CS191 – Fall 2014
Homework 6: due in lecture Oct. 29th

1. ***Gaussian integral.*** In lecture 14 we needed a generalized Gaussian integral to evaluate the dephasing rate of a qubit subject to an uncertain Hamiltonian. In this problem you will calculate the value of this integral, which is quite commonly encountered in physics and engineering. Show that:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}-i\omega t}d\omega = e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t}$$

   Hint: completing the square for the term in the exponent may be useful.

2. ***Generation of the dephasing (or phase-flip) process.*** Show that after evolution for a fixed time, $T > 0$, by the uncertain Hamiltonian from section II A of lecture 14:

$$H = \frac{\omega}{2}\sigma_z,$$

   with $\omega \sim \mathcal{N}(0, \sigma^2)$, the resulting map on an *arbitrary* initial density matrix ρ_0 is given by the dephasing process:

$$\mathcal{E}(\rho_0) = p\rho_0 + (1-p)\sigma_z\rho_0\sigma_z$$

   What is p as a function of T, σ^2?

3. ***Properties of the Lindblad master equation.*** Prove that the Lindblad master equation,

$$\frac{d}{dt}\rho(t) \;=\; -\frac{i}{\hbar}[h_0 + h_{LS}, \rho(t)] + \sum_{k=1}^{K}\gamma_k\left(L_k\rho(t)L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho(t) - \frac{1}{2}\rho(t)L_k^\dagger L_k\right),$$

   preserves the trace and Hermiticity of $\rho(t)$.

   Hint: $\rho(t+dt) = \rho(t) + dt\left(\frac{d}{dt}\rho(t)\right)$. Assuming $\rho(t)$ is Hermitian and trace 1, do these properties also hold true for $\rho(t+dt)$ when $\frac{d}{dt}\rho(t)$ is specified by the Lindblad master equation?

4. ***Action of channels on the Bloch vector.*** Recall that we can write any one qubit state in the form

$$\rho = \frac{1}{2}\left(I_2 + x\sigma_x + y\sigma_y + z\sigma_z\right),$$

   with $x = \mathrm{tr}\,(\rho\sigma_x), y = \mathrm{tr}\,(\rho\sigma_y), z = \mathrm{tr}\,(\rho\sigma_z)$. The vector $\vec{v} = (x, y, z)$ is called the Bloch vector and is a useful three-dimensional representation of the state. The length of the Bloch vector is $r = \sqrt{x^2 + y^2 + z^2}$, and pure states have $r = 1$. In terms of an arbitrary one-qubit density matrix

$$\rho = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},$$

   with $a + c = 1$, the Bloch vector elements are $x = \mathrm{Re}\{b\}, y = -\mathrm{Im}\{b\}, z = a - c$.

   The goal of this problem is to give you some intuition about how some common one qubit processes transform states by examining their action on the Bloch vector.

   (a) Write the Bloch vector elements of the output state of the phase-flip channel in terms of the Bloch vector elements of its input state. That is, let

$$\rho = p\rho_0 + (1-p)\sigma_z\rho_0\sigma_z.$$

   Then if ρ_0 has Bloch vector $\vec{v}_0 = (x_0, y_0, z_0)$, compute the Bloch vector of ρ.

   (b) Write the Bloch vector elements of the output state of the bit-flip channel in terms of the Bloch vector elements of its input state.

   (c) Write the Bloch vector elements of the output state of a depolarizing channel in terms of the Bloch vector elements of its input state.