A Recap of Predicate Logic (Part 1)

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Quick Overview of Sets

A set is a collection of objects.

- The objects in a set are called its elements.
- We write $x \in A$ to indicate that x is an element of set A.
- We write $x \notin A$ to indicate that x is **not** an element of set A.

Some very common sets:

- $\mathbb{Z}$, the set of *integers*
  $1 \in \mathbb{Z} \quad -5 \in \mathbb{Z} \quad 2.1 \notin \mathbb{Z}$
- $\mathbb{R}$, the set of *real numbers*
  $-5 \in \mathbb{R} \quad 2.1 \in \mathbb{R} \quad \sqrt{3} \in \mathbb{R} \quad \sqrt{-1} \notin \mathbb{R}$
- $\emptyset$, the empty set (also written as: $\{\}$)
  The set that contains no elements

Ways of Expressing Sets

List notation

- $\{1, 4, 5, 10, 17\}$
- $\{red, green, blue\}$

Set-builder notation

- $\{y : y \in \mathbb{Z} \text{ and } y \leq 100\}$
  "The set of those y such that y is in set $\mathbb{Z}$ and $y \leq 100$"
- $\{x : x \in \mathbb{R} \text{ and } x^3 = x\}$
  "The set of those x such that x is in set $\mathbb{R}$ and $x^3 = x$"

Note: You'll often see variations in this notation, such as:

- $\{y \in \mathbb{Z} : y \leq 100\}$
  "The set of those y in $\mathbb{Z}$ such that $y \leq 100$"
- $\{2x + 1 : x \in \mathbb{Z}\}$
  "The set of those numbers $2x + 1$ such that $x \in \mathbb{Z}$" (aka odd integers)

On to Predicates

Definition

A predicate is a declarative statement whose truth values depends on one or more variables.

Examples:

"x is tall" *"y is an athlete"* "$z \leq 100$" *"w likes t"*

When values are substituted in for the free (or unbound) variables, the result is a proposition.

Example

Suppose $Q(x)$ denotes "$x > 15$".

- $Q(3)$ is a proposition with truth value F.
- $Q(20)$ is a proposition with truth value T.

Universal Quantification

Suppose $p(x)$ is a predicate whose only unbound variable is x.

- The statement $\forall x \in U,\ p(x)$ is read as:
  "For all elements x in the set U, $p(x)$ is true".
- This statement is defined to be true if and only if,
  for every value v in the set U, the proposition $p(v)$ is true.
- The quantifier $\forall$ "binds" x, which means:
  x is bound (and no longer free) in the full statement.

Example

Let $\mathcal{S} = \{Ava, Bo, Cam, Dee\}$, and $C(x)$ be "x was in class today".

- $\forall x \in S,\ C(x)$ is true iff $C(Ava) \wedge C(Bo) \wedge C(Cam) \wedge C(Dee)$ is true.
  (Thus, it's true iff all four students were in class today.)

Existential Quantification

Suppose $p(x)$ is a predicate whose only unbound variable is x.

- The statement $\exists x \in U$ such that $p(x)$ is read as:
  "There exists an element x in set U such that $p(x)$ is true."
- This statement is defined to be true if and only if,
  there exists at least one value v in the set U for which the proposition $p(v)$ is true.
- The quantifier $\exists$ "binds" x, which means:
  x is bound (and no longer free) in the full statement.

Example

Let $\mathcal{S} = \{Ava, Bo, Cam, Dee\}$, and $C(x)$ be "x was in class today".

- $\exists x \in U$ such that $p(x)$ is true iff
  $C(Ava) \vee C(Bo) \vee C(Cam) \vee C(Dee)$ is true.
  (Thus, it's true iff at least one of the four students was in class today.)

A Thought Experiment

Suppose $W = \{1, 3, 5, 7, 9\}$ and $J(x)$ is a predicate over W.

We've already seen:

$$\begin{aligned} \forall x \in W,\ J(x) &\equiv J(1) \wedge J(3) \wedge J(5) \wedge J(7) \wedge J(9) \\ \exists x \in W \text{ such that } J(x) &\equiv J(1) \vee J(3) \vee J(5) \vee J(7) \vee J(9) \end{aligned}$$

Therefore, we can see:

$$\begin{aligned} \neg(\forall x \in W,\ J(x)) &\equiv \neg(J(1) \wedge J(3) \wedge J(5) \wedge J(7) \wedge J(9)) \\ &\equiv \neg J(1) \vee \neg J(3) \vee \neg J(5) \vee \neg J(7) \vee \neg J(9) \\ &\equiv \exists x \in W \text{ such that } \neg J(x) \\ \neg(\exists x \in W \text{ such that } J(x)) &\equiv \neg(J(1) \vee J(3) \vee J(5) \vee J(7) \vee J(9)) \\ &\equiv \neg J(1) \wedge \neg J(3) \wedge \neg J(5) \wedge \neg J(7) \wedge \neg J(9) \\ &\equiv \forall x \in W,\ \neg J(x) \end{aligned}$$

$\forall$ and $\exists$ are Duals

For all sets U and predicates $p(x)$:

$$\begin{aligned} \neg(\forall x \in U,\ p(x)) &\equiv \exists x \in U \text{ such that } \neg p(x) \\ \neg(\exists x \in U \text{ such that } p(x)) &\equiv \forall x \in U,\ \neg p(x) \end{aligned}$$

A very special case:

$$\begin{aligned} \exists x \in \emptyset \text{ such that } p(x) &\equiv F \\ \forall x \in \emptyset,\ p(x) &\equiv T \end{aligned}$$

Three Common Idioms (Plus a Non-idiom)

$\forall y \in U,\ (P(y) \wedge Q(y))$

For every y in set U, both $P(y)$ and $Q(y)$ are true.
("Every element in U satisfies both P and Q.")

$\exists y \in U$ such that $(P(y) \wedge Q(y))$

There is some y in set U such that both $P(y)$ and $Q(y)$ are true.
("Some element in U satisfies both P and Q.")

$\forall y \in U,\ (P(y) \rightarrow Q(y))$

For every y in set U, if $P(y)$ is true, then $Q(y)$ is true.
("Every element in U that satisfies P also satisfies Q.")

$\exists y \in U$ such that $(P(y) \rightarrow Q(y))$ (Not generally useful)

There exists at least one y in set U such that $P(y) \rightarrow Q(y)$ is true.
("There is some element y in U for which either $\neg P(y)$ or $Q(y)$ is true.")

Let's Take a Break from Slides

Our Task:

Translate between predicate logic and English.

Nested Quantifiers

Suppose we have the following:

- S is the set of all SU students
- P is the set of all SU professors
- $L(x, y)$ is the predicate "x likes y"

What's the difference between the following?

- $\forall y \in S,\ \exists w \in P$ such that $L(y, x)$
  For each SU student y, there is an SU professor w such that y likes w. (Every SU student likes some SU professor.)
- $\exists w \in P$ such that $\forall y \in S,\ L(y, x)$
  There is an SU professor w such that, for all SU students y, y likes w. (There is a particular SU professor whom every SU student likes.)

More on Nested Quantifiers

As a simple example, suppose:

- $X = \{a, b, c\},\ Y = \{1, 2\}$

$\forall x \in X,\ \exists y \in Y$ such that $A(x, y)$

$$\begin{aligned} &\equiv\ (\exists y \in Y \text{ such that } A(a, y)) \ \wedge\ (\exists y \in Y \text{ such that } A(b, y)) \\ &\qquad\qquad \wedge\ (\exists y \in Y \text{ such that } A(c, y)) \\ &\equiv\ (A(a, 1) \vee A(a, 2)) \ \wedge\ (A(b, 1) \vee A(b, 2)) \ \wedge\ (A(c, 1) \vee A(c, 2)) \end{aligned}$$

$\exists y \in Y$ such that $\forall x \in X,\ A(x, y)$

$$\begin{aligned} &\equiv\ \forall x \in X,\ A(x, 1) \ \vee\ \forall x \in X,\ A(x, 2) \\ &\equiv\ (A(a, 1) \wedge A(b, 1) \wedge A(c, 1)) \ \vee\ (A(a, 2) \wedge A(b, 2) \wedge A(c, 2)) \end{aligned}$$

Order of Nested Quantifiers

As we saw, the order of quantifiers generally matters:

- Always read from the outside in.
- If all quantifiers are $\forall$, their relative ordering does not matter.
- If all quantifiers are $\exists$, their relative ordering does not matter.

Let's Take Another Break from Slides

Our Task:

Translate between predicate logic and English, using nested quantifiers.