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# Decoherence Effects on the Entangled States in a Noisy Quantum Channel

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Decoherence in a two-qubit system caused by interaction with its environment is an interesting problem to study for quantum communication. We present noise models for the two-qubit states by using Kraus operators for each noisy channel; especially bit flip, bit-phase flip, phase flip, phase damping, amplitude damping, and depolarizing channels. By analyzing the possible quantum information changes of the six noisy channels, we use quantum entropy to discuss variations of their fidelity and mixedness. In addition, in order to compare the degree of entanglement between the initial and the output, we investigate the concurrence.

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## I. INTRODUCTION

With the rapid progress of quantum information theory, there have been studied on the realizations of quantum communication, such as quantum teleportation [1], cryptography [2], and entanglement [3]. A crucial requirement of quantum communication is the capability for transmitting states coherently. For quantum communication, a real system is disturbed by noise due to the interaction with its surrounding environment [4]. The interaction between the system and its environment introduces decoherence to the system, which is a process of the undesired correlations between the system and the environment when the system evolves [5]. Therefore, the communication accomplished under noisy channels [6] may not be faithful because the receiver may obtain partial or corrupted information different from sender's information.

The quantum noise process is represented by mapping $\rho \Rightarrow \mathcal{E}(\rho)$, where $\mathcal{E}$ is a super-operator [7] that makes the initial state $\rho$ evolve to the final state $\mathcal{E}(\rho)$. In general, the communication process of an open system can be represented by the operator-sum representation [6]:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{1}$$

where $E_i$ are Kraus operator elements for the super operation $\mathcal{E}$, and are trace-preserving, $\sum_i E_i^\dagger E_i = I$. Nielson and Chuang [8] simplified the noise process by using the Kraus decomposition for a single qubit [7] and presented operators for each noise such as bit, bit-phase, and phase flips, phase and amplitude dampings, and depolarization. The other approaches describing the two-bit noisy channel include cases of the quantum Liouville equation [9] and the non-Markovian master equation [10]. Accomplishment of a desired quantum communication requires information over a two-qubit state [11]. In order to study any realistic quantum communication, one has to analyze noise effects at least for two qubits. In the present paper, we propose theoretical models of noises affecting a two-qubit system by using an individual Kraus operator for each qubit, and we show such channels having the same kinds of noises as bit flip, bit-phase flip, phase flip, depolarizing, amplitude damping, and phase damping channels by way of simple examples in Sec. II. Then, in Sec. III, we investigate the changes in the information on the channels by using measures such as the fidelity [12], the von Neumann entropy [13] and the concurrence [14].

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## II. MODEL

We consider the noise process for a two-qubit state $\rho_{AB}$ where $\rho_{AB} = |\psi\rangle_{AB\,BA}\langle\psi|$. We assume that two qubits, A and B, are apart where each qubit is coupled with its environment so that the coupling might make several types of noise in the two-qubit system. The noises could be represented by local unitary operators $E^A$ or $E^B$. Hence, we can simply represent a noisy channel with the operator-sum representation

$$\mathcal{E}(\rho_{AB}) = \sum_{i,j} E_i^A \otimes E_j^B \rho_{AB} E_i^{A\dagger} \otimes E_j^{B\dagger}, \tag{2}$$

where $E_i^A$ and $E_j^B$ are single qubit operator elements [8] satisfying $\sum_{i,j}(E_i^A \otimes E_j^B)^\dagger (E_i^A \otimes E_j^B) = I^A \otimes I^B$. Surely, each operator related to a qubit $A$, $B$ could be any kind of noise operator since two qubits are seperated and an arbitrary noise might occur. However, we simply show six cases having the same noise on two qubits because these cases should be enough to represent the attributes of a decohered two-qubit system.

First of all, flip channels, such as bit, phase, and bit-phase flips, have $E_i^k$ represented by

$$\begin{aligned} E_0^k &= \sqrt{1-p_k}\, I, \\ E_1^k &= \sqrt{p_k}\,\sigma_i \begin{pmatrix} i=z & for\ bit\ flip \\ i=x & for\ phase\ flip \\ i=y & for\ bit-phase\ flip \end{pmatrix} \end{aligned} \tag{3}$$

where $p_k$ are probabilities to be flipped with $k \in \{A,\ B\}$. The depolarizing process includes all of Pauli matrices

$$\begin{aligned} E_0^k &= \sqrt{1-\frac{3p_k}{4}}\, I, \\ E_1^k &= \sqrt{\frac{p_k}{4}}\sigma_x,\ E_2^k = \sqrt{\frac{p_k}{4}}\sigma_y,\ E_3^k = \sqrt{\frac{p_k}{4}}\sigma_z \end{aligned} \tag{4}$$

with depolarizing probabilities $p_k$. In the damping cases, $E_i^k$ are

$$E_0^k = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_k} \end{bmatrix},\ E_1^k = \begin{bmatrix} 0 & \sqrt{r_k} \\ 0 & 0 \end{bmatrix} \tag{5}$$

for the amplitude damping process with amplitude damping probabilities $r_k$ and

$$E_0^k = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_k} \end{bmatrix},\ E_1^k = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{r_k} \end{bmatrix} \tag{6}$$

for the phase damping process with phase damping probabilities $r_k$. The above six noise models contain all three phenomena; none of the qubits, only one qubit, or both qubits are decohered by noise. By adjusting the noise parameters $p_A$ and $p_B$ for flip and depolorizing channels and $r_A$ and $r_B$ for damping channels, we can obtain the output states $\mathcal{E}(\rho)$ for each noise case.

In general, the pure states are written by

$$\begin{aligned} |\psi^\pm\rangle &= \cos\varphi|01\rangle \pm \sin\varphi|10\rangle, \\ |\phi^\pm\rangle &= \cos\varphi|00\rangle \pm \sin\varphi|11\rangle \end{aligned} \tag{7}$$

corresponding to the four Bell states [15]. Let us choose one of the above four states as the initial state, *i.e.*, $|\Psi\rangle_{in} = |\phi^+\rangle_{in}$. The quantum operator formalism with a density operator is convenient for describing the noisy process. The density of the initial state $\rho_{in}$ is simply represented by

$$\rho_{in} = \begin{bmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{bmatrix}, \tag{8}$$

with

$$\begin{aligned} \rho_{11} &= \cos^2\varphi,\quad \rho_{44} = \sin^2\varphi, \\ \rho_{14} &= \rho_{41} = \cos\varphi\sin\varphi. \end{aligned}$$

With this initial state, we can obtain output states, $\mathcal{E}(\rho_{in})$, from Eq. (2) for each noise channel. We denote the final states for the six cases as $\rho^I$ for the bit flip, $\rho^{II}$ for the bit-phase flip and $\rho^{III}$ for the phase flip channels, $\rho^{IV}$ for the phase damping, $\rho^V$ for the amplitude damping, and $\rho^{VI}$ for the depolarizing channels.

The density matrices of the final states can be categorized by three groups, $\rho^{I,II}$, $\rho^{III,IV}$, and $\rho^{V,VI}$. First, when the initial state is decohered by bit flip and bit-phase flip noise channels, the two final states have similarity in that the matrix elements of the states have the same values except for the sign difference between two off-diagonal elements $\rho_{23}$ and $\rho_{32}$:

$$\rho^{I,II} = \begin{bmatrix} \rho_{11}^{I,II} & 0 & 0 & \rho_{14}^{I,II} \\ 0 & \rho_{22}^{I,II} & \rho_{23}^{I,II} & 0 \\ 0 & \rho_{31}^{I,II} & \rho_{33}^{I,II} & 0 \\ \rho_{41}^{I,II} & 0 & 0 & \rho_{44}^{I,II} \end{bmatrix}, \tag{9}$$

with

$$\begin{aligned} &\rho_{11}^{I,II} = (1-p_A)(1-p_B)\cos^2\varphi + p_A p_B \sin^2\varphi, \\ &\rho_{22}^{I,II} = (1-p_A)p_B\cos^2\varphi + p_A p_B \sin^2\varphi, \\ &\rho_{33}^{I,II} = p_A(1-p_B)\cos^2\varphi + (1-p_A)p_B \sin^2\varphi, \\ &\rho_{44}^{I,II} = p_A p_B\cos^2\varphi + (1-p_A)(1-p_B) \sin^2\varphi, \\ &\rho_{14}^{I,II} = \rho_{41}^{I,II} = \{(1-p_A)(1-p_B) + p_A p_B\}\cos\varphi\sin\varphi, \\ &\rho_{23}^{I} = \rho_{32}^{I} = \{p_A(1-p_B) + (1-p_A)p_B\}\sin\varphi\cos\varphi, \\ &\rho_{23}^{II} = \rho_{32}^{II} = -\rho_{32}^{I} = -\rho_{23}^{I}. \end{aligned}$$

Second, both $\rho^{III}$ decohered by phase flip and $\rho^{IV}$ decohered by phase damping have just four elements in the matrices, and two elements of the two matrices are identical:

$$\rho^{III,IV} = \begin{bmatrix} \rho_{11}^{III,IV} & 0 & 0 & \rho_{14}^{III,IV} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41}^{III,IV} & 0 & 0 & \rho_{44}^{III,IV} \end{bmatrix}, \tag{10}$$

with

$$\begin{aligned} \rho_{11}^{III,IV} &= \cos^2\varphi, \quad \rho_{44}^{III,IV} = \sin^2\varphi, \\ \rho_{14}^{III} = \rho_{41}^{III} &= (1-2p_A)(1-2p_B)\cos\varphi\sin\varphi, \\ \rho_{14}^{IV} = \rho_{41}^{IV} &= \sqrt{(1-r_A)(1-r_B)}\cos\varphi\sin\varphi. \end{aligned} \tag{11}$$

Last, the final states affected by amplitude damping and depolarizing noises have all four diagonal terms and two off-diagonal elements:

$$\rho^{V,VI} = \begin{bmatrix} \rho_{11}^{V,VI} & 0 & 0 & \rho_{14}^{V,VI} \\ 0 & \rho_{22}^{V,VI} & 0 & 0 \\ 0 & 0 & \rho_{33}^{V,VI} & 0 \\ \rho_{41}^{V,VI} & 0 & 0 & \rho_{44}^{V,VI} \end{bmatrix}, \tag{12}$$

with

$$\begin{aligned} &\rho_{11}^{V} = \cos^2\varphi + r_A r_B \sin^2\varphi, \quad \rho_{22}^{V} = (1-r_A)r_B\sin^2\varphi, \\ &\rho_{33}^{V} = r_A(1-r_B)\sin^2\varphi, \quad \rho_{44}^{V} = (1-r_A)(1-r_B)\sin^2\varphi, \\ &\rho_{14}^{V} = \rho_{41}^{V} = \sqrt{(1-r_A)(1-r_B)}\sin\varphi\cos\varphi, \\ &\rho_{11}^{VI} = \frac{1}{4}\{(2-p_A)(2-p_B)\cos^2\varphi + p_A p_B \sin^2\varphi\}, \\ &\rho_{22}^{VI} = \frac{1}{4}\{(2-p_A)p_B\cos^2\varphi + p_A(2-p_B)\sin^2\varphi\}, \\ &\rho_{33}^{VI} = \frac{1}{4}\{p_A(2-p_B)\cos^2\varphi + (2-p_A)p_B\sin^2\varphi\}, \\ &\rho_{44}^{VI} = \frac{1}{4}\{p_A p_B\cos^2\varphi + (2-p_A)(2-p_B)\sin^2\varphi\}, \\ &\rho_{14}^{VI} = \rho_{41}^{VI} = (1-p_A)(1-p_B)\sin\varphi\cos\varphi. \end{aligned}$$

In order to identify the states affected by these noisy channels, we simply summarize the final states for special values of the noise probabilities in Table. 1. The states are useful for understanding the results of several measures referred to later in Sec. III.

The final states under the flip noises return to one of the entangled pure states when either qubit of the system is affected by maximal noises, $p_A = 1$ and/or $p_B = 1$. In cases of other noises, such as damping and depolarizing noises, the final states are all disentangled states at maximal noises. Especially, the depolarizing channel always changes the initial state to a mixed state when noise exists; besides, the state is maximally mixed when both qubits interact with the environment. When the state is initially separable pure as $\varphi = 0$, the system is surprisingly not affected by such noises as the phase flip, the phase damping, and the amplitude damping noises and remains in the initial state.

Table 1. Comparison of the final states $\varepsilon(\rho_{in})$ on the four special noise probabilities, $\{(p_A, p_B), (r_A, r_B)\} \in \{(0,1), (1,0), (1,1)\}$, when the initial state is $\rho_{in} = |\phi^+\rangle\langle\phi^+|$. Here, $|\psi'^{\pm}\rangle = \sin\varphi|01\rangle \pm \cos\varphi|10\rangle$ and $|\phi'^{\pm}\rangle = \sin\varphi|00\rangle \pm \cos\varphi|11\rangle$. Also, $\triangle = \cos^2\varphi|00\rangle\langle 00| + \sin^2\varphi|11\rangle\langle 11|$, $\nabla = \cos^2\varphi|00\rangle\langle 00| + \sin^2\varphi|10\rangle\langle 10|$, and $\nabla' = \cos^2\varphi|00\rangle\langle 00| + \sin^2\varphi|01\rangle\langle 01|$.

| Interaction with environment | $\rho^I$ | $\rho^{II}$ | $\rho^{III}$ | $\rho^{IV}$ | $\rho^V$ | $\rho^{VI}$ |
|---|---|---|---|---|---|---|
| Qubit B | $\|\psi^+\rangle$ | $\|\psi^-\rangle$ | $\|\phi^-\rangle$ | $\triangle$ | $\nabla$ | Mixed |
| Qubit A | $\|\psi'^+\rangle$ | $\|\psi'^-\rangle$ | | | $\nabla'$ | |
| Both qubits | $\|\phi'^+\rangle$ | $\|\phi'^+\rangle$ | $\|\phi^+\rangle$ | | $\|00\rangle$ | |

## III. DECOHERENCE EFFECTS ON THE INFORMATION

We analyze information on the outcomes derived from the formalism in Sec. II by investigating such measures as fidelity, entropy, and concurrence, which show how a pure state changes in each noise process.

### 1. Fidelity

The change of the state $\rho_{in}$, composed of system A and system B, can be quantified by the fidelity [12],

$$F(\rho_{in}, \mathcal{E}(\rho_{in})) = Tr\{\rho_{in}\mathcal{E}(\rho_{in})\}. \tag{13}$$

To analyze this measure, we derive the fidelities between the initial and the final states related to the decoherence of these six noisy channels. Depending on the noise probabilities $p_A$, $p_B$ and $\varphi$ of the initial state, these six fidelities are quantified as follows:

$$\begin{aligned} &F^I = F^{II} \\ &= (1-p_A)(1-p_B) + p_A p_B \frac{1-\cos 4\varphi}{2}, \end{aligned} \tag{14}$$

$$F^{III} = 1 - \frac{1}{2}(p_A + p_B - 2p_A p_B)(1-\cos 4\varphi), \tag{15}$$

$$F^{IV} = 1 - \frac{1}{2}(1 - \sqrt{(1-r_A)(1-r_B)})\sin^2 2\varphi, \tag{16}$$

$$\begin{aligned} F^V = &(\cos^2\varphi + \sqrt{(1-r_A)(1-r_B)}\sin^2\varphi)^2 \\ &+ \frac{1}{4} r_A r_B \sin^2 2\varphi, \end{aligned} \tag{17}$$

$$\begin{aligned} F^{VI} = 1 - &\frac{1}{8}\{6p_A p_B - (p_A + p_B - 2p_A p_B) \\ &\times (5 - \cos 4\varphi)\}. \end{aligned} \tag{18}$$

To understand the general features, we show plots of the fidelities representatively for one of the flip channels, for one of the damping channels, and for the depolarizing channel. Those are shown in Fig. 1 in which the initial

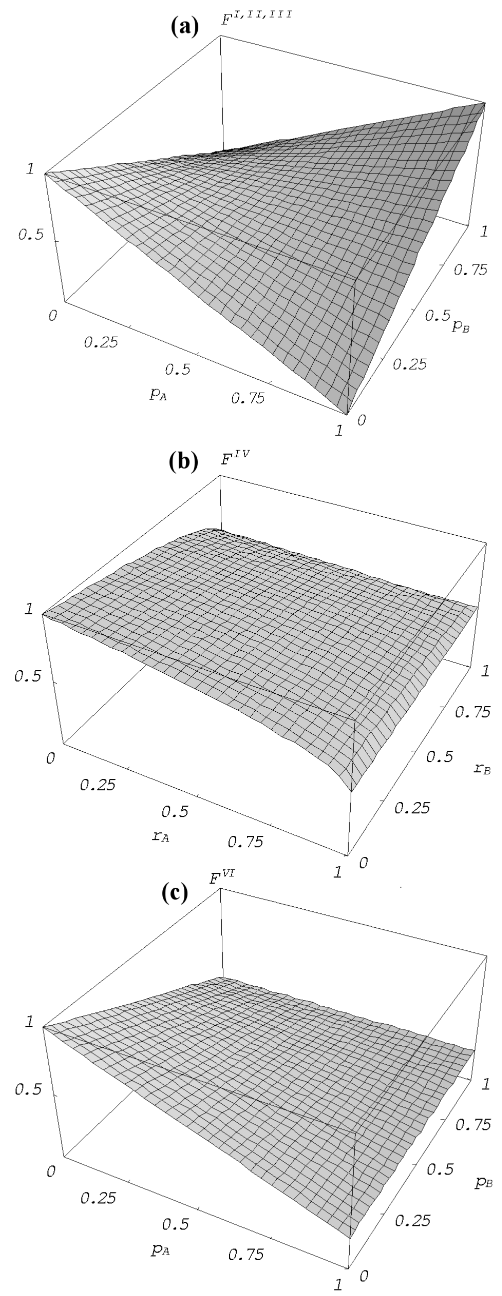


Fig. 1. Fidelity $F(\mathcal{E}(\rho_{in}))$ for $\varphi = \pi/4$ as a function of the noise parameters $p_A$, $p_B$ and $r_A$, $r_B$ (a) for the three flip channels, (b) for the phase damping channel, and (c) for the depolarizing channel. $F^{IV} = 1/2$ when $r_A = 0$ and/or $r_B = 1$ in (b), and $F^{VI} = 1/4$ when $p_A = 0$ and/or $p_B = 1$ in (c).

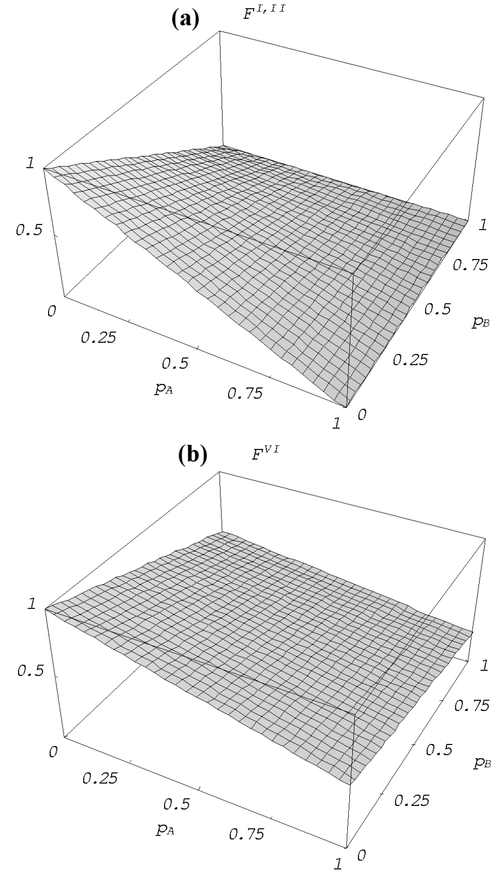


Fig. 2. Fidelity $F(\mathcal{E}(\rho_{in}))$ for $\varphi = 0$ as a function of the noise parameters $p_A$, $p_B$ for (a) the bit and bit-phase flip channels and (b) for the depolarizing channel. $F^{VI} = 1/4$ when $p_A = p_B = 1$, and $F^{VI} = 1/2$ when $p_A = 1$ and $p_B = 0$ (or vice versa) in (b).

state is maximally entangled as $\theta = \pi/4$ and in Fig. 2 in which the initial state is separable as $\theta = 0$.

In the flip channels, the fidelities are identically given by

$$F^{I,II,III} \;=\; 1 - p_A - p_B + 2p_Ap_B \tag{19}$$

only when the initial state is maximally entangled. For other initial states, the three flip channels are divided into two groups, the bit flip and the bit-phase flip channels and the phase flip channel. That is, the fidelities for two flip cases such as the bit flip and the bit-phase flip channels are always the same regardless of the initial state. We may say that the origins of the outputs $\rho^I$ and $\rho^{II}$ are different, yet their probabilities are equivalent due to the symmetry between $\sigma_x$ and $\sigma_y$ with respect to the $z$-axis. In addition to similarity, we can easily find that the values of $F^{I,II,III}$ for flip-type channels are symmetric by varying the flip probabilities $p_A$, $p_B$ as shown in Fig. 1. At $p_A = 1$ and $p_B = 0$ (or vice versa), the final states are orthogonal to the initial state because one of the two qubits is completely flipped. Besides, $F^{I,II}$ dramatically changes under noise of only one qubit as shown in Fig. 1(a) when the initial state is maximally entangled, but the state is also affected by noise of both qubits when the initial state is less or not entangled as shown in Fig. 2(a). Contrary to these flip channels, the phase flip channel increases the fidelity as the initial state becomes less entangled and does not affect the fidelity as $F^{III} = 1$ for all values of the probabilities when the initial state is a separable state.

$F^{IV}$ of the phase damping channel, shown in Figs. 1 and 2, smoothly decreases in proportion to increasing phase damping probabilities regardless of the number of qubits affected by noise. When one of the damping probabilities is maximum, $F^{IV} = 1/2$ and the final state is mixed as $\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. Similar to the phase flip channel, the phase damping channel increases the fidelity as the initial state becomes less entangled, which is also the case for the amplitude damping channel. That is, the fidelity of the two-qubit system is not changed by the phase flip, the phase damping, and the amplitude damping noises when the system is in a separable state.

$F^{VI}$ of the depolarizing channel is plotted in Fig. 1(c) in which the lowest value is 1/4, which means the state is maximally mixed as $I_2/4$. As shown Fig. 2(b), the fidelities, when only one qubit is affected by the noise, are different depending on the degree of entanglement of qubits in a initial state. When the initial state is separable, the fidelity of the system is two times higher than when the initial state is maximally entangled at $p_A = 1$ and $p_B = 0$ (or vice versa).

### 2. von Neumann Entropy

We continue to discuss the decoherence effect by measuring the mixedness of the final state with the von Neumann entropy. The quantum von Neumann entropy of a quantum state $\rho$ is defined by

$$S(\rho) \equiv -Tr\{\rho \log \rho\} = -\sum_x \lambda_x \log \lambda_x, \tag{20}$$

where $\lambda_x$ are eigenvalues of $\rho$ [13]. The quantum entropy measures the information, especially, the amount of mixedness of the system. If the initial state is pure, such as the input state $\rho_{in}$, the entropy can be regarded as an entropy exchange [6] by

$$S_e(\rho_{in}) = S(\rho_{in}, \mathcal{E}) = S(W) = -Tr\{W \log W\}, \tag{21}$$

with $W_{ij} = Tr\{E_i \rho_{in} E_j^+\}$. The $S_e(\rho_{in})$ is a measure of the information exchanged between the system and the environment. Hence, we consider the entropy not only as the degree of the mixedness but also as the quantity of the noise generated by the action of the environment.

Since $\rho_{in}$ is in a pure state, $S(\rho_{in}) = 0$. If the entropy $S(\mathcal{E}(\rho_{in}))$ is positive after evolution, the system is not a pure state anymore, but a mixed state having a quantum correlation with environment. In order to calculate the entropy, we derive the eigenvalues of the final states on the noisy channels. For the bit flip and the bit-phase flip channels, $\rho^I$ and $\rho^{II}$ have the same eigenvalues and are rank 4. Among the other four channels, however, the phase flip and the phase damping channels are rank 2, and the other two are rank 4.

The entropies of the noisy channels are shown with noise probabilities $p_A$, $p_B$ and $r_A$, $r_B$ in Fig. 3 when $\rho_{in}$ is maximally entangled and are shown in Fig. 4 when $\rho_{in}$ is separable. The $S^{I,II,III}$ of the flip channels are the same when the initial state is maximally entangled, and $S^{I,II}$ are the same regardless of the initial state, which properties are equivalent to the measure of the fidelity. For the two-qubit system, the maximally mixed state has $S = 2$ according to $S_{max} = N \log_2 N$ where $N$ is the

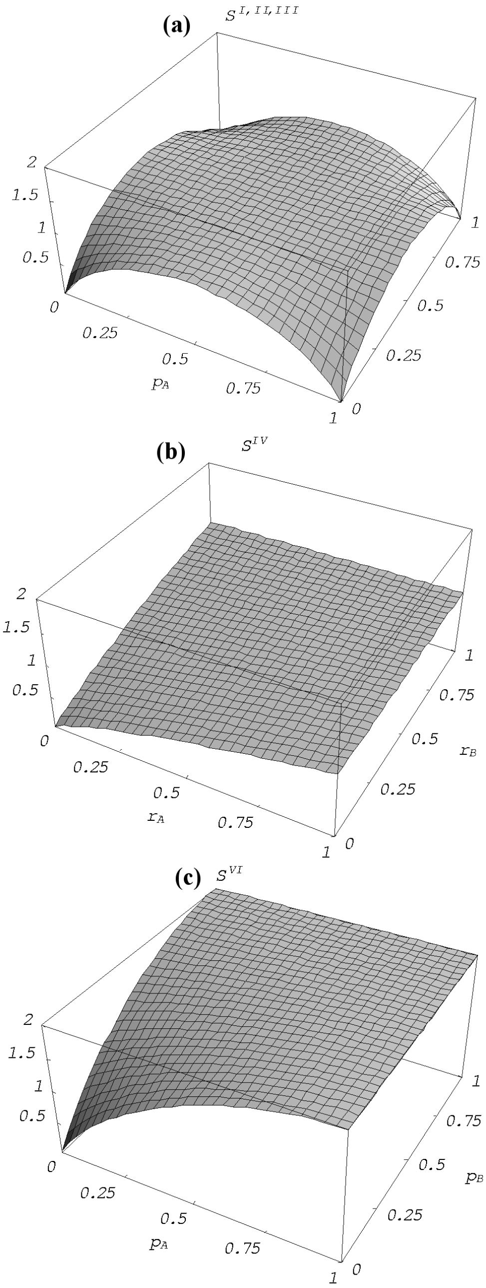


Fig. 3. Quantum entropy $S(\mathcal{E}(\rho_{in}))$ for $\varphi = \pi/4$ as a function of the noise parameters $p_A$, $p_B$ and $r_A$, $r_B$ (a) for the flip noise channels, (b) the phase damping channel, and (c) the depolarizing channel. $S^{I,II,III} = 1$ when $p_A = 1/2$ and/or $p_B = 1/2$ in (a), and $S^{IV} = 1$ when $r_A = 1$ and/or $r_B = 1$ in (b).

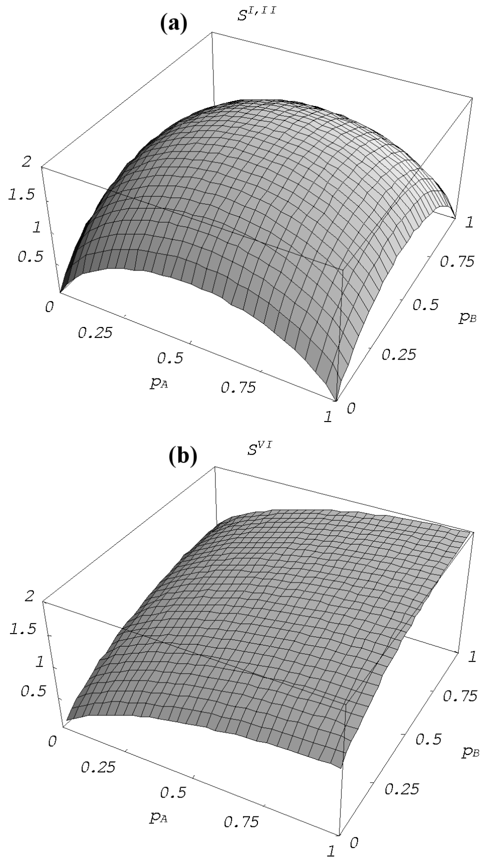


Fig. 4. Quantum entropy $S(\mathcal{E}(\rho_{in}))$ for $\varphi = 0$ as a function of the noise parameters $p_A$, $p_B$ (a) for the bit and bit-phase flip channels and (b) the depolarizing channel. $S^{I,II} = 2$ when $p_A = 1/2$ and/or $p_B = 1/2$ in (a), and $S^{VI} = 1$ when $p_A = 1$ and $p_B = 0$ (or vice versa) in (b).

number of qubits. Especially, the bit flip, the bit-phase flip, and the depolarizing channels have a maximum of $S = 2$, although the other channels only have a maximum of $S = 1$, which shows that noises related to $\sigma_x$ make the mixedness of the system higher. The damping and the depolarizing channels show that their entropy values are opposite their fidelity values; beside, we can recognize that the entropy increases as the fidelity decreases for these channels. Unlike these channels, the flip channels have symmetry of entropy not only on diagonal lines of the $p_Ap_B$ plane but also on the $p_A = 1/2$ or the $p_B = 1/2$ lines. That is why flip noises decohere the system weakly above certain values of the noise probabilities, which are $p_A = 1/2$ or $p_B = 1/2$.

### 3. Concurrence

Comparing with the fidelity and the entropy discussed in the previous sections, we study changes in the degree of entanglement. In order to measure the degree of the entanglement between two qubits of the system, we use the entanglement of formation [14] for $2\times 2$ mixed states, including pure states, which is defined as

$$E_f(\rho) = H(\frac{1+\sqrt{1-C(\rho)^2}}{2}). \tag{22}$$

Here, $H$ is the Shannon entropy [16], $H(x) = -x\log x - (1-x)\log(1-x)$, and $C$ is the concurrence [17],

$$C(\rho) = max\{0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{23}$$

where $\lambda_i$'s are the square roots of eigenvalues of the $\rho\tilde{\rho}$ matrix, set in decreasing order, with a spin-flipped state $\tilde{\rho} = (\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$. We can simply show the degree of entanglement by means of the concurrence instead of the entanglement of formation since the two measures have the same range, 0 to 1.

The concurrence of the initial state $\rho_{in}$ is $C(\rho_{in}) = \sin^2 2\varphi$. Under the noises, we show how the degree of entanglement of the two qubits changes for the maximally entangled initial state which has $C(\rho_{in}) = 1$. For noisy channels, except for the phase damping channel, the concurrence $C(\mathcal{E}(\rho_{in}))$ shown in Fig. 5 has a reversed figure compared to that for the entropy shown in Fig. 3. That is, the more the system interacts with the environment as $S \to 2$, the more the two qubits loose entanglement as $C \to 0$. The phase damping channel has $C^{IV} = 0$ when the system is an initially separable pure state as $S = 0$. It is well known that an initially separable state cannot generate any entanglement through a noise process.

The entanglement changes of the system under a noise process are as follows: In the cases of the flip channels, the state is one of the Bell states when $p_A = 1$ and/or $p_B = 1$; however, the state is mixed, having no entanglement, when $p_A = 1/2$ and/or $p_B = 1/2$. For the cases of two damping channels, $C^{IV,V}$ monotonously decreases with increasing damping probabilities $r_A$, $r_B$; finally, the entanglement disappears when one of the two damping probabilities is 1. In the depolarizing channel, $C^{VI}$ decreases to zero as the noise probabilities $p_A$, $p_B$ increase up to near the dashed line, satisfying $\sqrt{p_A^2 + p_B^2} = 2/3$, shown in Fig. 5(c); $C^{VI} = 0$ for probabilities above the dashed line. In view of the six noisy channels, the depolarizing noise is the most powerful noise for losing the entanglement of two-qubit system.

## IV. SUMMARY

We have modelled the noise process for a two-qubit system decohered by bit, bit-phase, and phase flips, by phase and amplitude dampings, and by depolarization. A generalization of the Kraus operator-sum representation can be extended to a multi-qubit system in which each qubit of a quantum channel is coupled with its environment by a local unitary operation formed in $\mathcal{E}(\rho_{ABC\cdots}) = \sum_{i,j,k,\cdots=0}^{1} E_i^A \otimes E_j^B \otimes E_k^C \cdots \rho_{ABC\cdots} E_i^{A\dagger} \otimes E_j^{B\dagger} \otimes E_k^{C\dagger} \cdots$.

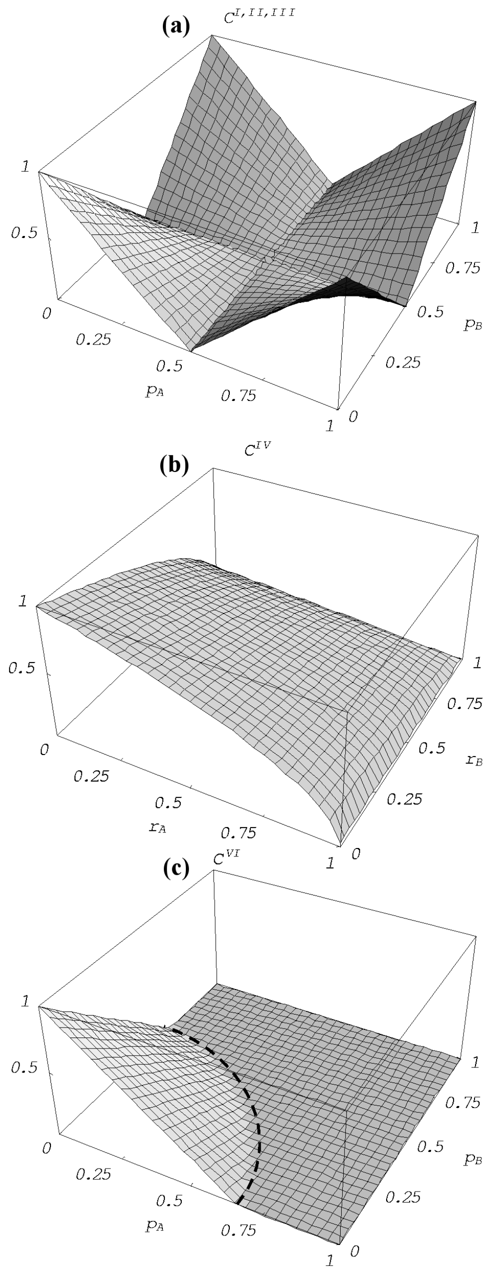


Fig. 5. Concurrence $C(\mathcal{E}(\rho_{in}))$ as a function of the noise parameters $p_A$, $p_B$ and $r_A$, $r_B$ for $\varphi = \pi/4$ for (a) the flip noise channels, (b) the phase damping channel, and (c) the depolarizing channel. The minimum value of the concurrence is $C = 0$ when $p_A = 1/2$ and/or $p_B = 1/2$ in (a), when $r_A = 1$ and/or $r_B = 1$ in (b), and when $p_A$ and $p_B$ are above the dashed line, which represents $\sqrt{p_A^2 + p_B^2} = 2/3$, in (c).

Using this formalism, for a initially pure state of a two-qubit system, we measured the degree of decoherence by investigating information properties such as the fidelity, the entropy, and the concurrence. The fidelity, the von Neumann entropy, and the concurrence show how much the state of the system changes under the noisy channels. Particularly, the entropy represents how much noise occurs in the system due to interactions with the environment. Contrary to a single-qubit system, the entropy in a two-qubit system has $S = 2$ maximally. The next measure, the concurrence shows the degree of entanglement, which behaves oppositely to the entropy. That is, the state loses entanglement as the state becomes more mixed. Through these measures, we may say following: The flip channels decohere the state of the two-qubit system maximally under half noises; however, the state is less decohered for noises above half noises. When the system is a maximally entangled state, the amounts of noise are the same in three flip channels because their probabilities are the same although the origins of the three flip noises are different. Indeed, we show that the damping channels almost directly lose the entanglement of the system in proportion to the damping probabilities and that the depolarizing noise completely loses the entanglement of the system for noises above certain noises. In view of the influence of noise, the depolarizing channel is more powerful in removing the entanglement of the two qubits of the system than any other channel.

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