

CS189: Introduction to Machine Learning

Homework 2

Due: September 24, 2015 at 11:59pm

Instructions:

- Homework 2 is completely a written assignment, no coding involved.
- Please write (legibly!) or typeset your answers in the space provided. If you choose to typeset your answers, please use this template file ([hw2.tex](#)), provided on bCourses announcement page. If there is not enough space for your answer, you can continue your answer on a separate page (for example : You might want to append pages in Questions 6,7,8).
- Submit a pdf of your answers to <https://gradescope.com> under Homework 2. A photograph or scanned copy is acceptable as long as it is clear with good contrast. You should be able to see CS 189/289 on gradescope when you login with your primary e-mail address used in bCourses. Please let us know if you have any problems accessing gradescope.
- While submitting to Gradescope, you will have to select the region containing your answer for each of the question. Thus, write the answer to a question (or given part of the question) at one place only.
- Start early and don't wait until last minute to submit the assignment as the submission procedure might take sometime too.

About the Assignment:

- This assignment tries to refresh the concepts of probability, linear algebra and matrix calculus.
- Questions 1 to 6 are dedicated to deriving fundamental results related to these concepts. You might want to refer your math class textbooks for help.
- Questions 7,8 discuss few applications of these concepts in machine learning.
- Hope you will enjoy doing the assignment !

Homework Party : Sept 21, 2-4pm in the Wozniak Lounge, SODA 430

Problem 1. A target is made of 3 concentric circles of radii $1/\sqrt{3}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle are given 4 points, shots within the next ring are given 3 points, and shots within the third ring are given 2 points. Shots outside the target are given 0 points.

Let X be the distance of the hit from the center (in feet), and let the p.d.f of X be

$$f(x) = \begin{cases} \frac{2}{\pi(1+x^2)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected value of the score of a single shot?

Solution:

Problem 2. Assume that the random variable X has the exponential distribution

$$f(x|\theta) = \theta e^{-\theta x} \quad x > 0, \theta > 0$$

where θ is the parameter of the distribution. Use the method of maximum likelihood to estimate θ if 5 observations of X are $x_1 = 0.9$, $x_2 = 1.7$, $x_3 = 0.4$, $x_4 = 0.3$, and $x_5 = 2.4$, generated i.i.d. (i.e., independent and identically distributed).

Solution:

Problem 3. The polynomial kernel is defined to be

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + c)^d$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $c \geq 0$. When we take $d = 2$, this kernel is called the quadratic kernel.

- (a) Find the feature mapping $\Phi(\mathbf{z})$ that corresponds to the quadratic kernel.
- (b) How do we find the optimal value of d for a given dataset?

Solution:

Def: Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. We say that A is positive definite if $\forall x \in \mathbb{R}^n$, $x^\top Ax > 0$. Similarly, we say that A is positive semidefinite if $\forall x \in \mathbb{R}^n$, $x^\top Ax \geq 0$.

Problem 4. Let $x = [x_1 \ \cdots \ x_n]^\top \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- (a) Give an explicit formula for $x^\top Ax$. Write your answer as a sum involving the elements of A and x .
- (b) Show that if A is positive definite, then the entries on the diagonal of A are positive (that is, $a_{ii} > 0$ for all $1 \leq i \leq n$).

Solution:

Problem 5. Let B be a positive semidefinite matrix. Show that $B + \gamma I$ is positive definite for any $\gamma > 0$.

Solution:

Problem 6 : Derivatives and Norms. Derive the expression for following questions.
Do not write the answers directly.

- (a) Let $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$. Derive $\frac{\partial(\mathbf{x}^T \mathbf{a})}{\partial \mathbf{x}}$.
- (b) Let \mathbf{A} be a $n \times n$ matrix and \mathbf{x} be a vector in \mathbb{R}^n . Derive $\frac{\partial(\mathbf{x}^T \mathbf{A} \mathbf{x})}{\partial \mathbf{x}}$.
- (c) Let \mathbf{A}, \mathbf{X} be $n \times n$ matrices. Derive $\frac{\partial \text{Trace}(\mathbf{X} \mathbf{A})}{\partial \mathbf{X}}$.
- (d) Let \mathbf{X} be a $m \times n$ matrix, $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$. Derive $\frac{\partial(\mathbf{a}^T \mathbf{X} \mathbf{b})}{\partial \mathbf{X}}$.
- (e) Let $\mathbf{x} \in \mathbb{R}^n$. Prove that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2$. Here $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ and $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$.

Solution:

Problem 7 : Application of Matrix Derivatives.

Let \mathbf{X} be a $n \times d$ data matrix, \mathbf{Y} be the corresponding $n \times 1$ target/label matrix and $\mathbf{\Lambda}$ be the diagonal $n \times n$ matrix containing weight of each example. Expanding them, we

have $\mathbf{X} = \begin{bmatrix} (\mathbf{x}^{(1)})^T \\ (\mathbf{x}^{(2)})^T \\ \vdots \\ (\mathbf{x}^{(n)})^T \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{bmatrix}$ and $\mathbf{\Lambda} = \text{diag}(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)})$

where $\mathbf{x}^{(i)} \in \mathbb{R}^d$, $y^{(i)} \in \mathbb{R}$, and $\lambda^{(i)} > 0 \quad \forall \quad i \in \{1 \dots n\}$. \mathbf{X} , \mathbf{Y} and $\mathbf{\Lambda}$ are fixed and known.

In the remaining parts of this question, we will try to fit a weighted linear regression model for this data. We want to find the value of weight vector \mathbf{w} which best satisfies the following equation $y^{(i)} = \mathbf{w}^T \mathbf{x}^{(i)} + \epsilon^{(i)}$ where ϵ is noise. This is achieved by minimizing the weighted noise for all the examples. Thus, our risk function is defined as follows:

$$\begin{aligned} R[\mathbf{w}] &= \sum_{i=1}^n \lambda^{(i)} (\epsilon^{(i)})^2 \\ &= \sum_{i=1}^n \lambda^{(i)} (\mathbf{w}^T \mathbf{x}^{(i)} - y^{(i)})^2 \end{aligned}$$

- Write this risk function $R[\mathbf{w}]$ in matrix notation, i.e., in terms of \mathbf{X} , \mathbf{Y} , $\mathbf{\Lambda}$ and \mathbf{w} .
- Find the value of \mathbf{w} , in matrix notation, that minimizes the risk function obtained in Part (a). You can assume that $\mathbf{X}^T \mathbf{\Lambda} \mathbf{X}$ is full rank matrix. Hint: You can use the expression derived in Q-6(b).
- What will be the answer for questions in Parts (a) and (b) if you add L_2 regularization (i.e., shrinkage) on \mathbf{w} ? The L_2 regularized risk function, for $\gamma > 0$, is

$$R[\mathbf{w}] = \sum_{i=1}^n \lambda^{(i)} (\mathbf{w}^T \mathbf{x}^{(i)} - y^{(i)})^2 + \gamma \|\mathbf{w}\|_2^2$$

Hint: You can make use of the result in Q-5.

- What role does the regularization (i.e., shrinkage) play in fitting the regression model and how? You can observe the difference in expressions for \mathbf{w} obtained in Parts (c) and (d), and argue.

Solution:

Problem 8: Classification. Suppose we have a classification problem with classes labeled $1, \dots, c$ and an additional doubt category labeled as $c + 1$. Let the loss function be the following:

$$\ell(f(x) = i, y = j) = \begin{cases} 0 & \text{if } i = j \quad i, j \in \{1, \dots, c\} \\ \lambda_r & \text{if } i = c + 1 \\ \lambda_s & \text{otherwise} \end{cases}$$

where λ_r is the loss incurred for choosing doubt and λ_s is the loss incurred for making a misclassification. Note that $\lambda_r \geq 0$ and $\lambda_s \geq 0$.

Hint : The risk of classifying a new datapoint as class $i \in \{1, 2, \dots, c + 1\}$ is

$$R(\alpha_i|x) = \sum_{j=1}^{j=c} \ell(f(x) = i, y = j)P(\omega_j|x)$$

- (a) Show that the minimum risk is obtained if we follow this policy: (1) choose class i if $P(\omega_i|x) \geq P(\omega_j|x)$ for all j and $P(\omega_i|x) \geq 1 - \lambda_r/\lambda_s$, and (2) choose doubt otherwise.
- (b) What happens if $\lambda_r = 0$? What happens if $\lambda_r > \lambda_s$?

Solution: