Final Exam Solutions

1 Chicago Seven / Conspiracy Eight (2×15 pts)

1.1 LP solve (15 pts)

Consider the following linear program:

$$\min x_1 + x_2$$
$$x_1 + 2x_2 \geq 4$$
$$3x_1 + x_2 \geq 5$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

- (10 pts) Solve the linear program and find the:
  - $Value =$ ________________
  - $(x_1, x_2) =$ ________________

- (5 pts) Write the dual of the linear program in the following form:

$$\begin{aligned} \min y^T b \\ y^T A \geq c^T \\ y \geq 0 \end{aligned}$$

$A =$

$b =$

$c =$

1.2 Time to Play (15 pts)

Given an undirected, unweighted graph, with each node having a certain value, consider the following game.

- All nodes are initially *unmarked* and your score is 0.
- First, you choose an unmarked node u. You look at the neighbors of u and add to your score the sum of the values of the *marked* neighbors v of u.
- Then, mark u.
- You repeat the last two steps for as many turns as you like (you *do not* have to mark all the nodes. Each node can be marked at most once).

For instance, suppose we had the graph $A - B - C$ with A, B, C having values $3, 2, 3$ respectively. Then, the optimal strategy is to mark A then C then B giving you a score of $0 + 0 + 6$. We can check that no other order will give us a better score.

- (5 pts) Suppose all the node values are nonnegative. Give an efficient algorithm to determine the order to mark nodes to maximize your score. Justify your answer.

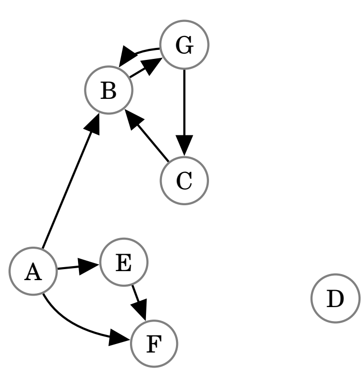
- (10 pts) Now, node values can be negative. Show this problem is NP-hard by giving a reduction from INDEPENDENT SET.

2 Short Questions (5 points each)

1. Does 5 have a multiplicative inverse modulo 111? If not, prove that there is no inverse. If there is an inverse, find it.

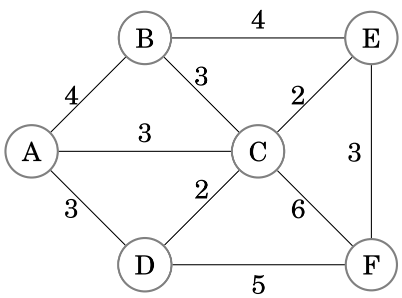
2. Show that if $n \times n$ matrices can be squared in time $O(n^c)$, then any two $n \times n$ matrices can be multiplied in time $O(n^c)$.

3. Partition the following directed graph into strongly connected components.



4. Given an **unweighted** undirected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, give an $O(n(n+m))$ algorithm to compute *all* pairwise graph distances (shortest path lengths measured by number of edges) between every pair of nodes.

5. What is a minimal spanning tree of the following weighted undirected graph? What is its weight?



6. Given a sequence of n integers $a_1, \ldots, a_n$ give an $O(n^2)$ algorithm to find the longest increasing-then-decreasing subsequence. For example for the sequence $1, 8, 3, 5, 9, 6, 7, 11, 6, 4, 8, 2, 1$ the longest increasing-then-decreasing sequence is $1, 3, 5, 6, 7, 11, 6, 4, 2, 1$.

7. We want to sort n distinct items where comparisons may be faulty, that is for elements i, j, we may have the wrong answer to the query $a[i] < a[j]$. The goal is to find an order $a[1], a[2], \ldots, a[n]$ that maximize the number of $i < j$ with $a[i] < a[j]$. Give a simple deterministic $O(n^2)$ algorithm that will achieve an approximation ratio of $\frac{1}{2}$.

8. If we measure the qubit $\frac{1}{\sqrt{2}} |0\rangle - \frac{i}{\sqrt{2}} |1\rangle$, what is the chance that the measurement will be 0?

3 (**True** $\vee$ **False**) (2×15 **pts**)

No need to justify. Mark the correct answer (if both TRUE and FALSE are marked the answer is incorrect)

1. In the range $[1, 10^{10}]$, there are more Carmichael numbers than there are prime numbers.
2. The recurrence relation $T(n) = n^2 + T(\lfloor\sqrt{n}\rfloor), T(1) = 1$ satisfies $T(n) = O(n^2 \log n)$.
3. If u and v are two vertices of a graph, then running DFS on the graph could never result in $pre(u) < pre(v) < post(u) < post(v)$.
4. Let $G = (V, E)$ be a directed graph where the only edges with negative weights are outgoing edges of a vertex $s \in V$. Running Dijkstra's algorithm on G starting at s will give the correct shortest path lengths from s to all other vertices in G.
5. Every connected weighted graph has a unique minimal spanning tree.
6. Consider an instance of the knapsack problem *without replacement* with 8 items whose weight and values are
$$(1kg, 1\$), (2kg, 4\$), (2kg, 3\$), (3kg, 6\$), (5kg, 5\$), (5kg, 3\$), (7kg, 3\$), (9kg, 9\$)$$
with total weight $W = 10kg$ has optimal value $15\$$.
7. Let $G = (V, E)$ be a directed graph with positive edge weights, with edge $(u, v) \in E$. During a full run of the Ford-Fulkerson maximum flow algorithm, the residual edge (v, u) can be added to the graph no more than $|V| + |E|$ times.
8. Given a graph $G = (V, E)$ with integer edge weight values (positive, zero, and/or negative), finding the shortest simple path (a simple path does not revisit any vertices) from a vertex $s \in V$ to $t \in V$ is NP-hard.
9. Given any instance of the set cover problem, a greedy algorithm that chooses an unchosen set with the largest number of uncovered elements at each iteration will always produce a solution consisting of at most twice as many sets as the optimal solution.
10. RSA would be secure against quantum computers (if such computers could be built).
11. If we strictly increase the capacity of all edges in *one* min $s - t$ cut, then we will strictly increase the capacity of the maximum $s - t$ flow (you can can assume all capacities are integer).
12. Given that there is one unique maximum $s - t$ flow in a graph G, there is one unique minimum $s - t$ cut in G.
13. If an edge e is not used in any max $s - t$ flow, then that edge will not be in any min $s - t$ cut.
14. Finding a starting vertex for a linear program to run simplex can be solved by running simplex on a modified linear program.
15. Suppose that problem A is NP-hard and has a known polynomial approximation algorithm with an approximation ratio of 2. Then, for all problems B which reduce to A, we can construct a polynomial approximation algorithm for problem B with approximation ratio at most 2.