A Recap of Predicate Logic (Part 3)

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Interpretations: A First Example

Some interpretations that make $\forall x \in U,\ (p(x) \rightarrow q(x))$ true:

1. $U = \mathbb{Z}$, $p(x)$ is "$x \geq 0$", $q(x)$ is "$x \leq x * 10$"

   For every integer n, if $p(n)$ is true (i.e., n is non-negative), then $q(n)$ is true (i.e., $n \leq n * 10$). Thus, $\underline{\forall x \in U,\ (p(x) \rightarrow q(x))}$ is true.

2. $U = \{red, blue\}$, $p(red) = q(red) = q(blue) = T$, $p(blue) = F$

   Under this interpretation, both of the following are true:

   $$p(red) \rightarrow q(red)$$
   $$p(blue) \rightarrow q(blue)$$

   Therefore, $\underline{\forall x \in U,\ (p(x) \rightarrow q(x))}$ is true.

3. $U = \emptyset$, $p(x)$ and $q(x)$ can be anything

   $\forall x \in \emptyset,\ (p(x) \rightarrow q(x))$ is vacuously true.

Interpretations: A Second Example

Some interpretations that make $\forall x \in U,\ (p(x) \rightarrow q(x))$ false:

1. $U = \mathbb{Z}$, $p(x)$ is "x is even", $q(x)$ is "$x \leq x * 10$"

   -2 is even, but it's not the case that $-2 \leq -20$, which means $p(-2) \rightarrow q(-2)$ is false. Hence $\underline{\forall x \in U,\ (p(x) \rightarrow q(x))}$ is false.

2. $U = \{red, blue\}$, $p(red) = q(red) = q(blue) = F$, $p(blue) = T$

   Under this interpretation, $p(blue) \rightarrow q(blue)$ is false. Therefore, $\underline{\forall x \in U,\ (p(x) \rightarrow q(x))}$ is false.

Logical Equivalence for Predicate Logic

Definition

Two predicate-logic formulas φ_1 and φ_2 are logically equivalent (written $\varphi_1 \equiv \varphi_2$) exactly when:

> *Each has the same truth values as the other, regardless of the choice of quantified sets or the interpretations of the predicates.*

Examples we've seen before:

- $\neg(\exists y \in U \text{ such that } Q(y)) \equiv \forall y \in U,\ \neg Q(y)$
- $\forall x \in U,\ (p(x) \wedge q(x)) \quad \equiv \quad (\forall x \in U,\ p(x)) \wedge (\forall x \in U,\ q(x))$

How to Show Two Formulas are Not Logically Equivalent

To show that φ_1 and φ_2 are not logically equivalent:

It suffices to come up with one specific set of interpretation(s) for which the formulas have different truth values.

Example

Consider the following two formulas:

$$\exists y \in U \text{ such that } \forall z \in W,\ Q(y,z)$$
$$\forall z \in W,\ (\exists y \in U \text{ such that } Q(y,z))$$

One way to show they're not logically equivalent:

Take $U = W = \mathbb{Z}$, and let $Q(x,w)$ be "$x \leq w$".

- $\underline{\exists y \in U \text{ such that } \forall z \in W,\ Q(y,z)}$ is false: no integer is less than or equal to *every* integer.
- $\underline{\forall z \in W,\ (\exists y \in U \text{ such that } Q(y,z))}$ is true: for each integer z, there is *some* integer that is less than or equal to z.

How to Show that an Argument Form is Not Valid

The following argument form is not valid:

$$\exists x \in U \text{ such that } (P(x) \to Q(x))$$
$$\exists y \in U \text{ such that } P(y)$$
$$\therefore \exists w \in U \text{ such that } Q(w)$$

How do we show that it's not valid?

Find a specific set U and interpretations for P and Q such that:

1. $\exists x \in U$ such that $(P(x) \to Q(x))$ is true, and
2. $\exists y \in U$ such that $P(y)$ is true, and
3. $\exists w \in U$ such that $Q(w)$ is false.

How to Show that an Argument Form is Not Valid (2)

How do we show that it's not valid?

Find a specific set U and interpretations for P and Q such that:

1. $\exists x \in U$ such that $(P(x) \to Q(x))$ is true, and
2. $\exists y \in U$ such that $P(y)$ is true, and
3. $\exists w \in U$ such that $Q(w)$ is false.

Example

Let $U = \{1, 2\}$, and let $P(1), Q(1), Q(2)$ be false and $P(2)$ be true. Then:

- $P(1) \to Q(1)$ is true; thus $\underline{\exists x \in U \text{ such that } (P(x) \to Q(x))}$ is true.
- $P(2)$ is true, and thus $\underline{\exists y \in U \text{ such that } P(y)}$ is true.
- However, both $Q(1)$ and $Q(2)$ are false, which means $\underline{\exists w \in U \text{ such that } Q(w)}$ is false.

How to Show that an Argument Form is Not Valid (3)

How do we show that it's not valid?

Find a specific set U and interpretations for P and Q such that:

- $\exists x \in U$ such that $(P(x) \to Q(x))$ is true
- $\exists y \in U$ such that $P(y)$ is true
- $\exists w \in U$ such that $Q(w)$ is false

Another Example

Let $U = \mathbb{Z}$, let $P(x)$ be "x is odd", and let $Q(x)$ be "$x = x + 1$". Then:

- Since $P(2)$ is false, $P(2) \to Q(2)$ is vacuously true. Thus, $\underline{\exists x \in U \text{ such that } (P(x) \to Q(x))}$ is true.
- $P(1)$ is true, and thus $\underline{\exists y \in U \text{ such that } P(y)}$ is true.
- However, there is no integer w such that $w = w + 1$. Therefore, $\underline{\exists w \in U \text{ such that } Q(w)}$ is false.

Recap: Universal Instantiation

Principle of Specification (aka Universal Instantiation)

If the premises

$$\forall x \in U,\ p(x)$$
$$a \in U$$

both hold true, then the conclusion

$$p(a)$$

also holds true.

Example

Let C be set of people in CIS 275; let $A(x)$ be "x will get an A". Suppose these premises are both true:

$$\forall x \in C,\ A(x) \qquad \text{and} \qquad Zeke \in C.$$

Then $A(Zeke)$ must also be true.

Recap: Using Universal Instantiation in a Formal Proof

Show that the following is a valid argument:

$$\forall n \in W,\ (p(n) \to q(n))$$
$$5 \in W$$
$$p(5)$$
$$\therefore q(5)$$

1.	$\forall n \in W,\ p(n) \to q(n)$	Given
2.	$5 \in W$	Given
3.	$p(5)$	Given
4.	$p(5) \to q(5)$	1,2 Universal Instantiation
5.	$q(5)$	3,4 Modus Ponens

Recap: Universal Generalization

Principle of Generalization (aka Universal Generalization)

From the following steps:

$$(i) \quad \text{Take an arbitrary element } a \in U$$
$$(ii) \quad \text{Establish that } p(a) \text{ holds}$$

the conclusion

$$\forall x \in U,\ p(x)$$

can be obtained.

What do we mean by arbitrary?

We are introducing a name (e.g., a) to represent a generic element of U: we assume nothing about a except that it possesses the general properties of elements of U.

Recap: Using Universal Generalization in a Formal Proof

Show that the following is a valid argument:

$$\forall x \in U,\ p(n) \to q(n)$$
$$\forall x \in U,\ p(x)$$
$$\therefore \forall x \in U,\ q(x)$$

1.	$\forall x \in U,\ p(x) \to q(x)$	Given
2.	$\forall x \in U,\ p(x)$	Given
3.	Let $v \in U$ be arbitrary.	Assumption
4.	$p(v) \to q(v)$	1,3 Universal Instantiation
5.	$p(v)$	2,3 Universal Instantiation
6.	$q(v)$	4,5 Modus Ponens
7.	$\forall x \in U,\ q(x)$	3, 6 Universal Generalization

Recap: A Bogus Use of Universal Generalization

Here's a bogus proof:

1.	$5 < 17$	Given
2.	$5 \in \mathbb{Z}$	Fact
3.	$\forall x \in \mathbb{Z},\ x < 17$	1, 2 Universal Generalization

Note:

- Step 3 is not allowed: 5 is a specific element of $\mathbb{Z}$, not arbitrary.

The problem does not disappear by renaming:

1.	$b < 17$	Given
2.	$b \in \mathbb{Z}$	Given
3.	$\forall x \in \mathbb{Z},\ x < 17$	1, 2 Universal Generalization

Existential Generalization (Not in textbook)

Existential Generalization

If the premises $p(a)$ and $a \in U$ both hold true, then the conclusion

$$\exists x \in U \text{ such that } p(x)$$

can be obtained.

Example

Suppose we know that $Q(17)$ and $17 \in \mathbb{Z}$. We can therefore deduce that

$$\exists x \in \mathbb{Z} \text{ such that } Q(x)$$

Existential Instantiation (Not in textbook)

Existential Instantiation

If the premise

$$\exists x \in U \text{ such that } p(x)$$

holds and a is a fresh name (i.e., not previously used in the proof), then the conclusion

$$p(a)$$

can be deduced.

What is the idea here?

We are introducing the name a as a way to refer to the specific value of U for which $p(a)$ holds.

IMPORTANT: the name a is not considered to be arbitrary.

Using Existential Instantiation in a Formal Proof

Show that the following is a valid argument:

$$\begin{array}{l} \forall x \in U,\ P(x) \rightarrow Q(x) \\ \exists y \in U \text{ such that } P(y) \\ \therefore \exists w \in U \text{ such that } Q(w) \end{array}$$

1.	$\forall x \in U,\ P(x) \rightarrow Q(x)$	Given
2.	$\exists y \in U$ such that $P(y)$	Given
3.	$P(c)$	2, Existential Instantiation (fresh $c \in U$)
4.	$P(c) \rightarrow Q(c)$	1, Universal Instantiation (since $c \in U$)
5.	$Q(c)$	3,4 Modus Ponens
6.	$\exists w \in U$ such that $Q(w)$	5, Existential Generalization (since $c \in U$)

Example of a Bogus Proof

Consider an attempt to prove the following (which is not valid):

$$\exists x \in U \text{ such that } (P(x) \to Q(x))$$
$$\exists y \in U \text{ such that } P(y)$$
$$\therefore \exists w \in U \text{ such that } Q(w)$$

1.	$\exists x \in U$ such that $(P(x) \to Q(x))$	Given
2.	$\exists y \in U$ such that $P(y)$	Given
3.	$P(k)$	2, Existential inst (fresh $k \in U$)
4.	$P(k) \to Q(k)$	1, Existential instantiation
5.	$Q(k)$	3,4 Modus ponens
6.	$\exists w \in U$ such that $Q(w)$	5, Existential generalization

Notes:

- Step 4 is not allowed: we can not use k for this EI step, because k is no longer fresh.

Another Bogus Proof

Consider an attempt to prove the following (which is not valid):

$$\exists x \in U \text{ such that } P(x)$$
$$\therefore \forall x \in U,\ P(x)$$

1.	$\exists x \in U$ such that $P(x)$	Given
2.	$P(k)$	1, Existential instantiation (fresh $k \in U$)
3.	$\forall x \in U,\ P(x)$	2, Universal generalization

Notes:

- Step 3 is not allowed: UG does not allow us to generalize from an element that was introduced through EI.