A Recap of Propositional Logic

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Propositions

Propositions are declarative statements that are either true or false (but not both), such as:

- It is snowing right now in Syracuse, NY.
- Syracuse University's athletic mascot is a piece of fruit.
- CIS 275 will be your favorite class ever.
- There is a positive integer that is less than every other positive integer.
- Every even integer greater than 2 can be expressed as the sum of two primes.

(We might not know whether it's true or false, but we know that it's true or false.)

Some statements that are not propositions:

- What courses are you taking this semester?
- Tell me about your computing background.
- This statement is false.

Syntax Versus Semantics

Propositional logic is a formalism for reasoning about the truth values of (compound) propositions.

But first, an informal quiz: Which number is bigger?

$$\Huge 3 \qquad\qquad \small 5$$

- If we're talking about syntax (i.e., what's written), then the numeral 3 above is clearly bigger/taller/thicker than the numeral 5.
- If we're talking about semantics (i.e., what the symbols mean), then the value 5 is greater than the value 3.

This distinction between syntax and semantics is vital for studying any language, including logics.

Propositional Logic: Syntax

The syntax of propositional logic relies on two sets:

- Propositional variables are used to represent propositions.
  Convention for this course: Lowercase letters $p, q, r, s, t, \ldots$
- Compound propositions are built from other propositions, using logical connectives:

$\neg p$	negation	("not p")
$p \wedge q$	conjunction	("p and q")
$p \vee q$	disjunction	("p or q")
$p \rightarrow q$	conditional	("p implies q", "if p then q")
$p \leftrightarrow q$	biconditional	("p if and only if q")

A sample compound proposition:

$$p \vee ((q \wedge \neg t) \rightarrow p)$$

Propositional Logic: Semantics

The semantics of propositional logic involves two truth values:

T ("true")
F ("false")

The truth value of a compound proposition depends upon the interpretation of the variables that occur in it:

> *The truth value of $(p \wedge q) \rightarrow r$ depends on the truth values of p, q, and r.*

Truth tables provide a syntax-directed means for calculating the meanings of propositions.

Truth Tables for Propositional Logic

p
T
F

φ	$\neg\varphi$
T	F
F	T

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$
T	T	T
T	F	F
F	T	F
F	F	F

φ_1	φ_2	$\varphi_1 \vee \varphi_2$
T	T	T
T	F	T
F	T	T
F	F	F

φ_1	φ_2	$\varphi_1 \rightarrow \varphi_2$
T	T	T
T	F	F
F	T	T
F	F	T

φ_1	φ_2	$\varphi_1 \leftrightarrow \varphi_2$
T	T	T
T	F	F
F	T	F
F	F	T

A Sample Truth Table

A truth table for the compound proposition $(p \wedge q) \rightarrow (p \leftrightarrow q)$:

p	q	$p \wedge q$	$p \leftrightarrow q$	$(p \wedge q) \rightarrow (p \leftrightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	T	T

Things to note:

- Each row corresponds to a particular interpretation of the variables. For n distinct variables, we require 2^n rows.
- Every subterm of the original proposition has its own column.
- $(p \wedge q) \rightarrow (p \leftrightarrow q)$ is a tautology: its value is always T, regardless of the interpretation.

Another Sample Truth Table

A truth table for the compound proposition $(p \rightarrow r) \wedge (q \vee \neg r)$:

p	q	r	$p \rightarrow r$	$\neg r$	$q \vee \neg r$	$(p \rightarrow r) \wedge (q \vee \neg r)$
T	T	T	T	F	T	T
T	T	F	F	T	T	F
T	F	T	T	F	F	F
T	F	F	F	T	T	F
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	T	T

- We now have $2^3 = 8$ rows.
- $(p \rightarrow r) \wedge (q \vee \neg r)$ is a contingency: its value is sometimes T and sometimes F, depending on the interpretation.
- A proposition whose value is always F is called a contradiction.

Some Comments on Conditionals

The conditional $p \rightarrow q$ translates into English as:

- If p, then q.
- Whenever p, then q.
- p only if q.
- q if p.

Consider a conditional $p \rightarrow q$:

- Its converse is $q \rightarrow p$.
- Its inverse is $\neg p \rightarrow \neg q$.
- ⋆ Its contrapositive is $\neg q \rightarrow \neg p$.

Important properties (verify with truth tables!):

- A conditional and its contrapositive always have the same truth value.
- A conditional's converse and inverse always have the same truth value.

Valid Logical Arguments

What makes for a valid argument?

- By argument, we mean:

  a line of reasoning

- By valid, we mean:

  Whenever all of the premises (aka hypotheses) are true, the conclusion must also be true.

  That is, we will never conclude something false from true premises.

  (If any of the premises is false, the conclusion can be either true or false; the argument remains valid.)

Because truth tables are impractical for this purpose, we use logical rules.

Logical Equivalences

Two propositions A and B are logically equivalent provided that they have the same truth values for all possible interpretations (i.e., when $A \leftrightarrow B$ is a tautology).

Example: $p \to q$ and $\neg p \vee q$ are logically equivalent:

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

We write $A \equiv B$ (or $B \equiv A$) to indicate that the propositions A and B are logically equivalent.

Key Logical Equivalences

Identity:

$$p \wedge T \equiv p$$
$$p \vee F \equiv p$$

Absorption:

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

Domination:

$$p \vee T \equiv T$$
$$p \wedge F \equiv F$$

Negation:

$$p \vee \neg p \equiv T$$
$$p \wedge \neg p \equiv F$$

Idempotent:

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

Double negation:

$$p \equiv \neg\neg p$$

More Key Logical Equivalences

Commutativity:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

Associativity:

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$
$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

Implication:

$$p \rightarrow q \equiv \neg p \vee q$$

Distributivity:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

De Morgan's laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Logical Implications

Proposition A logically implies proposition B (equivalently: B is a logical consequence of A) provided that every interpretation that makes A true also makes B true.

For example, $p \wedge q$ logically implies $p \vee q$:

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

The only interpretation that makes $p \wedge q$ true also makes $p \vee q$ true.

We write $A \Rightarrow B$ to indicate that A logically implies B.

Example: $(p \wedge q) \Rightarrow (p \vee q)$.

More on Logical Implications

More generally, the propositions $A_1, \ldots, A_n$ together logically imply B provided that any interpretation that makes every A_i true also makes B true. (Equivalently: When $(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to B$ is a tautology)

Example: $p \to q$ and $\neg q$ together logically imply $\neg p$:

p	q	$p \to q$	$\neg q$	$\neg p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The only interpretation that makes both $p \to q$ and $\neg q$ true also makes $\neg p$ true.

Still More on Logical Implications

The textbook writes

$$\begin{array}{l} A_1 \\ \vdots \\ A_n \\ \therefore B \end{array}$$

to indicate that the propositions $A_1, \ldots, A_n$ logically imply proposition B.

If you look at other sources, you may see something like

$$\frac{A_1 \quad \ldots \quad A_n}{\therefore B}$$

to denote the same thing.

Key Logical Implications ("Valid Argument Forms")

Direct Implication

p
$p \to q$ ("Modus Ponens")
$\therefore q$

Transitivity of $\to$

$p \to q$
$q \to r$ ("Hypothetical Syllogism")
$\therefore p \to r$

Contrapositive Implication

$p \to q$
$\neg q$ ("Modus Tollens")
$\therefore \neg p$

Two Separate Cases

$p \lor q$
$\neg p \lor r$ ("Resolution")
$\therefore q \lor r$

Some More Logical Implications

Eliminating a Possibility

$p \lor q$
$\neg p$ ("Disjunctive Syllogism")
$\therefore q$

Obtaining And

p
q ("Conjunction")
$\therefore p \land q$

In Particular

$p \land q$
$\therefore p$ ("Simplification")

Obtaining Or

p
$\therefore p \lor q$ ("Addition")

Substitution of Equivalent

$p \leftrightarrow q$
p ("Equivalence")
$\therefore q$

Formal Proofs

The logical equivalences and logical implications together form the derivation rules for propositional logic.

A formal proof is a sequence of annotated statements in the logic, where each statement is one of the following:

- An assumption (annotated by "Assumption" or "Given")
- Obtained by applying one of the derivation rules to previous statements in that sequence (annotated with appropriate rule)

In this class: All assumptions should appear at the very beginning of the formal proof.

A Sample Proof Sequence

Let's prove the Disjunctive Syllogism rule to be valid:

$$\begin{array}{l} p \vee q \\ \neg p \\ \therefore q \end{array}$$

A formal proof of its validity:

1.	$p \vee q$	Given
2.	$\neg p$	Given
3.	$p \leftrightarrow \neg\neg p$	Double negation
4.	$\neg\neg p \vee q$	1, 3 Substitution of Equivalent
5.	$\neg p \rightarrow q$	4, Implication
6.	q	2,5 Modus ponens

You'll often see steps 3 and 4 combined into a single step:

3'.	$\neg\neg p \vee q$	1 Double Negation, Equivalence
4'.	$\neg p \rightarrow q$	3', Implication
5'.	q	2,4' Modus ponens

Putting it All Together

Consider the following argument:

- If it does not rain or if it is not foggy, then the sailing race will be held.
- If the sailing race is held, then the trophy will be awarded.
- The trophy was not awarded.
- Therefore, it rained.

Let's come up with a formal proof that validates the argument:

- Use the first three statements as givens (aka premises or hypotheses).
- Conclude the fourth statement.

Putting it All Together: Solution

Use the following propositions:

$$\begin{aligned} r &: \textit{It rains} \\ f &: \textit{It is foggy} \\ s &: \textit{The sailing race is held} \\ t &: \textit{The trophy is awarded} \end{aligned}$$

1. $(\neg r \vee \neg f) \to s$	Given
2. $s \to t$	Given
3. $\neg t$	Given
4. $(\neg r \vee \neg f) \to t$	1,2 Hypothetical Syllogism
5. $\neg(\neg r \vee \neg f)$	3,4 Modus Tollens
6. $\neg\neg r \wedge \neg\neg f$	5 DeMorgan's
7. $\neg\neg r$	6, Simplification
8. r	7, Double Negation, Equivalence