CAAM 471/571: Traveling Salesman Project

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1 Overview

1.1 Problem Statement

We were tasked to solve the Traveling Salesman Problem using the branch-and-cut method, utilizing gurobi to solve only linear programming relaxations of integer programs.

Given a graph $G = (N, E)$ with nodes N and edges E, and an associated cost c_e for each edge, the goal of the TSP is to find the least costly path which visits each node exactly once and returns to the starting node (a Hamiltonian cycle).

The TSP can be formulated as the following integer program:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(\{n\})} x_e = 2, \quad \forall n \in N \\ & \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall S \subset N, S \neq \emptyset \\ & x_e \in \{0, 1\}, \quad \forall e \in E \end{aligned} \tag{1}$$

where x_e is a decision variable indicating whether or not the associated edge is part of the tour, and $\delta(S)$ is the set of edges in the cut of node set S. The first set of constraints in (1) ensures that each node is entered exactly once and then exited exactly once. This leaves open the possibilty of subtours, which are eliminated by the second set of constraints.

1.2 Approach

The TSP can be solved by using the branch-and-cut variant of the branch-and-bound method.

Branch-and-bound divides a problem into subproblems, solves each subproblem, and then determines which, if any, of the subproblems could potentially yield a result which is better than the best-known result. Subproblems which cannot potentially yield improvements are discarded, leaving more time to investigate the other, more promising, subproblems.

In the case of an LP relaxation to an IP problem, branch-and-bound can be used to implement variable fixing, where some non-integral x_e is forced to take on an integral value. In the case of the TSP, variable fixing results in one non-integral solution being turned into two subproblems, one for $x_e = 0$ and one for $x_e = 1$.

Branch-and-cut works similarly to branch-and-bound, but helps to avoid the potentially exponental growth in the number of subproblems. It does this by adding constraints (cuts) to models with non-integral x_e and re-optimizing them instead of automatically branching. If no new cuts can be found, branch-and-cut will degrade to normal branch-and-bound.

Our solver consists of three main parts, which will be desribed in the following sections:

1. a *solver* which manages branch-and-cut related activities
2. a *graph class* implements graph-related algorithms
3. a collection of *cut generating algorithms*

2 Solver

2.1 Top-Level Solver

The top-level solver creates the initial LP relaxation model and adds it to the model pool.

Until the pool is empty, it pulls a model from the pool, processes it, and adds any resulting models into the pool.

Processing ends when the model pool is empty.

(Note that code in listings may have been modified cosmetically for display purposes. Refer to the accompanying source for full details.)

```
## tsp_solver.py
class TspBranchAndCut(object):
    def solve(self):
        initial_model = self.create_initial_model()
        self.add_model_to_pool(model=initial_model, obj_lb=-float('inf'))

        while not self.model_pool_is_empty():
            model = self.remove_next_model_from_pool()

            for obj_lb,new_model in self.process_model(model):
                self.add_model_to_pool(model=new_model, obj_lb=obj_lb)
```

2.2 The Initial Model

The initial LP relaxation of (1) is:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(\{n\})} x_e = 2, \quad \forall n \in N \\ & 0 \le x_e \le 1, \quad \forall e \in E \end{aligned} \tag{2}$$

where the decision variables x_e are now allowed to take any value between zero and one, and the subtour constraints have been removed (they will gradually be re-introduced as the algorithm progresses).

```
## tsp_solver.py
class TspBranchAndCut(object):
    def create_initial_model(self):
        model = grb.Model('tsp')
        xx = model.addVars(self.edges,
            lb    = 0.0,
            ub    = 1.0,
            vtype = grb.GRB.CONTINUOUS,
            name  = 'xx',
            obj   = self.cost_by_edge
        )
        degree_constrs = model.addConstrs(
          (xx.sum(node,'*') + xx.sum('*',node) == 2.0 for node in
              self.nodes),
          'degree'
```

```
        )
        model.update()
        return model
```

2.3 Model Processing

Models are processed using a variation of the branch-and-bound technique called branch-and-cut.

Once a model has been pulled from the model pool, it is optimized using gurobi.

If the model is infeasible, or if it cannot possibly yield a new best tour because its LP relaxation lower bound is greater than the current best tour cost, it is discarded (the 'bound' part of branch-and-cut) and processing stops.

If the current solution is a valid tour (and therefore integral) with a cost less than that of the current best tour, the current solution becomes the new best tour and processing stops.

Otherwise the current solution is either non-integral or not a tour, so, if possible, new constraints are added to the model (the 'cut' part of branch-and-cut) and it is re-optimized. If no cuts could be added, new variable fixing models (where some non-integral x_e is forced to take on a value of 0 or 1) are created and returned to the caller (the 'branch' part of branch-and-cut).

```
## tsp_solver.py
class TspBranchAndCut(object):
    def process_model(self, model):
        new_model_info = []
        while True:
            model.update()
            model.optimize()

            if self.solution_is_infeasible(model):
                break

            if not self.solution_can_become_new_best(model):
                break

            if self.solution_is_tour(model):
                if self.solution_is_new_best(model):
                    self.update_best(model)
                break
```

```python
        if self.add_cuts_to_model(model):
            continue

        branch_models = self.create_branch_models(model)
        for branch_model in branch_models:
            new_model_info.append( (model.getAttr('ObjVal'),
                branch_model) )

        break

    return new_model_info
```

2.4 Adding Cuts (Constraints)

Several algorithms were developed for finding violated constraints and adding them to models. These algorithms are called sequentially, and once a new constraint is added, the model updated and re-optimized.

Details of the constraint-generating algorithms can be found in section 4.

```python
## tsp_solver.py
class TspBranchAndCut(object):
    def add_cuts_to_model(self, model):
        constraints_were_added = False

        if self.add_comb_constraints(model):
            constraints_were_added = True
        elif self.add_integral_subtour_constraints(model):
            constraints_were_added = True
        elif self.add_nonintegral_subtour_constraints(model):
            constraints_were_added = True
        elif self.add_objective_constraints(model):
            constraints_were_added = True
        # elif self.add_gomory_constraints(model):
        #     constraints_were_added = True

        return constraints_were_added
```

2.5 Variable Fixing (Branching)

Variable fixing (branching) is implemented by finding the first non-integral solution variable x_e and creating two new models: one forcing x_e to take on exactly 0, and one forcing x_e to take on exactly 1. The newly-created models are then returned to the caller, where they will be added to the model pool for further processing.

```
## tsp_solver.py
class TspBranchAndCut(object):
    def create_branch_models(self, model):
        best_idx = best_var = best_val = None

        for idx,mvar in enumerate(model.getVars()):
            val = mvar.getAttr('X')
            if abs(val - int(val)) != 0.0:
                if (best_val is None) or (abs(val - 0.5) < best_val):
                    best_val = abs(val - 0.5)
                    best_var = mvar
                    best_idx = idx

        model1 = grb.Model.copy(model)
        m1var  = model1.getVarByName(best_var.getAttr('VarName'))
        model1.addConstr(m1var == 0.0)
        model1.update()

        model2 = grb.Model.copy(model)
        m2var  = model2.getVarByName(best_var.getAttr('VarName'))
        model2.addConstr(m2var == 1.0)
        model2.update()

        return (model1, model2)
```

3 Graph Class

3.1 Tour Detection

Tours are detected by (a) verifying that $x_e \in \{0, 1\} \forall e \in E$, (b) ensuring that every node has a degree of two, and (c) ensuring that there is only one connected component (no subtours).

```
## graph.py
class Graph(object)
    def is_tour(self, solution):
        ##
        ## Check that the solution is a "binary" vector,
        ## and that number of selected edges is the same as
```

```
## the number of nodes
##

num_selected_edges = 0
for idx,value in enumerate(solution):
    if not value in [0.0,1.0]:
        return False

    if value == 1.0:
        num_selected_edges += 1

if num_selected_edges != len(self.nodes):
    return False

##
## Form a mapping from a node to nodes connected
## by selected edges, and ensure that each node
## has degree 2.
##

selected_edges_by_node = {node: set() for node in self.nodes}
for idx,value in enumerate(solution):
    if value == 1.0:
        edge = self.edge_by_idx[idx]
        (node1,node2) = edge
        selected_edges_by_node[node1].add(edge)
        selected_edges_by_node[node2].add(edge)

for node,selected_edges in selected_edges_by_node.items():
    if len(selected_edges) != 2:
        return False

##
## Find all connected components. If there is only one,
## we have a tour.
##

node_components, edge_components =
     self.binary_connected_components(solution)
if len(node_components) != 1:
    return False

return True
```

3.2 Connected Components

Connected components can be detected by doing a breadth-first search of previously-unseen nodes. When no unseen nodes can be reached, all seen nodes form a connected component. Nodes in the current connected component can be removed from the graph, and the process repeated, until every node belongs to a connected component.

In our implementation, we return connected components in both node and edge forms.

```python
## graph.py
class Graph(object)
    def connected_components(self):
        connected_components_nodes = []
        connected_components_edges = []

        unvisited_nodes = set(self.nodes)
        while len(unvisited_nodes) != 0:
            queue = deque([unvisited_nodes.pop()])

            visited_nodes = set()
            visited_edges = set()
            while len(queue) != 0:
                cur_node = queue.popleft()

                if cur_node in unvisited_nodes:
                    unvisited_nodes.remove(cur_node)

                visited_nodes.add(cur_node)

                for edge in self.edges_by_node[cur_node]:
                    visited_edges.add(edge)

                    (node1,node2) = edge
                    if node1 not in visited_nodes:
                        queue.append(node1)
                    if node2 not in visited_nodes:
                        queue.append(node2)

            connected_components_nodes.append(visited_nodes)
            connected_components_edges.append(visited_edges)

        return connected_components_nodes, connected_components_edges
```

3.3 Min-Cut

The Stoer-Wagner[1] algorithm is used to find min-cuts in a graph. The algorithm works by repeatedly removing the "least attached" node (based on its cut) from the current graph by merging it with the next "least attached" node. When all nodes have been merged, the smallest encountered cut is the min-cut for the graph.

In our variation of this algorithm, we accumulate all encountered cuts so that they can be processed using whatever cut threshold the caller wants.

```
## graph.py
class Graph(object)
    def find_min_cut(self):
        merged_graph = Graph(nodes=self.nodes, edges=self.edges,
            weight_by_edge=self.weight_by_edge)

        min_cut_value = float("inf")
        min_cut_nodes = []

        all_cuts = []

        while merged_graph.num_nodes > 1:

            penultimate_node = None
            last_node        =
                merged_graph.nodes[random.randint(0,len(merged_graph.nodes)-1)]

            node_set = set([last_node])

            while True:
                new_node =
                    merged_graph._find_next_connected_node(target_nodes=node_set)
                if new_node is None:
                    break
                penultimate_node = last_node
                last_node        = new_node
                node_set.add(new_node)

            current_cut_value = 0
            current_cut_nodes = [last_node]
            for other_node in merged_graph.nodes_by_node[last_node]:
                current_cut_value +=
                    merged_graph.weight_by_edge[(last_node, other_node)]

            merged_graph =
                merged_graph._merge_nodes(node1=penultimate_node,
                node2=last_node)
```

```
    all_cuts.append((current_cut_value, [item for item in
        flatten(current_cut_nodes, list_or_tuple)]))

    if current_cut_value < min_cut_value:
        min_cut_value = current_cut_value
        min_cut_nodes = current_cut_nodes

all_cuts = sorted(all_cuts, key=lambda x: x[0])

return all_cuts
```

4 Constraint (Cut) Algorithms

4.1 Subtour

The subtour constraints in the original TSP IP formulation (1) cannot be enumerated in practice for large TSP problems, which is why they are dropped in the initial LP relaxation (2).

Given a solution $\boldsymbol{x}^*$ to an LP relaxation, it is possible to find and re-apply a violated subtour constraint by examining min-cuts on the resulting graph: if the min-cut is less than two, then some subtour contraint has been violated. By dividing the nodes into two subsets (S and $N \setminus S$) as determined by the edges in the min-cut, the following constraint can be constructed:

$$\sum_{e \in S} x_e \geq 2 \tag{3}$$

Two algorithms were developed to find cuts based on this property: One works for binary $\boldsymbol{x}^*$ and the other works for any $\boldsymbol{x}^*$. Both utilize the same underlying graph ultilities.

Subtour contraints are universal: they apply to any TSP model regardless of what other decisions (e.g., variable fixing) might have been made. As a result, when a subtour constraint is found, it is added to *all* models (not just the current model). This avoids searching for the identical constraint multiple times.

4.1.1 Integral

If $\boldsymbol{x}^*$ is binary ($x_e \in \{0,1\} \forall e \in E$) then finding a min-cut is the same as finding a cut for some connected component (assuming there is more than one). This cut will obviously have value zero (since $\boldsymbol{x}^*$ is binary), which is below the subtour constraint threshold of two.

```
## tsp_solver.py
class TspBranchAndCut(object):
    def add_integral_subtour_constraints(self, model):
        graph, xx = self.convert_model(model)
        connected_component_nodes, connected_component_edges =
            graph.binary_connected_components(solution=xx)

        if len(connected_component_nodes) == 1:
            return False

        for cur_model in all_models:
            mvars = cur_model.getVars()

            for cc in connected_component_nodes:
                ## translate the edge indices into model variables
                cut_edges = graph.get_cut_edges(nodes=cc)
                var_idxs = sorted([self.idx_by_edge[edge] for edge in
                    cut_edges])
                cvars = [mvars[idx] for idx in var_idxs]
                coeffs = [1.0 for idx in var_idxs]

                cur_model.addConstr(grb.quicksum(cvars) >= 2.0,
                    'subtour-integral')

                constraints_were_added = True

        return constraints_were_added
```

4.1.2 Non-Integral

Instead of using connected components like the integral approach, this algorithm looks for any min-cut whose value is less than two. In fact, it accumulates *all* of the violated subtour constraints it encounters for the given solution.

```
## tsp_solver.py
class TspBranchAndCut(object):
```

```python
def add_nonintegral_subtour_constraints(self, model):
    graph, xx = self.convert_model(model)

    modified_graph, xx = self.convert_model(model)
    all_cuts = []
    while still_adding_new_cuts:
        cur_cuts = modified_graph.find_min_cut()
        all_cuts.extend(cur_cuts)

        if cur_cuts[0][0] > 0.0:
            break

        new_nodes  = set(modified_graph.nodes) - set(cur_cuts[0][1])
        new_edges  = set()
        new_weights = {}
        for node in new_nodes:
            for edge in modified_graph.edges_by_node[node]:
                if (edge[0] in new_nodes) and (edge[1] in new_nodes):
                    new_edges.add(edge)
                    new_weights[edge] =
                        modified_graph.weight_by_edge[edge]

        modified_graph = Graph(nodes=new_nodes, edges=new_edges,
             weight_by_edge=new_weights)

    all_cuts = sorted(all_cuts, key=lambda x: x[0])
    if len(all_cuts) == 0:
        return False
    if all_cuts[0][0] >= 2.0:
        return False

    for cur_model in all_models:
        for var_idxs in non_dup_cuts:
            add_constr_to_model(model=cur_model, var_idxs=var_idxs)

    return constraints_were_added
```

4.2 Comb (Blossom) Inequalities

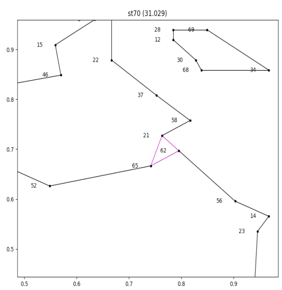
Given an optimal tour on $G = (N, E)$, it can be shown that, for some 'handle' node set $H \subset N$ and 'tooth' edges $\{T_1, \ldots, T_k\} = \delta(H)$, that:

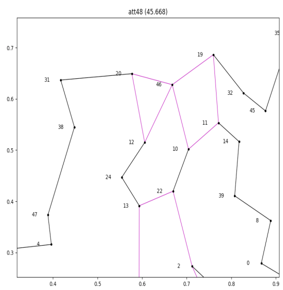
$$\sum_{e \in \delta(H)} x_e + \sum_{i=1}^{k} \sum_{e \in \delta(T_i)} x_e \geq 3k + 1 \tag{4}$$

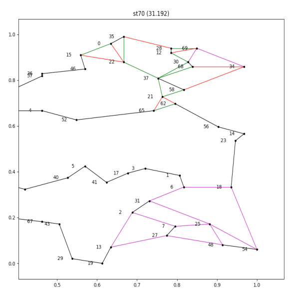
if k is odd.

Equation (4) is called the *comb inequality* and can be used to find subtour constraint violations.

The code is too involved to show here, but details can be found in [2],[3]. The following figures show combs found during the processing of datasets for this project.







4.3 Objective

In our particular flavor of the TSP, all edge costs c_e are integral. This implies that the objective value of a valid TSP tour must also be integral because all x_e decision variables must be integral.

This makes it possible to create a new constraint whenever the LP relaxation objective value is non-integral by just rounding it up:

$$\sum_{e \in E} c_e x_e >= \left\lceil \sum_{e \in E} c_e x_e^* \right\rceil \tag{5}$$

where x_e^* are the current LP relaxation decision variables.

It turns out that gurobi allows constraints to be modified, so only one objective constraint is ever present at any given time. Also, because this constraint relies on locally-made decisions (like variable fixing), it is applied to the current model only. Some precautions were taken to avoid applying this constraint inappropriately due to floating-point rounding issues.

```
## tsp_solver.py
class TspBranchAndCut(object):
    def add_objective_constraints(self, model):
        obj_val = model.getAttr('ObjVal')
        ceil_obj_val = math.ceil(obj_val)
        floor_obj_val = math.floor(obj_val)

        if (abs(obj_val - floor_obj_val) < 1e-6) or (abs(obj_val -
            ceil_obj_val) < 1e-6):
```

```
        return False

    target_constr_name = 'objective-roundup'

    target_constr = None
    for constr in model.getConstrs():
        if constr.getAttr('ConstrName') == target_constr_name:
            target_constr = constr
            break

    if target_constr is not None:
        target_constr.setAttr('RHS', ceil_obj_val)
    else:
        mvars = model.getVars()

        var_idxs = sorted([self.idx_by_edge[edge] for edge in
            self.edges])
        cvars = [mvars[idx].getAttr('Obj')*mvars[idx] for idx in
            var_idxs]

        model.addConstr(grb.quicksum(cvars) >= ceil_obj_val,
            'objective-roundup')

    return True
```

4.4 Gomory

Gomory mixed-integer constraints can be generated from the final tableau that results from the simplex algorithm.

We chose to implement these constraints based on [4], partly because the derivation was rather clear and partly because it deals with the case of general lower- and upper-bounds on variables.

The code worked fine on all datasets except pr76, where it resulted in a sub-optimal tour. This was possibly due to accumulation of rounding errors, but steps were taken to mitigate that problem proved ineffective.

The code is included in our project, but was disabled for the final runs.

5 Results

The algorithms outlined above were able to solve all assigned TSP datasets.

dataset	time (sec)	optimal tour cost
att48	0.84	10628
berlin22	0.06	7542
gr21	0.01	2707
hk48	0.49	11461
pr76	4263.32	108159
st70	4.24	675
ulysses22	0.04	7013

Images and details of each tour can be found in section 7.

6 Future Improvements

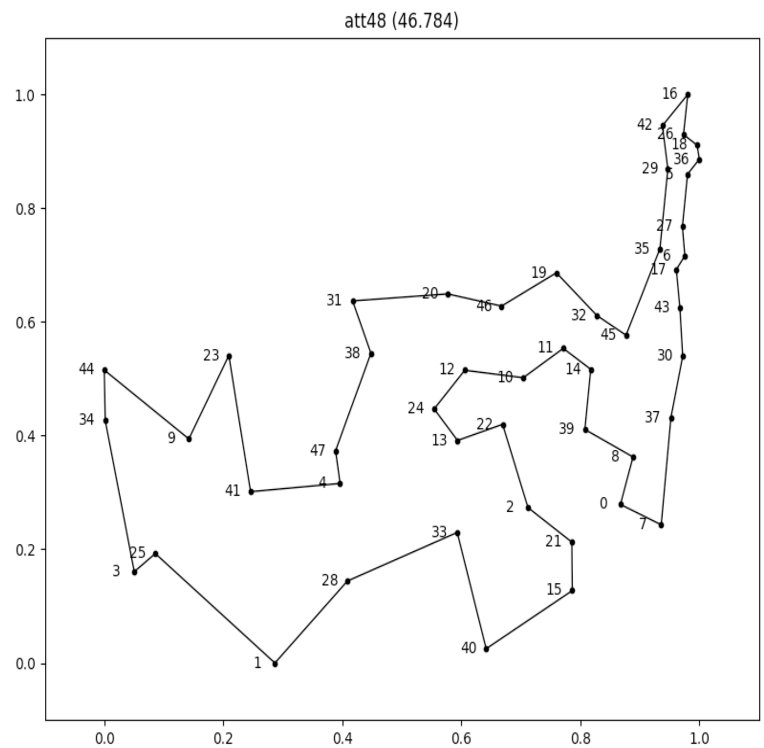
Clearly there is room for improvement on dataset pr76.

We performed an analysis to determine whether or not using the Held-Karp algorithm could drive models toward the optimum value more quickly, but the Held-Karp bounds on these datasets were not particularly close to the optima.

We also implemented Gomory mixed-integer cuts, which were very successful on all datasets except pr76, which we suspect might be due to the accumulation of rounding errors. The Gomory cuts could solve all other datasets even with all other constraints removed (except, of course, integral subtour detection) quite quickly (on the order of a few seconds at most). The code for these cuts is included, though it was disabled for these runs. Using purely integer-based cuts might work better for this type of problem.

7 Result Details

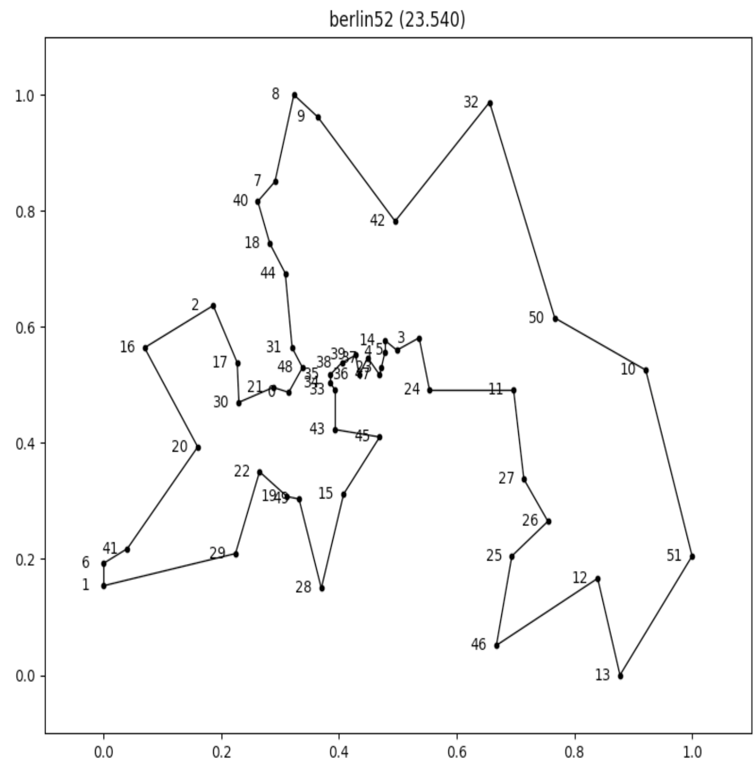
7.1 att48



0	7	178
7	37	312
37	30	183
30	43	139
43	17	112
17	6	54
6	27	85
27	5	153
5	36	66
36	18	42
18	26	64
26	16	117
16	42	139
42	29	127
29	35	233
35	45	286

```
45  32  134
32  19  207
19  46  246
46  20  225
20  31  393
31  38  169
38  47  317
47  4   97
4   41  370
41  23  403
23  9   292
9   44  401
44  34  146
34  3   451
3   25  102
25  1   585
1   28  381
28  33  474
33  40  356
40  15  393
15  21  140
21  2   207
2   22  262
22  13  192
13  24  133
24  12  169
12  10  242
10  11  186
11  14  129
14  39  175
39  8   214
8   0   147
The cost of the best tour is: 10628.0
```

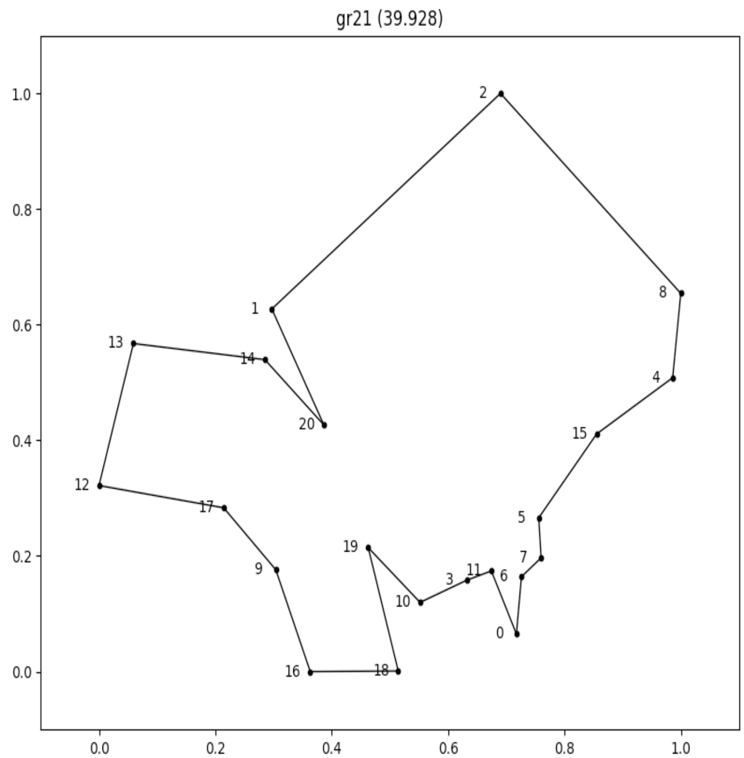
7.2 berlin52



0	21	46
21	30	104
30	17	80
17	2	135
2	16	217
16	20	253
20	41	290
41	6	76
6	1	45
1	29	390
29	22	179
22	19	94
19	49	35
49	28	191
28	15	201
15	45	156
45	43	131
43	33	80
33	34	21

```
34  35  15
35  38  43
38  39  43
39  36  41
36  37  43
37  47  49
47  23  16
23  4   32
4   14  25
14  5   40
5   3   70
3   24  109
24  11  245
11  27  182
27  26  110
26  25  126
25  46  186
46  12  324
12  13  206
13  51  319
51  10  399
10  50  285
50  32  475
32  42  365
42  9   308
9   8   83
8   7   183
7   40  64
40  18  92
18  44  75
44  31  151
31  48  50
48  0   64
The cost of the best tour is: 7542.0
```

7.3 gr21

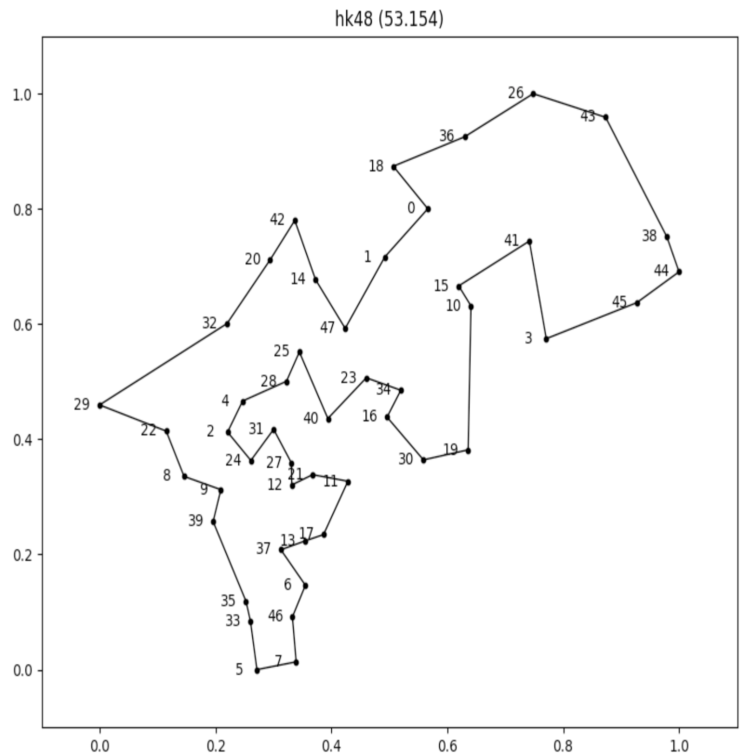


0	6	110
6	7	29
7	5	36
5	15	125
15	4	125
4	8	120
8	2	295
2	1	355
1	20	140
20	14	105
14	13	170
13	12	190
12	17	180
17	9	77
9	16	150
16	18	87
18	19	155
19	10	100
10	3	63

3	11	27
11	0	68

The cost of the best tour is: 2707.0

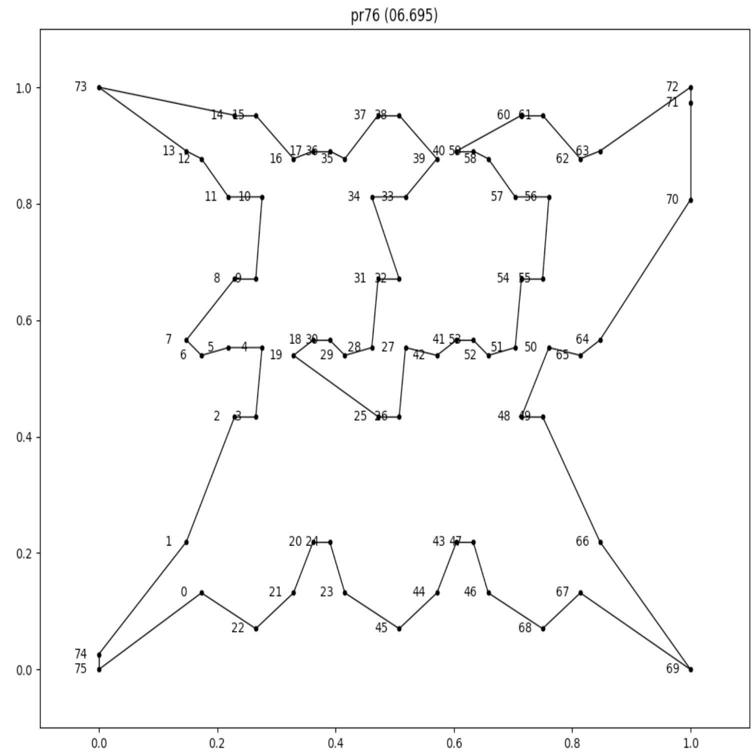
7.4 hk48



0	1	273
1	47	335
47	14	236
14	42	252
42	20	189
20	32	318
32	29	669
29	22	326
22	8	197
8	9	177
9	39	130
39	35	345

```
35  33  83
33  5   188
5   7   182
7   46  177
46  6   138
6   37  178
37  13  115
13  17  90
17  11  238
11  21  165
21  12  100
12  27  90
27  31  154
31  24  161
24  2   155
2   4   140
4   28  217
28  25  129
25  40  291
40  23  238
23  34  166
34  16  123
16  30  235
30  19  214
19  10  563
10  15  96
15  41  371
41  3   387
3   45  443
45  44  229
44  38  145
38  43  534
43  26  347
26  36  357
36  18  346
18  0   229
The cost of the best tour is: 11461.0
```

7.5 pr76



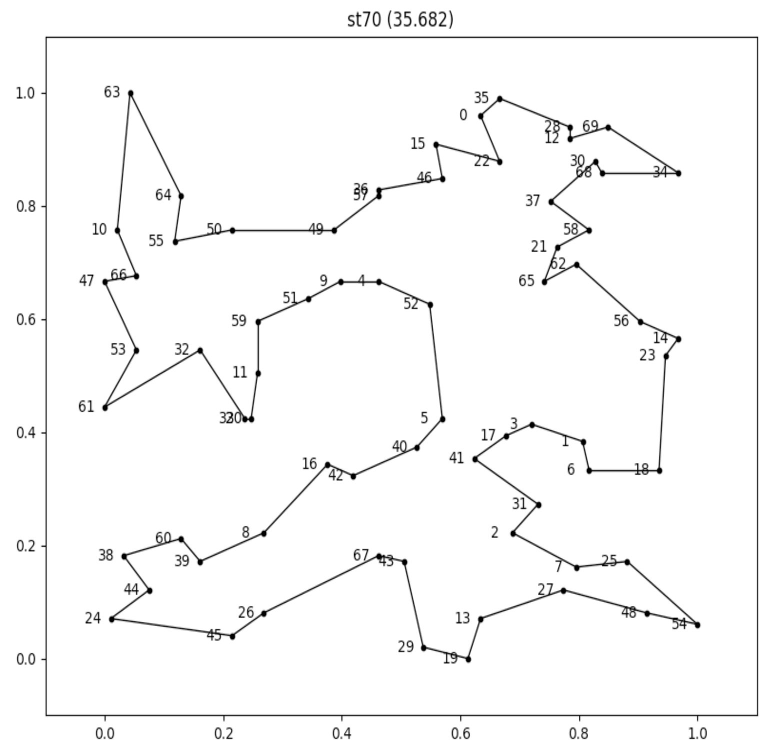
0	22	1931
22	21	1433
21	20	1193
20	24	550
24	23	1118
23	45	1931
45	44	1433
44	43	1193
43	47	550
47	46	1118
46	68	1931
68	67	1433
67	69	3946
69	66	3905
66	49	3100
49	48	700
48	50	1629
50	65	1053
65	64	716

64	70	4070
70	71	1900
71	72	300
72	63	3250
63	62	667
62	61	1512
61	60	700
60	40	2261
40	59	550
59	58	522
58	57	1164
57	56	1118
56	55	1617
55	54	700
54	51	1364
51	52	906
52	53	583
53	41	550
41	42	716
42	27	1053
27	26	1369
26	25	700
25	19	3046
19	18	716
18	30	550
30	29	583
29	28	906
28	31	1364
31	32	700
32	34	1841
34	33	1118
33	39	1280
39	38	1512
38	37	700
37	35	1390
35	36	522
36	17	550
17	16	667
16	15	1512
15	14	700
14	73	4533
73	13	3158
13	12	522
12	11	1164
11	10	1118
10	9	1617
9	8	700
8	7	2000
7	6	583
6	5	906

5	4	1115
4	3	1369
3	2	700
2	1	2926
1	74	3640
74	75	300
75	0	3716

The cost of the best tour is: 108159.0

7.6 st70

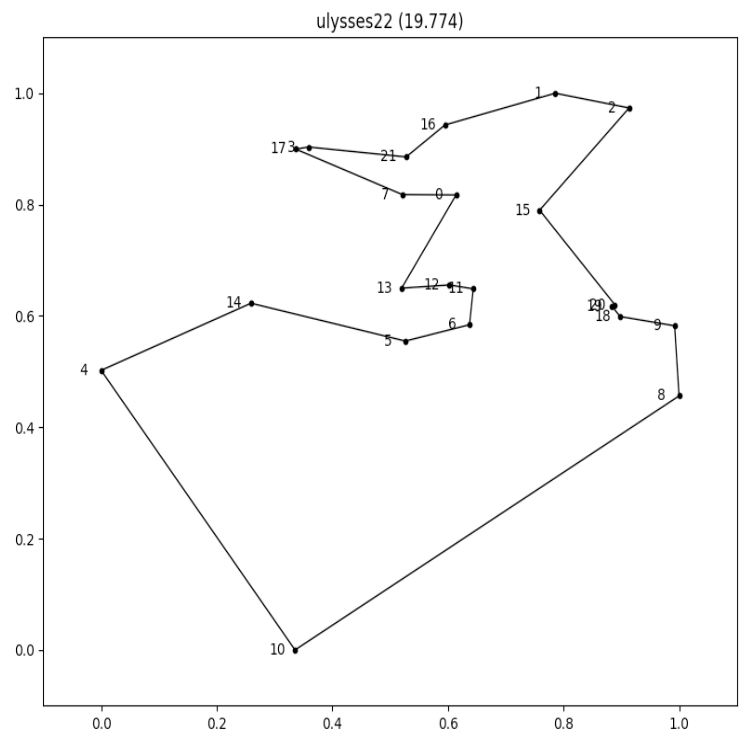


0	22	9
22	15	10
15	46	6
46	36	10
36	57	1
57	49	9
49	50	16

50	55	9
55	64	8
64	63	20
63	10	24
10	47	9
47	66	5
66	53	13
53	61	11
61	32	18
32	33	14
33	20	1
20	11	8
11	59	9
59	51	9
51	9	6
9	4	6
4	52	9
52	5	20
5	40	6
40	42	11
42	16	4
16	8	16
8	39	11
39	60	5
60	38	9
38	44	7
44	24	8
24	45	19
45	26	6
26	67	21
67	43	4
43	29	15
29	19	7
19	13	7
13	27	14
27	48	14
48	54	8
54	25	16
25	7	8
7	2	12
2	31	6
31	41	13
41	17	6
17	3	4
3	1	9
1	6	5
6	18	11
18	23	20
23	14	4
14	56	7

```
56  62  14
62  65  6
65  21  6
21  58  6
58  37  8
37  30  10
30  68  2
68  34  12
34  69  14
69  12  6
12  28  2
28  35  12
35  0   4
The cost of the best tour is: 675.0
```

7.7 ulysses22




```
0    7    60
```

```
7   17  278
17  3   37
3   21  171
21  16  148
16  1   246
1   2   126
2   15  499
15  20  486
20  19  14
19  18  33
18  9   96
9   8   328
8   10  1387
10  4   1504
4   14  401
14  5   308
5   6   115
6   11  177
11  12  68
12  13  52
13  0   479
The cost of the best tour is: 7013.0
```

References

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[3] George L. Nemhauser and Laurence A Wolsey. *Integer and Combinatorial Optimization*. John Wiley and Sons, Inc, 1999, pp.270-281.

[4] Bruno Dutertre and Leonardo de Moura. *Integrating Simplex with DPLL(T)*. SRI International, CSL Technical Report SRI-CSL-06-01, May 23, 2006.