Poisson Probability Discussion

> Source Code: https://github.com/kevinbird61/stochastic-calculus-and-probability-model/tree/master/poisson_distribution
>
> **Before starting to read this article, please install chrome extension: `Github with MathJax` , to ensure the correctness of formula format.**

Improvement

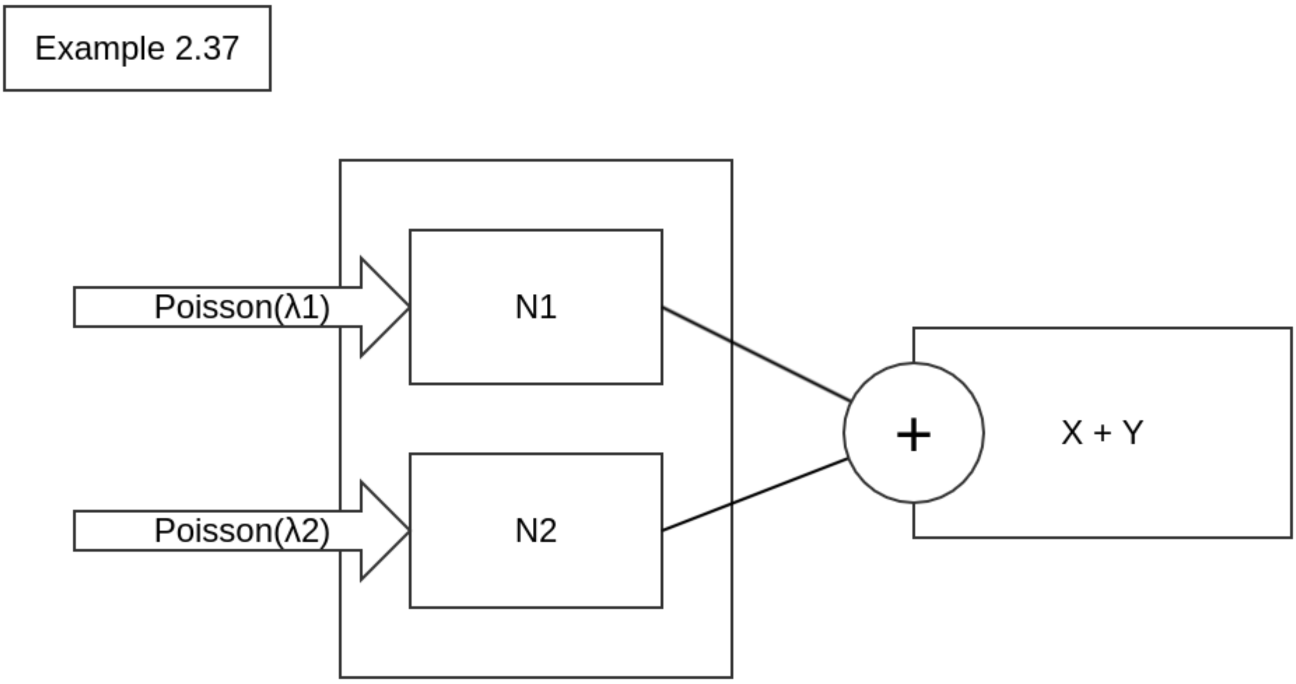
After `example 2.5` , `3.31` , the program has been refactor a lot, make code reusable.

- Consider simulation need to be tested with several different input, to **accelerate** the arguments parsing process, I construct `parse_arg` class to deal with this problem. See more about `parse_arg` .

Discuss two *network model*

Example 2.37 (Merge)

- It will be implemented in `part_a.cc`



Mathematic Model

- Because N1,N2 is **independent**, so we know N1=>

$$N_1|N = n \sim Binomial(n, p)$$

, which N2:

$$N_2|N = n \sim Binomial(n, 1-p)$$

, Both N1,N2 is a sum of `n` independent *Bernoulli(p)* random variables, with `Binomial(N,P)` , N and P represent **Number and Probability**.
- We have:

$$\begin{aligned} P_{N_1}(k) &= \sum_{n=0}^{\infty} P(N_1 = k|N = n) \cdot P_{N1}(n) \\ &= \sum_{n=k}^{\infty} C_k^n \cdot p^k(1-p)^{n-k} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \sum_{n=k}^{\infty} \frac{p^k(1-p)^{n-k}\lambda^n}{k!(n-k)!} \\ &= \frac{e^{-\lambda} \cdot (\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p)^{n-k})}{(n-k)!} \\ &= \frac{e^{-\lambda} \cdot (\lambda p)^k}{k!} \cdot e^{\lambda(1-p)} \\ &= \frac{e^{-\lambda p} \cdot (\lambda p)^k}{k!}, for\ k = 0, 1, 2, ... \end{aligned}$$

- So that we conclude that

$$
N_1 \sim Poisson(\lambda \cdot p)
$$
$$
N_2 \sim Poisson(\lambda \cdot (1-p))
$$

$$
which\ N_1\ and\ N_2\ are\ independent,
$$
$$
so\ P_{N_1+N_2}\ will\ be\ :
$$
$$
P_{N_1+N_2}(n,m) = P_{N_1}(n) \cdot P_{N_2}(m)
$$

- Consider the formula:

$$
P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)
$$
$$
= \sum_{k=0}^{n} P(X=k) \cdot P(Y=n-k)
$$

So that `Merging Poisson Process` can be:

- **Directly** calculate the *S*=*X*+*Y* with:

$$
P(X+Y=n) = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1+\lambda_2)^n
$$

- **Separately** calculate *X* and *Y* with:

$$
\begin{aligned}
P(X=k) &= \tfrac{e^{-(\lambda_1)}}{n!} \cdot (\lambda_1)^n, \\
P(Y=n-k) &= \tfrac{e^{-(\lambda_2)}}{(n-k)!} \cdot (\lambda_2)^{n-k}
\end{aligned}
$$

, and need to consider the summation, from **k=0~n**:

$$
\sum_{k=0}^{n} \dots
$$

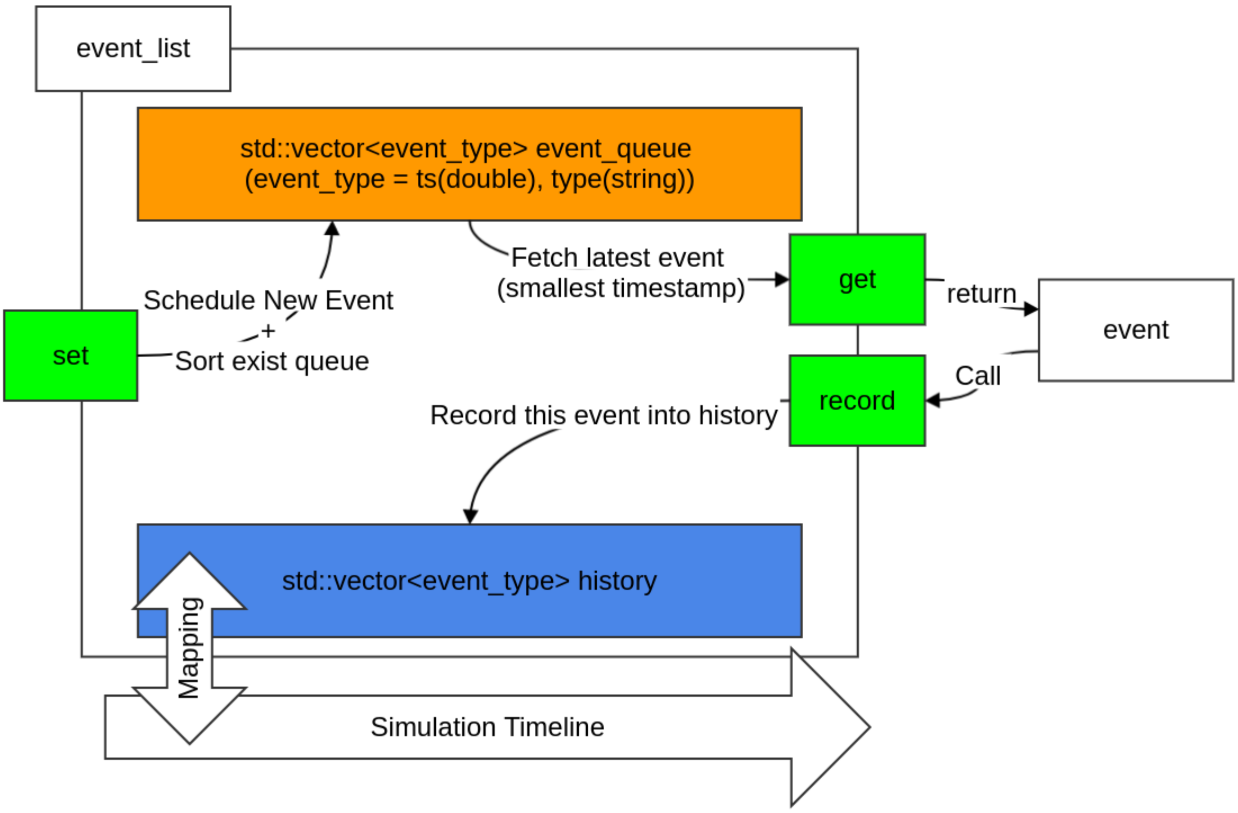
Simulation Model

- We can use exponential distribution:

$$
f(x) = \lambda \cdot e^{-\lambda \cdot x}
$$

  , which let `x` be a random number to get a **random variable** from exponential distribution.
- In my implementation, I use C++ STL (`standard library`) - `<random>` to do this.

Implementation Detail

- `Step 1` , I using self-defined class - `event_list` as my event queue. See more about `event_list`.
- `Step 2` , scheduling 2 individual event: `X` , `Y` into event queue for initialization, then we can start our simulation. End condition is the number you can set in arguments before starting program by `-s` .
- `Step 3` , pop out the element from `event_list` , and depend on its type (e.g. is `X` or `Y` ?) to schedule next event with **exponential random variable** as timestamp and push back into `event_list` . And the old event will be record into this `event_list` object (treat like a event history, sort by its timestamp.). *Do this routine until reaching the number we set by specifying `-s` .*
  - 

- `Step 4` , after event scheduling process has been done, we now can count the ratio of event arrival in each time scale.
  - For example, between timestamp `0.0~1.0` , we get `5` event arrival during this time scale; And `1.0~2.0` , we get `4` as event arrival.
  - Now, assume `2.0` is the end point of simulation, we now have 2 result: `X=5` and `X=4` , both have 1 occurance.
  - Then we can say: P(X=5)=1/(1+1)= `0.5` = `50%` =P(X=4) !
- `Step 5` , and now we have the history record in object of `event_list` , which record the type of each event, then we can pop it out and get the `P(X)` , `P(Y)` and `P(X+Y)` , with specified value of time scale:

$$time\ scale = 1$$

, which

$$rate\ parameter = \lambda\ , scale\ parameter =\ 1/\lambda = \beta$$

- Because *rate parameter* means in this time scale (which indicate as `1` above), how many event will happen. So we can use `1` to measure this simulation is fitting with poisson distribution or not.

- **`Final`** , Then we can count the arrival rate in this time scale to finish our simulation!

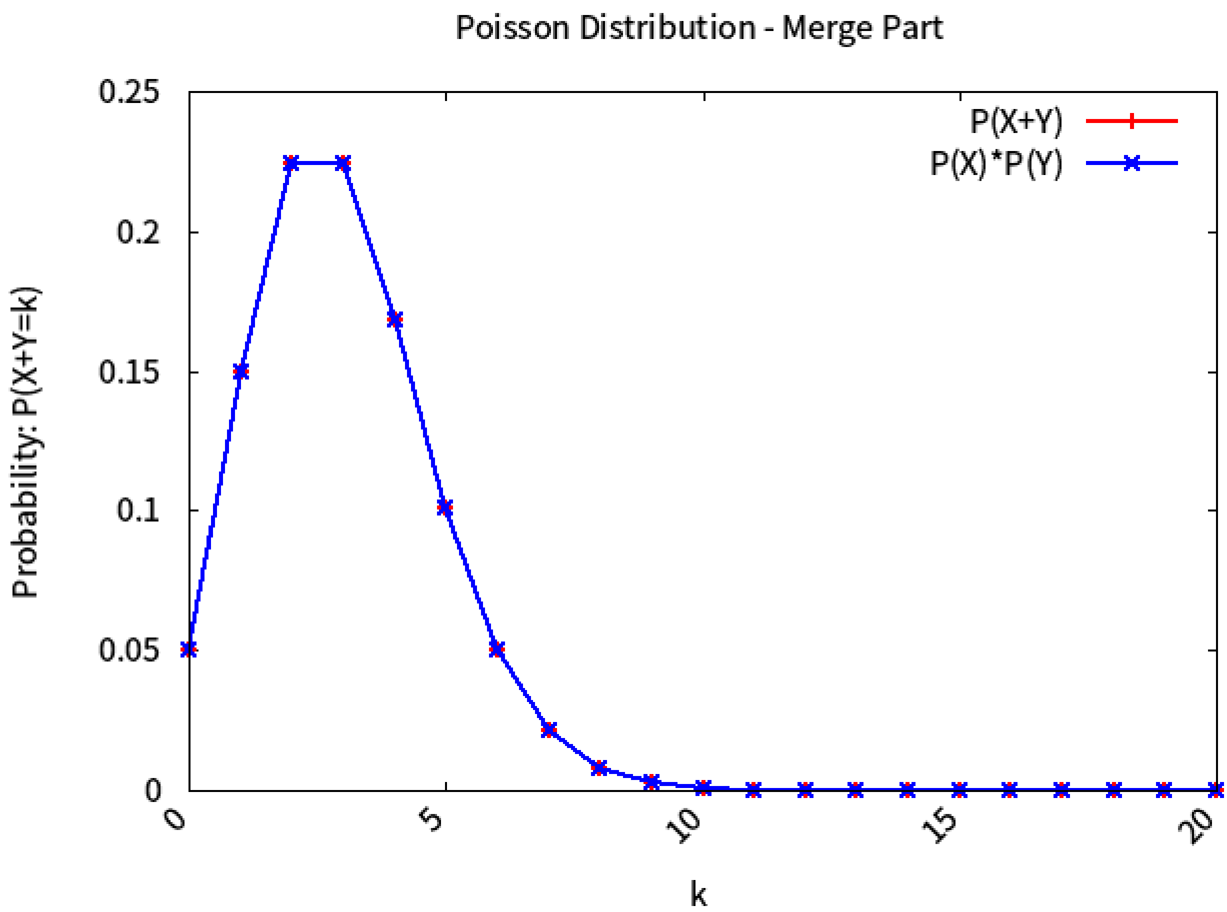
Result

- So we need to compare `simulation` and `mathematic` model:
  - run with command `make && make plot` to run the program and plot:

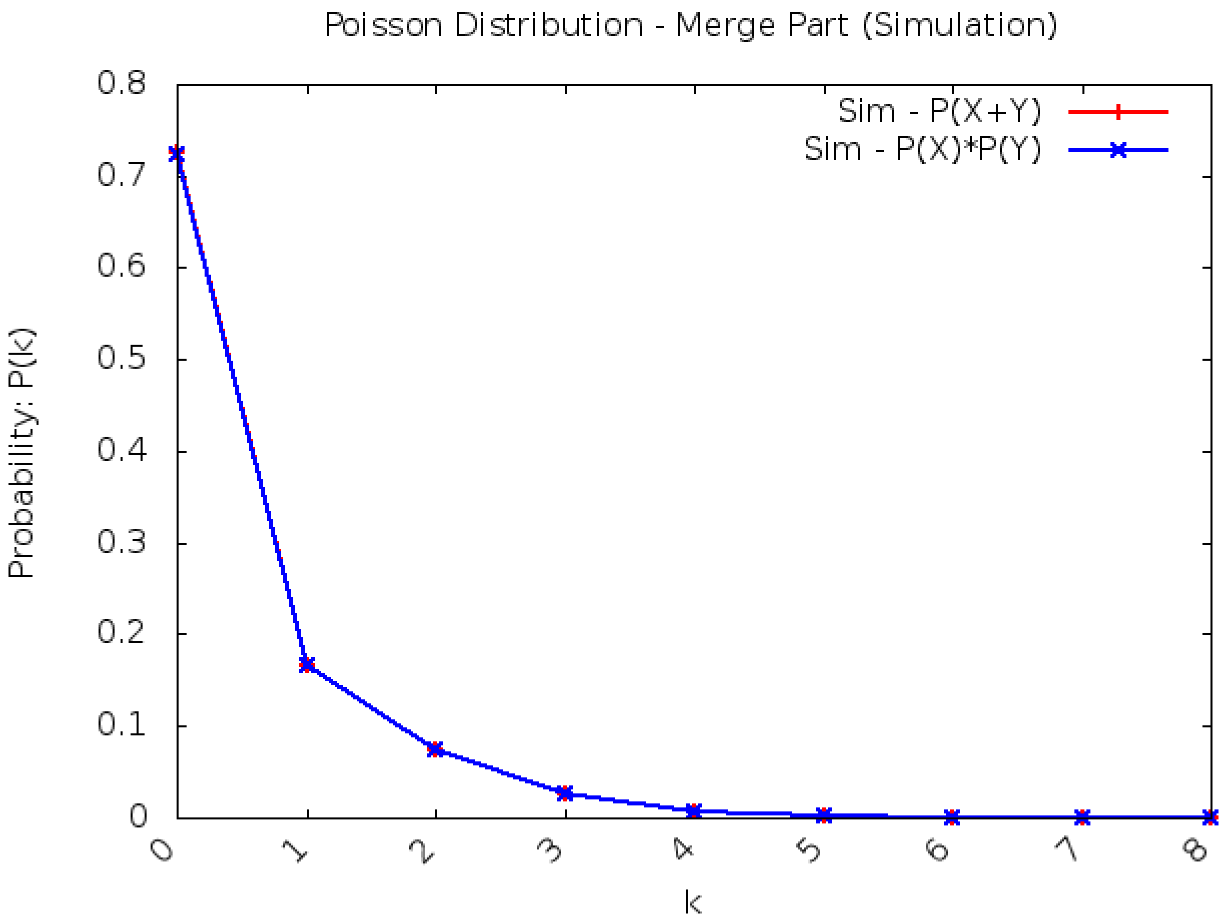
$$k = 20,\ \lambda_X = 1,\ \lambda_Y = 2$$

, also if you want to adjust, please using `./part_a.out -h` to see more.

  - **Mathematic Model**



  - **Simulation Model**

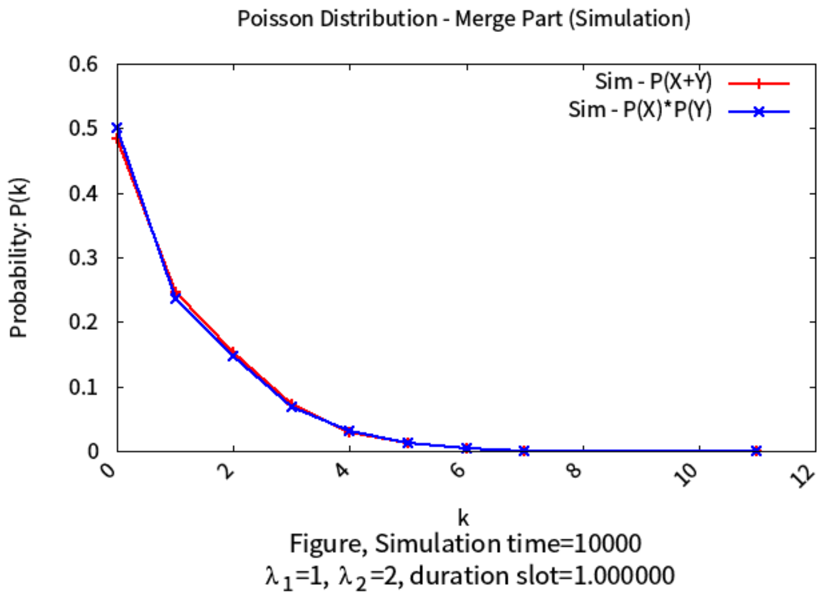
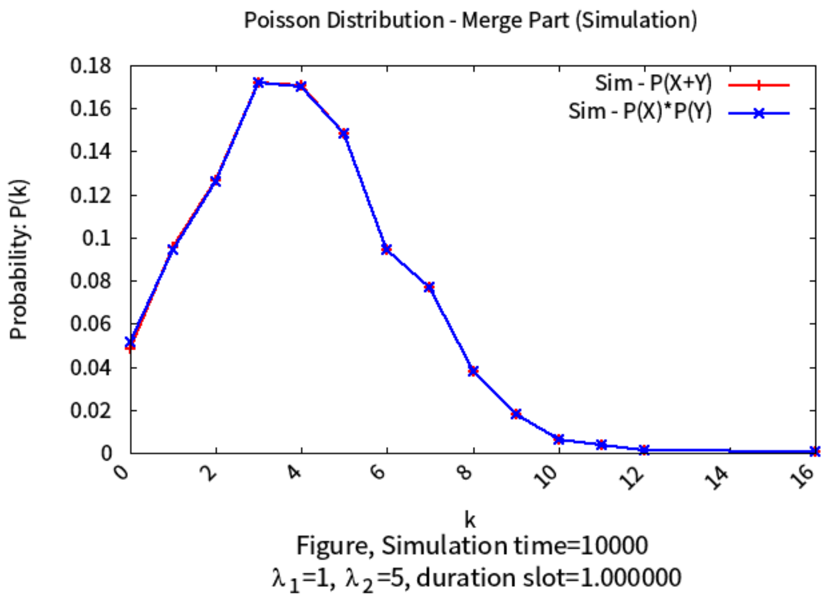


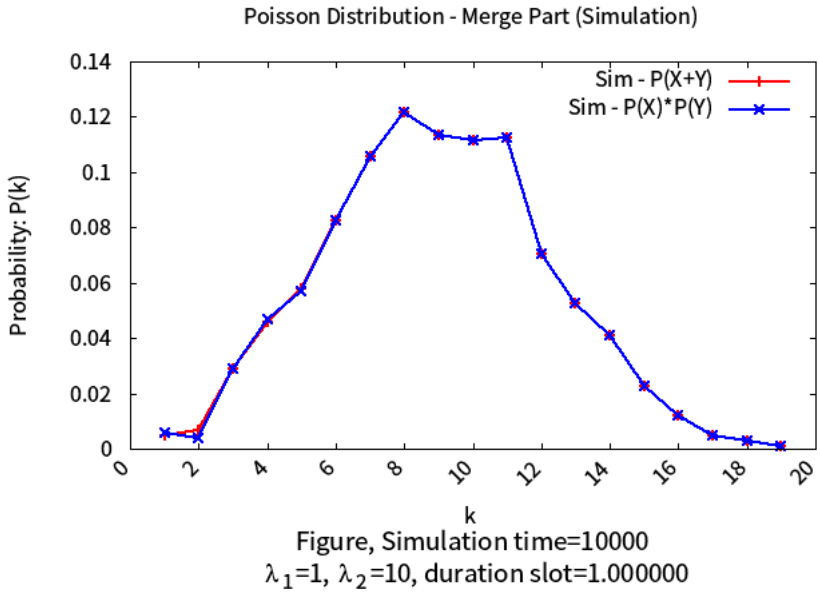
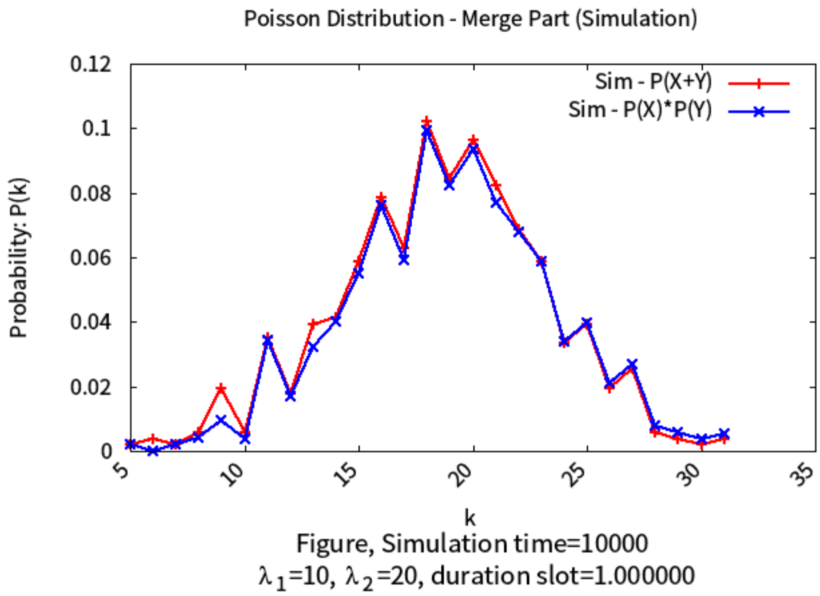
- We can see, both `mathematic` and `simulation` model all have the same curve in **`P(X+Y)`** and **`P(X)*P(Y)`**

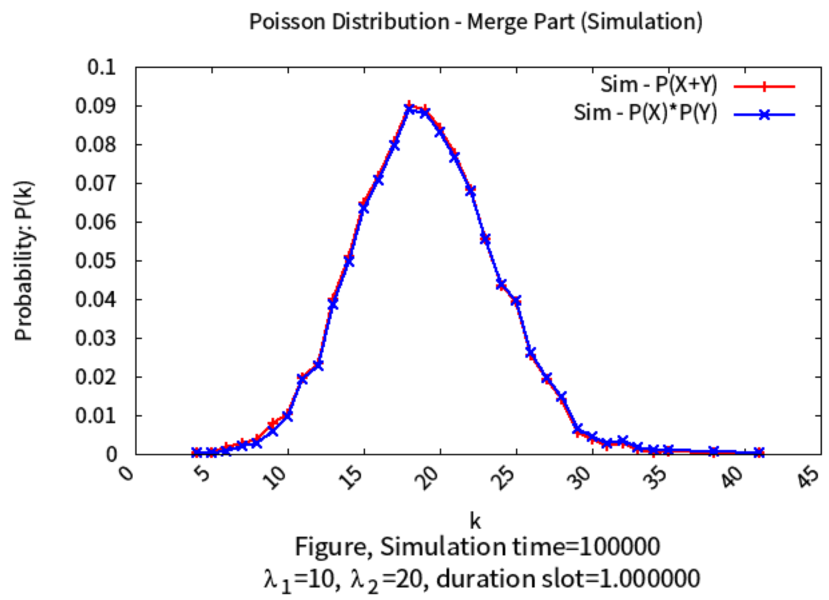
Different Case

- After we have finished the `part_a.cc` and compile it to get the executable file, we now can use it to run **`multiple testcase`** - ***test_a.sh***

case	simulation times	λ_1	λ_2	result

case	simulation times			result
1	10000	1	2	
2	10000	1	5	

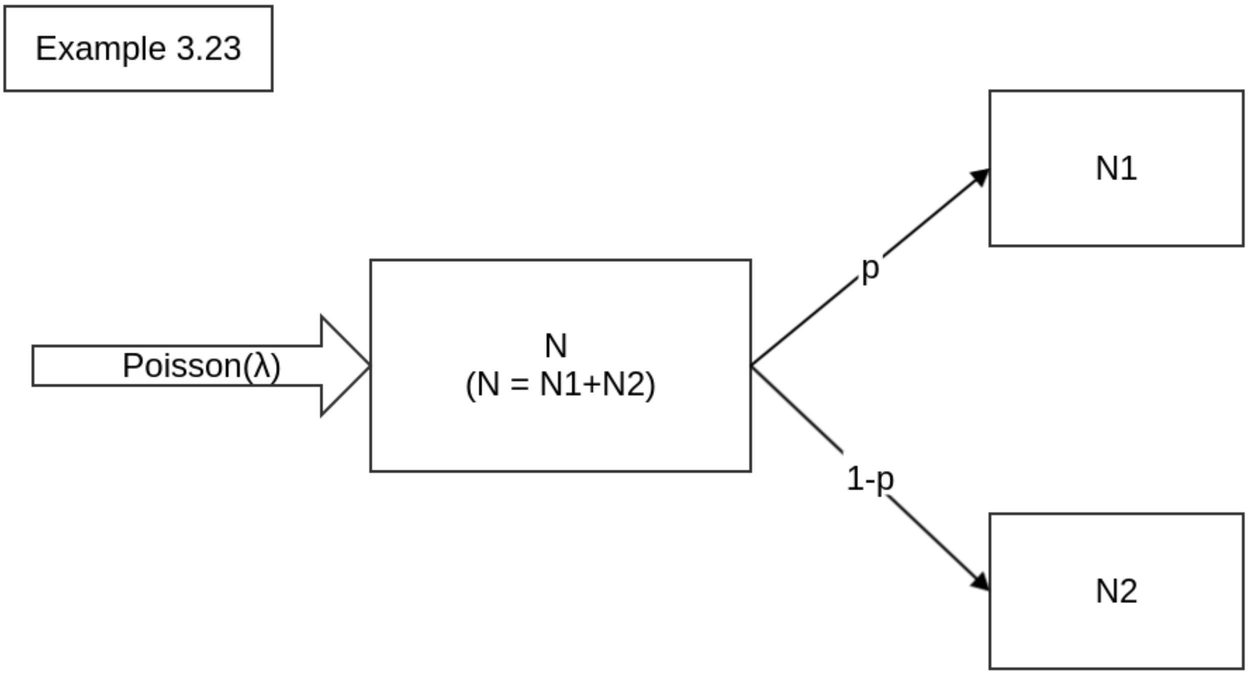
case	simulation times			result
3	10000	1	10	
4	10000	10	20	

case	simulation times			result
5	100000	10	20	

- Parameters:
  - `simulation times` represent the **number** of total event in simulation process.
  - `lambda_1` represent the lambda in `X`.
  - `lambda_2` represent the lambda in `Y`.
- As the result shown above, we can see `P(S=X+Y)` is almost perfectly match with `P(X)*P(Y)`; And we can see in case 4, these 2 curves are quite not matching with each other; But after *increase the total event number*, then we can see these 2 curves are matching again.

Example 3.23 (Split)

- It will be implemented in `part_b.cc`



In this part, we can see Part-B is the inverse process of Part-A (e.g. `Poisson Process Merge`). Part-B is the `Poisson Process Split` , which separate one arrival queue into 2 different set of queue, with specified `probability` (p) to transform from original one to these 2 different set.

Mathematic Model

From the formula, we can have the equation:

$$P(X+Y) = P(\lambda \cdot p_x) \cdot P(\lambda \cdot (1-p_x))$$

, which

$$P(X) = P(\lambda \cdot p_x),\ P(Y) = P(\lambda \cdot (1-p_x))$$

So in mathematic part, we can construct this equation by program. See detail in part_b.cc.

Simulation Model

As the same concept in Part-A, we use a event queue to represent the entire simulation.

The **differences** between them are:

- `lambda_1` and `lambda_2` become `lambda * p` and `lambda * (1-p)`
- When each arrival event occur, we need to using a random number (`0.0` ~ `1.0`) to decide this event type (e.g. become " `X` " or " `Y` "), and as same as `Step 3` in `Part-A` , assign an exponential random variable as timestamp to this event, and then schedule it into event list.

- And we can use the same step of `Step 4` in `Part-A`, to get the probability of each number of event occur during specified time scale: 1 (**Which represent "in this time slot, how many event will occur"**)
- **Most important part,** in `Part B` there have need to create three event queue, `N`, `LX`, `LY` respectively.
  - `N = N ~ Poisson (λ)`, is using to generate the `X (derive from N)` and `Y (derive from N)` with probability `p` and `1-p`
  - `LX` represent the independent Poisson (λ*p), compare with `X (derive from N)`.
  - `LY` represent the independent Poisson (λ*(1-p)), compare with `Y (derive from N)`.
  - The other calculation are similar with above.
- With all the statistics required, we can count the arrival rate in this time scale to finish our simulation!

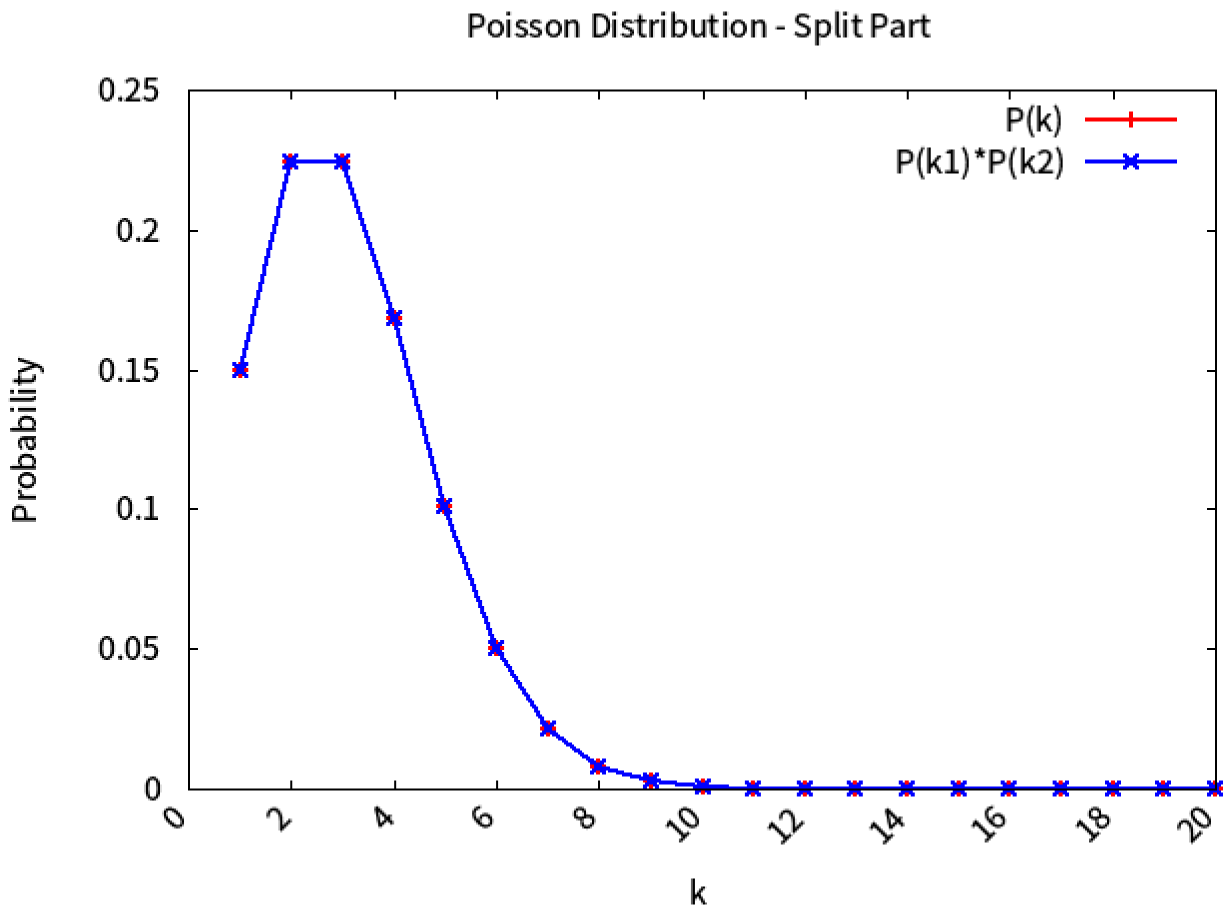
Result

- Run `make && make plot` with statistics:

$$k = 20,\ \lambda = 3,\ p = 0.5$$

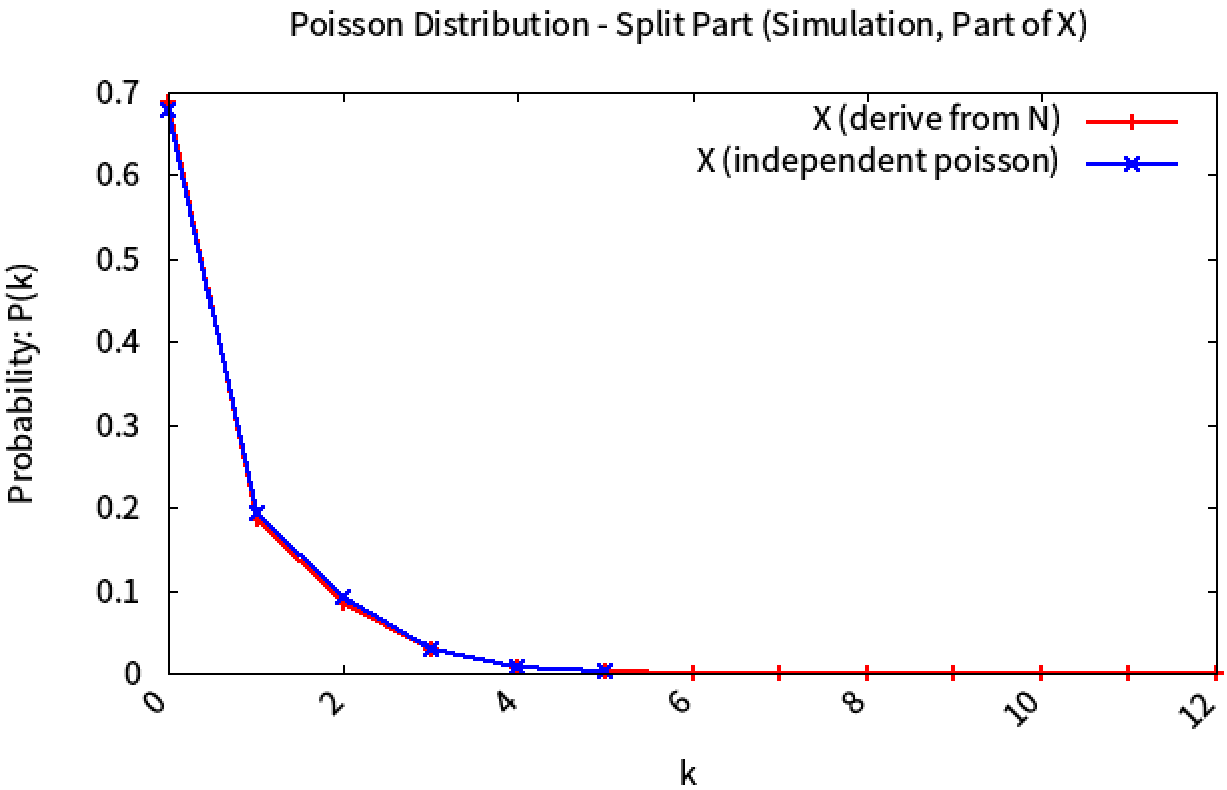
, also if you want to adjust, please using `./part_b.out -h` to see more.

- **Mathematic Model**



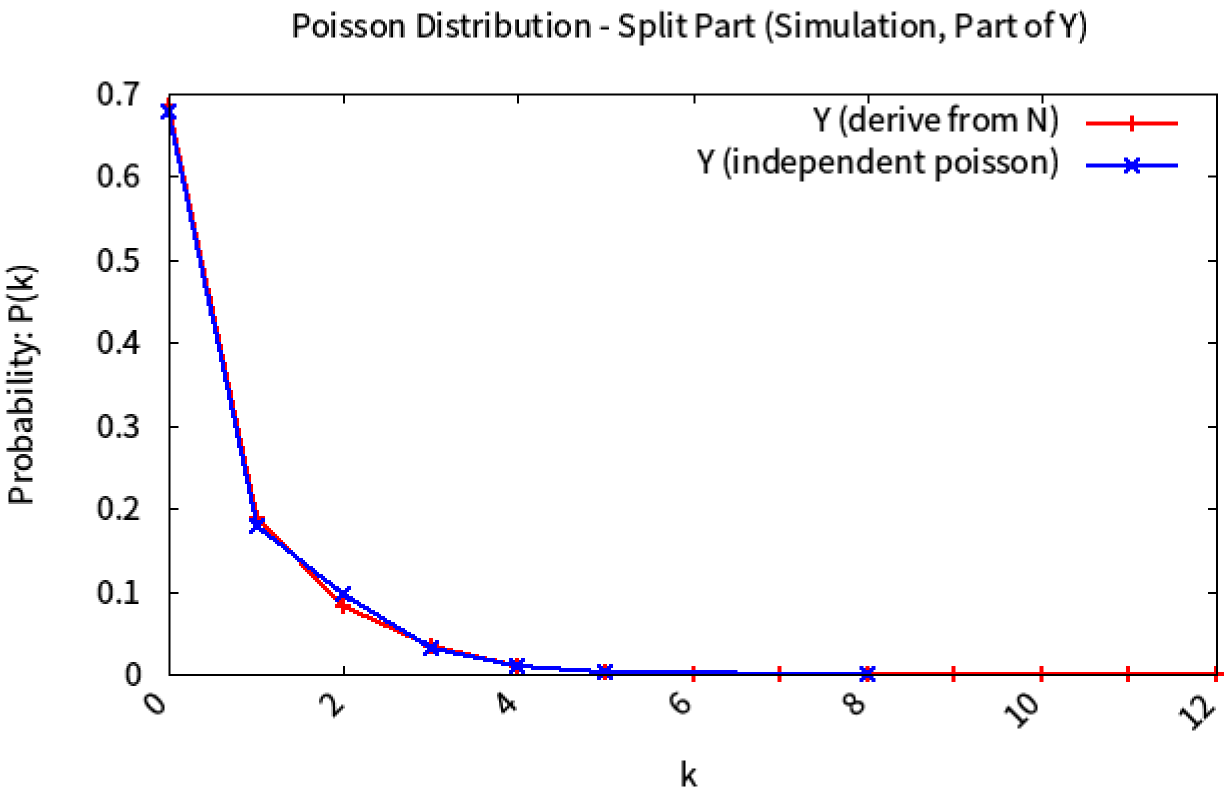
- **Simulation Model** (with `10000` event)

- `X (derive from N)` compare with `LX (Poisson (λ*p))`



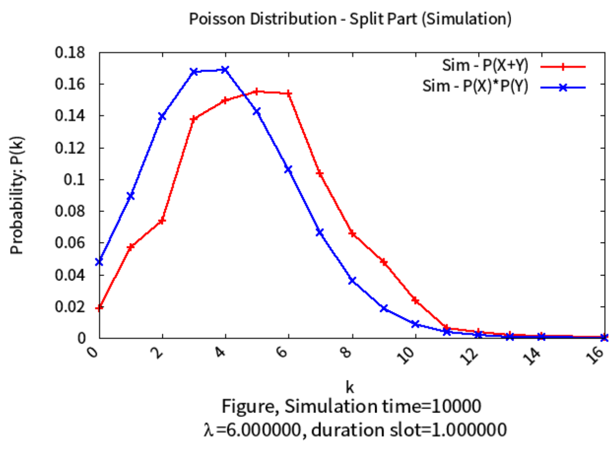
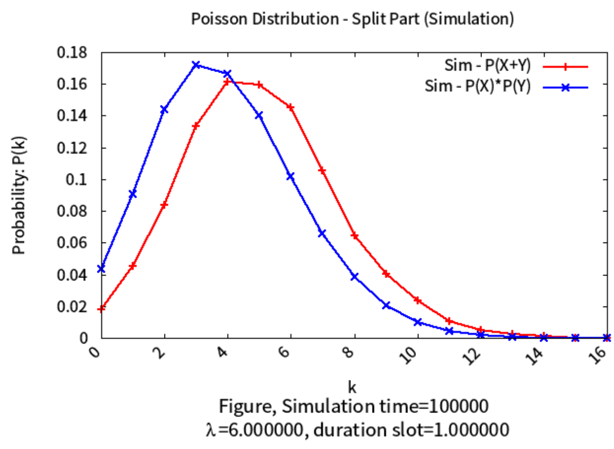
Figure, Simulation time=10000
λ=3.000000, duration slot=1.000000

- `Y (derive from N)` compare with `LY (Poisson (λ*(1-p)))`



Figure, Simulation time=10000
λ=3.000000, duration slot=1.000000

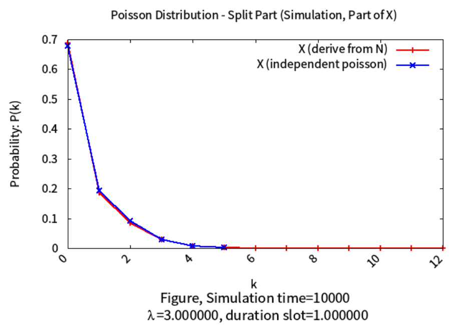
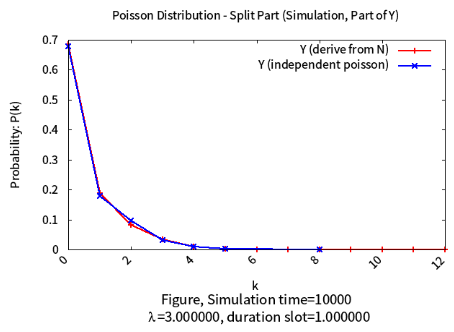
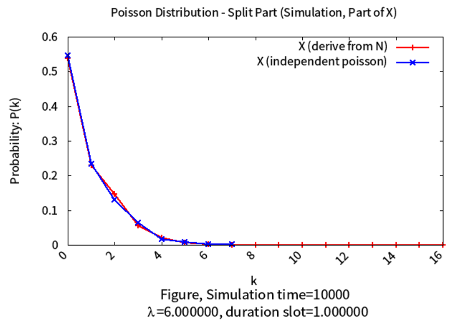
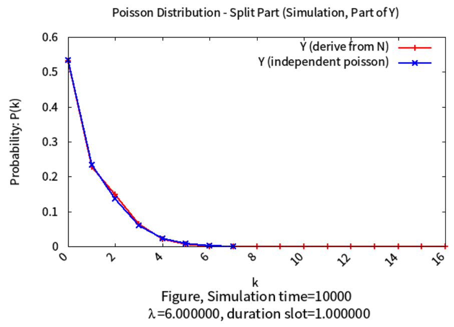
- Also, you can compare `P(X+Y)` with `P(X)*P(Y)` , too. But there are lots of bias between them.

$sim = 10000, \lambda = 6$	$sim = 100000, \lambda = 6$
	

And next part we will adjust the parameter and compare the results.

Different Case

- After we have finished the `part_b.cc` and compile it to get the executable file, we now can use it to run **`multiple testcase`** - ***test_b.sh***

simulation times	λ	P	`X , LX`	`Y, LY`
10000	3	0.5		
10000	6	0.3		

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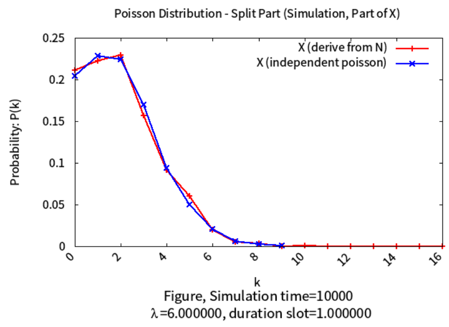
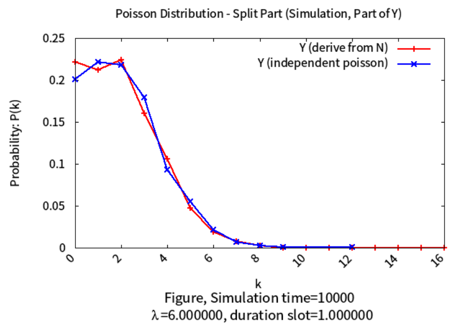
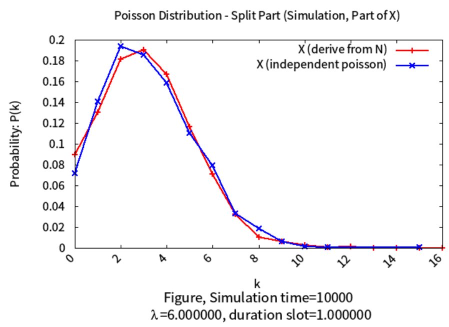
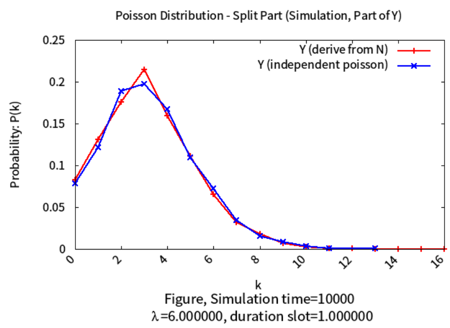
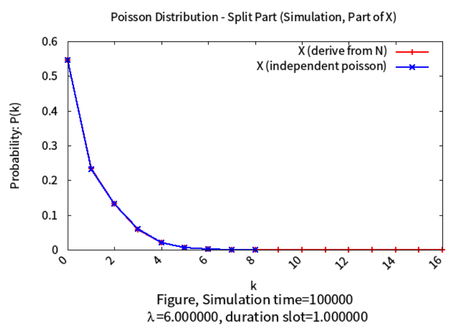
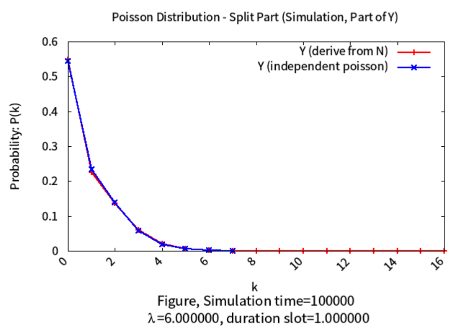
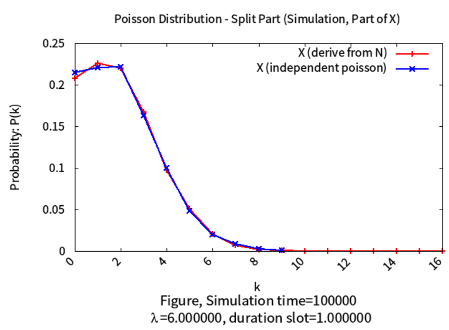
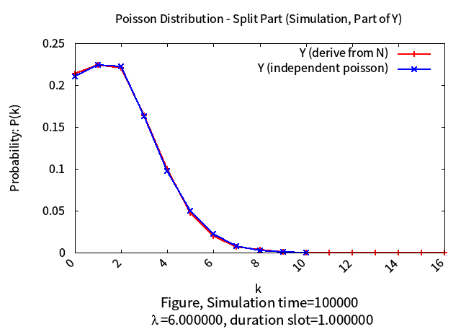
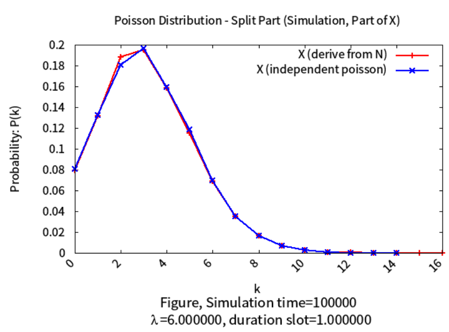
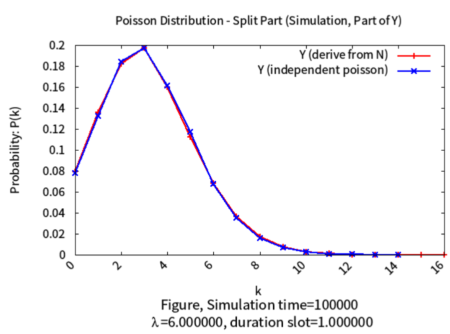
<tr><th>simulation times</th><th></th><th></th><th>X , LX</th><th>Y, LY</th></tr>
<tr><td>10000</td><td>6</td><td>0.5</td><td></td><td></td></tr>
<tr><td>10000</td><td>6</td><td>0.7</td><td></td><td></td></tr>
<tr><td>100000</td><td>6</td><td>0.3</td><td></td><td></td></tr>
<tr><td>100000</td><td>6</td><td>0.5</td><td></td><td></td></tr>
<tr><td>100000</td><td>6</td><td>0.7</td><td></td><td></td></tr>
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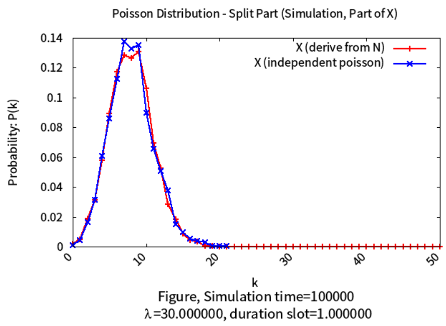
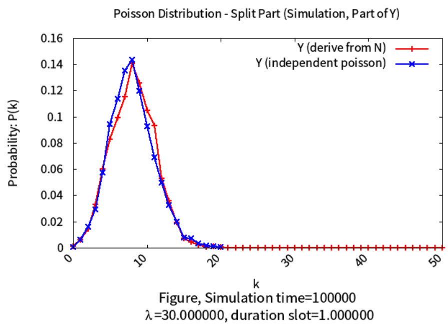
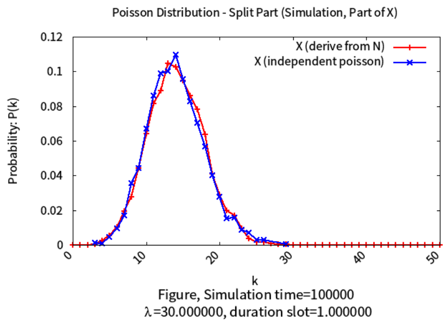
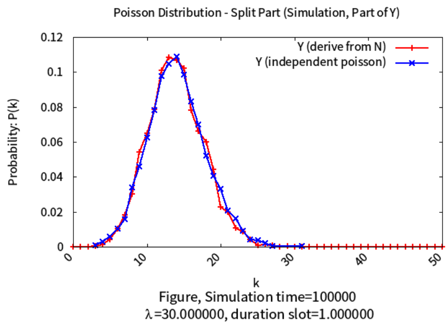
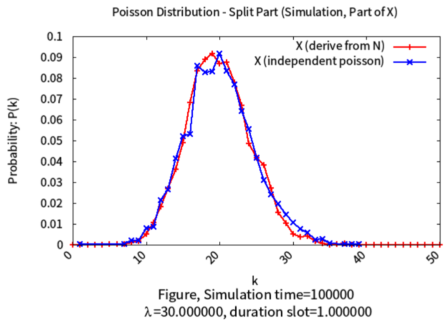
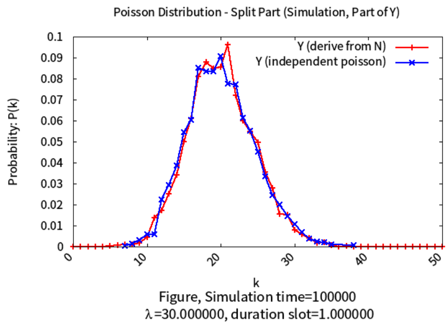
simulation times			X , LX	Y, LY
100000	30	0.3		
100000	30	0.5		
100000	30	0.7		

- Parameters:
  - `simulation times` represent the **number** of total event in simulation process.
  - `P` represent the probability of event from `N` deriving to `X` .
  - `X,LX` : result compare with `X (derive from N)` (e.g. using λ directly) and `LX (independent Poisson)` (e.g. $\lambda \cdot p$)
  - `Y,LY` : result compare with `Y (derive from N)` (e.g. using λ directly) and `LY (independent Poisson)` (e.g. $\lambda \cdot (1-p)$)
- As the result above, we can see `X, LX` , `Y, LY` are almost matching respectively !
  - And you can see case of `lambda=6` , when we increase the `simulation times` from 10000 to 100000, the result will be more matching.
  - Otherwise, it will have a large oscillation between 2 curves.

Reference

- Basic Concept of Poisson Process

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