Shallow Water Waves

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Abstract

We study the nature of waves in shallow water, measure the rate of decay, and observe the properties of solitons.

Introduction

To measure the resonant frequencies of these water waves, we slowly sweep the range from 0 Hz to 10Hz. This allows us to find where the frequency peaks over the range of frequencies. We must, however, do this range in smaller ranges, reseting the max range to the last known resonant frequency to get a more precise recording for the next range. To measure the decay constant, we drive the force at a known, well-measured resonant frequency, and then abruptly stop the force. This provides us with good data as the frequency decays back to the still water state. We perform this on more than one resonant frequency to measure the decay constants. To observe solitons, we use a popsicle stick to instigate one. Although, I, personally, was not able to create one of my own, my other lab partner could.

Theory

The surface waves of water are dispersive, which means that the phase velocity of the wave depends on the frequency. The frequency ω can be calculated from the wavenumber k and the height of the channel h with the following expression:

$$\omega^2 = gk \tanh kh \tag{1}$$

For very long wavelengths, where $kh << 1$, the hyperbolic tangent function can be approximated as $\tanh kh \approx kh$. Then the phase velocity will become:

$$c_{phase} = \frac{\omega}{k} = \sqrt{kh} \tag{2}$$

As the frequency increases, the dispersion must be included in the phase velocity calculation. Also, the wavelength becomes close enough to the capillary length of the water to change the results. Thus, surface tension restoring forces must also be included as a third-order term:

$$\omega^2 = \left(gk + \frac{\sigma}{\rho}k^3\right)\tanh kh \tag{3}$$

where σ is the surface tension, and ρ is the liquid density. The liquid density of water is simply $1g/cm^3$. To calculate the surface tension of water, we use Sears and Zemansky's equation:

$$\sigma = \frac{\rho g r h}{2\cos\theta} \tag{4}$$

However, we use a wetting agent called Photo Flo to reduce the angle θ to 0, simplifying the equation to:

$$\sigma = \frac{1}{2}\rho g r h \tag{5}$$

We use a capillary tube, of which the radius is easily calculated using the colume of a cylinder, measure the height at which the water flows up into the tube, and calculate the surface tension this way.

Apparatus and Procedure

In this experiment, we set up a long, narrow channel, with a driving force on one end, and a recording device on the other. We fill it with water and change the frequency of the "speaker" over different ranges to determine the resonant frequencies. Our recording device is simply two metal wires, which record a voltage between the two. They are as close to the end as can be without touching the end of the channel. The voltage between the two metal rods depends on how much water is between the two allowing them to interact with each other. As the water level rises, the voltage becomes higher, and as the water level lowers, the voltage level lowers. This allows us to perform a fast Fourier Transform to determine the amplitude of the water at any point in time. We also had to of course zero the voltage between the two rods, in still water.

For the solitons, we simply set the frequency at a known resonance, and then used a popsicle stick to start the soliton.

Data and Analysis

Resonant Frequencies

Below are plots of my three ranges of data, 0.5 to 3.5, 3.5 to 7.0, and 7.0 to 10.0. One can notice that there are slight outliers at the end of each data range, though these do not affect the measurements I took.

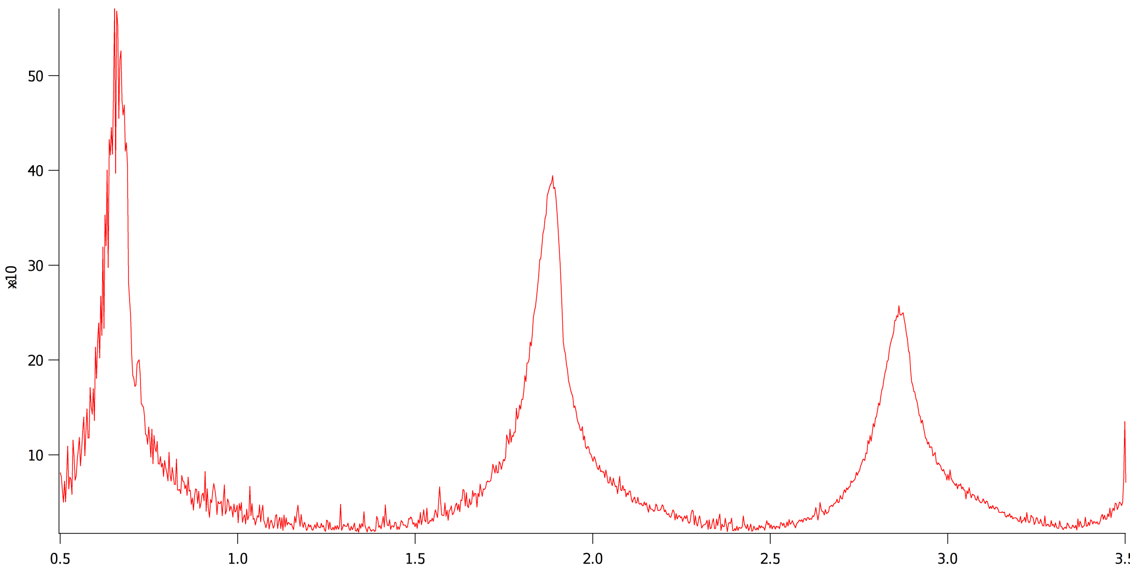


Figure 1: Amplitude (m) vs. Frequency (Hz), 0.5 to 3.5 Hz

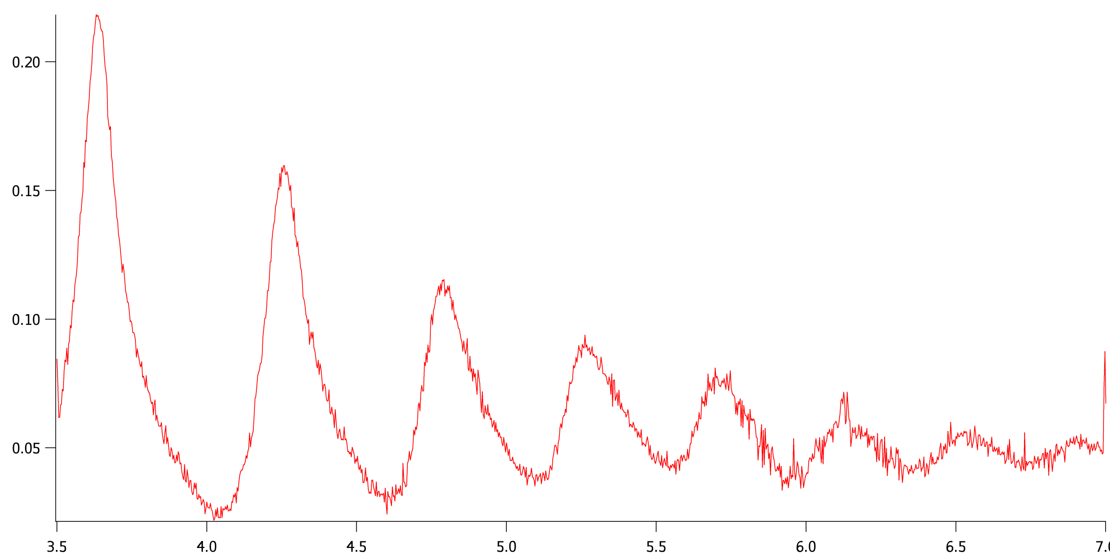


Figure 2: Amplitude (m) vs. Frequency (Hz), 3.5 to 7.0 Hz

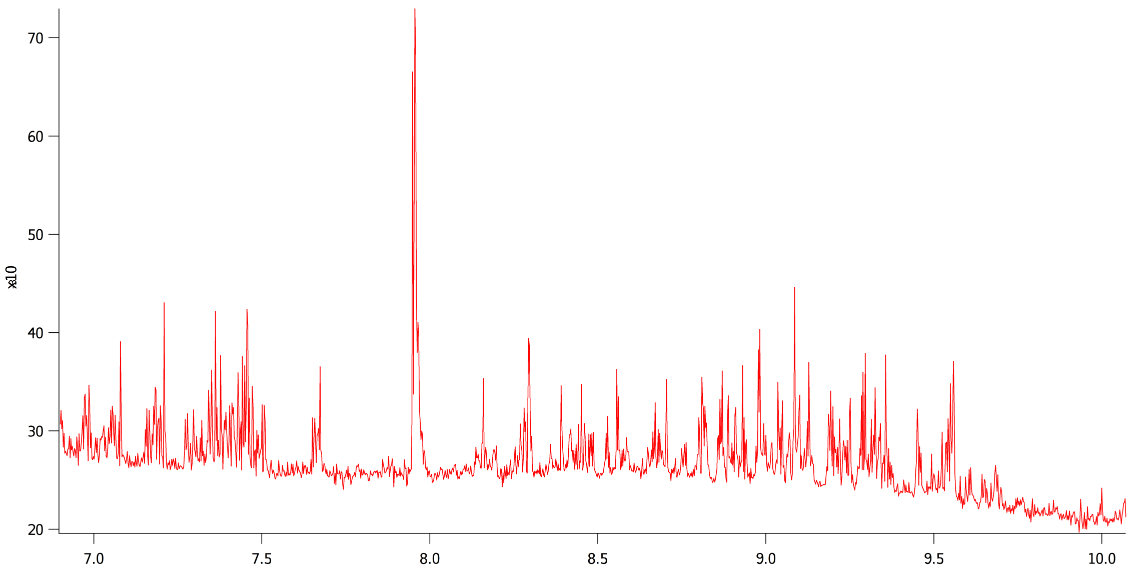


Figure 3: Amplitude (m) vs. Frequency (Hz), 7.0 to 10.0 Hz

One can quickly see that the third range is most definitely not nearly clear enough to gather data from. Also, the spike around the 8.0 Hz mark is also very suspect. The resonant frequencies and their corresponding measured phase velocities are listed in the table below:

n	Wavenumber k (rads/m)	Frequency ω (Hz)	Measured Phase Velocity c_p (m/s)
1	8.377	0.65908	0.07867
3	25.132	1.8867	0.07507
5	41.888	2.8682	0.06847
7	58.643	3.6369	0.06201
9	75.398	4.259	0.05649
11	92.153	4.7918	0.05200
13	108.909	5.268	0.04837
15	125.664	5.7115	0.04545
17	142.419	6.1253	0.043009
19	159.174	6.5479	0.041136

Below is a plot of the measured phase velocity versus the wavenumber:

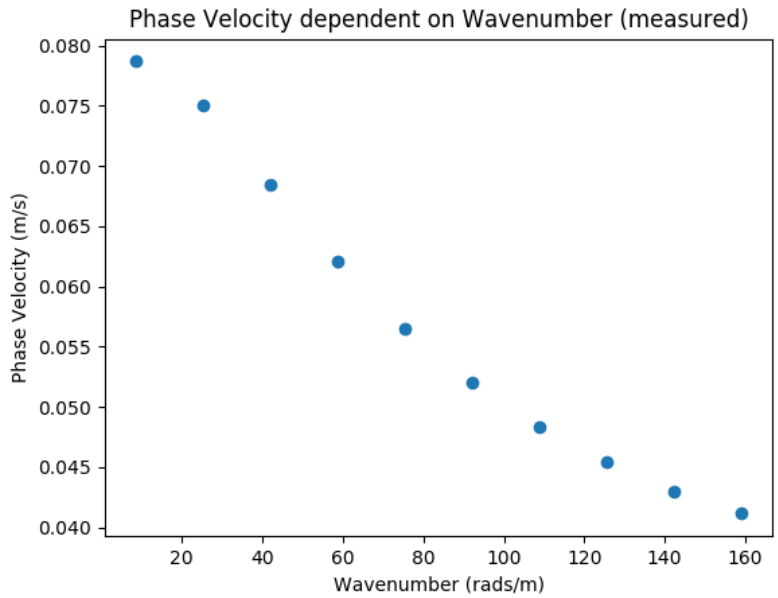


Figure 4: Wavenumber vs. Phase Velocity, measured values

Below is a plot depicting the calculated version of the phase velocity using the equations stated in the introduction. There are two plots, one with surface tension, and the other without.

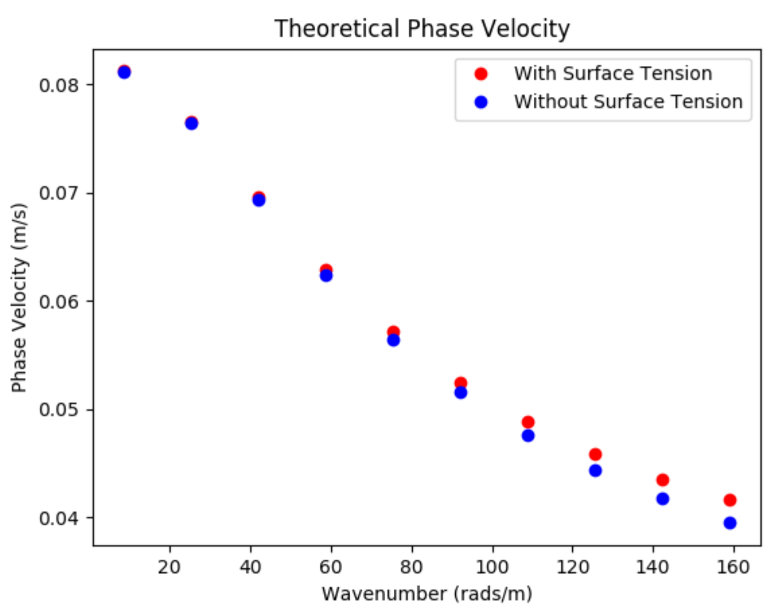
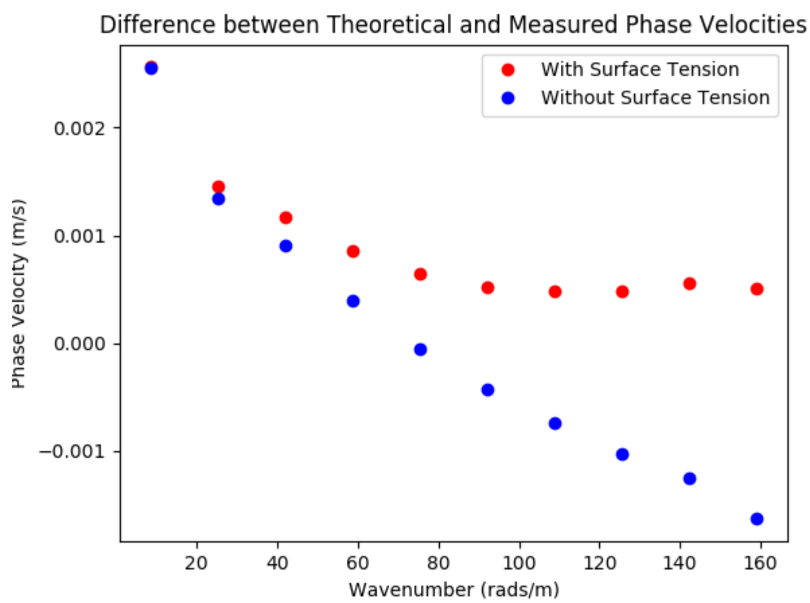


Figure 5: Theoretical Phase Velocities, with and without surface tension.

As we can see, the surface tension doesn't begin to affect the calculations significantly until around the fourth or fifth wavenumber. A better way to see the difference between how necessary the surface tension factor is, is to compare the difference between the theoretical and measured values.



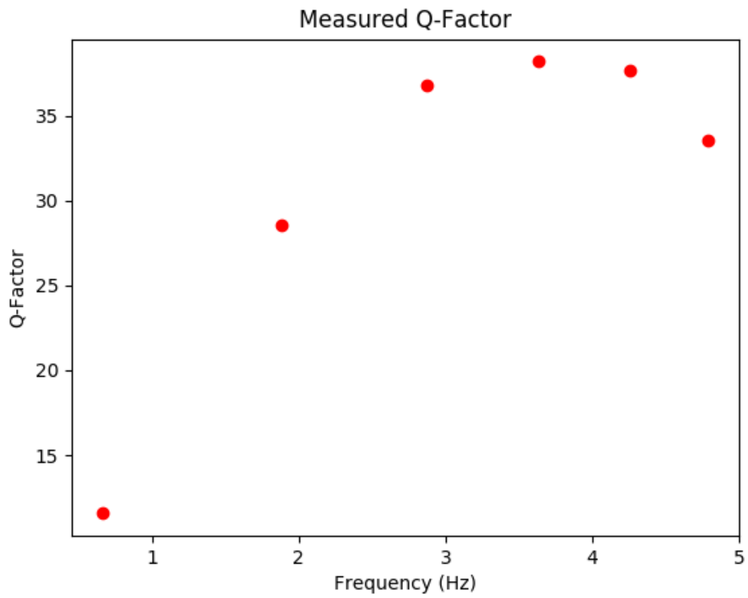
In the plot above, we can see that around the fifth wavenumber, the theoretical value with the surface tension factor seems to level off, whereas, the value without the surface tension continues its trend downwards. The ideal situation would be where the difference between the theoretical and calculated values are 0, but the fact that the value with the surface tension levels off, means that the error in our experiment can be contributed to some constant factor. Were we to continue taking data, far past these values, one can notice that using the calculation without the surface tension term would lead to a high amount of error, whereas, with the surface tension factor, we would have a small error. This proves that surface tension does matter a significant amount, but only in the later wavenumbers. Though the calculation involving the surface tension involves more calculations, if one were doing this on a computer, it is not much of a hassle to add an extra term, and should be done even if one were calculating lower wavenumbers. Nevertheless, the error between the two theoretical values is still around 1-2% which is quite good.

Q-Factor

The Q-Factor is a quality unit. It is dimensionless. It is defined as:

$$Q = \frac{A_0}{\Delta A} \tag{6}$$

where A is the unit of which the graph is depicting. A_0 is the apex of the peak, and ΔA is the difference between the two A values at $\sqrt{2}/2$ of the apex's value.



In our case, we can see that our Q-factor got better as our measurements increased. It seems to also start declining after the fifth value, but that could just be a discrepancy. By looking at this graph, we can see that our best quality peak is the fourth wavenumber.

We also measured the decay rate of five different resonant frequencies. To do this, we measured the amplitude as a function of time. The decay can be expressed as:

$$A(t) = A_0 e^{-\alpha t} \tag{7}$$

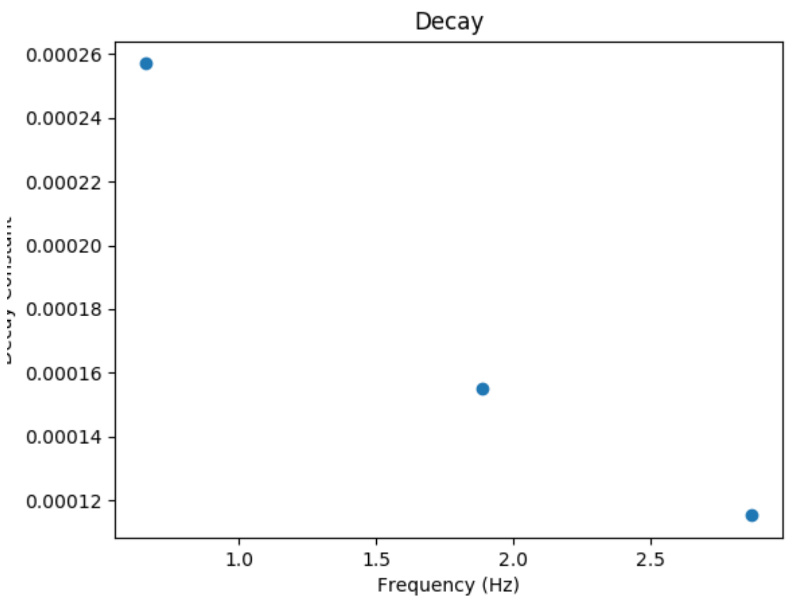
We can change the axis of this plot to be log scale to remove the exponential factor. Thus the slope of the best fit line would be the decay constant α. From this decay constant we can also determine the Q-factors of these decay plots by using:

$$Q = \frac{\omega_0}{2\alpha} \tag{8}$$

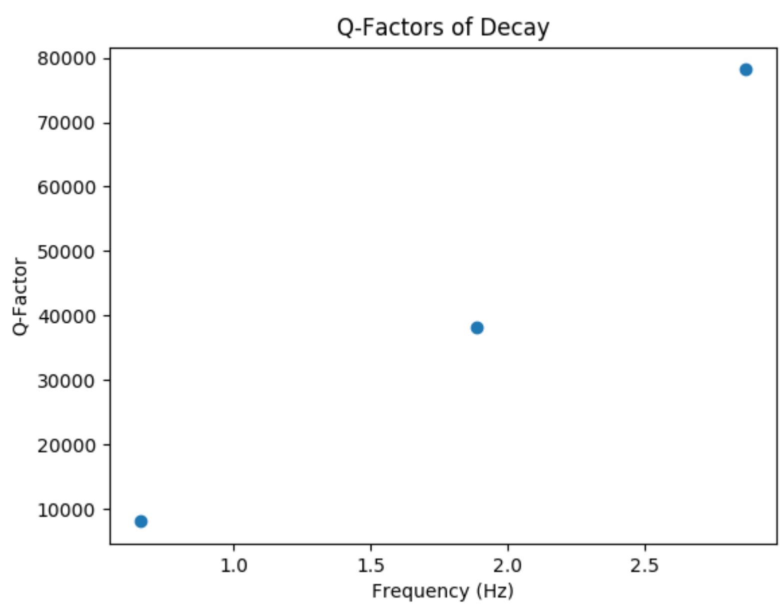
We can compare these Q factors to the Q factors of the sweep data to determine which have the better quality. In the table below, I show the frequencies we decayed and the decay constants for each.

Frequency (Hz)	Decay Constant α
0.68	0.00025703
1.92	0.000155139
2.94	0.000115283
3.57	0.0013404
4.25	0.004078

There was a problem with our last two data points for the decay constant. This was due to the fact that our data was centered below zero, and trying to do a natural log of zero causes the data to have problems. To counteract this, I tried shifting the data up by enough to move the data above zero. Of course, I tried to add this factor into the equation to remove after, when solving for α. Apparently, this didn't work as well as I thought it would.



Having to remove the last two data points, we can see that as the frequency increases, the decay constant decreases. This means that the time it takes the equalize the amplitude is much longer, which makes natural sense.



Our Q-Factors of our decay are rather high, but that is simply because the decay constants are really low. What really matters is comparing these Q-Factors to one another and to the sweep data. We can see that there is an upward trend, even though we only have three viable data points. We can also see that the Q Factor is multiple orders of ten higher than the Q-factors of the sweep data.

Solitons

A soliton is a solitary wave packet that keeps its shape. In our case, it is stationary, and perpendicular to the driving force. This soliton would retain its shape (once we created it). We ended up being able to make multiple in the channel, however not nearly enough to be impressive.

The soliton is interesting because it is not something that is intuitive. After reading the article attached to the lab manual, one can understand why a soliton exists. The energy used to excite the soliton is trapped in the soliton, because the velocity in the "wings" of the solitons are below the cutoff velocity. And in the center of the soliton, where the cutoff velocity is surpassed, the soliton makes it way back to the other wing, where it is stopped again.

Error Analysis

The uncertainty of the radius of the capillary tubes depends on the uncertainty in the measured height of the tube. Thus:

$$\delta r = \sqrt{\left(\frac{\partial r}{\partial h}\delta h\right)^2} = 0.002m \tag{9}$$

The uncertainty in the angular frequency depends on the uncertainty in the frequency which is collected from our data:

$$\delta\omega = \delta f = 0.0001Hz \tag{10}$$

The uncertainty of the wavenumber k depends on the uncertainty in our measurement of the length of the channel L:

$$\delta k = 0.007rads/m \tag{11}$$

The uncertainty in the phase velocity depends on the uncertainty in the angular frequency and the wavenumber, for the first frequency:

$$\delta c_p = \sqrt{(\omega\delta\omega)^2 + \left(\frac{1}{k^2}\delta k\right)^2} = 0.0001m/s \tag{12}$$

This can be extrapolated to find the uncertainty for each phase velocity.

Conclusions

In conclusion, we learned a lot about the dispersive waves in shallow water. More specifically, we learned about how the resonant frequencies of shallow water only allow for odd-number wavenumbers, because of the boundary conditions on either end of the apparatus. We also learned about the quality factor, and it reinforces what data points are better than others. The decay allowed us to see a natural wave decay in real time. It was also interesting to learn about how the surface tension actually affected our calculations so much, but that is mainly due to how shallow and narrow our experiment was. It would be interesting to compare the surface tension effect to something larger.

These ideas can be applied to the real world, such as waves crashing onto the beach, or waves in swimming pools. Besides real world examples, it was intersting to learn about solitons – things I never knew existed until this experiment. Besides just solitary solitons we observed in this lab, researching about other types of solitons, particularly ones that travel as a wavepacket, reminded me of the "wave pool" at any good water park. I remember spending lots of time in those things as a kid, and next time I encounter one, I'll be sure to think about the mechanics going on, creating the soliton and propagating it.

References

[1] Experiment 2 Lab Manual and Appendicies