(3,4,1)
(4,0,1)	(4,1,0)	(4,2,3)	(4,3,2)

This table is a condensed way to write each person's strategy. Let $x \in (1, 2, 3)$. Notice that if we ignore the xth entry in each ordered triple, then the 20 ordered pairs that remain are the 20 possibilities for what person x sees and hears. So, based on the ordered pair of what person x sees and hears, he guesses the xth entry in the appropriate ordered triple. We call a strategy that guarantees $n-1$ correct answers to be a **successful strategy**.

3.2.4 The General Problem for $k = n + 2$

So, we have boiled the line of sages problem into a simpler problem. Let $x \in (0, 1, \cdots, n-1)$. Find $\frac{(n+2)!}{3!}$ ordered n-tuples such that if we remove the xth entry in each ordered n-tuple, then the $\frac{(n+2)!}{3!}$ ordered $(n-1)$-tuples that remain are all of the possibilities that person x may see and hear. In other words, the $(n-1)$-tuples are all $(a_1, a_2, \cdots, a_{n-1})$ such that $(i \neq j \implies a_i \neq a_j)$ and $a_i \in (0, 1, \cdots, n+1)$.

3.2.5 Using Latin Squares to solve $n = 3$

Theorem 3.4
$HAT_{LS}(3, k) = 2$.

Consider the problem of 3 people, 5 colors. We would like to make 20 3-tuples such that when the ith element of each tuple is removed, the remaining 2-tuples are distinct. In other words, we would like to fill in the 20 blanks here. Note that if any 2 numbers in the same row are the same, then when the 1st element of each tuple is removed, there will be some identical 2-tuples. Similarly, if any 2 numbers int he same column are the same, then when the 2nd element of each tuple is removed, there will be some identical 2-tuples.

	(1,0,___)	(2,0,___)	(3,0,___)	(4,0,___)
(0,1,___)		(2,1,___)	(3,1,___)	(4,1,___)
(0,2,___)	(1,2,___)		(3,2,___)	(4,2,___)
(0,3,___)	(1,3,___)	(2,3,___)		(4,3,___)
(0,4,___)	(1,4,___)	(2,4,___)	(3,4,___)	

Seeing a form like this suggests converting this into a Latin square that we have to fill in:

0				
	1			
		2		
			3	
				4

As an example, look at the cell in the top right. Either a 1, 2, or 3 can be placed in that cell in the strategy table, as well as in the Latin square. This suggests that the problem of filling in a k by k Latin square, where the diagonal is already filled as above, is isomorphic to the Line of Sages problem for 3 people and k colors.

For odd k, the Latin square has a linear construction. If we create a "coordinate axis" around the Latin square, then we can fill in the cells by the function $f(x, y) \equiv 2x - y \pmod{k}$. The case for $k = 5$ colors is shown below.

	x	0	1	2	3	4
y						
0		0	2	4	1	3
1		4	1	3	0	2
2		3	0	2	4	1
3		2	4	1	3	0
4		1	3	0	2	4

For even k, the construction is not linear, but is based off the odd k construction. We take the Latin square of order $k - 1$, where $f(x, y) = 2x - y$ as before. We will take $k = 6$ as an example. First, we will show that the colored numbers are an arrangement of $0, 1, \cdots, k - 1$.

0	2	4	1	3	
4	1	3	0	2	
3	0	2	4	1	
2	4	1	3	0	
1	3	0	2	4	

The colored numbers are in cells of the form $(i, i + 1 \pmod{k})$, where $i \in 0, 1, \cdots, k - 1$. The number in the cell is

$$f(i, i + 1 \pmod{k}) \equiv 2i - (i + 1) \equiv i - 1 \pmod{k}$$

As i ranges from 0 to $k - 1$, $i - 1 \pmod{k}$ also ranges from 0 to $k - 1$. Therefore the colored numbers are an arrangement of $0, 1, \cdots, k - 1$. What this means is that we can now copy these colored numbers into a new row and column, and that new row and column will still be Latin:

0	2	4	1	3	2
4	1	3	0	2	3
3	0	2	4	1	4
2	4	1	3	0	0
1	3	0	2	4	1
1	2	3	4	0	

Finally, we replace the original colored numbers with k's (in this case, 5's) and also put a k in the lower right corner, which finishes the construction. Thus, we have shown that $HAT_{LS}(3, k) \geq 2$.

0	5	4	1	3	2
4	1	5	0	2	3
3	0	2	5	1	4
2	4	1	3	5	0
5	3	0	2	4	1
1	2	3	4	0	5

Also, we know $HAT_{LS}(3, k) \leq 2$, because the adversary can always force the first person in line to be incorrect. Thus, we have shown that $HAT_{LS}(3, k) = 2$, as desired.

3.2.6 Computer Results

The Line of Sages problem has a better time complexity than the $\mathrm{HAT_S}(G, k)$ problems. Therefore, we created a Java program that counts the number of possible successful strategies for $HAT_{LS}(n, k)$.

		Colors (k)								
	X	1	2	3	4	5	6	7	8	9
People (n)	1	x	0	0	0	0	0	0	0	0
	2	x	x	2	9	44	265	1854	14833	133496
	3	x	x	x	2	48	10752	49390080	?	?
	4	x	x	x	x	2	108	?	?	?
	5	x	x	x	x	x	2	180	?	?
	6	x	x	x	x	x	x	2	1+	?
	7	x	x	x	x	x	x	x	2	?
	8	x	x	x	x	x	x	x	x	2
	9	x	x	x	x	x	x	x	x	x

This table shows the number of possible successful strategies for n people and k colors.

The row for $n = 2$ people is actually the same as the number of derangements of k numbers. A **derangement** is a permutation of $\{0, 1, 2, \cdots, k\}$ such that no number stays in its original spot.

The strategy notation, which is simply k ordered pairs, makes the isomorphism to derangements apparent. We start by writing all possible $(n-1)$-tuples, which is $\{0, 1, 2, \cdots, k\}$.

Then, we fill in the spaces such that no ordered pairs tell the people to guess the same number. In other words– the strategy is a derangement of $\{0, 1, 2, \cdots, k\}$!

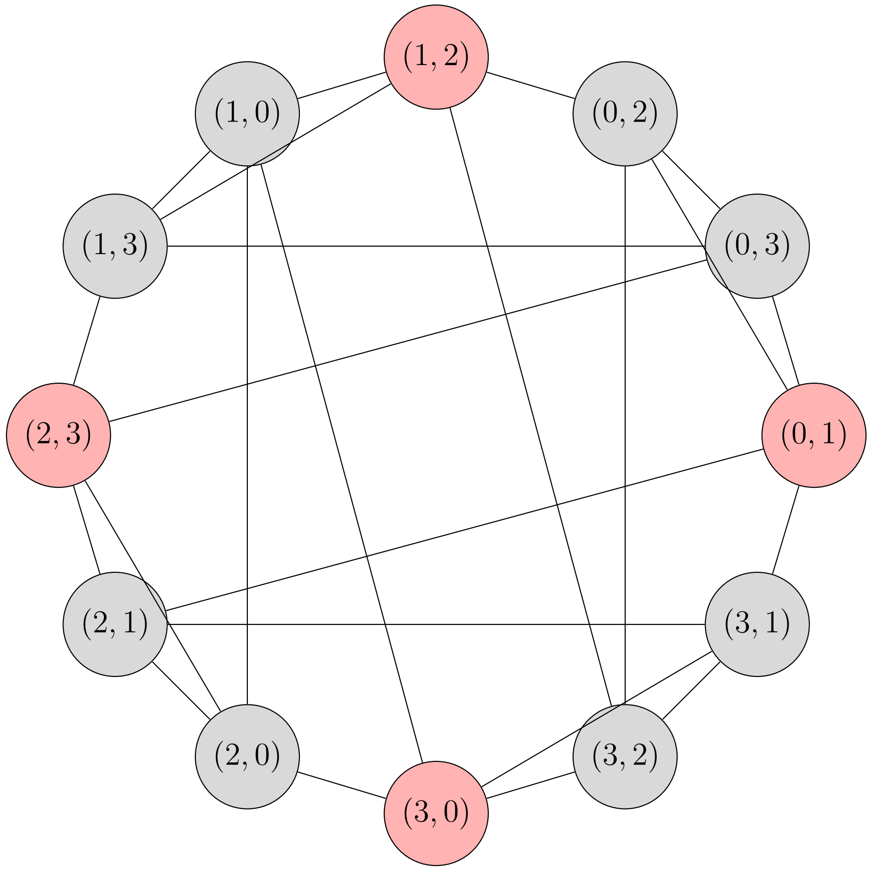
(0,___)
(1,___)
(2,___)
(3,___)
(4,___)

Another inspired result from the computer results is that the number of successful strategies for $HAT_{LS}(n, k)$ is divisible by $k-n$. Consider the tuple $(0, 1, 2, \cdots, n-1, \_\_\_)$ in the strategy. There are $k - n$ possible numbers to fill in. Due to symmetry, for each of the $k - n$ possible numbers, there are the same number of strategies after this tuple is filled in. Therefore, the number of successful strategies is divisible by $k - n$.

Our program suggests that $HATS_{LS}(n, k) = n - 1$ in general, since a successful strategy exists for all values of (n, k) where the program terminated. However, for larger (n, k), the program takes far too long. For example, $HAT_{LS}(3, 7)$ took 1 day to run, and we believe $HAT_{LS}(3, 8)$ will take the order of months to terminate.

3.2.7 Application to Graph Theory

The Line of Sages problem is also isomorphic to finding the maximal independent set of a graph. Each vertex is a n-tuple, and vertices share an edge if having both in the same strategy would create an ambiguity. This is equivalent to a Hamming graph for $n = 2$, where vertices share an edge if they differ by exactly one coordinate. In the figure below, the red vertices are an independent set, and thus form a successful strategy for $HAT_{LS}(2, 4)$.



So, the Line of Sages problem is isomorphic to finding an independent set, as well as finding Latin Squares (or n-dimensional cubes), an interesting result.

4 References

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