A novel differentiable unification of least absolute deviations and least squares

Kevin Burke | University of Limerick

Science Foundation Ireland sfi For what's next

LEAST *SQUARES* VS *ABSOLUTE DEVIATIONS*

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Least *squares*

$$\min_{\beta} \sum_i \left(y_i - x_i^T \beta \right)^2$$

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Model

$$y_i = x_i^T \beta + \sigma \varepsilon_i$$

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Laplace

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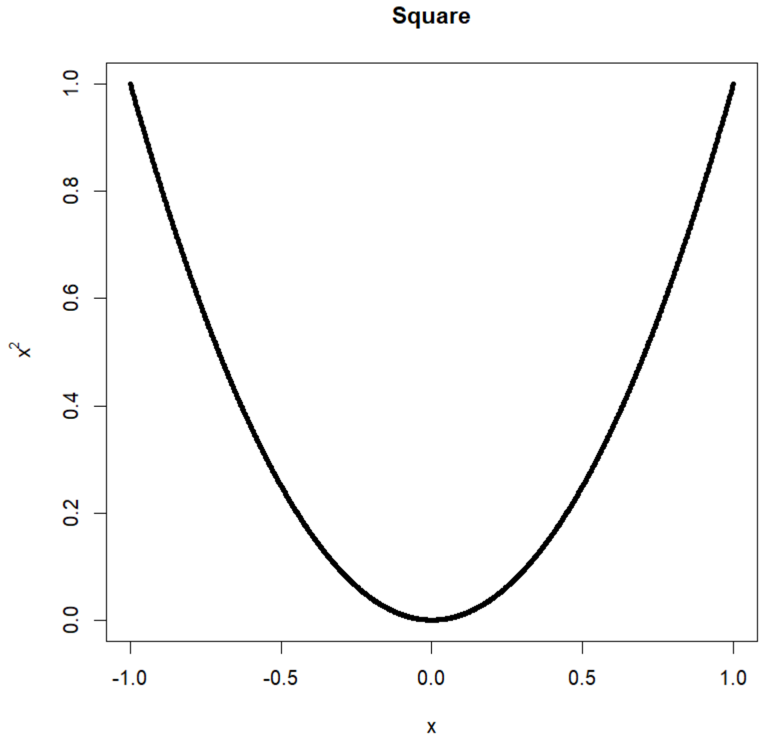
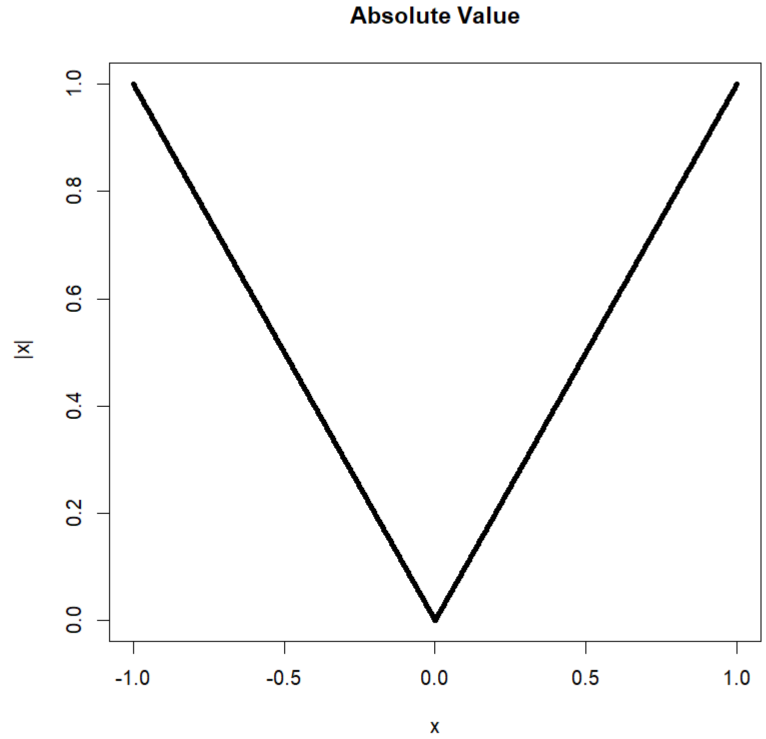
SQUARE VS ABSOLUTE VALUE FUNCTION





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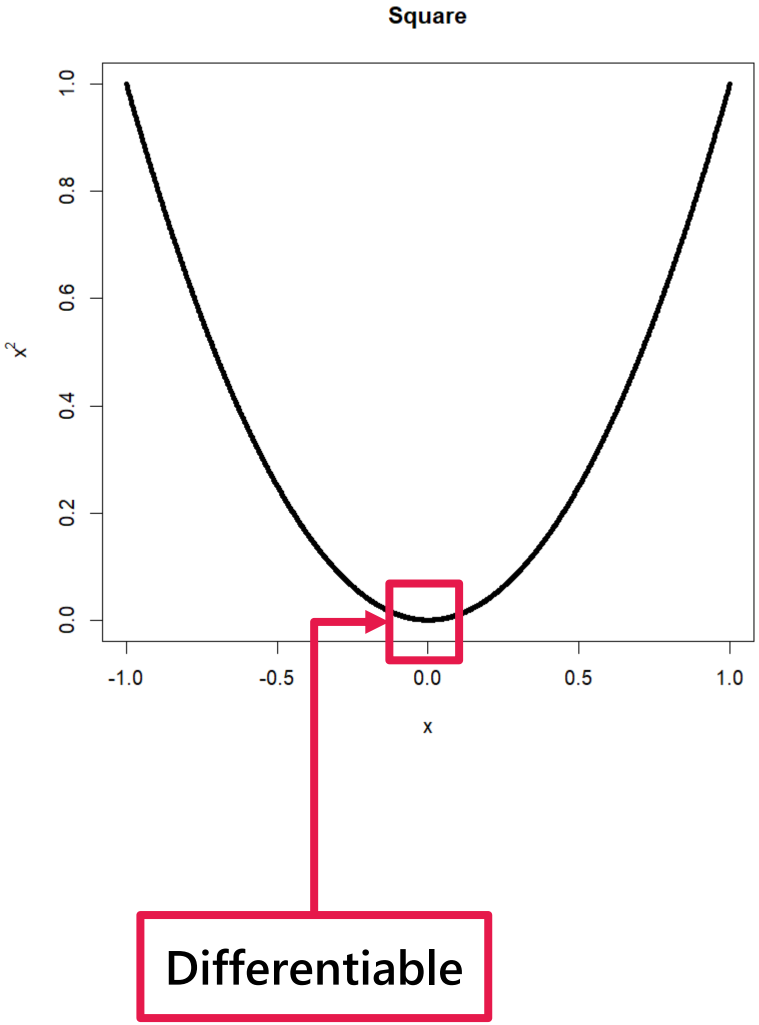

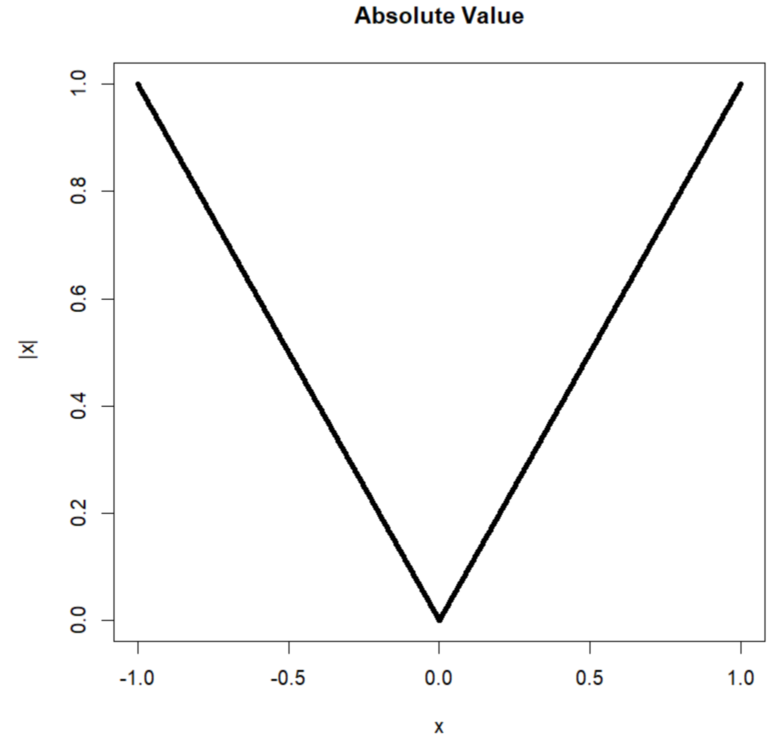

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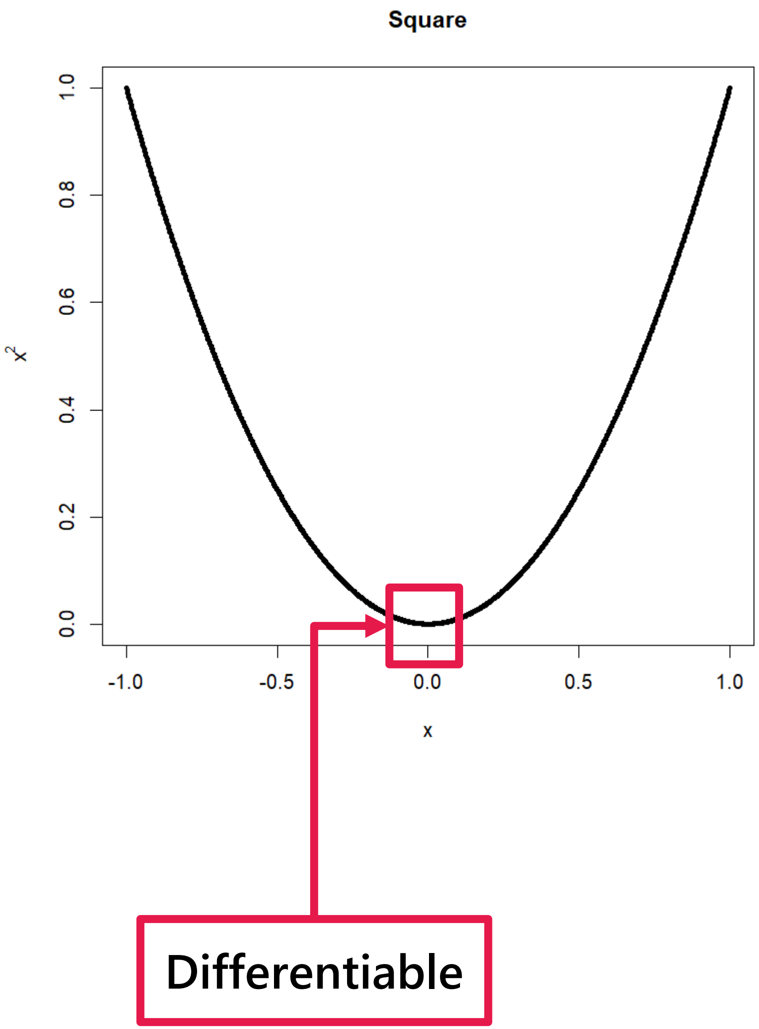

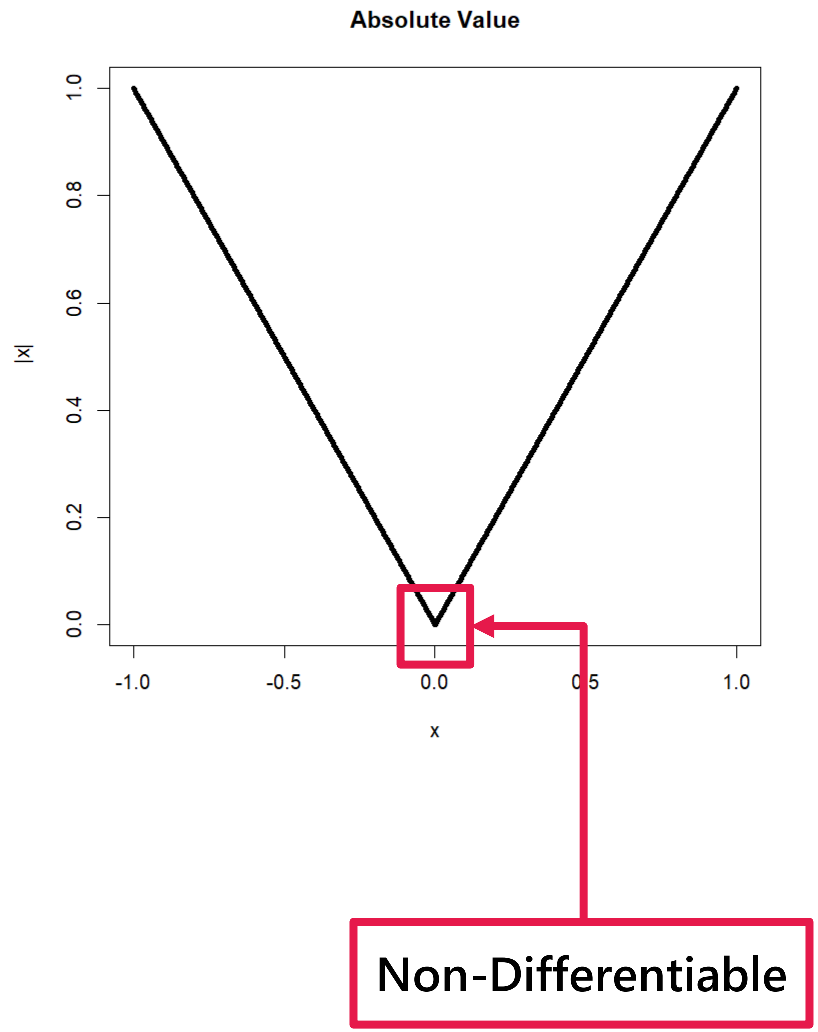

DIFFERENTIABLE APPROXIMATION

$$a_\tau(x) = \sqrt{x^2 + \tau^2} - \tau$$

Jaouimaa, Ha, & Burke (2019). Penalized Variable Selection in Multi-Parameter Regression Survival Modelling. arXiv.
Burke & Patilea (2021). A likelihood-based approach for cure regression models. TEST.

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(differentiable)

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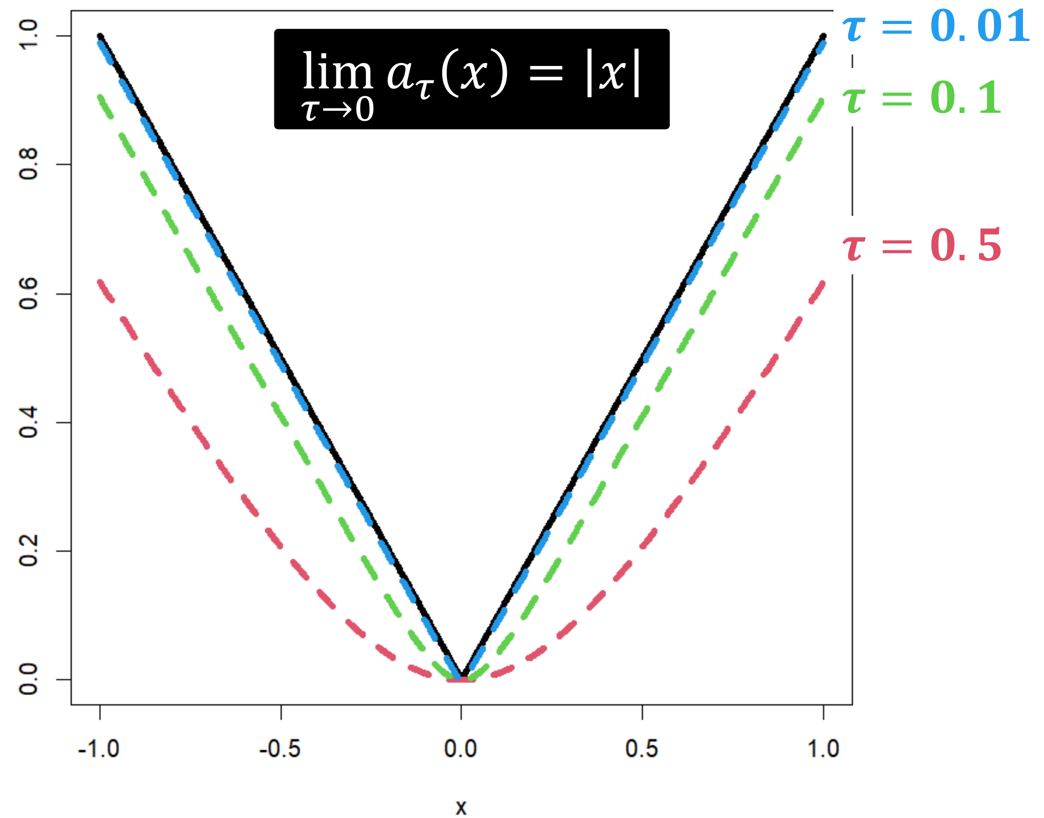

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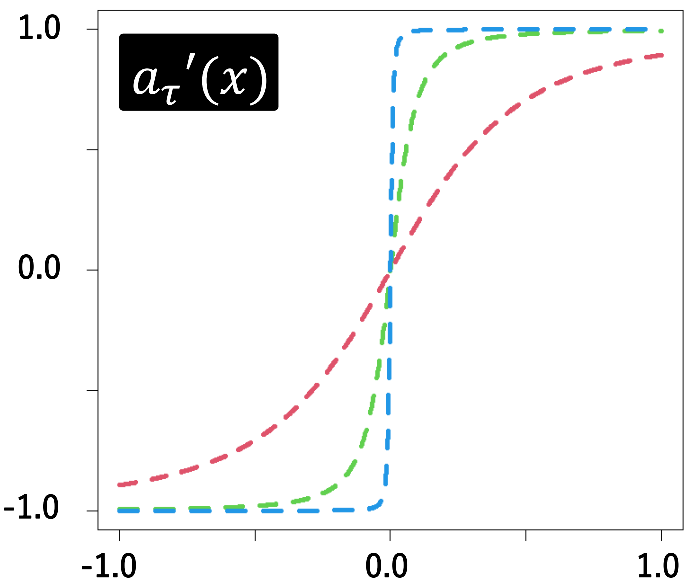

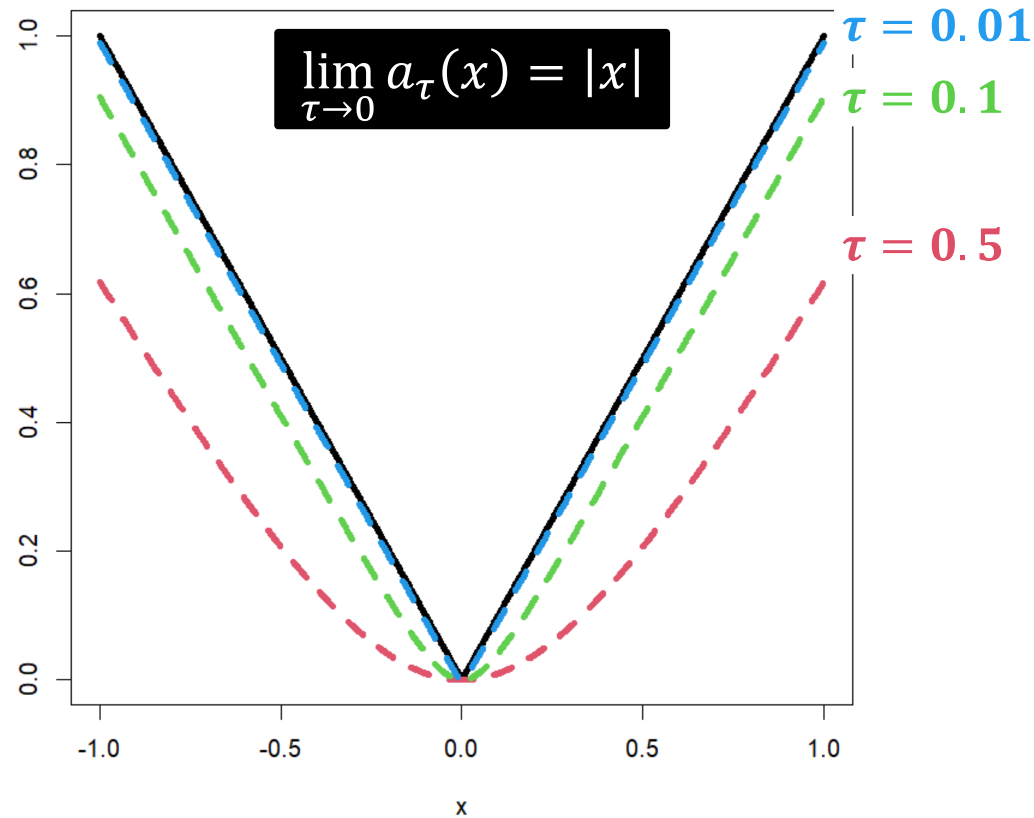

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SMOOTH LAPLACE DISTRIBUTION

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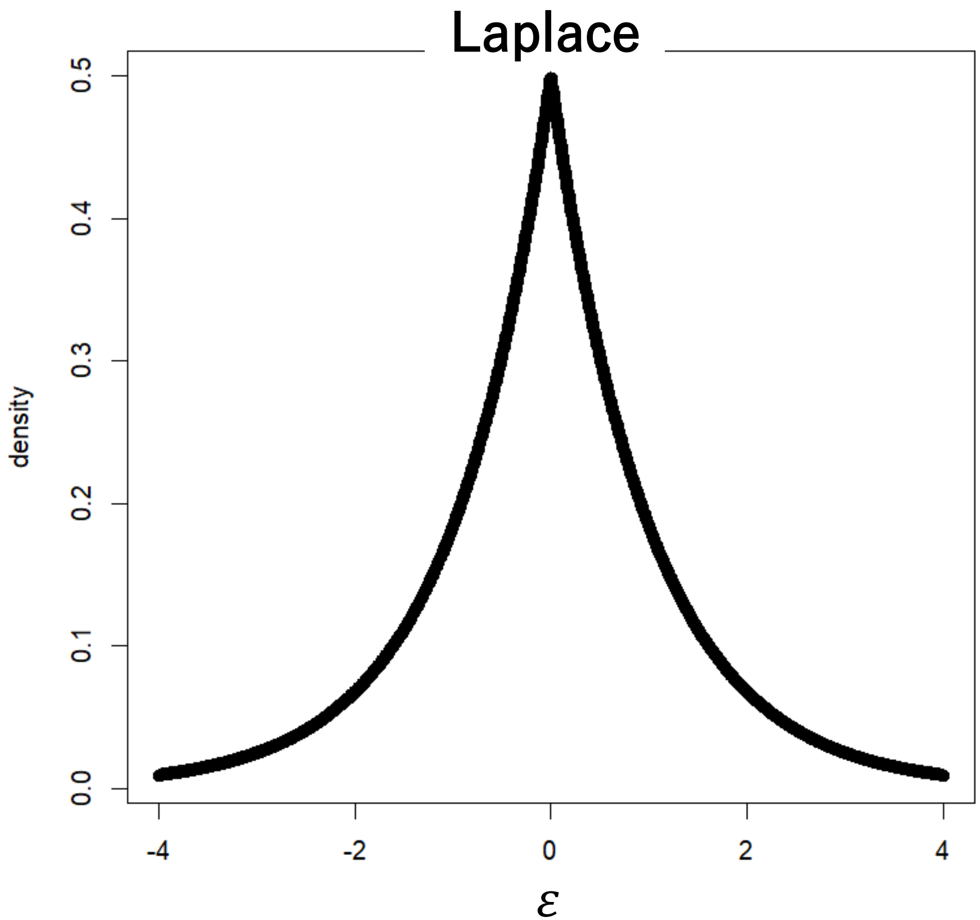

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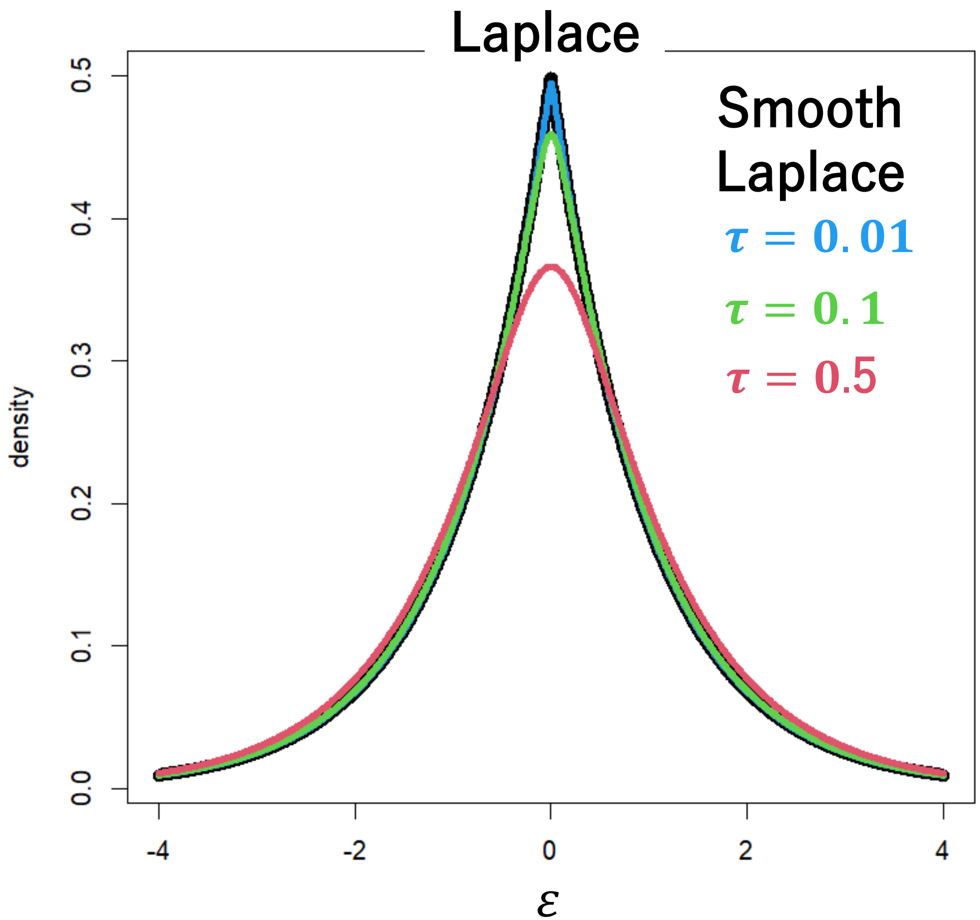

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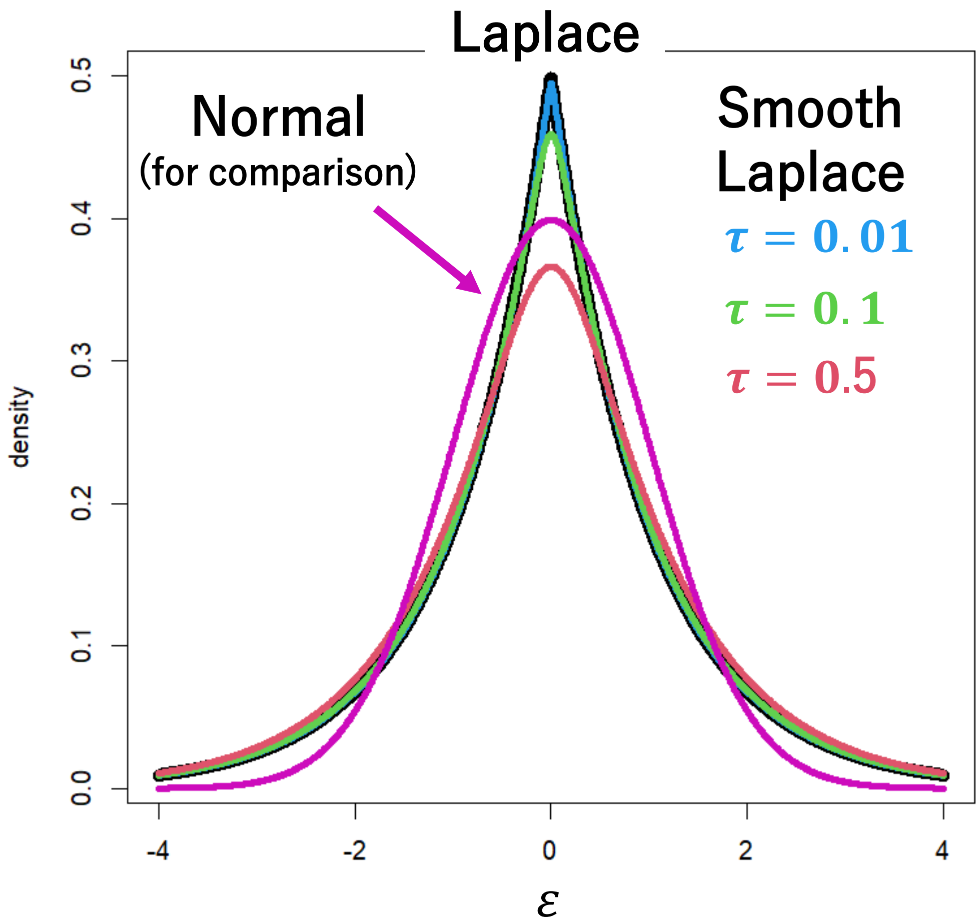

LIKELIHOOD ESTIMATION

- Log-likelihood function

$$\ell(\beta,\sigma) = n\log c_\tau - n\log\sigma - \sum_i a_\tau\left(\frac{y_i - x_i^T\beta}{\sigma}\right)$$

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- Differentiable in β and σ
- Standard, gradient-based optimisation can proceed, e.g., **`nlm`**

STACK LOSS DATA FIT

- "`stackloss`": data on industrial process for oxidising ammonia to nitric acid
- **Response:** `stack_loss` (inefficiency)
- **Inputs:** `air_flow`, `water_temp`, `acid_conc`

Brownlee (1960) Statistical Theory and Methodology in Science and Engineering

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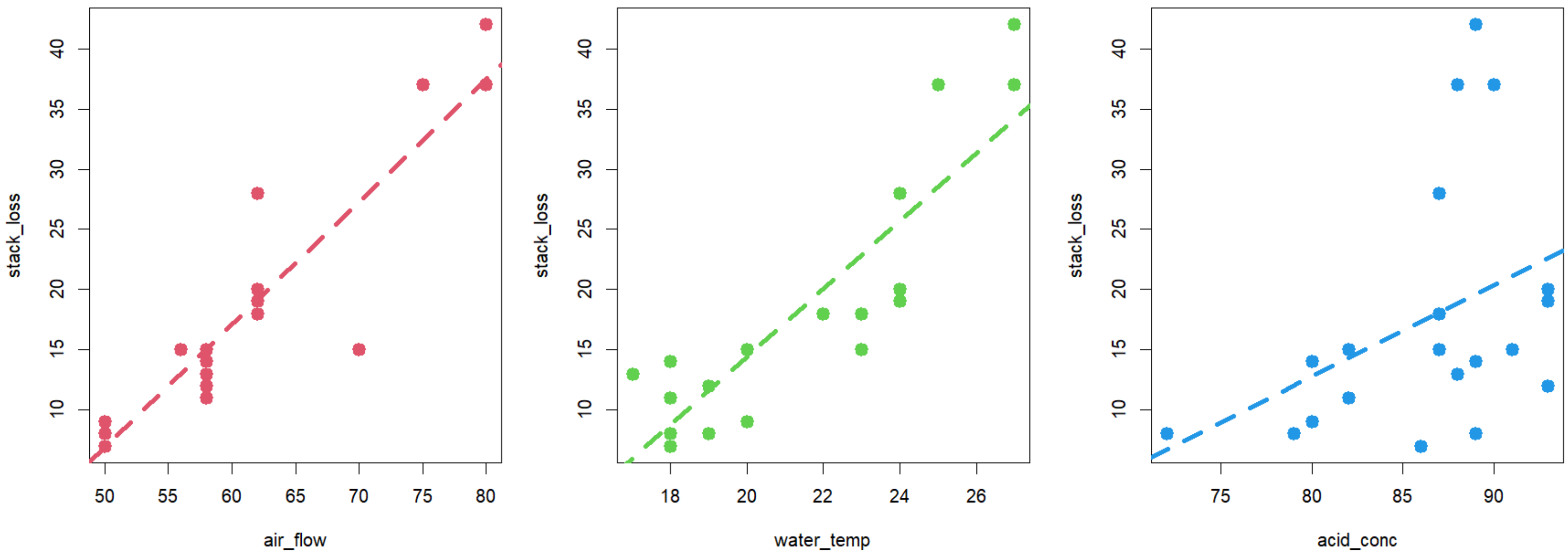

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`air_flow`	0.83	0.83
`water_temp`	0.57	0.57
`acid_conc`	-0.06	-0.06

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	LAD	0.01	0.1	0.5
`air_flow`	0.83	0.83	0.83	0.83
`water_temp`	0.57	0.57	0.59	0.69
`acid_conc`	-0.06	-0.06	-0.07	-0.10

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`L1pack::lad` | τ | larger τ | even larger τ

	LAD	0.01	0.1	0.5	2	5	10
`air_flow`	0.83	0.83	0.83	0.83	0.81	0.77	0.75
`water_temp`	0.57	0.57	0.59	0.69	0.92	1.09	1.18
`acid_conc`	-0.06	-0.06	-0.07	-0.10	-0.12	-0.14	-0.14

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..tending to least squares?

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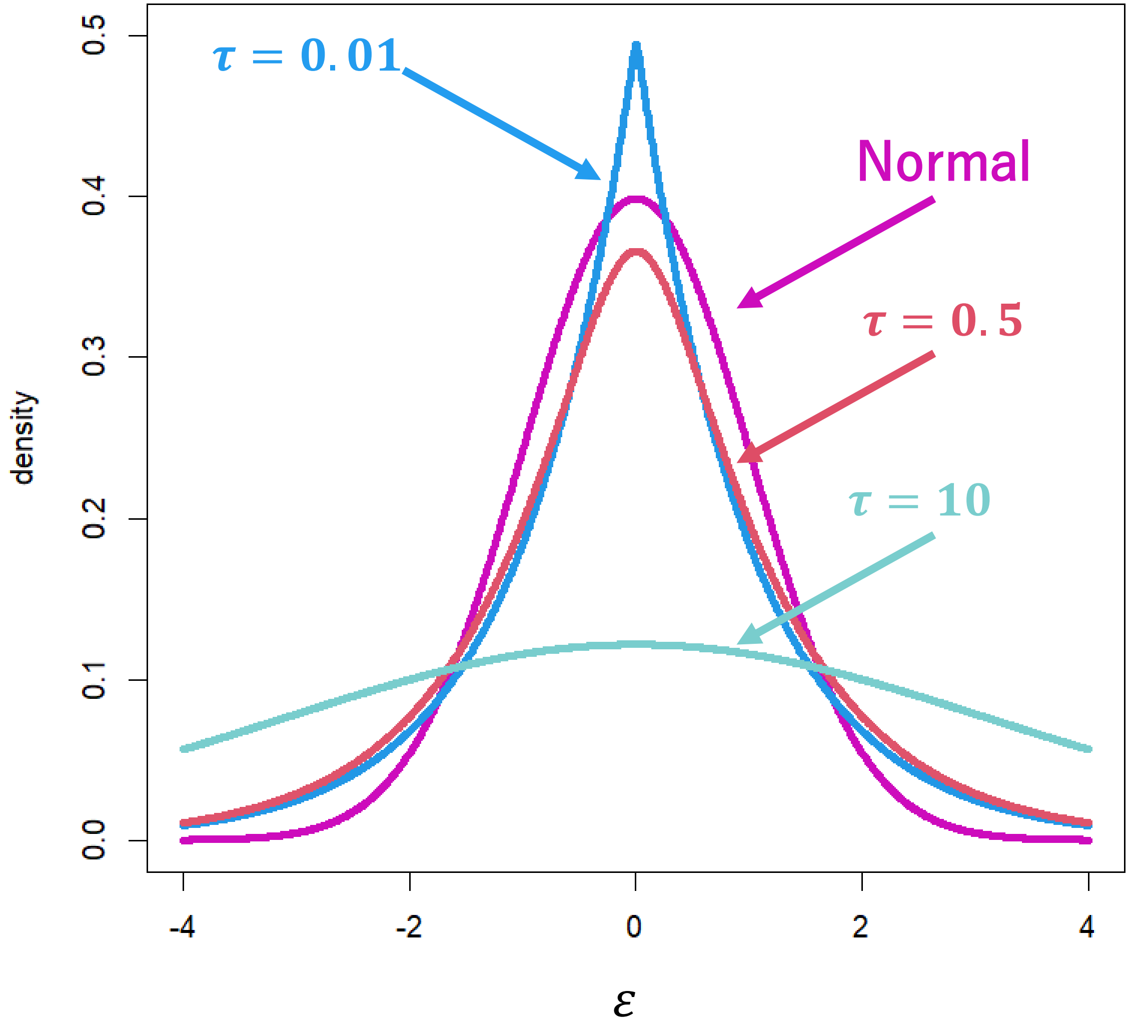
INCREASING τ



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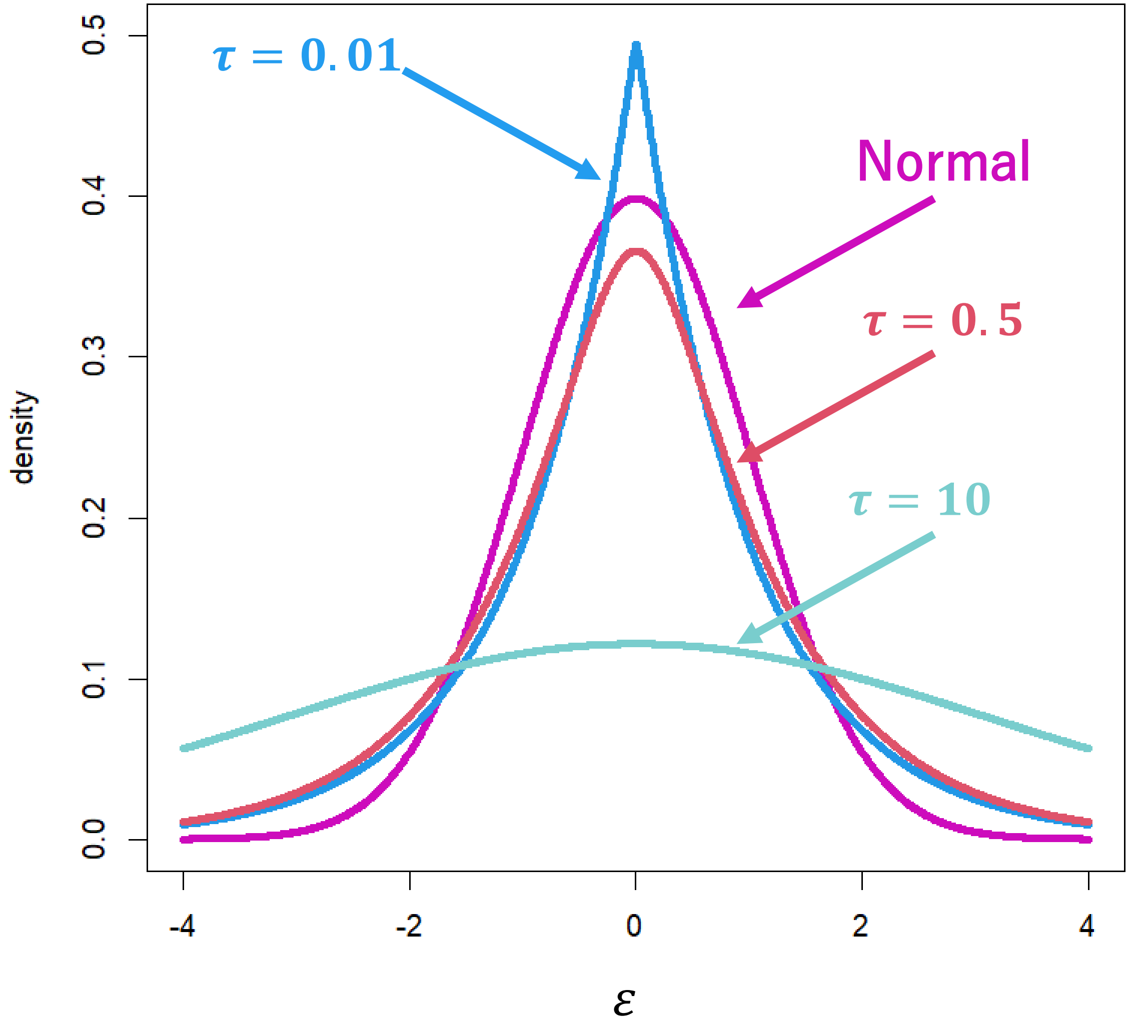

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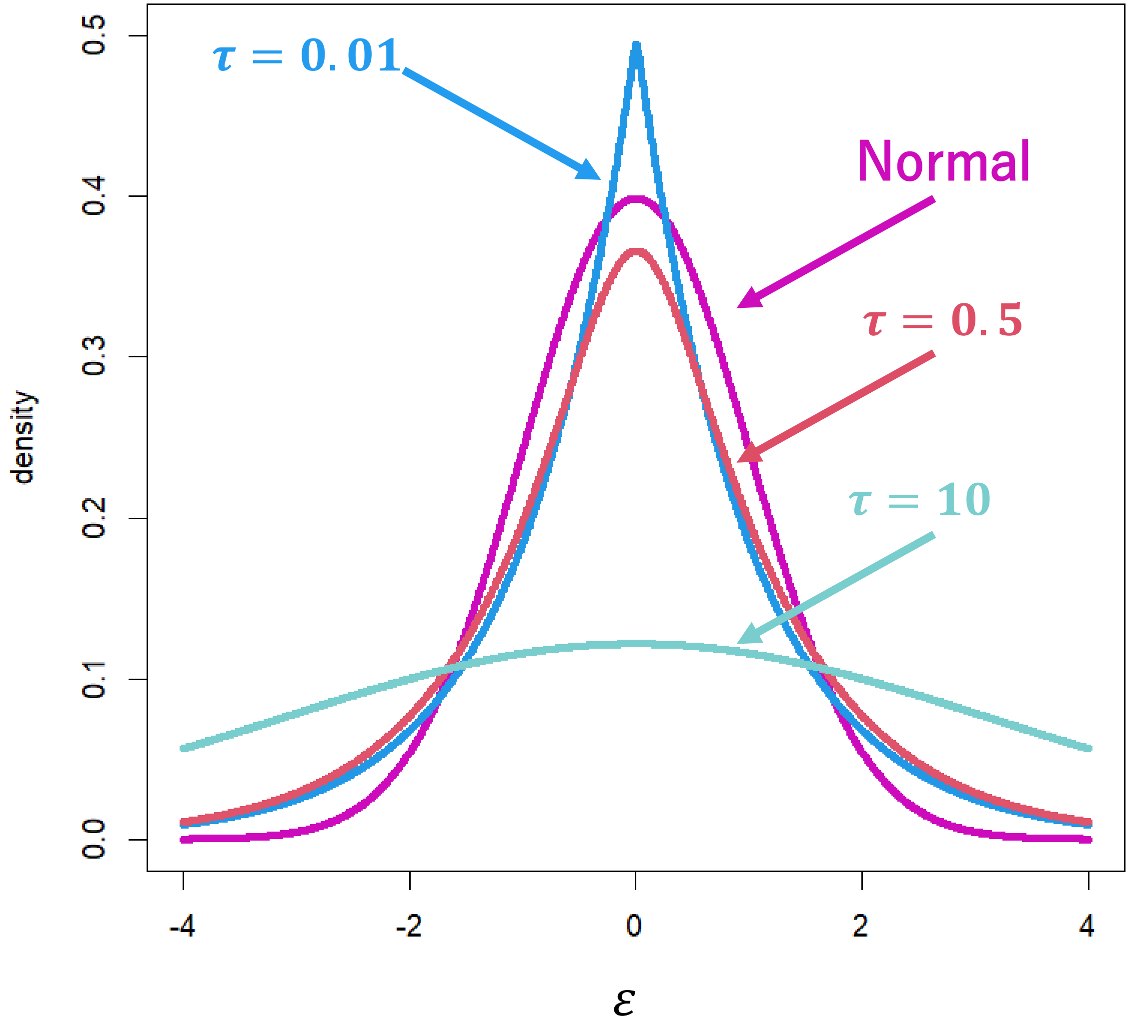

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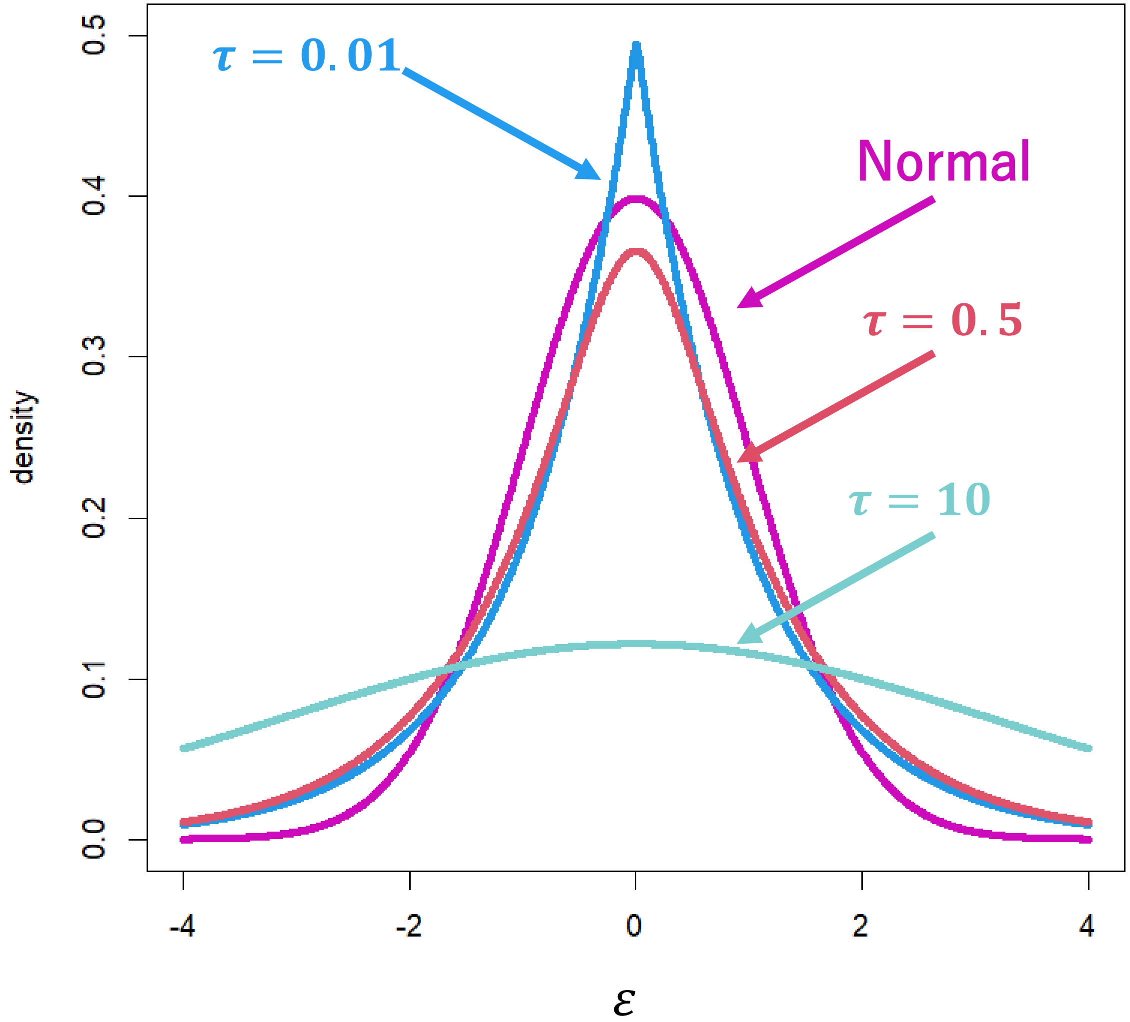

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Large τ

$a_\tau(\varepsilon) \sim \varepsilon^2$?

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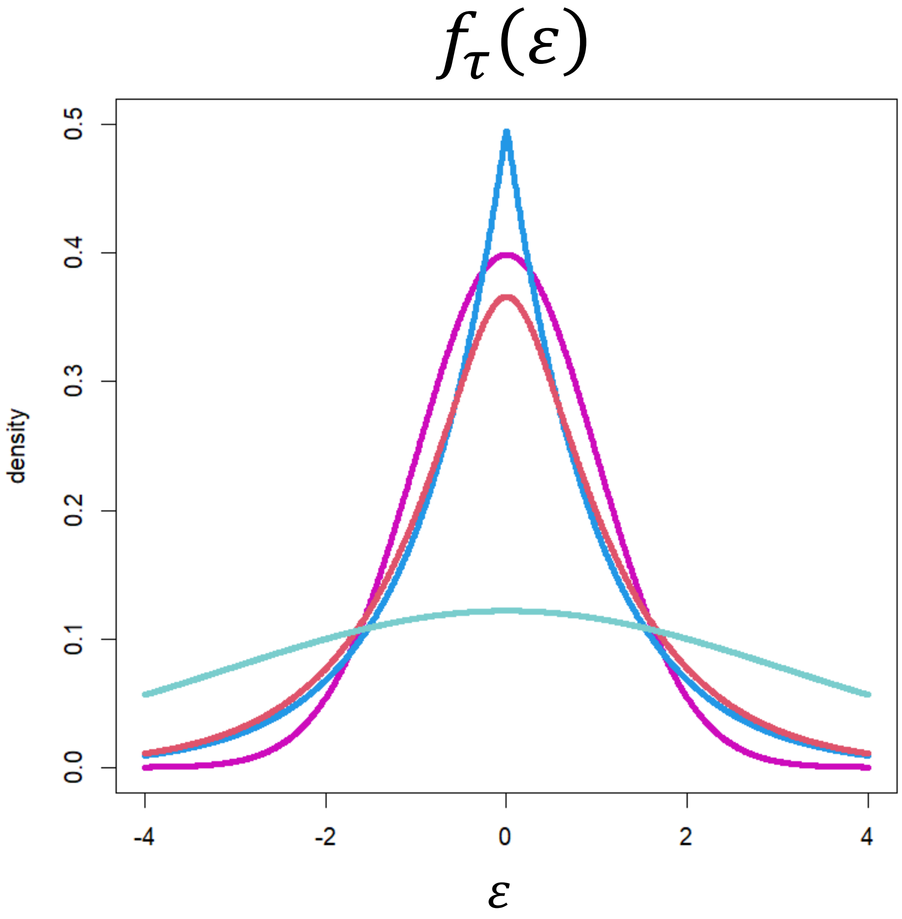
  - $\lim\limits_{\tau \to 0} \tilde{f}_\tau(\varepsilon) =$ Laplace
  - $\lim\limits_{\tau \to \infty} \tilde{f}_\tau(\varepsilon) =$ Gaussian

NEW PARAMETERISATION



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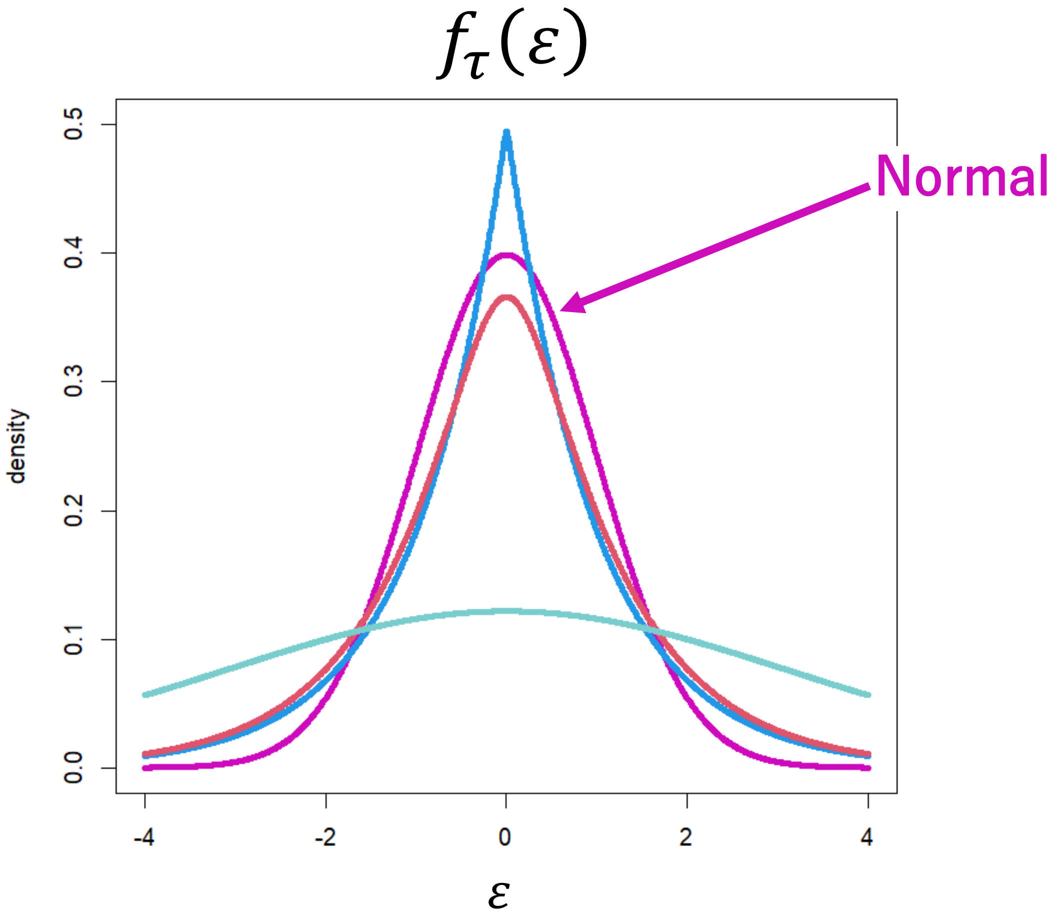

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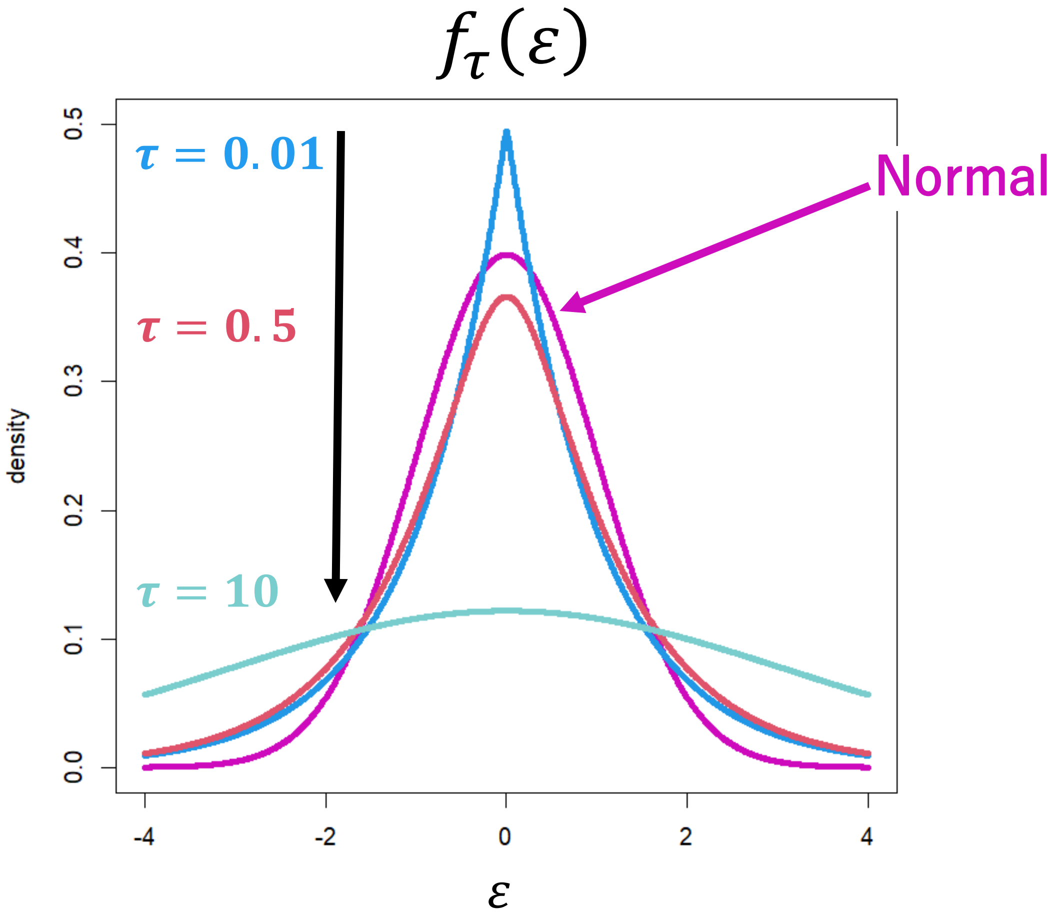

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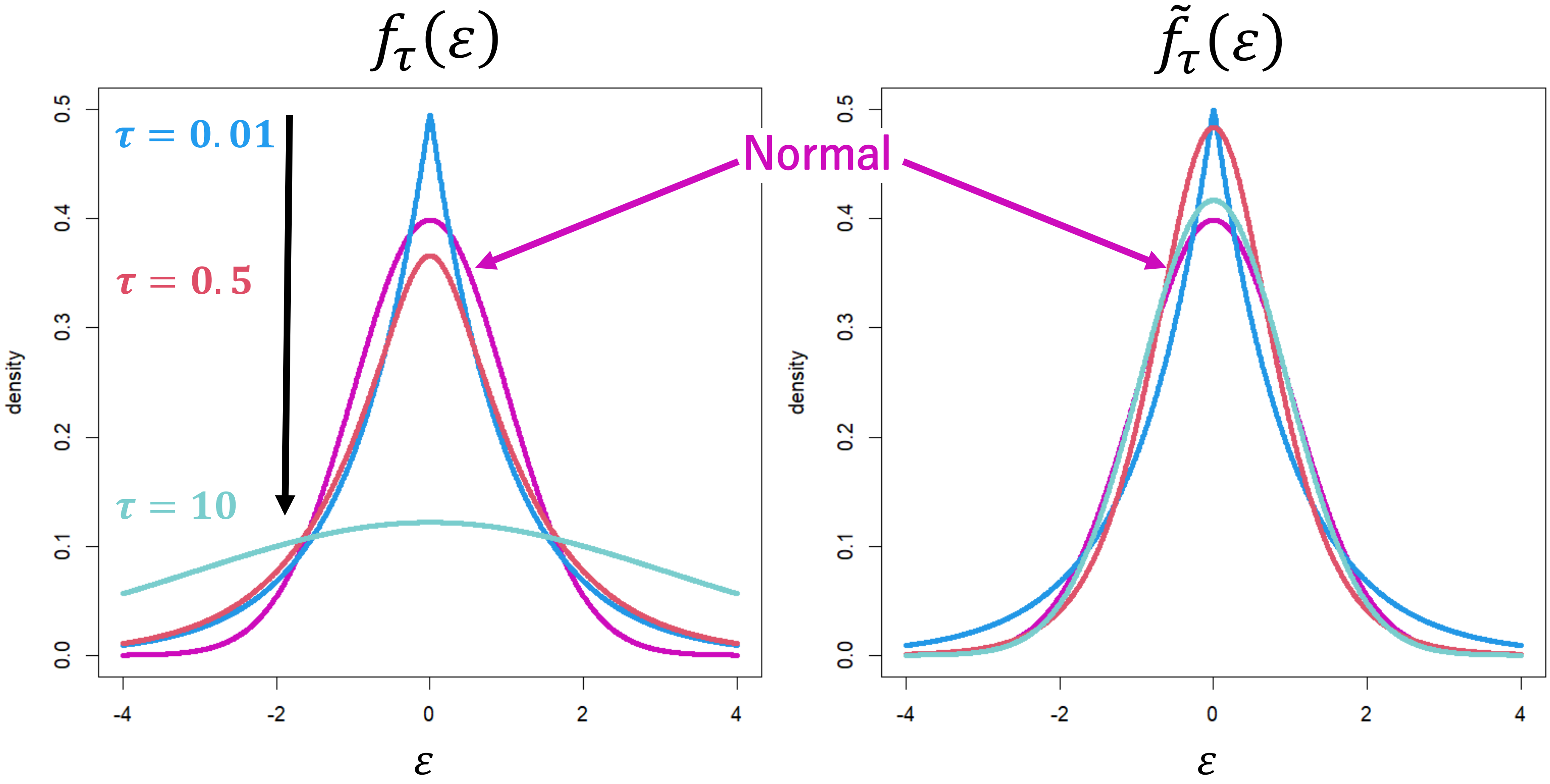

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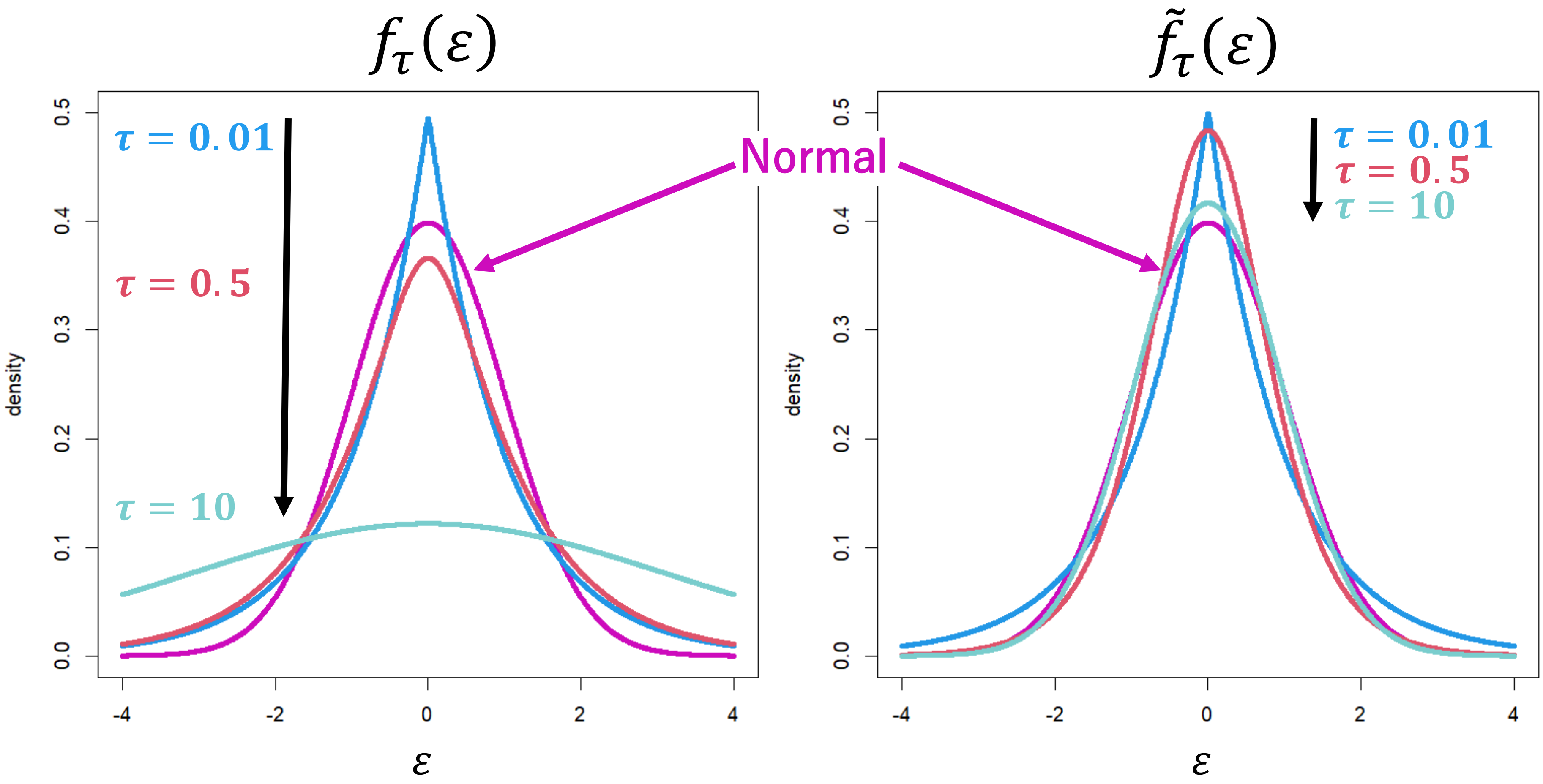
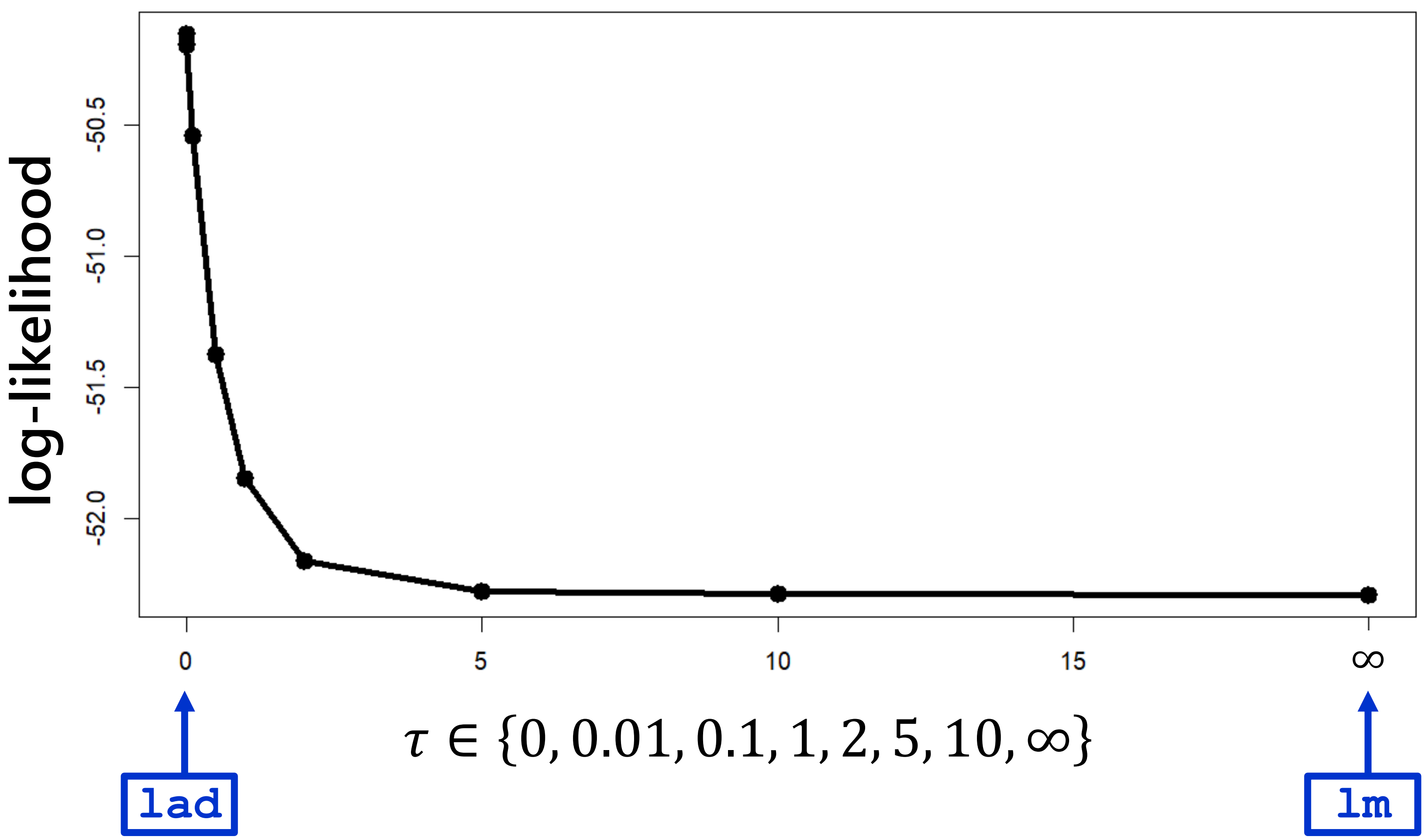


STACKLOSS: LOG-LIKELIHOOD



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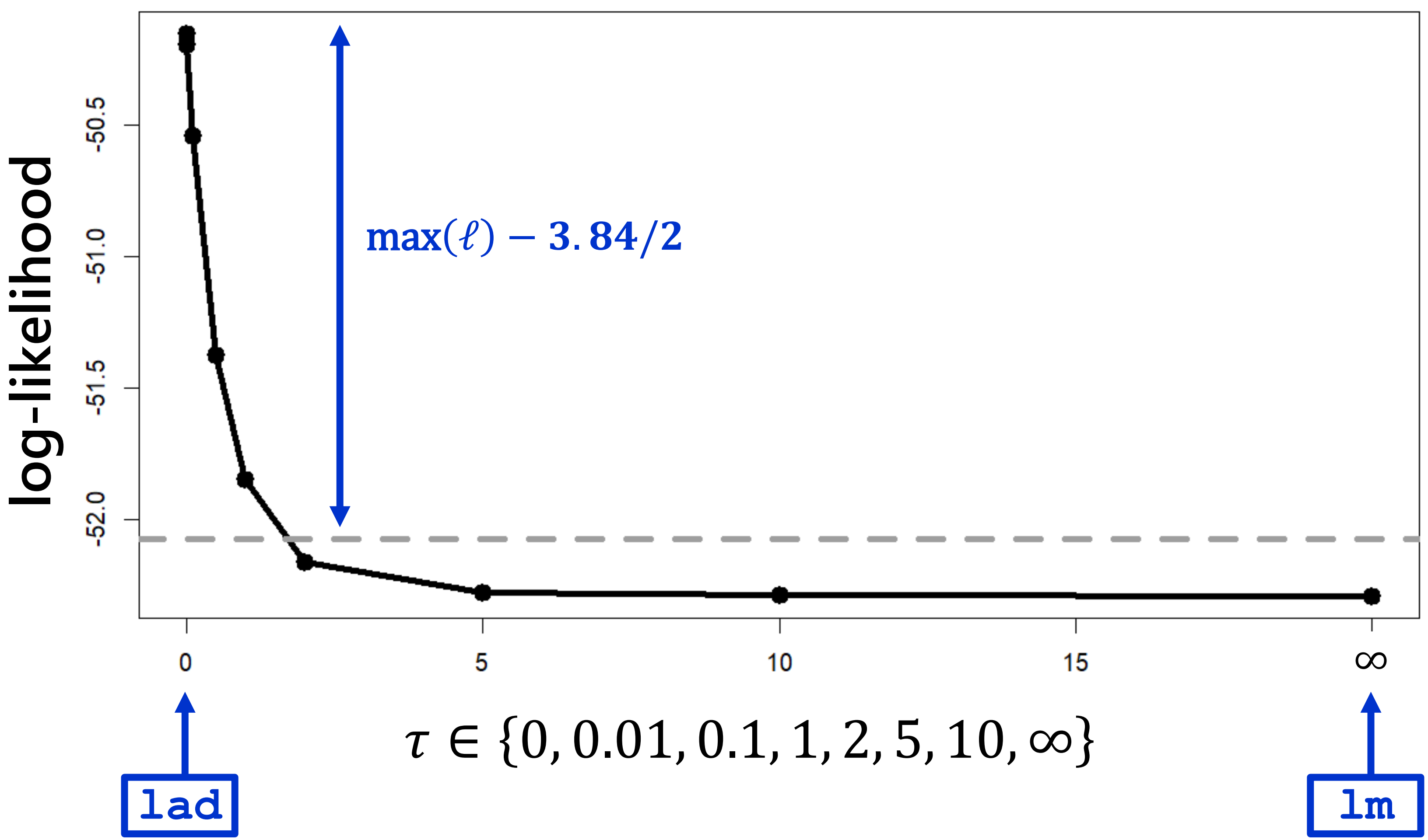
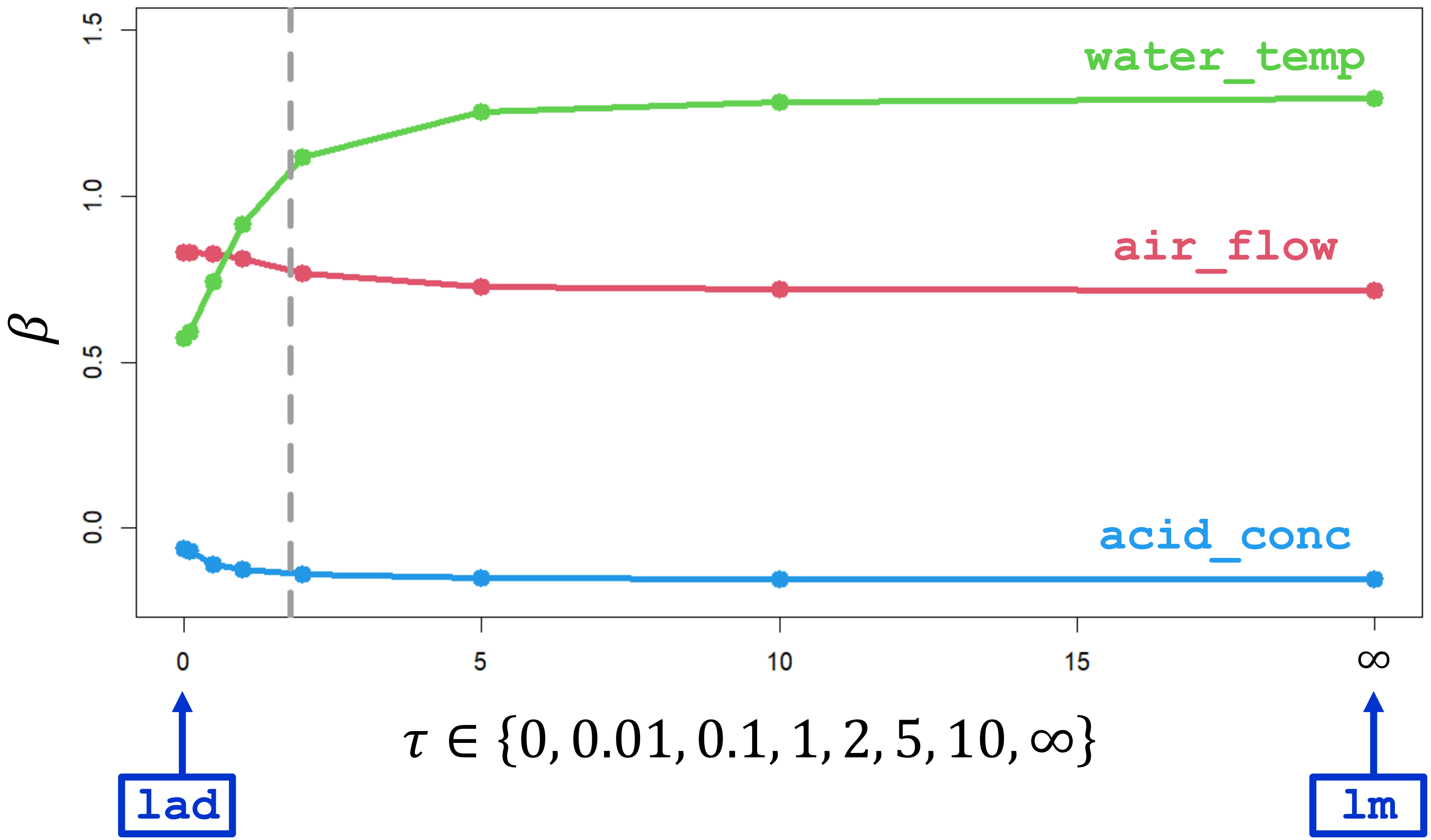
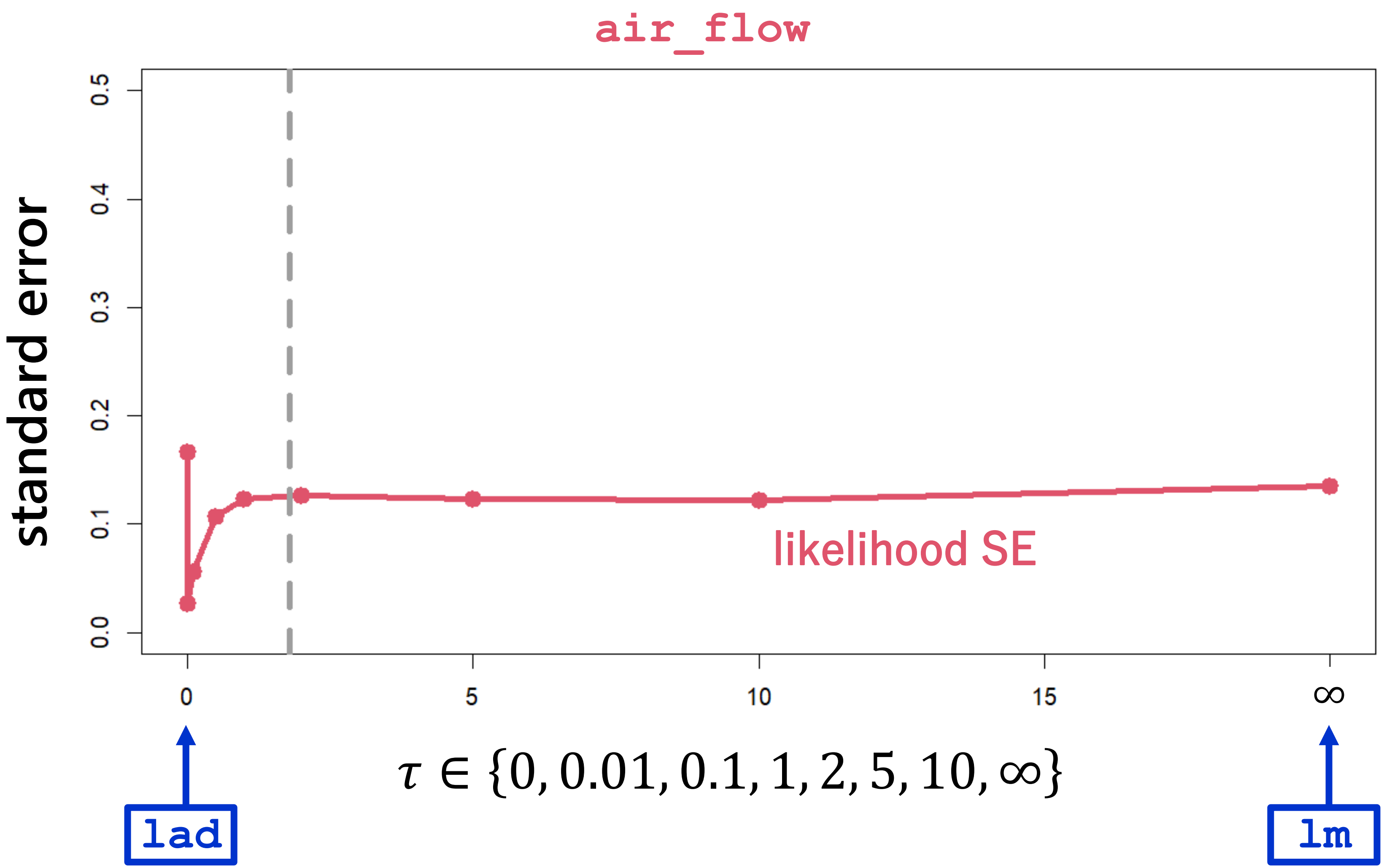


$\tau \in \{0, 0.01, 0.1, 1, 2, 5, 10, \infty\}$

STACKLOSS: BETA COEFFICIENTS



STACKLOSS: STANDARD ERRORS



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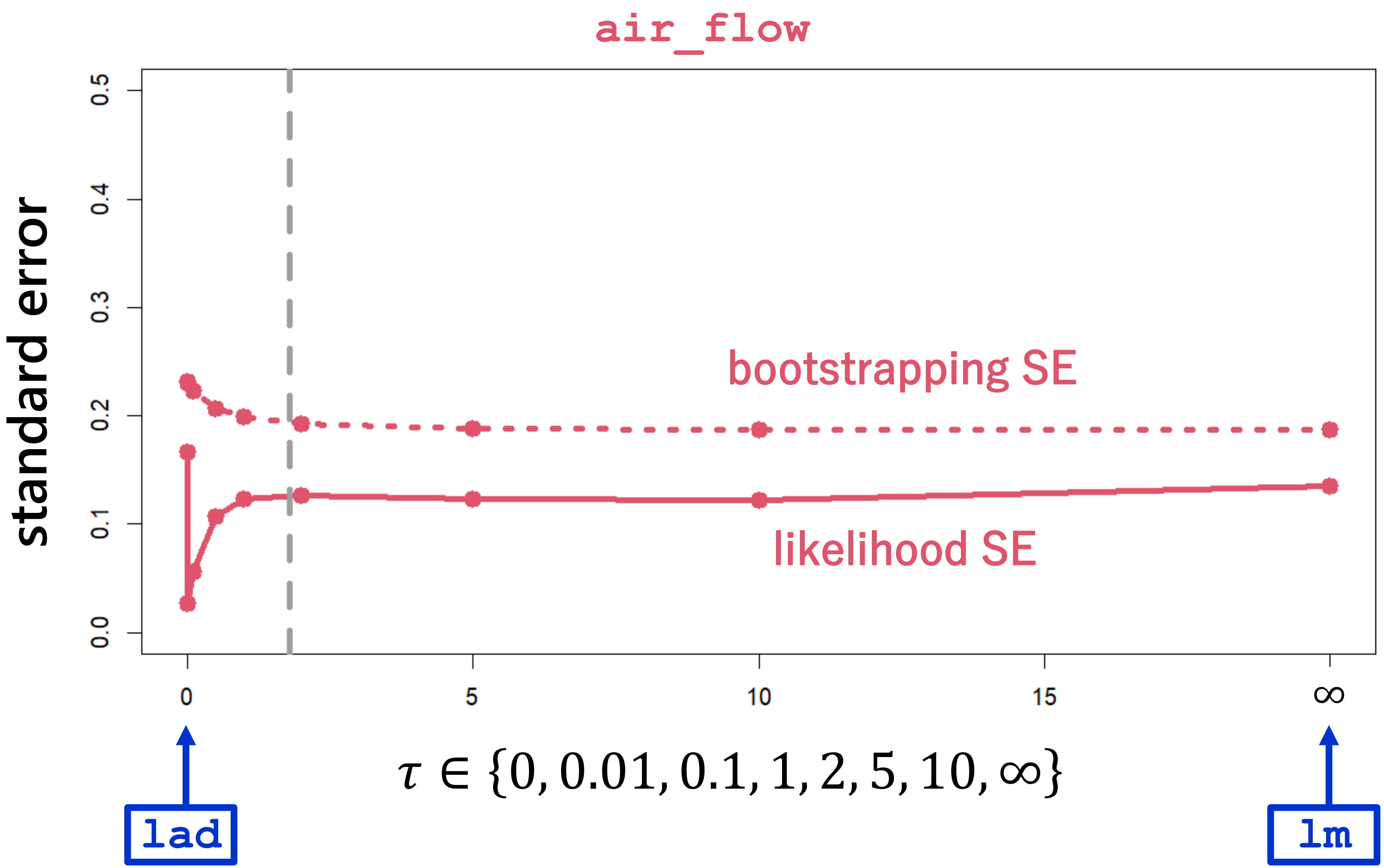

$$\tau \in \{0, 0.01, 0.1, 1, 2, 5, 10, \infty\}$$

SIMULATION: SE ESTIMATION

- $y = \beta_0 + \mathbf{1}x_1 + \mathbf{0.5}x_2 + \mathbf{0}x_3 + \sigma\varepsilon_\tau,$
- $\boldsymbol{\tau} \in \{\mathbf{0.01}, \mathbf{0.1}\}, \quad x_j \sim N(0,1), \quad n \in \{100, 500, 1000, 5000\}$

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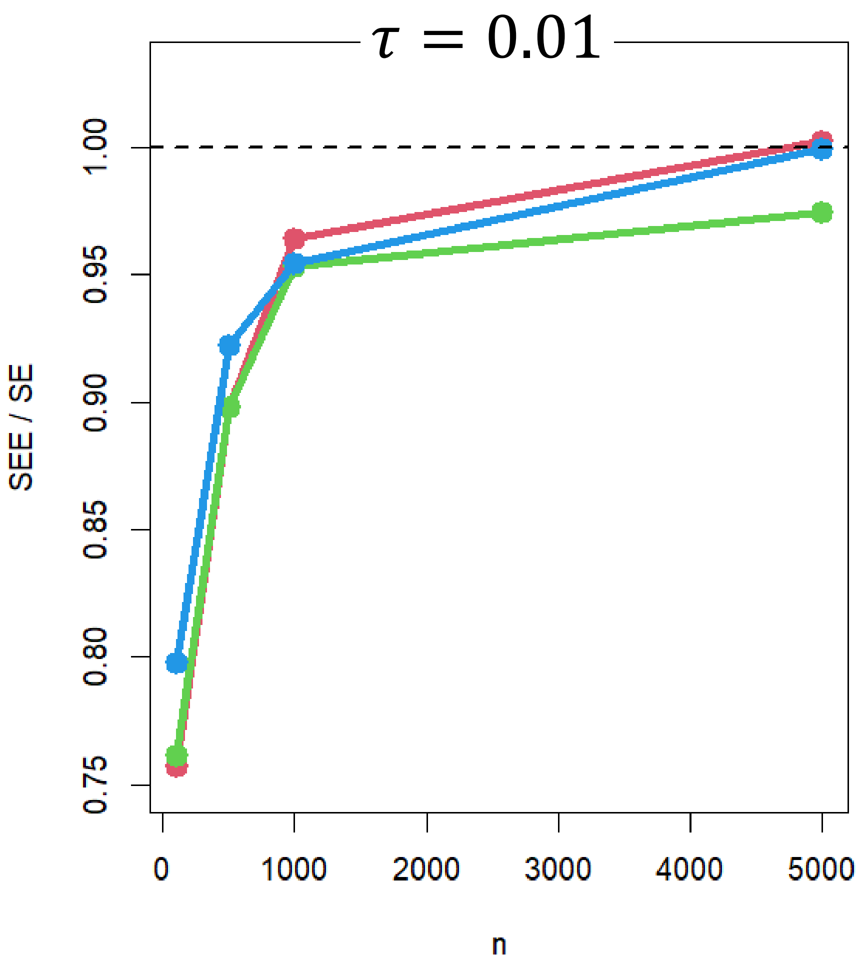

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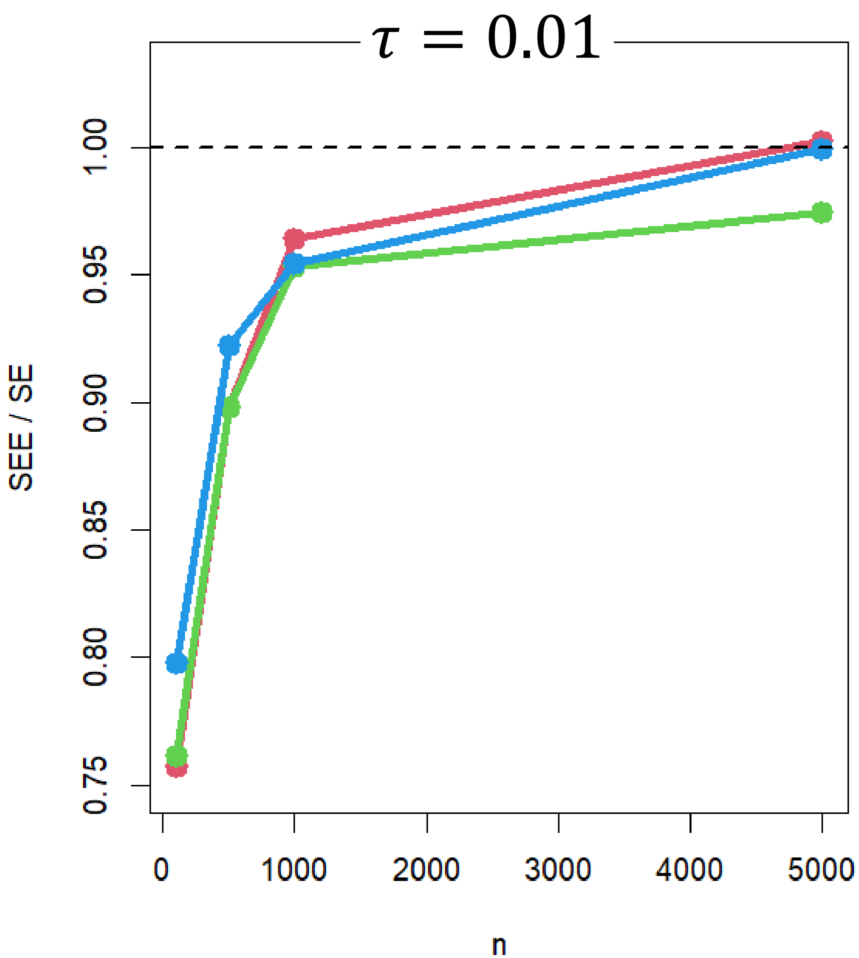
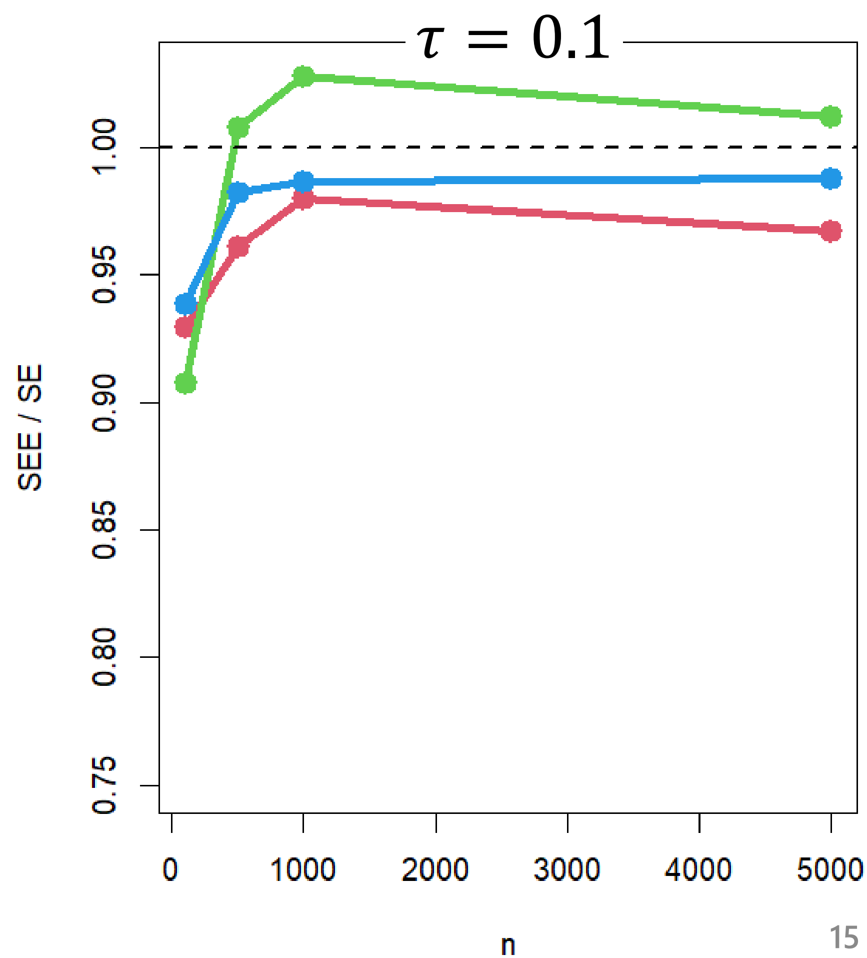
- $y = \beta_0 + \mathbf{1}x_1 + \mathbf{0.5}x_2 + \mathbf{0}x_3 + \sigma\varepsilon_\tau$,
- $\boldsymbol{\tau} \in \{\mathbf{0.01}, \mathbf{0.1}\}$, $x_j \sim N(0,1)$, $n \in \{100, 500, 1000, 5000\}$





SIMULATION: 95% CI COVERAGE

- $y = \beta_0 + \mathbf{1}x_1 + \mathbf{0.5}x_2 + \mathbf{0}x_3 + \sigma\varepsilon_\tau,$
- $\boldsymbol{\tau} \in \{\mathbf{0.01}, \mathbf{0.1}\}, \quad x_j \sim N(0,1), \quad n \in \{100, 500, 1000, 5000\}$

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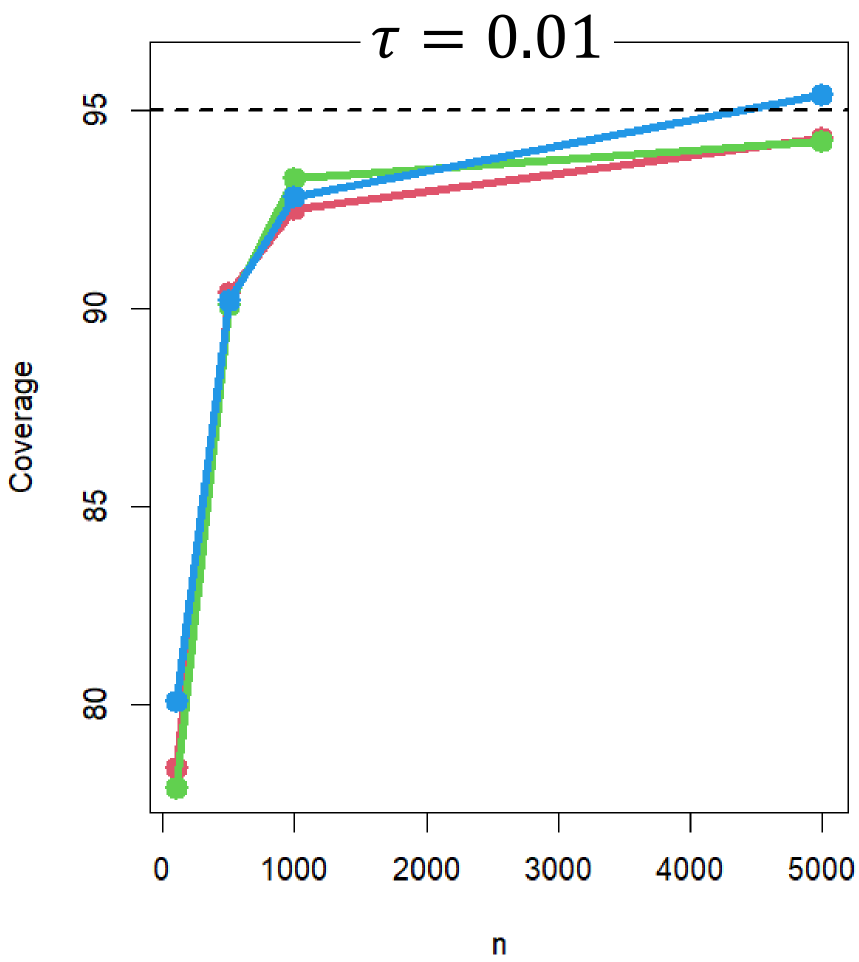
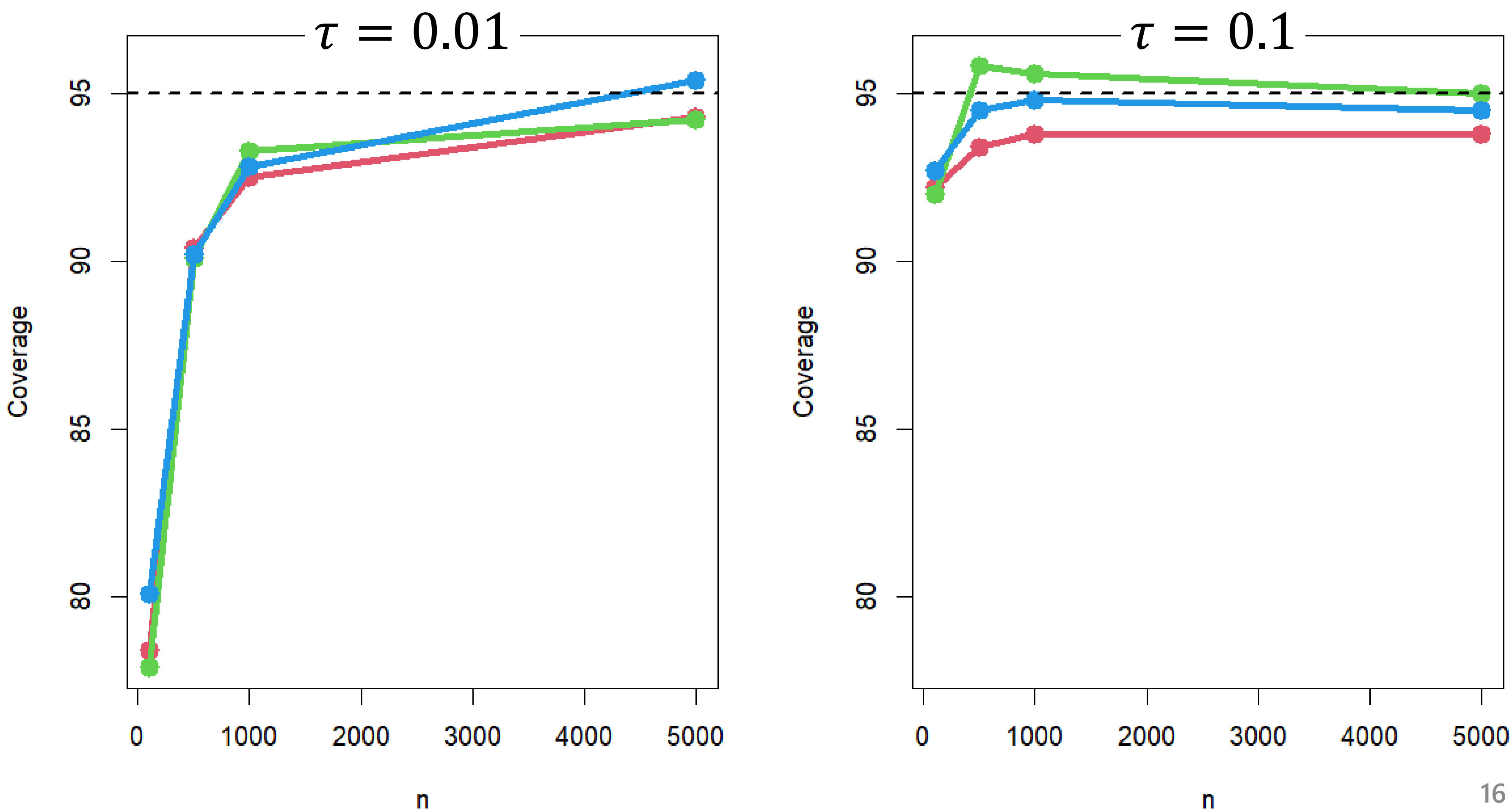


SIMULATION: 95% CI COVERAGE

- $y = \beta_0 + \mathbf{1}x_1 + \mathbf{0.5}x_2 + \mathbf{0}x_3 + \sigma\varepsilon_\tau,$
- $\boldsymbol{\tau} \in \{\mathbf{0.01}, \mathbf{0.1}\}, \quad x_j \sim N(0,1), \quad n \in \{100, 500, 1000, 5000\}$



SUMMARY

- New differentiable approximation to L1 regression / Laplace
- Extension includes L2 regression / Gaussian
- Smoothly joins two common regression approaches
- SEs can be improved
- References
  - O'Neill & **Burke** (2022). Robust Distributional Regression with Automatic Variable Selection. arXiv.
  - O'Neill & **Burke** (2021) Variable Selection Using a Smooth Information Criterion for Multi-Parameter Regression Models. arXiv.
  - **Burke** & Patilea (2021). A likelihood-based approach for cure regression models. TEST.
  - Jaouimaa, Ha, & **Burke** (2019). Penalized Variable Selection in Multi-Parameter Regression Survival Modelling. arXiv.
  - **Also see: kevinburke.ie and arxiv.org/a/burke_k_1**

Session EC814 ***Room:* S-1.04**
Variable selection *Sunday 18.12.2022* 08:15 - 09:55
Chair: Asaf Weinstein *Organizer:* CMStatistics
B1717: M. ONeill, K. Burke
Distributional regression models with automatic variable selection