A Recap of Predicate Logic (Part 2)

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Recap: Some Important Logical Equivalences

For all sets U and predicates $p(x)$:

$$\begin{aligned} \neg(\forall x \in U,\ p(x)) &\equiv \exists x \in U \text{ such that } \neg p(x) \\ \neg(\exists x \in U \text{ such that } p(x)) &\equiv \forall x \in U,\ \neg p(x) \end{aligned}$$

A very special case:

$$\begin{aligned} \exists x \in \emptyset \text{ such that } p(x) &\equiv F \\ \forall x \in \emptyset,\ p(x) &\equiv T \end{aligned}$$

Two additional logical equivalences:

- $\forall x \in U,\ (p(x) \wedge q(x)) \equiv (\forall x \in U,\ p(x)) \wedge (\forall x \in U,\ q(x))$
- $\exists x \in U$ such that $(p(x) \vee q(x)) \equiv$ $(\exists x \in U \text{ such that } p(x)) \vee (\exists x \in U \text{ such that } q(x))$

Recap: Common Idioms

$\forall y \in U, \ (P(y) \land Q(y))$

For every y in set U, both $P(y)$ and $Q(y)$ are true.
("Every element in U satisfies both P and Q.")

$\exists y \in U$ such that $(P(y) \land Q(y))$

There is some y in set U such that both $P(y)$ and $Q(y)$ are true.
("Some element in U satisfies both P and Q.")

$\forall y \in U, \ (P(y) \to Q(y))$

For every y in set U, if $P(y)$ is true, then $Q(y)$ is true.
("Every element in U that satisfies P also satisfies Q.")

$\exists y \in U$ such that $(P(y) \to Q(y))$ (Not generally useful)

There exists at least one y in set U such that $P(y) \to Q(y)$ is true.
("There is some element y in U for which either $\neg P(y)$ or $Q(y)$ is true.")

Nested Quantifiers

Suppose we have the following:

- S is the set of all SU students
- P is the set of all SU professors
- $L(x, y)$ is the predicate "x likes y"

What's the difference between the following?

- $\forall y \in S, \ \exists w \in P$ such that $L(y, w)$
  For each SU student y, there is an SU professor w such that y likes w.
  (Every SU student likes some SU professor.)
- $\exists w \in P$ such that $\forall y \in S, \ L(y, w)$
  There is an SU professor w such that, for all SU students y, y likes w.
  (There is a particular SU professor whom every SU student likes.)

More on Nested Quantifiers

As a simple example, suppose:

- $X = \{a, b, c\}$, $Y = \{1, 2\}$

$\forall x \in X,\ \exists y \in Y$ such that $A(x, y)$

$$\begin{aligned} &\equiv\ (\exists y \in Y \text{ such that } A(a, y)) \ \wedge\ (\exists y \in Y \text{ such that } A(b, y)) \\ &\qquad\qquad \wedge\ (\exists y \in Y \text{ such that } A(c, y)) \\ &\equiv\ (A(a,1) \vee A(a,2)) \ \wedge\ (A(b,1) \vee A(b,2)) \ \wedge\ (A(c,1) \vee A(c,2)) \end{aligned}$$

$\exists y \in Y$ such that $\forall x \in X,\ A(x, y)$

$$\begin{aligned} &\equiv\ (\forall x \in X,\ A(x,1)) \ \vee\ (\forall x \in X,\ A(x,2)) \\ &\equiv\ (A(a,1) \wedge A(b,1) \wedge A(c,1)) \ \vee\ (A(a,2) \wedge A(b,2) \wedge A(c,2)) \end{aligned}$$

Order of Nested Quantifiers

As we saw, the order of quantifiers generally matters:

- Always read from the outside in.
- If all quantifiers are $\forall$, their relative ordering does not matter.
- If all quantifiers are $\exists$, their relative ordering does not matter.

Let's Take Another Break from Slides

Our Task:

Translate between predicate logic and English, using nested quantifiers.

Universal Instantiation

Principle of Specification (aka Universal Instantiation)

If the premises

$$\forall x \in U, \; p(x)$$
$$a \in U$$

both hold true, then the conclusion

$$p(a)$$

also holds true.

Example

Let C be set of people in CIS 275; let $A(x)$ be "x will get an A".
Suppose these premises are both true:

$$\forall x \in C, \; A(x) \qquad \text{and} \qquad Zeke \in C.$$

Then $A(Zeke)$ must also be true.

Using Universal Instantiation in a Formal Proof

Show that the following is a valid argument:

$$\begin{array}{l} \forall n \in W, \ (p(n) \to q(n)) \\ 5 \in W \\ p(5) \\ \therefore q(5) \end{array}$$

1.	$\forall n \in W, \ p(n) \to q(n)$	Given
2.	$5 \in W$	Given
3.	$p(5)$	Given
4.	$p(5) \to q(5)$	1,2 Universal Instantiation
5.	$q(5)$	3,4 Modus Ponens

Universal Generalization

Principle of Generalization (aka Universal Generalization)

From the following steps:

(i) Take an arbitrary element $a \in U$

(ii) Establish that $p(a)$ holds

the conclusion

$$\forall x \in U, \ p(x)$$

can be obtained.

What do we mean by arbitrary?

We are introducing a name (e.g., a) to represent a generic element of U: we assume nothing about a except that it possesses the general properties of elements of U.

Using Universal Generalization in a Formal Proof

Show that the following is a valid argument:

$$\forall x \in U,\ p(n) \to q(n)$$
$$\forall x \in U,\ p(x)$$
$$\therefore \forall x \in U,\ q(x)$$

1.	$\forall x \in U,\ p(x) \to q(x)$	Given
2.	$\forall x \in U,\ p(x)$	Given
3.	Let $v \in U$ be arbitrary.	Assumption
4.	$p(v) \to q(v)$	1,3 Universal Instantiation
5.	$p(v)$	2,3 Universal Instantiation
6.	$q(v)$	4,5 Modus Ponens
7.	$\forall x \in U,\ q(x)$	3, 6 Universal Generalization

A Bogus Use of Universal Generalization

Here's a bogus proof:

1.	$5 < 17$	Given
2.	$5 \in \mathbb{Z}$	Fact
3.	$\forall x \in \mathbb{Z},\ x < 17$	1, 2 Universal Generalization

Note:

- Step 3 is not allowed: 5 is a specific element of $\mathbb{Z}$, not arbitrary.

The problem does not disappear by renaming:

1.	$b < 17$	Given
2.	$b \in \mathbb{Z}$	Given
3.	$\forall x \in \mathbb{Z},\ x < 17$	1, 2 Universal Generalization