Problem 1

I will first solve this problem in Dirac notation, and then I will attach a Mathematica printout where I've done the matrix algebra.

(a)

$$\langle 0|+\rangle = \langle 0|(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle) = \frac{1}{\sqrt{2}}\langle 0|1\rangle + \frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{1}{\sqrt{2}}$$

(b)

$$|0\rangle\langle +| = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1|)$$

(c)

$$\langle +|-\rangle = \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 0|0\rangle) = \frac{1}{2}(1 - 0 + 0 - 1) = 0$$

(d)

$$|+\rangle\langle -| = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

(e)

$$|+\rangle \otimes |-\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$$

Problem 1, Matrix Form

```
plus = (1 / Sqrt[2]) {{1}, {1}};

minus = (1 / Sqrt[2]) {{1}, {-1}};

zero = {{1}, {0}};
```

(a)

In[12]:=
```
Transpose[zero].plus // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \end{pmatrix}$$

(b)

In[16]:=
```
zero.Transpose[plus] // MatrixForm
```

Out[16]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

(c)

In[18]:=
```
Transpose[plus].minus // MatrixForm
```

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 \end{pmatrix}$$

(d)

In[19]:=
```
plus.Transpose[minus] // MatrixForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

(e)

In[20]:=
```
KroneckerProduct[plus, minus] // MatrixForm
```

Out[20]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Problem 2

The first two parts of this I will do with Mathematica. The third part I will type up in LaTeX.

```
sx = {{0, 1}, {1, 0}}; sy = {{0, -i}, {i, 0}}; sz = {{1, 0}, {0, -1}};
```

(a) Write down the eigenvalues and eigenstates for each Pauli matrix

Look at X first. The first eigenvector, and corresponding eigenvalue is: (Note, I added the normalization by hand because *Mathematica* doesn't do it by default)

In[39]:=
```
(1 / Sqrt[2]) Eigenvectors[sx][[1]] // MatrixForm
Eigenvalues[sx][[1]]
```

Out[39]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[40]= -1

In the above, the eigenvector is the first line and the corresponding eigenvalue is on the second. The second eigenvector, eigenvalue combination is

In[41]:=
```
(1 / Sqrt[2]) Eigenvectors[sx][[2]] // MatrixForm
Eigenvalues[sx][[2]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[42]= 1

For Y, the combination eigenvectors and eigenvalues

In[45]:=
```
(1 / Sqrt[2]) Eigenvectors[sy][[1]] // MatrixForm
Eigenvalues[sy][[1]]
(1 / Sqrt[2]) Eigenvectors[sy][[2]] // MatrixForm
Eigenvalues[sy][[2]]
```

Out[45]//MatrixForm=

$$\begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[46]= -1

Out[47]//MatrixForm=

$$\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[48]= 1

For Z, we have

```
In[53]:= Eigenvectors[sz][[1]] // MatrixForm
         Eigenvalues[sz][[1]]
         Eigenvectors[sz][[2]] // MatrixForm
         Eigenvalues[sz][[2]]
```

Out[53]//MatrixForm=

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Out[54]= -1

Out[55]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Out[56]= 1

(b) Verify some relations

Show that X^2 = Y^2 = Z^2 = I

```
In[58]:= sx.sx // MatrixForm
         sy.sy // MatrixForm
         sz.sz // MatrixForm
```

Out[58]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[59]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[60]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show XY = iZ

```
In[65]:= sx.sy // MatrixForm
         i * sz // MatrixForm
```

Out[65]//MatrixForm=

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Out[66]//MatrixForm=

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show [X,Y] = 2iZ. Remember, [A, B] = AB - BA

```
In[69]:= sx.sy - sy.sx // MatrixForm
         2 * i * sz // MatrixForm
```

Out[69]//MatrixForm=

$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

Out[70]//MatrixForm=

$$\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

(c) Verify the Euler identity $e^{i\theta\hat{n}\cdot\vec{\sigma}} = \cos(\theta)I + i\sin(\theta)\hat{n}\cdot\vec{\sigma}$

Compute $(\hat{n}\cdot\vec{\sigma})^{2m}$

Let's look first at the base case, $m = 1$.

$$\begin{aligned}(\hat{n}\cdot\vec{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 \\ &= (n_x^2 + n_y^2 + n_z^2)I + n_xn_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_xn_z(\sigma_x\sigma_z + \sigma_z\sigma_x) + n_yn_z(\sigma_y\sigma_z + \sigma_z\sigma_y) \\ &= I\end{aligned}$$

Where we have used the facts that $\sigma_i^2 = I$, $n_x^2+n_y^2+n_z^2 = 1$, and $\sigma_i\sigma_j+\sigma_j\sigma_i = 0$ for $i \neq j$ (you didn't prove this earlier, but it's easy to prove to yourself with matrix multiplication). So we are left with

$$(\hat{n}\cdot\vec{\sigma})^{2m} = I$$

Compute $(\hat{n}\cdot\vec{\sigma})^{2m+1}$

Now that we have the above, this part is easy:

$$(\hat{n}\cdot\vec{\sigma})^{2m+1} = (\hat{n}\cdot\vec{\sigma})^{2m}(\hat{n}\cdot\vec{\sigma}) = \hat{n}\cdot\vec{\sigma}$$

Expand the left and right hand sides and show they are equal

Recall

$$e^x = \sum \frac{x^n}{n!}$$

$$\begin{aligned}e^{i\theta\hat{n}\cdot\vec{\sigma}} = \sum \frac{(i\theta\hat{n}\cdot\vec{\sigma})^n}{n!} &= \sum \frac{(i\theta\hat{n}\cdot\vec{\sigma})^{2m}}{(2m)!} + \sum \frac{(i\theta\hat{n}\cdot\vec{\sigma})^{2m+1}}{(2m+1)!} \\ &= \sum \frac{(i\theta)^{2m}}{(2m)!}I + \sum \frac{(i\theta)^{2m+1}}{(2m+1)!}\hat{n}\cdot\vec{\sigma} \\ &= \cos(\theta)I + i\sin(\theta)(\hat{n}\cdot\vec{\sigma})\end{aligned}$$

3. Combining quantum states

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \\ |\phi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\end{aligned}$$

(a) $|\psi\rangle \otimes |\phi\rangle$

$$\begin{aligned}|\psi\rangle \otimes |\phi\rangle =& \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\ =& \frac{1}{2}|0\rangle \otimes |0\rangle + \frac{i}{2}|0\rangle \otimes |1\rangle - \frac{i}{2}|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle \\ =& \frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle\end{aligned}$$

(b) $\langle\psi| \otimes \langle\phi|$

$$\begin{aligned}\langle\psi| \otimes \langle\phi| =& \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|\right) \otimes \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) \\ =& \frac{1}{2}\left(\langle 00| - i\langle 01| + i\langle 10| + \langle 11|\right)\end{aligned}$$

Notice that in the first line, when going from $|\psi\rangle \to \langle\psi|$ we don't just make the kets into bras, we also apply complex conjugation of the coefficients.

(c) $(\sigma_x \otimes I)(|\psi\rangle \otimes |\phi\rangle)$

$$\begin{aligned}(\sigma_x \otimes I)(|\psi\rangle \otimes |\phi\rangle) =& \sigma_x|\psi\rangle \otimes I|\phi\rangle \\ =& \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) \\ =& \frac{1}{2}(|10\rangle + i|11\rangle - i|00\rangle + |01\rangle)\end{aligned}$$

Where we used the fact that $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$.

(d) $(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\phi\rangle)$

$$\begin{aligned}(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\phi\rangle) =& \sigma_x|\psi\rangle \otimes \sigma_x|\phi\rangle \\ =& \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle\right) \\ =& \frac{1}{2}(|11\rangle + i|10\rangle - i|01\rangle + |00\rangle)\end{aligned}$$

Problem 4 Entangled states

$$\begin{aligned}|T, 0\rangle =& \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\ |S\rangle =& \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle\end{aligned}$$

(b) Show these two states are not separable

Suppose $|S\rangle$ were separable. Then, there would exist single qubit states $|\psi\rangle$ and $|\phi\rangle$ such that

$$|\psi\rangle \otimes |\phi\rangle = |S\rangle$$

Without loss of generality, let

$$\begin{aligned}|\psi\rangle =& a|0\rangle + b|1\rangle \\ |\phi\rangle =& c|0\rangle + d|1\rangle\end{aligned}$$

for some undetermined coefficients. Then:

$$|\psi\rangle \otimes |\phi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

But since this must equal $|S\rangle$, we must have $ac = 0$ and $bd = 0$ since $|S\rangle$ has no component of $|00\rangle$ or $|11\rangle$. Then either $a = 0$ or $c = 0$. If $a = 0$, we have

$$|\psi\rangle \otimes |\phi\rangle = bc|10\rangle + bd|11\rangle$$

If $c = 0$, we have

$$|\psi\rangle \otimes |\phi\rangle = ad|01\rangle + bd|11\rangle$$

It is now clear by inspection that in either case, no matter the choice of the remaining coefficients, we cannot make this equal to $|S\rangle$. This proof holds without modification for $|T, 0\rangle$.

(b) Show $|S\rangle$ is invariant under global rotation

For any single qubit operator U, let

$$\begin{aligned}U|0\rangle =& a|0\rangle + b|1\rangle \\ U|1\rangle =& c|0\rangle + d|1\rangle\end{aligned}$$

Now:

$$\begin{aligned}(U \otimes U)|S\rangle =& \frac{1}{\sqrt{2}}(U \otimes U)(|01\rangle - |10\rangle) \\ =& \frac{1}{\sqrt{2}}((a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle) - (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle)) \\ =& \frac{1}{\sqrt{2}}(ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle - ac|00\rangle - bc|01\rangle - ad|10\rangle - bd|11\rangle) \\ =& \frac{1}{\sqrt{2}}((ad - bc)|01\rangle - (ad - bc)|10\rangle \\ =& \frac{ad - bc}{\sqrt{2}}(|01\rangle - |10\rangle) \\ =& (ad - bc)|S\rangle\end{aligned}$$

We are almost there! Remember that we require that all quantum states be normalized to 1. (More formally, $U \otimes U$ is a unitary operator, one property of which is that it preserves the norm of vectors it acts on.) This means that $|ad - bc|^2 = 1$. This means that $ad - bc = e^{i\theta}$ for some θ. Thus, our final state is $e^{i\theta}|S\rangle$, which is just an overall phase times $|S\rangle$. In quantum mechanics, we don't care about overall phases acting on our state vectors, so we make the identification $e^{i\theta}|S\rangle \sim |S\rangle$.

Problem 5 Dynamics

(a) The $|0\rangle, |1\rangle$ basis

Let $|\psi(t)\rangle = a|0\rangle + b|1\rangle$. Then $H|\psi(t)\rangle = \alpha(a|1\rangle + b|0\rangle)$. $\frac{\partial}{\partial t}|\psi(t)\rangle = \dot{a}|0\rangle + \dot{b}|1\rangle$. Here, the dots represent differentiation with respect to time. So our differential equation reads

$$i\hbar(\dot{a}|0\rangle + \dot{b}|1\rangle) = \alpha(a|1\rangle + b|0\rangle)$$

We can set the coefficients of each vector equal on each side of the equation, yielding the following coupled equations

$$\frac{\partial a}{\partial t} = -i\frac{\alpha}{\hbar}b$$
$$\frac{\partial b}{\partial t} = -i\frac{\alpha}{\hbar}a$$

Thus, recognize that the differential equation for the coefficients is a matrix equation of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\alpha}{\hbar}\sigma_x \begin{pmatrix} a \\ b \end{pmatrix}$$

Recall that the solution to such an equation is given by:

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = e^{i\alpha t/\hbar\sigma_x} \begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

Now we can use the result from Problem 2, and write

$$e^{i\alpha t/\hbar\sigma_x} = \cos(\alpha t/\hbar)I + i\sin(\alpha t/\hbar)\sigma_x$$

. Plugging in our initial condition $(a(0) = 1, b(0) = 0)$, we have

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = (\cos(\alpha t/\hbar)I + i\sin(\alpha t/\hbar)\sigma_x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Or

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t/\hbar) \\ i\sin(\alpha t/\hbar) \end{pmatrix}$$

We can, if you'd like, go back to Dirac notation and write

$$|\psi(t)\rangle = \cos(\alpha t/\hbar)|0\rangle + i\sin(\alpha t/\hbar)|1\rangle$$

(b) The $|+\rangle, |-\rangle$ basis

If we wanted, we could rewrite the Schrodinger equation in the the $|+\rangle, |-\rangle$ basis, but since we've already done the work of solving, let's just take the answer from the previous problem and write it in the new basis.

$$\begin{aligned}|0\rangle =& \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ |1\rangle =& \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\end{aligned}$$

Plugging in to the solution from part (a), we get

$$\begin{aligned}|\psi(t)\rangle =& \frac{1}{\sqrt{2}}(\cos(\alpha t/\hbar) + i\sin(\alpha t/\hbar))|+\rangle + \frac{1}{\sqrt{2}}(\cos(\alpha t/\hbar) - i\sin(\alpha t/\hbar))|-\rangle \\ =& \frac{1}{\sqrt{2}}e^{i\alpha t/\hbar}|+\rangle + \frac{1}{\sqrt{2}}e^{-i\alpha t/\hbar}|-\rangle\end{aligned}$$

1 The secular approximation

1.1 Rotating frame

Define

$$|\psi'(t)\rangle = e^{iH_1 t/\hbar}|\psi(t)\rangle$$

We will seek an equation governing time evolution for $|\psi'(t)\rangle$. First, let's write the ordinary Schrodinger equation in terms of $|\psi'(t)\rangle$:

$$\begin{aligned}
i\hbar\frac{d}{dt}|\psi(t)\rangle =&(H_0+H_1)|\psi(t)\rangle \\
i\hbar\frac{d}{dt}(e^{-iH_1t/\hbar}|\psi'(t)\rangle) =&(H_0+H_1)e^{-iH_1t/\hbar}|\psi'(t)\rangle \\
i\hbar(-iH_1/\hbar e^{-iH_1t/\hbar}|\psi'(t)\rangle + e^{-iH_1t/\hbar}\frac{d}{dt}|\psi'(t)\rangle) =&(H_0+H_1)e^{-iH_1t/\hbar}|\psi'(t)\rangle \\
(H_1|\psi'(t)\rangle + i\hbar\frac{d}{dt}|\psi'(t)\rangle) =&e^{iH_1t/\hbar}(H_0+H_1)e^{-iH_1t/\hbar}|\psi'(t)\rangle \\
i\hbar\frac{d}{dt}|\psi'(t)\rangle =&(e^{iH_1t/\hbar}H_0e^{-iH_1t/\hbar} + e^{iH_1t/\hbar}H_1e^{-iH_1t/\hbar} - H_1)|\psi'(t)\rangle \\
i\hbar\frac{d}{dt}|\psi'(t)\rangle =&e^{iH_1t/\hbar}H_0e^{-iH_1t/\hbar}|\psi'(t)\rangle
\end{aligned}$$

In the physics literature, this is transformation is often referred to as the "interaction picture", and the last line is the Schrodinger equation in the interaction picture. The $e^{iH_1t/\hbar}H_0e^{-iH_1t/\hbar}$ is the interaction picture Hamiltonian. Several times in the above derivation I have used the fact that H_1 and $e^{-iH_1t/\hbar}$ commute, and therefore I am free to move the matrix exponentials across the H_1 whenever I would like. This is because the matrix exponential of H_1 is a power series only of H_1, and we expect H_1 to commute with any polynomial function of H_1. This is **not** the case for H_0, and I am not allowed to move $e^{-iH_1t/\hbar}$ across H_0.

Now let's see if we can do anything with this. matrix product. Remember

$$\begin{aligned}
H_0 =&J(2\sigma_+^e\otimes\sigma_-^p + 2\sigma_-^e\otimes\sigma_+^p + \sigma_z^e\otimes\sigma_z^p) \\
H_1 =&(1/2)g\mu_B B\sigma_z - (1/2)g\mu_N B\sigma_z
\end{aligned}$$

Let's work on the interaction picture Hamiltonian $e^{iH_1t/\hbar}H_0e^{-iH_1t/\hbar}$:

$$e^{iH_1t/\hbar}H_0e^{-iH_1t/\hbar} = e^{(i/2\hbar)gB(\mu_B\sigma_z^e-\mu_N\sigma_z^p)t}J(2\sigma_+\otimes\sigma_- + 2\sigma_-\otimes\sigma_+ + \sigma_z\otimes\sigma_z)e^{(-i/2\hbar)gB(\mu_B\sigma_z^e-\mu_N\sigma_z^p)t}$$

Well that looks complicated. Let's take this piece by piece.

$$\begin{aligned}
e^{(i/2\hbar)gB(\mu_B\sigma_z^e-\mu_N\sigma_z^p)t}\sigma_+\otimes\sigma_-e^{(-i/2\hbar)gB(\mu_B\sigma_z^e-\mu_N\sigma_z^p)t} = \\
e^{(i/2\hbar)gB\mu_B\sigma_z^et}\sigma_+^ee^{(-i/2\hbar)gB\mu_B\sigma_z^et}\otimes e^{(-i/2\hbar)gB\mu_N\sigma_z^et}\sigma_-^pe^{(i/2\hbar)gB\mu_N\sigma_z^et} = \\
e^{i\omega_et}\sigma_+^e\otimes e^{i\omega_pt}\sigma_-^p = \\
e^{i(\omega_e+\omega_p)t}\sigma_+^e\otimes\sigma_-^p
\end{aligned}$$

Here, I have made use of the identity $e^{i\alpha\sigma_z}\sigma_+ e^{-i\alpha\sigma_z} = e^{2i\alpha}\sigma_+$, and the analogous equation for σ_-. You should prove this for yourself. It is not difficult, and can be done by direct matrix multiplication. I have also defined here $\omega_e = g\mu_B B/\hbar$ and $\omega_p = g\mu_N B/\hbar$.

The next term is similar, and we find:

$$e^{(i/2\hbar)gB(\mu_B\sigma_z^e - \mu_N\sigma_z^p)t}\sigma_- \otimes \sigma_+ e^{(-i/2\hbar)gB(\mu_B\sigma_z^e - \mu_N\sigma_z^p)t} = e^{-i(\omega_e+\omega_p)t}\sigma_-^e \otimes \sigma_+^p$$

The last part is a bit different, however. We note that $e^{(i/2\hbar)gB(\mu_B\sigma_z^e - \mu_N\sigma_z^p)t}$ commutes with $\sigma_z^e \otimes \sigma_z^p$. Therefore:

$$e^{(i/2\hbar)gB(\mu_B\sigma_z^e - \mu_N\sigma_z^p)t}\sigma_z^e \otimes \sigma_z^p e^{(-i/2\hbar)gB(\mu_B\sigma_z^e - \mu_N\sigma_z^p)t} = \sigma_z^e \otimes \sigma_z^p$$

Putting it all together then, our Schrodinger equation for $|\psi'(t)\rangle$ reads

$$i\hbar\frac{d}{dt}|\psi'(t)\rangle = J(2e^{i(\omega_e+\omega_p)t}\sigma_+^e \otimes \sigma_-^p + 2e^{-i(\omega_e+\omega_p)t}\sigma_-^e \otimes \sigma_+^p + \sigma_z^e \otimes \sigma_z^p)|\psi'(t)\rangle$$

1.2 Neglecting the fast oscillating terms

The frequencies ω_e and ω_p depend linearly on B. For large B, $\omega_e + \omega_p \gg J/\hbar$, and thus the effects of the $\sigma_+ \otimes \sigma_-$ terms will very quickly average to zero and can thus be neglected in comparison to the $\sigma_z^e \otimes \sigma_z^p$ term. Of course, this will not be satisfied if B is not large enough. Making this approximation, we have

$$i\hbar\frac{d}{dt}|\psi'(t)\rangle \approx J\sigma_z^e \otimes \sigma_z^p|\psi'(t)\rangle$$

This is known as the rotating wave approximation.

1.3 Going back to the lab frame

The way we go back to the lab frame is to do the inverse thing we did in the first section. We take our evolution equation for $|\psi'(t)\rangle$, and replace it with $|\psi(t)\rangle$, using the definition $|\psi(t)\rangle = e^{-iH_1 t/\hbar}|\psi'(t)\rangle$.

$$\begin{aligned}
i\hbar\frac{d}{dt}|\psi'(t)\rangle &\approx J\sigma_z^e \otimes \sigma_z^p|\psi'(t)\rangle \\
i\hbar\frac{d}{dt}(e^{iH_1 t/\hbar}|\psi(t)\rangle) &= J\sigma_z^e \otimes \sigma_z^p e^{iH_1 t/\hbar}|\psi(t)\rangle \\
-H_1 e^{iH_1 t/\hbar}|\psi(t)\rangle + i\hbar\frac{d}{dt}|\psi(t)\rangle &= J\sigma_z^e \otimes \sigma_z^p e^{iH_1 t/\hbar}|\psi(t)\rangle \\
i\hbar\frac{d}{dt}|\psi(t)\rangle &= (H_1 + J\sigma_z^e \otimes \sigma_z^p)|\psi(t)\rangle
\end{aligned}$$

In the last line, I made use again of the fact that H_1 commutes with $\sigma_z^e \otimes \sigma_z^p$, by multiplying both sides of the equation with $e^{-iH_1 t/\hbar}$. In any case, the last line is the equation we seek: the term $H1 + J\sigma_z^e \otimes \sigma_z^p = (1/2)g\mu_B B\sigma_z^e - (1/2)g\mu_N B\sigma_z^p + J\sigma_z^e \otimes \sigma_z^p$ is the Hamiltonian in the secular approximation. Thus, in this approximation, the evolution of $|\psi(t)\rangle$ is described by this Hamiltonian.

2 Spin echo

$H = KB\sigma_z$

2.1 What is the state of the qubit after time τ?

We know that $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = e^{-iKBt\sigma_z/\hbar}|\psi(0)\rangle$. $|\psi(0)\rangle = |+\rangle$. We also proved a thing in the previous homework about matrix exponentials of this form. We get:

$$
\begin{aligned}
|\psi(\tau)\rangle = & \\
e^{-iKB\tau\sigma_z/\hbar}|+\rangle =& (\cos(KB\tau/\hbar)I + i\sin(KB\tau/\hbar)\sigma_z)|+\rangle \\
=& \frac{1}{\sqrt{2}}((\cos(KB\tau/\hbar)I + i\sin(KB\tau/\hbar)\sigma_z)(|0\rangle + |1\rangle) \\
=& \frac{1}{\sqrt{2}}((\cos(KB\tau/\hbar) + i\sin(KB\tau/\hbar))|0\rangle + (\cos(KB\tau/\hbar) - i\sin(KB\tau/\hbar))|1\rangle) \\
=& \frac{1}{\sqrt{2}}(e^{iKB\tau/\hbar}|0\rangle + e^{-iKB\tau/\hbar}|1\rangle)
\end{aligned}
$$

where I used the fact that $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$.

2.2 Now apply σ_x at time τ

We can read off from the matrix form of σ_x that $\sigma_x|0\rangle = |1\rangle$ and $\sigma_x|1\rangle = |0\rangle$. Thus,

$$\sigma_x|\psi(\tau)\rangle = \frac{1}{\sqrt{2}}(e^{iKB\tau/\hbar}|1\rangle + e^{-iKB\tau/\hbar}|0\rangle$$

2.3 Now the qubit is allowed to evolve under H for an additional time τ. What is the final qubit state?

We now apply the same evolution to the new state. You could do it again, exactly as above. A faster way to get the same answer is to notice that under this Hamiltonian, $|0\rangle$ acquired a phase $e^{iKB\tau/\hbar}$, and $|1\rangle$ acquired a phase $e^{-iKB\tau/\hbar}$. This is always true because it is the action of the time evolution operator on the basis. Thus, we can just apply these phase factors to our state again, and

get

$$\begin{aligned}|\psi(2\tau)\rangle =& \frac{1}{\sqrt{2}}(e^{-iKB\tau/\hbar}e^{iKB\tau/\hbar}|1\rangle + e^{iKB\tau/\hbar}e^{-iKB\tau/\hbar}|0\rangle \\ =& \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ =& |+\rangle\end{aligned}$$

Thus. this "echo" returns our state to the original state, regardless of the magnitude of the magnetic field.

3 Circuits

Most of these can shown with just matrix multiplication. I will attach a Mathematica sheet with that algebra. By way of commentary, I only want to point out that, when talking about two qubits, if apply some gate, U to one qubit, and do nothing to the other, the appropriate matrix to write down is either $U \otimes I$ or $I \otimes U$ depending on which qubit is being acted on. The math is worked out in the last page of this document.

4 Single qubit gates

I will use the notation from the Lecture 4 notes on spin resonance for this part.

$$H_1 = -\frac{1}{2}g\mu_B(B_0\sigma_z + B_1\sin(\omega t)\sigma_x)$$

4.1 What unitary operator is generated by applying H_1 for a time τ?

From the lecture notes, we have the rotating frame Hamiltonian

$$H'(t) = \left(\frac{\hbar\omega_L}{2}\sigma_z + e^{i\omega_L\sigma_z t/2}H_1(t)e^{-i\omega_L\sigma_z t/2}\right)$$

We do the same trick as in the notes, writing

$$\begin{aligned}\sin(\omega t)\sigma_x =& \frac{1}{2}((\sin(\omega t)\sigma_x + \cos(\omega t)\sigma_y) + (\sin(\omega t)\sigma_x\cos(\omega t)\sigma_y)) \\ =& \frac{1}{2}(e^{i\omega t - i\pi/2}\sigma_+ + e^{-i\omega t + i\pi/2}\sigma_- + e^{-i\omega t + i\pi/2}\sigma_+ + e^{i\omega t - i\pi/2}\sigma_-)\end{aligned}$$

We now have to apply

$$e^{i\omega_L\sigma_z t/2}(e^{i\omega t - i\pi/2}\sigma_+ + e^{-i\omega t + i\pi/2}\sigma_- + e^{-i\omega t + i\pi/2}\sigma_+ + e^{i\omega t - i\pi/2}\sigma_-)e^{-i\omega_L\sigma_z t/2}$$

to obtain

$$\begin{aligned} e^{i(\omega_L+\omega)t-i\pi/2}\sigma_+ + e^{-i(\omega_L+i\omega)t+i\pi/2}\sigma_- + e^{i(\omega_L-\omega)t+i\pi/2}\sigma_+ + e^{-i(\omega-\omega_L)t-i\pi/2}\sigma_- \\ \approx e^{i(\omega_L-\omega)t+i\pi/2}\sigma_+ + e^{-i(\omega-\omega_L)t-i\pi/2}\sigma_- \\ = i\sigma_+ - i\sigma_- \\ = -\sigma_y \end{aligned}$$

In the last two lines I assumed resonance, *e.g.* $\omega = \omega_L$, and in the second line I applied the rotating wave approximation.

This is the only part that differs from the lecture notes. Thus, we obtain

$$\begin{aligned} H'(t) &= \left(\frac{\hbar\omega_L}{2}\sigma_z - \frac{\hbar\omega_L}{2}\sigma_z - \frac{\hbar g\mu_B B_1}{2}\sigma_y\right) \\ &= -\frac{\hbar g\mu_B B_1}{2}\sigma_y \end{aligned}$$

The unitary operation this generates (in the rotating frame) is just $e^{-iHt/\hbar} = e^{ig\mu_B B_1 \tau\sigma_y/(2\hbar)}$

4.2 What Hamiltonian can we use to generate $U = \cos\theta\sigma_x + \sin(\theta)\sigma_y$

For this, we can again draw inspiration from our old friend the Euler identity for Pauli operators, noting that, with $\hat{n} = (\cos(\theta), \sin(\theta), 0)$, we have:

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = \cos\alpha I + i\sin(\alpha)(\cos\theta\sigma_x + \sin(\theta)\sigma_y)$$

We thus need a Hamiltonian of the form $A(\cos\theta\sigma_x + \sin(\theta)\sigma_y)$. In the context of NMR, we can take $H = \frac{\hbar g\mu_B B}{2}(\cos\theta\sigma_x + \sin(\theta)\sigma_y)$. To generate the unitary, we need to apply this Hamiltonian for a time to make $\alpha = \frac{g\mu_B B\tau}{2} = \pi/2$.

We are supposed to notice here that this problem is simply a generalization of the previous part. Imagine that our field had the phase ϕ, such that

$$H_1 = -\frac{1}{2}g\mu_B(B_0\sigma_z + B_1\sin(\omega t + \phi)\sigma_x)$$

This is what we solved in the last part with $\phi = \pi/2$. If we have a general phase ϕ, the only thing that would change in our above derivation is that, when we our interaction Hamiltonian would now be proportional to

$$\begin{aligned} & e^{i\phi}\sigma_+ + e^{-i\phi}\sigma_- \\ &= \cos\phi\sigma_x - \sin\phi\sigma_y \end{aligned}$$

Thus, by changing the phase of the driving field, we change the axis about which we rotate our qubit.

In[27]:=

Problem 3

First, let's build the matrices. For the controlled gates, we need to think about what happens to the basis vectors. For instance, CPHASE takes each vector to itself whenever the first qubit is 0, and applies Z if the first qubit is one. So |00> -> |00>, |01> -> |01>, |10> -> |10> and |11> -> -|11>.

In[59]:=
```
H = (1 / (Sqrt[2])) * {{1, 1}, {1, -1}}; (* Hadamard *)
H2 = KroneckerProduct[IdentityMatrix[2], H];
CPHASE = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}};
CNOT = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}};
(* CNOT with control on 2nd qubit *)
rCNOT = {{1, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}};
SWAP = {{1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}};
U = {{a, b}, {c, d}};
```

(a)

In[44]:=
```
H2.CPHASE.H2 // MatrixForm
CNOT // MatrixForm
```

Out[44]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

In[46]:=
```
H2.CNOT.H2 // MatrixForm
CPHASE // MatrixForm
```

Out[46]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(c)

In[57]:=
```
CNOT.rCNOT.CNOT // MatrixForm
SWAP // MatrixForm
```

Out[57]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[58]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d)

(i)

In[65]:=
```
KroneckerProduct[IdentityMatrix[2], U] // MatrixForm
```

Out[65]//MatrixForm=

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

(ii)

In[66]:=
```
KroneckerProduct[U, IdentityMatrix[2]] // MatrixForm
```

Out[66]//MatrixForm=

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

(iii) For the controlled U, we apply U to the basis elements whenever the first qubit is in |1>

In[67]:=
$$\mathbf{CU} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a} & \mathbf{c} \\ \mathbf{0} & \mathbf{0} & \mathbf{b} & \mathbf{d} \end{pmatrix} \mathbf{// MatrixForm}$$

Out[67]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}$$

(iv) Now for the control on the second qubit

In[68]:=
$$\mathbf{rCU} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} & \mathbf{0} & \mathbf{b} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} & \mathbf{0} & \mathbf{d} \end{pmatrix} \mathbf{// MatrixForm}$$

Out[68]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

C191 - Lecture 1

I. AN INCOMPLETE LIST OF THE AXIOMS OF QUANTUM MECHANICS

1. Every physical system is associated with a Hilbert space. A Hilbert space is a vector space (collection of objects that can be added together and multiplied by scalars) together with an inner product, $\langle a|b\rangle$ such that:
   (a) Complex conjugate relation: $\langle a|b\rangle = \langle b|a\rangle^*$
   (b) Linear: $\langle x_1 a_1 + x_2 a_2|b\rangle = x_1 \langle a_1|b\rangle + x_2 \langle a_2|b\rangle$
   (c) Positive definite: $\langle a|a\rangle \geq 0$
2. Every ray in the Hilbert space is associated with a state of the system. We sometimes call these rays *kets* - and give them labels such as, $|\psi\rangle$. Dual vectors are represented by *bras* and are labeled $\langle\psi|$. In the matrix representations, kets are column vectors and the bras are the conjugate transpose of the kets, making them row vectors.
3. Each measurement corresponds to Hermitian operator, $A = A^\dagger \equiv A^{\mathsf{T}*}$.
4. The possible results of the measurements are the eigenvalues of A.
5. The probability of observing a particular value is given by $\text{Prob}(a = a_n) = \frac{\langle\psi|P_{a_n}|\psi\rangle}{\langle\psi|\psi\rangle}$, where P_{a_n} is the projector onto the set of states which correspond to given measurement value (this is also known as the a_n eigenspace of A). The denominator in this expression is to handle the case where the state vector has not been properly normalized.

II. STATES

Let's say we have a three-dimensional Hilbert space spanned by the orthogonal basis vectors, $|0\rangle, |1\rangle, |2\rangle$. Because they are orthogonal, their inner products are: $\langle 0|1\rangle = \langle 0|2\rangle = \langle 1|2\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = \langle 2|2\rangle = 1$. One particular state in this Hilbert space is

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ i/\sqrt{3} \end{pmatrix}$$

This state is an example of *superposition*, where the state of the system is a sum of basis states. We have written the state using the bra-ket notation as well as using a column-vector representation of the state where the entries in the column vector correspond to the coefficients in front of the basis vectors in the bra-ket notation. The coefficients in front of the basis vectors are called *amplitudes*, and the probability of finding the system in any given basis state is given by the modulus squared of the amplitude of this basis vector. In this case the probabilities are all 1/3 - you should check this! That quantum mechanics predicts probabilities is the first major differences between it and classical mechanics. In classical mechanics, complete knowledge of the state of the system could tell you exactly what the result of any measurement would be. But quantum mechanics is fundamentally probabilistic and there does not appear to be any underlying deterministic theory.

Note that could also determine these probability using the axioms above. For instance - what is the probability of finding the system in the $|0\rangle$ state? The fifth axiom says that the probability is given in terms of a projector (we'll ignore the denominator since the state is normalized: $\langle\psi|\psi\rangle = 1$), and the the projector onto the $|0\rangle$ state is the operator:

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while the *bra* vector for the state is

$$\langle\psi| = |\psi\rangle^\dagger = \frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2| = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3} \end{pmatrix}$$

We can compute the inner product $\text{Prob}(0) = \langle\psi|P_0|\psi\rangle = \langle\psi\,|0\rangle\langle 0|\,\psi\rangle = |\langle 0|\psi\rangle|^2$ using either matrix notation:

$$\begin{aligned}\text{Prob}(0) &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\i/\sqrt{3}\end{pmatrix}\\ &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3}\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\0\\0\end{pmatrix}\\ &= 1/3\end{aligned}$$

or with braket notation:

$$\begin{aligned}\text{Prob}(0) &= \left(\frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2|\right)(|0\rangle\langle 0|)\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle\right)\\ &= \left(\frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2|\right)\left(\frac{1}{\sqrt{3}}|0\rangle\right)\\ &= 1/3\end{aligned}$$

We'll be very concerned with collections of two-level systems (qubits) from now on. With a single qubit, we usually label the states as $|0\rangle$ and $|1\rangle$. These states comprise the *standard* or *computational* basis. Another important basis is the *plus-minus* basis defined as $|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$.

A. Combining quantum systems

One feature of quantum systems that makes them (i) very difficult to simulate and (ii) powerful computationally is what happens when many small systems are combined. Consider a collection of three one-qubit Hilbert spaces, $\mathcal{H}_1$, each spanned by the vectors $\{|0\rangle, |1\rangle\}$. The Hilbert space of the combined system, denoted $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1$ or $\mathcal{H}_1^{\otimes 3}$, is spanned by the vectors: $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$. This space is therefore eight-dimensional. For each qubit that is added to the system, the dimension doubles. Simulating the exact dynamics of a collection of 30 qubits requires our keeping track of $2^{30} \simeq 10^9$ complex numbers (one amplitude for each basis vector). This combination is called a *tensor product*. The tensor product is linear, so

$$\begin{aligned}(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) &= ac|0\rangle\otimes|0\rangle + ad|0\rangle\otimes|1\rangle + bc|1\rangle\otimes|0\rangle + bd|1\rangle\otimes|1\rangle\\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle\end{aligned}$$

Here for convenience we've written, for example, $|0\rangle \otimes |0\rangle = |00\rangle$, the tensor product notation as being implied.

Note that this is different from classical systems, which combine by *direct sum*, where the dimension of the resulting space is the *sum* of the constituent spaces. This can be understood by considering a collection of gas molecules. For each particle, we need only keep track its position (3 real numbers) and its velocity (3 real numbers). Adding a particle to our collection then requires us to keep track of an additional 6 terms. Simulating classical mechanics is a problem whose difficulty increases only like a polynomial in the number of particles, whereas simulating quantum mechanics is a problem whose difficulty increases exponentially in the number of particles. We'll see this separation of polynomial vs exponential difficulty later in the course.

B. Measuring qubits in different bases

Suppose we have a state specified in the standard basis

$$|\phi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

the probability of measuring 0 or 1 can be read off directly as 2/3 and 1/3, respectively. But suppose we had a measurement device which could only distinguish between states in the plus-minus basis? What then is the probability of each? There are a few ways to proceed. Firstly, we could rewrite the state in the plus minus basis and read off the

coefficients. From the definition of the plus-minus basis above, we see that $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ and $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$. Therefore,

$$\begin{aligned}
|\phi\rangle &= \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \\
&= \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= \left(\sqrt{\frac{2}{6}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\sqrt{\frac{2}{6}} - \frac{1}{\sqrt{6}}\right)|-\rangle
\end{aligned}$$

So the probabilities of measuring + and - are $\left(\sqrt{\frac{2}{6}} + \frac{1}{\sqrt{6}}\right)^2 \simeq 0.97$ and $\left(\sqrt{\frac{2}{6}} - \frac{1}{\sqrt{6}}\right)^2 \simeq 0.03$ respectively. Alternatively, we can use the axioms. For projection onto a single state, we saw that we simply need to take the modulus squared of the inner product of the given state with the target state. So:

$$\begin{aligned}
\text{Prob}_+ &= |\langle +|\phi\rangle|^2 \\
&= \left|\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\langle 0|1\rangle + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\langle 1|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\langle 1|1\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} \times 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \times 0 + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} \times 0 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \times 1\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\right|^2 \\
&\simeq 0.97
\end{aligned}$$

C191 - Lecture 2 - Quantum states and observables

I. ENTANGLED STATES

We saw last time that quantum mechanics allows for systems to be in superpositions of basis states. Many of these superpositions possess a uniquely quantum feature known as *entanglement*. As an example, consider the two-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Because this is a two qubit state, I can separate the two qubits, giving one to Alice and the other to Bob. If Alice measures her qubit, she will have a 50% chance of measuring 0 and 50% chance of measuring 1. But if Alice measures a 1, then necessarily Bob will measure a 1, even if the qubits are removed from causal contact from one another (many light seconds away, for instance). Contrast this with the state

$$|phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

Now Alice will measure 0 with 100% probability, but Bob's measurement will still be 0 or 1 with 50% probability each. This state is called *separable* because it may be written as

$$|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|0\rangle\,1 = \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle = |0\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

Entangled states are those which can never be written this way. We will see later that entanglement is an extremely important resource for quantum computation.

II. DENSITY MATRICES

Quantum states are inherently probabilistic, but how do we handle situations where there is additional *classical* uncertainty? This could arise, for instance, if we were given a quantum state that, with probability p_1 was prepared in some state $|\psi\rangle$, and with probability p_2 was prepared in a state $|\phi\rangle$. The expectation value of an operator, A over some state, $|\psi\rangle$, is defined as,

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$

But recall what an expectation value is: it the sum of all possible measurement outcomes weighted by their probabilities. So if we have some classical uncertainty, we can just build that into the definition of the expectation value,

$$\begin{aligned}\langle A\rangle &= p_1\langle A\rangle_\psi + p_2\langle A\rangle_\phi \\ &= p_1\langle\psi|A|\psi\rangle + p_2\langle\phi|A|\phi\rangle\end{aligned}$$

At this point we introduce a mathematical object known as the *resolution of the identity*. We can take a complete set of states, $\{|n\rangle\}$ and take their outer products, $\mathcal{I} = \sum_n |n\rangle\langle n|$. This is equal to the identity. We can see this in a simple example of a two level system: multiply any two level state on the left by $|0\rangle\langle 0| + |1\rangle\langle 1|$ and you will see that you get the same state back. In matrix form, this can be written as:

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},$$

which is the identity matrix. Inserting the resolution of the identity into the above expression, we have

$$\begin{aligned}\langle A\rangle &= p_1\sum_n \langle\psi|n\rangle\langle n|A|\psi\rangle + p_2\sum_n\langle\psi|n\rangle\langle n|A|\phi\rangle \\ &= p_1\sum_n\langle n|A|\psi\rangle\langle\psi|n\rangle + p_2\sum_n\langle n|A|\phi\rangle\langle\psi|n\rangle \\ &= \sum_n\langle n|A\left(p_1|\psi\rangle\langle\psi| + p_2|\phi\rangle\langle\phi|\right)|n\rangle\end{aligned}$$

We call the quantity in the parentheses the *density matrix*

$$\rho = p_1 |\psi\rangle\langle\psi| + p_2 |\phi\rangle\langle\phi|$$

and we can interpret $\sum_n \langle n| B |n\rangle$ as the trace of the matrix B. This can be seen again by considering a two level system:

$$\begin{aligned}\sum_n \langle n| B |n\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= b_{00} + b_{11} \\ &= \operatorname{Tr} B\end{aligned}$$

So we have

$$\langle A\rangle = \operatorname{Tr} \rho A$$

We'll see density matrices frequently when we begin to talk about errors in quantum states. These errors occur when the controls we use to create states are imperfect and introduce uncertainty into the system.

III. HERMITIAN OPERATORS

One of the axioms of quantum mechanics is that every observable corresponds to a Hermitian operator. Let's review some properties of these objects

1. By definition, a Hermitian operator is equal to its conjugate transpose. $A = A^{\mathsf{T}*} = A^\dagger$.
2. Hermitian operators have real eigenvalues. For some eigenstate, $A |u\rangle = a |u\rangle$. Multiply on the left by $\langle u|$ and we have

   $$\langle u|A|u\rangle = a \langle u|u\rangle = a$$

   Now take the conjugate-transpose of both sides.

   $$\begin{aligned}\langle u|A|u\rangle &= a^\dagger \\ |u\rangle^\dagger A^\dagger \langle u|^\dagger &= a^* \\ \langle u| A |u\rangle &= a^*\end{aligned}$$

   Which implies that $a = a^*$, so the eigenvalue must be real.
3. Eigenvectors are orthogonal if they possess different eigenvalues. Now additionally assume we have another eigenvector with a different eigenvalue, $A |v\rangle = b |v\rangle$. Then with A acting to the right, we have

   $$\langle v| A |u\rangle = a \langle v|u\rangle$$

   but with A acting to the left we have

   $$\langle v| A |u\rangle = b \langle v|u\rangle$$

   this means that $a \langle v|u\rangle = b \langle v|u\rangle$. If $a \neq b$, then $\langle v|u\rangle = 0$, so the vectors are orthogonal.
4. Hermitian operators may be expressed in terms of their *spectral representation*:

   $$A = \sum_n a_n |n\rangle\langle n|$$

   Where $A |n\rangle = a_n |n\rangle$. This is called a spectral representation because the collection of eigenvalues is often referred to as the *spectrum* of an operator.
5. Commuting observables possess a simultaneous eigenbasis. This proof has two parts: i) Two operators that share an eigenbasis must commute, and ii) two operators that commute must share an eigenbasis. I'll do the first one here, the second one is worth working out yourselves. Assume we have two operators which share an eigenbasis, $A |n\rangle = a_n |n\rangle$ and $B |n\rangle = b_n |n\rangle$. Then

   $$AB |n\rangle = A(b_n |n\rangle) = b_n A |n\rangle = a_n b_n |n\rangle$$

   $$BA |n\rangle = B(a_n |n\rangle) = a_n B |n\rangle = a_n b_n |n\rangle$$

   So if $AB |n\rangle = BA |n\rangle$ on every state $|n\rangle$, then $AB = BA$, and we say that the operators commute.

A. Commutators

The commutator of two hermitian operators, A, B is defined as $[A, B] = AB - BA$. Here are some of their properties,

1. The are linear in both elements

$$[c_1 A_1 + c_2 A_2, B] = c_1[A_1, B] + c_2[A_2, B]$$
$$[A, c_1 B_1 + c_2 B_2] = c_1[A, B_1] + c_2[A, B_2]$$

2. The are antisymmetric

$$[A, B] = -[B, A]$$

3. They satisfy the Jacboi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

4. They satisfy the Leibnitz identity

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

IV. TENSOR PRODUCTS OF OPERATORS

When we combine quantum systems, the state vectors live in a Hilbert space that is the tensor product of the two constituent Hilbert spaces. If each qubit is in a particular state, then the combined state is the tensor product of the individual states. And if each qubit is operated on by a particular operator, then the combined operator is the tensor product of the individual operators.

1. Tensor products are linear

$$(a_1 A_1 + a_2 A_2) \otimes B = a_1 A_1 \otimes B + a_2 A_2 \otimes B$$

2. Each term in a tensor product acts on its own component

$$(A \otimes B)(|mn\rangle) = (A \otimes B)(|m\rangle \otimes |n\rangle) = A|m\rangle \otimes B|n\rangle$$

3. Multiplication of operators

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

4. The matrix representation of tensor products is, for states:

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \\ a_1 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$$

For operators,

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \otimes \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{01} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \\ a_{10} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} & a_{11} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{01}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{01}b_{10} & a_{11}b_{11} \end{pmatrix}$$

A. Complete set of commuting observables

For any Hilbert space, we can specify a (not unique!) set of commuting observables, $\{A, B, \ldots\}$. If each eigenstate is associated with a unique set of eigenvalues over this set, the set is called a *complete set of commuting observables* or CSCO. Taking a qubit as an example, we can measure the Pauli Z operator,

$$\mathsf{Z} = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is a somewhat trivial example, because there is only one operator in the set, but we can nonetheless associate each eigenstate, $|0\rangle$ and $|1\rangle$ with its unique eigenvalue, $+1$ and -1, respectively. For multiple qubits, we can take tensor products of the Z with the identity operator,

$$\mathcal{I} = \sigma_i = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For example, take a six qubit system and the state $|b_1 b_2 b_3 b_4 b_5 b_6\rangle$, where b_i is a binary digit. We will consider the expectation values of the following operators:

$$\begin{aligned} \mathsf{Z}_1 &= \mathsf{ZIIIII} \\ \mathsf{Z}_2 &= \mathsf{IZIIII} \\ \mathsf{Z}_3 &= \mathsf{IIZIII} \\ &\vdots \end{aligned}$$

All of these operators commute, which can be proven by direct computation:

$$\begin{aligned} [\mathsf{Z}_1, \mathsf{Z}_2] &= \mathsf{Z}_1\mathsf{Z}_2 - \mathsf{Z}_2\mathsf{Z}_1 \\ &= (\mathsf{ZIIIII})(\mathsf{IZIIII}) - (\mathsf{IZIIII})(\mathsf{ZIIIII}) \\ &= (\mathsf{ZI} \otimes \mathsf{IZ} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I}) - (\mathsf{IZ} \otimes \mathsf{ZI} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I}) \\ &= (\mathsf{Z} \otimes \mathsf{Z} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I}) - (\mathsf{Z} \otimes \mathsf{Z} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I} \otimes \mathsf{I}) \\ &= 0 \end{aligned}$$

And their expectation values are

$$\begin{aligned} \langle \mathsf{Z}_1 \rangle &= \langle b_1 b_2 b_3 b_4 b_5 b_6 | \, \mathsf{ZIIIII} \, | b_1 b_2 b_3 b_4 b_5 b_6 \rangle \\ &= \langle b_1 | \, \mathsf{Z} \, | b_1 \rangle \, \langle b_2 | b_2 \rangle \, \langle b_3 | b_3 \rangle \, \langle b_4 | b_4 \rangle \, \langle b_5 | b_5 \rangle \, \langle b_6 | b_6 \rangle \\ &= (-1)^{b_1} \times 1 \times 1 \times 1 \times 1 \times 1 \\ &= (-1)^{b_1} \end{aligned}$$

Following the trend, we see that any eigenstate, $|b_1 b_2 b_3 b_4 b_5 b_6\rangle$ corresponds to a unique set of eiegnvalues, $\{(-1)^{b_i}\}_{i \in [1..6]}$ with respect to the CSCO listed above.

V. FUNCTIONS OF OPERATORS AND THE SCHRODINGER EQUATION

The Schrodinger equation is

$$i\hbar \frac{\partial}{\partial i} |\psi(t)\rangle = H(t) \, |\psi t\rangle$$

This looks very familiar to the ODE,

$$\frac{d}{dt} y(t) = a(t) y(t)$$

for which the solution is

$$y(t) = exp \int_0^t a(s) ds \, y(0).$$

Similarly, the formal solution to the Schrodinger equation is

$$|\psi(t)\rangle = \underset{\leftarrow}{\mathcal{T}} \exp\left(-i \int_0^t H(s)ds/\hbar\right) |\psi(0)\rangle$$

The time ordering operator $\underset{\leftarrow}{\mathcal{T}}$ is included to handle the fact the $H(t)$ might not commute with $H(t')$. But for now we will assume that $H(t) = H$ is constant, so the time ordering complication goes away and we are left with:

$$|\psi(t)\rangle = \exp\left(-iHt/\hbar\right) |\psi(0)\rangle$$

But now we must interpret this exponential of an operator,

$$\exp\left(-iHt/\hbar\right).$$

In general, functions of operators can be interpreted in terms of the spectral representation of the operators. Recall that an operator A can be written in terms of its eigenvalues and eigenvectors as

$$A = \sum_n a_n |n\rangle\langle n|$$

A function f of this operator is then interpreted as

$$f(A) = \sum_n f(a_n) |n\rangle\langle n|$$

So the solution to the Schrodinger equation may be written as

$$|\psi(t)\rangle = \sum_n \exp\left(-iE_n t/\hbar\right) |n\rangle\langle n| \, |\psi(0)\rangle$$

Where the eigenvalues of the Hamiltonian operator, H are given the special symbol E_n and are interpreted as energies.

Spin

1 Introduction

For the past few weeks you have been learning about the mathematical and algorithmic background of quantum computation. Over the course of the next couple of lectures, we'll discuss the physics of making measurements and performing qubit operations in an actual system. In particular we consider nuclear magnetic resonance (NMR). Before we get there, though, we'll discuss a very famous experiment by Stern and Gerlach.

2 Spins as quantized magnetic moments

The quantum two level system is, in many ways, the simplest quantum system that displays interesting behavior (this is a very subjective statement!). Before diving into the physics of two-level systems, a little on the history of the canonical example: electron spin.

In 1921, Stern proposed an experiment to distinguish between Larmor's classical theory of the atom and Sommerfeld's quantum theory. Each theory predicted that the atom should have a magnetic moment (i.e., it should act like a small bar magnet). However, Larmor predicted that this magnetic moment could be oriented along any direction in space, while Sommerfeld (with help from Bohr) predicted that the orientation could only be in one of two directions (in this case, aligned or anti-aligned with a magnetic field). Stern's idea was to use the fact that magnetic moments experience a linear force when placed in a magnetic field gradient. To see this, not that the potential energy of a magnetic dipole in a magnetic field is given by:

$$U = -\vec{\mu} \cdot \vec{B}$$

Here, $\vec{\mu}$ is the vector indicating the magnitude and direction of the magnetic moment. The direction of the moment is analogous to the orientation of a bar magnet. This expression for the potential energy can also be used to derive a force that acts on the dipole (dipole is just another name for something that posesses a magnetic moment). Recall that the force is defined as the negative gradient of the potential:

$$F = -\nabla U = \nabla \left(\vec{\mu} \cdot \vec{B} \right)$$

Let's suppose that the magnetic field looks like $\vec{B} = B_0 z \hat{z}$; this field doesn't satisfy Maxwell's equations, but it makes the analysis easier. We get a force,

$$F = \nabla \left(\vec{\mu} \cdot B \right) = \nabla \left(\mu_z B z \right) = \mu B \cos(\theta) \, \hat{z}$$

The $\cos\theta$ term comes from the dot product. If, for example, the dipole is initially aligned with the field, it will experience an 'upward' force, and if it is antialigned, it will experience a 'downward' force.

Now consider a beam of dipoles passing through this field gradient. Larmor's classical theory predicts that the dipole moment could point in *any* direction, so the beam would spread out homogenously. The Bohr-Sommerfeld theory, though, predicts that the dipole moment can take only two values, aligned or anti-aligned with the field, so the beam would be *split* into two beams.

In 1922 Gerlach performed this experiment using silver atoms. (It turns out that electrons are a bad choice for this experiment because they are also affected by the Lorentz force, which is proportional to their

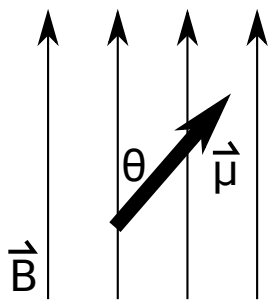


Figure 1: A magnetic moment in a magnetic field.

velocity. Any spread in the initial velocity causes a spread in the output that overwhelms the spin-gradient force.) Gerlach saw his beam split into two distinct beams, thereby demonstrating the spatial quantization of the magnetic moment and falsifying the Lorentz theory. In an interesting twist, the Sommerfeld theory was *also* incorrect, even though it predicted the correct result of this experiment. In 1925/1926, Uhlenbeck and Goudspit postulated that the electron carried its own spin magnetic moment independent of its orbital angular momentum.

In any case, the Stern-Gerlach experiment provides a toy model that we can use to learn about quantum two level systems. Let's make the language a but more precise by labeling some things:

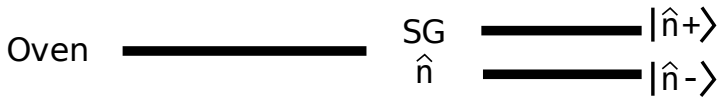


Figure 2: Schematic of Stern-Gerlach device.

In this diagram we have a cartoon picture of the Stern-Gerlach device. An oven produces a beam of particles which enters a reion with an inhomogenrous magnetic field, that gradient of which points in the $\hat{n}$ direction. The two beams that emerge we label $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$. These symbols are what we use to label a quantum state, and what is written inside gives some information about this particular state. However, we should stress that what we write inside the ket is simply a label that we give to help us remember how the state behaves. We could have just as easily called the two states $|\text{Bob}\rangle$ and $|\text{Alice}\rangle$. Here, though, $|\hat{n}+\rangle$ denotes the state that is directed upwards when the field gradient is along the vector $\hat{n}$, and $|\hat{n}-\rangle$ is the state directed downwards.

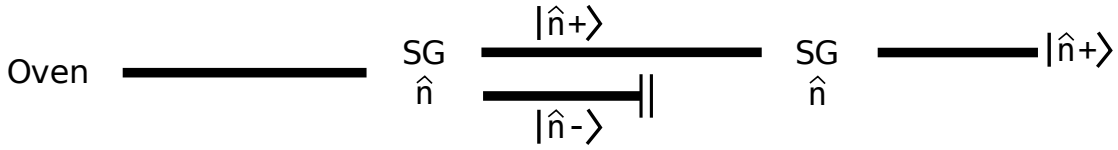


Figure 3: Cascaded Stern-Gerlach devices.

This all becomes much more interesting if we consider multiple, cascaded Stern-Gerlach devices. Let's

add another identical SG device, which we'll denote as SG($\hat{n}$), after the first, but we'll discard the $|\hat{n}-\rangle$ state.

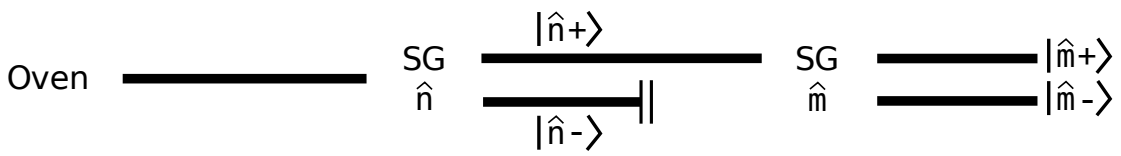


Figure 4: Cascaded Stern-Gerlach devices.

Notice that if we measure the output of the first SG($\hat{n}$) with a second SG($\hat{n}$) then we only get one beam out, the $|\hat{n}+\rangle$ state again. This shouldn't be surprising, as we have established with the first SG device that the dipole moment points along $\hat{n}$. But what would happen if we rotated the device?

Now we get two beams! The probability that $|\hat{n}+\rangle \to |\hat{m}\pm\rangle$ is found experimentally to be

$$P(|\hat{n}+\rangle \to |\hat{m}+\rangle) = \frac{1}{2}\left(1 + \hat{n}\cdot\hat{m}\right),$$

and also

$$P(|\hat{n}+\rangle \to |\hat{m}-\rangle) = \frac{1}{2}\left(1 - \hat{n}\cdot\hat{m}\right).$$

These probabilities can also be consdidered as the relatice intesitites of the two outgoing beams, $|\hat{m}\pm\rangle$, given an incoming beam $|\hat{n}+\rangle$.

Our task now is to seek a quantum mechanical description of this experiment. One way to do thisis to seach for the simplest description we can come up with, adding complexity only as we need to. Because the result of any measurement we can do is either "aligned" or "antialigned," the simplest model is the two-state system. We now consider the states $|\hat{n}\pm\rangle$ to be quantum states which are orthogonal to one another. Because we have two orthogonal states in a two-level space, these states form a basis. The states $|\hat{m}\pm\rangle$ also form a different basis. Let's pick a special basis, $|\hat{z}\pm\rangle$ and express all other states as linear combinations of these vectors. To indicate that this basis is special, let's relabel the vectors:

$$|\hat{z}+\rangle \to |0\rangle$$

$$|\hat{z}-\rangle \to |1\rangle$$

We can now represent any other state as a linear combination of the $\{|0\rangle, |1\rangle\}$ states:

$$|\hat{n}+\rangle = \alpha\,|0\rangle + \beta\,|1\rangle,$$

where α and β are complex numbers. Now that we know, roughly, how to express our states, we ned to figure out what the α and β are. We can start by considering the following double Stern-Gerlach experiment.

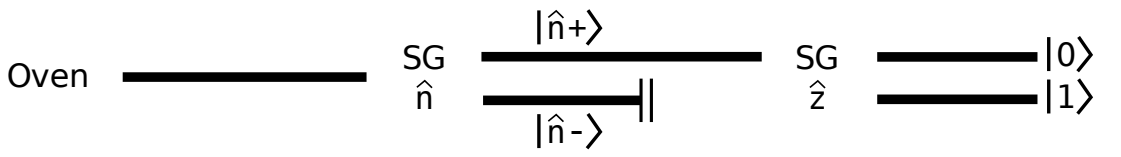


Figure 5: Cascaded Stern-Gerlach devices.

So, given the initial state $|\hat{n}+\rangle$, the probability of measuring $|0\rangle$ is

$$P_0(\hat{n}+) = |\langle 0|\hat{n}+\rangle|^2 = |\alpha \langle 0|0\rangle + \beta \langle 0|1\rangle|^2 = |\alpha|^2 = \frac{1}{2}(1 + \hat{n}\cdot\hat{z})$$

We have used that fact that $\langle 0|1\rangle = 0$ and $\langle 0|0\rangle = 1$. To make this a bit nicer, we are going to rewrite $\hat{n}\cdot\hat{z} = \cos\theta$, where θ is the angle between the two unit vectors $\hat{n}$ and $\hat{z}$. This angle is also equal to the spherical coordinate, θ, that corresponds to $\hat{n}$:

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Applying the trig identity, $(1+\cos\theta)/2 = \cos^2(\theta/2)$, we can say that:

$$|\alpha| = \cos(\theta/2)$$

A similar analysis shows that $|\beta| = \sin(\theta/2)$. So this simple argument has given us the magnitudes of α and β, but what about their phases? Because they are complex, they can be written as:

$$\alpha = |\alpha|\, e^{i\psi} \qquad\qquad \beta = |\beta|\, e^{i\chi}$$

giving,

$$|\hat{n}+\rangle = |\alpha|\, e^{i\psi}|0\rangle + |\beta|\, e^{i\chi}|1\rangle = \cos(\theta/2)e^{i\psi}|0\rangle + \sin(\theta/2)e^{i\chi}|1\rangle$$

However, a quantum state is only defined up to an overall phase, so we can multiply this state by $e^{-i\psi}$ to get

$$|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i(\chi-\psi)}|1\rangle \equiv \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle\,,$$

where $\phi = \chi - \psi$ is the relevant phase. So what is ϕ for a given $|\hat{n}+\rangle$? Let's look at $|\hat{x}+\rangle$ and $|\hat{y}+\rangle$:

$$|\hat{x}+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{i\phi_x}|1\rangle$$

$$|\hat{y}+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{i\phi_y}|1\rangle$$

If a double SG device is set up, with $\hat{x}$ first, then $\hat{y}$, the probability of seeing $|\hat{y}+\rangle$ given that the first beam emerged in $|\hat{x}+\rangle$, is $P(x\to y) = (1+\hat{x}\cdot\hat{y})/2 = 1/2$. But this is also the overlap between the two states:

$$\begin{aligned} P(\hat{x}\to\hat{y}) &= |\langle\hat{x}+|\hat{y}+\rangle|^2 \\ &= \left|\frac{1}{2}\langle 0|0\rangle + \frac{1}{2}e^{i(\phi_y-\phi_x)}\langle 1|1\rangle\right|^2 \\ &= \frac{1}{2} + \frac{1}{2}\cos(\phi_y - \phi_x) \\ &= \frac{1}{2} \end{aligned}$$

So, $\cos(\phi_y - \phi_x) = 0 \to \phi_y - \phi_x = \pi/2$. This implies that we can associate the phase angle, ϕ_n, with the second spherical coordinate, ϕ, in:

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

So, for any vector $\hat{n}$, $|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$, where (θ,ϕ) are the polar coordinates of the vector $\hat{n}$. But what about $|\hat{n}-\rangle$? The states $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$ must be orthogonal. This gives a unique solution, up to the standard quantum mechanical phase,

$$|\hat{n}-\rangle = \sin(\theta/2)|0\rangle + \cos(\theta/2)e^{-i\phi}|1\rangle$$

But this is the same as

$$|(-\hat{n})+\rangle = \cos((\pi-\theta)/2)\,|0\rangle + \sin((\pi-\theta)/2)e^{-i\phi}\,|1\rangle = |\hat{n}-\rangle$$

So, $|\hat{n}+\rangle$ is orthogonal to $|(-\hat{n})+\rangle$! This representation we have been using (θ, ϕ as parameters for a qubit state) is known as the Bloch Sphere representation. Every point on the Bloch Sphere (a unit sphere in $\mathbb{R}^3$) corresponds to a unique state, with the orthogonal state being represented by the antipodal point on the sphere.

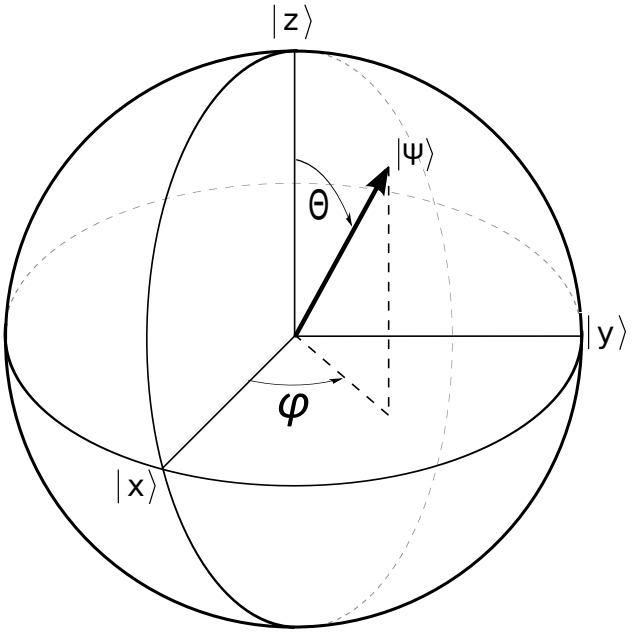


Figure 6: Bloch sphere representation of a qubit state.

We'll see a lot more of the Bloch Sphere representation over the course of the semester, so you'll have time to get accustomed to it.

Now that we've come up with a good way to represent the states in the Stern-Gerlach experiment, we need to come up with some way to mathematically describe the Stern-Gerlach devices (what we've been calling $SG(\hat{n})$. Before we can do this, however, we need to understand a little bit about how we measure quantum states.

Measurement is the assignment of a particular value to some attribute of the system under study. One of the postulates of quantum mechanics is that for every possible measurement you can do, there exists a Hermitian operator. Recall that a Hermitian operator, A, satisfies $A = A^\dagger$. The outcomes that are possible for the measurement are the eigenvalues of the operator. It might help to see an example:

Let's say we have some state: $|\hat{n}+\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\phi}\,|1\rangle$, and we want to measure its magnetic moment inthe $\hat{z}$ direction. We know from the discussion above that we should get one of two values, "aligned" or "antialigned." Because we're building a mathematical theory, let's call "aligned" $= 1$ and "antialigned" $= -1$. So now we have out eigenvalues. The eigenstates that correspond to these eigenvalues are the states $|\hat{z}+\rangle = |0\rangle$ and $|\hat{z}+\rangle = |1\rangle$. In order to represent out operators as matrices, we need to represent our states as vectors,

$$|\hat{z}+\rangle = |0\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\hat{z}-\rangle = |1\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this notation,

$$|\hat{n}+\rangle \longrightarrow \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \qquad |\hat{n}-\rangle \longrightarrow \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{i\phi} \end{pmatrix}$$

So, our task now is to find a matrix $S(\hat{z})$, that has eigenvalues ± 1 and eigenvectors $(1,0)^\mathsf{T}$ and $(0,1)^\mathsf{T}$. This is easy:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we need to find $S(\hat{n})$, the matrix corresponding to a measurement along $\hat{n}$. The eigenvalues are still (± 1), but the eigenvalues are still ± 1, but the eigenvectors have changed. We have already written down the eigenvectors, $|\hat{n}+\rangle$ and $|\hat{n}-\rangle$, so we can use the eigenvalue decomposition of a matrix,

$$S = P\Lambda P^{-1}$$

where Λ is the diagonal matrix of eigenvalues and P is the matrix of eigenvectors. Let's use this to explicitly construct $S(\hat{x})$. The eigenvectors are

$$|\hat{n}+\rangle \longrightarrow \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad |\hat{n}+\rangle \longrightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

So,

$$P_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which gives,

$$S(\hat{x}) = P_x \Lambda_x P_x^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This case was a little special, because $P_x = P_x^{-1}$. Doing the same for $S(\hat{y})$,

$$P_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

These three matrices we've just derived are very special. So special, in fact, that we are going to give them special names:

$$S(\hat{x}) \rightarrow \sigma_x \equiv X$$
$$S(\hat{y}) \rightarrow \sigma_y \equiv Y$$
$$S(\hat{z}) \rightarrow \sigma_z \equiv Z$$

Where $\sigma_{(x,y,z)}$ is the notation preferred by physicists and (X, Y, Z) is the notation preferred by computer scientists. These are the Pauli matrices, and you'll see them a lot this semester. A nice feature of them is that any matrix $S(\hat{n})$ can be written as

$$S(\hat{n}) = \hat{n} \cdot \vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$$

This is a nice exercise, and I recommend you try to show that you try to show that it's true.

Spin Resonance (ESR, NMR, μSR, etc.)

1 Electron Spin Resonance

Nearly every important concept in quantum computing can be illustrated with nuclear magnetic resonance (NMR). The first quantum factoring algorithm with implemented with NMR quantum computing, and it's the perfect platform to discuss decoherence and quantum control. Electron spin resonance (ESR), the topic we'll discuss, is almost exactly the same thing in principle.

Recall from the previous section that the Hamiltonian for a magnetic moment in a magnetic field is:

$$H = -\vec{\mu} \cdot \vec{B}$$

in that section we were concerned with the action of this Hamiltonian on the spatial part of the electron wavefunction. However, in this section we are going to neglect the spatial part and consider only the time evolution of the electron's spin. Hand wavingly, the spin angular momentum of an electron gives rise to a magnetic moment. We can make a (surprisingly) reasonable estimate of this magnetic moment by considering the electron to be a classical particle whirling around in a circle. The movement of the electron, a charged particle, is equivalent to a current loop. The magnetic moment of a current loop is given by the current

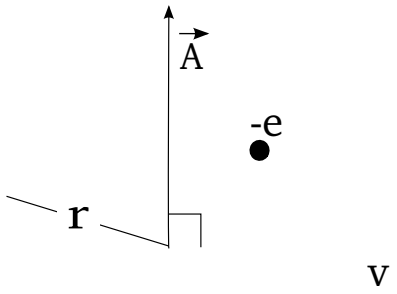


Figure 1: The magnetic moment associated with a whirling electron.

times the area of the loop. Applying this to the whirling electron, we have

$$\vec{\mu} = I\vec{A} = \frac{-e}{2\pi r/v}\pi r^2 \hat{A} = \frac{-e\, mvr\hat{A}}{2m} = \frac{-e\vec{L}}{2m}$$

We will replace the angular momentum vector, $\vec{L}$, with the spin angular momentum operator, $\vec{S} = \hbar\vec{\sigma}/2$. This gives a magnetic moment, $\vec{\mu} = \frac{1}{2}\left(\frac{-e\hbar}{2m}\right)\vec{\sigma} \equiv -\mu_B\vec{\sigma}/2$. This expression also defines the Bohr magneton, μ_B. Our classical derivation is surprisingly close to the actual answer. We're just off by a relativistic factor (if you're interested, look up the Dirac and Pauli equations), which we'll call g, and is roughly equal to 2,

$$\vec{\mu} = -\frac{1}{2}g\mu_B\vec{\sigma}$$

Now let's rewrite the Hamiltonian for a spin in a magnetic field using this expression we've just derived for the magnetic moment operator,

$$H = -\vec{\mu} \cdot \vec{B} = \frac{1}{2} g \mu_B \vec{\sigma} \cdot B \equiv \frac{1}{2} \hbar \gamma \vec{\sigma} \cdot \vec{B}$$

This expression defines $\gamma = g\mu_B/\hbar$, the gyromagnatic ratio.

So let's consider the time evolution of a state $|\psi(t)\rangle$ with the initial condition $|\psi(t)\rangle = |\hat{x}+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ when placed in a constant magnetic field, $\vec{B} = B_0 \hat{z}$. The time evolution is given by solving the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

Recall that for a constant Hamiltonian, the evolution can be solved for exactly:

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

Using the Hamiltonian and initial state above,

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\gamma B_0 \sigma_z/2} \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) \\ &= \frac{1}{\sqrt{2}} \left(e^{-i\gamma B_0 t/2} |0\rangle + e^{i\gamma B_0 t/2} |1\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle + e^{i\gamma B_0 t} |1\rangle \right) \end{aligned}$$

In the last equality we noted that we exploited our freedom to multiply the wavefunction by an arbitrary phase, in this case, $\exp(i\gamma Bt)$. So what does this look like on the Bloch sphere? Recall that an arbitrary qubit state can be written as $|\hat{n}+\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\exp(i\phi)|1\rangle$. The phase, ϕ, is evolving in time as $\phi(t) = \gamma Bt$. So the Bloch vector is processing around the applied magnetic field with a frequency $\omega_L = \gamma B_0$, known as the Larmor frequency.

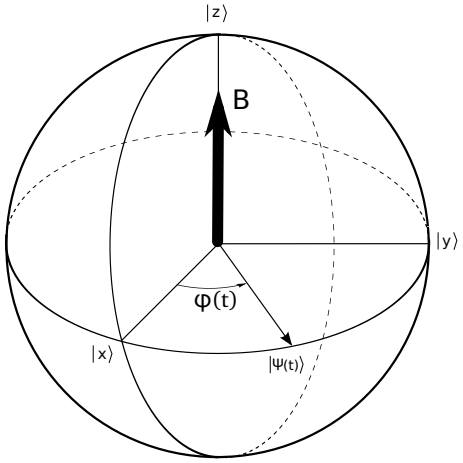


Figure 2: Bloch sphere representation of spin procession in constant field.

It's pretty easy to see that *any* state will precess about the field with the same frequency. Now we're going to consider a more complicated problem: in addition to the static applied field, $B_0\hat{z}$, we will apply an *oscillating* field, $B_1 \cos(\omega t)\hat{x}$. The Hamiltonian becomes:

$$H(t) = \frac{1}{2} \hbar \gamma \left(B_0 \sigma_z + B_1 \cos(\omega t) \sigma_x \right)$$

Note that, in general, $[H(t), H(t')] \neq 0$, so we cannot write the time evolution operator in the normal way, $U(t) = \exp\left(-i\int_0^t H(t')\,dt'\right)$. Instead, we would have to use something called the *time ordering operator*, $\underset{\leftarrow}{\mathcal{T}}$, but even that doesn't help us to put the evolution operator in closed form.

Instead we will take a different approach. What does the evolution of the state look like in a reference frame that is rotating at the Larmor frequency? To do this, we define a new state variable in this rotating frame, the *unwound* state,

$$|\psi'(t)\rangle = e^{i\omega_L\sigma_z t/2}\,|\psi(t)\rangle$$

Now does this state evolve in time?

$$\begin{aligned}
i\hbar\frac{d}{dt}|\psi'(t)\rangle &= (i\hbar)i\omega_L\sigma_z/2\,|\psi'(t)\rangle + e^{i\omega_L\sigma_z t/2}i\hbar\frac{d}{dt}|\psi(t)\rangle \\
&= -\frac{\hbar\omega_L\sigma_z}{2}|\psi'(t)\rangle + e^{i\omega_L\sigma_z t/2}H(t)\,|\psi(t)\rangle \\
&= -\frac{\hbar\omega_L\sigma_z}{2}|\psi'(t)\rangle + e^{i\omega_L\sigma_z t/2}H(t)e^{-i\omega_L\sigma_z t/2}e^{i\omega_L\sigma_z t/2}\,|\psi(t)\rangle \\
&= \left(-\frac{\hbar\omega_L\sigma_z}{2} + e^{i\omega_L t/2}H(t)e^{-i\omega_L\sigma_z t/2}\right)|\psi'(t)\rangle \\
&= H'(t)\,|\psi'(t)\rangle
\end{aligned}$$

Which defines the Hamiltonian in the rotating frame:

$$H'(t) = \left(-\frac{\hbar\omega_L}{2}\sigma_z + e^{i\omega_L\sigma_z t/2}H(t)e^{-i\omega_L\sigma_z t/2}\right)$$

We'll focus on this last term:

$$\begin{aligned}
e^{i\omega_L\sigma_z t/2}H(t)e^{-i\omega_L\sigma_z t/2} &= \frac{\hbar\gamma}{2}e^{i\omega_L\sigma_z t/2}\left(B_0\sigma_z + B_1\cos(\omega t)\,\sigma_x\right)e^{-i\omega_L\sigma_z t/2} \\
&= \frac{\hbar\gamma}{2}\left(B_0\sigma_z + e^{i\omega_L\sigma_z t/2}B_1\cos(\omega t)\sigma_x e^{-i\omega_L\sigma_z t/2}\right)
\end{aligned}$$

Let's rewrite the field as two counter rotating circular waves:

$$\begin{aligned}
\cos(\omega t)\sigma_x &= \frac{1}{2}\left((\cos(\omega t)\sigma_x + \sin(\omega t)\sigma_y) + (\cos(\omega t)\sigma_x - \sin(\omega t)\sigma_y)\right) \\
&= \left(\left(e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-\right) + \left(e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-\right)\right)
\end{aligned}$$

Here we have used the raising and lowering Pauli operators, $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. Now we need to calculate the transformation, $\exp(i\alpha\sigma_z)\sigma_\pm\exp(-i\alpha\sigma_z)$:

$$e^{i\alpha\sigma_z}\sigma_+e^{-i\alpha\sigma_z} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 0 & e^{2i\alpha} \\ 0 & 0 \end{pmatrix} = e^{2i\alpha}\sigma_+$$

$$e^{i\alpha\sigma_z}\sigma_+e^{-i\alpha\sigma_z} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e^{-2i\alpha} & 0 \end{pmatrix} = e^{-2i\alpha}\sigma_-$$

So,

$$\begin{aligned}
H'(t) &= -\frac{\hbar\omega_L}{2}\sigma_z + \frac{\hbar\gamma}{2}\left(B_0\sigma_z + B_1e^{i\omega_L\sigma_z t/2}\cos(\omega t)\,\sigma_x e^{-i\omega_L\sigma_z t/2}\right) \\
&= -\frac{\hbar\omega_L}{2}\sigma_z + \frac{\hbar\omega_L}{2}\sigma_z + \frac{\hbar\gamma}{2}\left(B_1e^{i\omega_L\sigma_z t/2}\left(\left(e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-\right) + \left(e^{i\omega t}\sigma_+ + e^{-i\omega t}\sigma_-\right)\right)e^{-i\omega_L\sigma_z t/2}\right) \\
&= \frac{\hbar\gamma}{2}B_1\left(\left(e^{i(\omega_L-\omega)t}\sigma_+ + e^{i(\omega-\omega_L)t}\sigma_-\right) + \left(e^{i(\omega+\omega_L)t}\sigma_+ + e^{-i(\omega+\omega_L)t}\sigma_-\right)\right)
\end{aligned}$$

If the frequency of the applied field is oscillating at the Larmor frequency, ω_L, then this simplifies:

$$H'(t) = \frac{\hbar\gamma}{2} B_1 \left((\sigma_+ + \sigma_-) + \left(e^{2i\omega_L t}\sigma_+ + e^{-2i\omega_L t}\sigma_- \right) \right)$$

However, the second term oscillates so fast that its effect on the qubit dynamics are negligible, so we drop it. This is known as the *rotating wave approximation*, or RWA. Setting the applied field frequency to the Larmor frequency is known as *resonance*. Our Hamiltonian is then,

$$H'(t) = \frac{\hbar\gamma}{2} B_1 \left(\sigma_+ + \sigma_- \right) = \frac{\hbar\gamma}{2} B_1 \sigma_x$$

This Hamiltonian causes an x rotation with frequency $\omega_R = \gamma B_1/2$, the Rabi frequency. In the lab frame, a π rotation looks like:

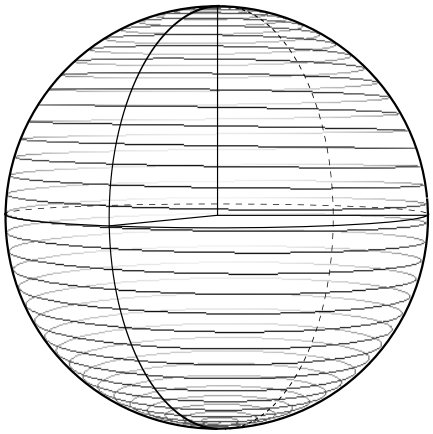


Figure 3: Lab frame Bloch sphere representation of Rabi oscillation.

Nuclear magnetic resonance is a beautiful subject with a rich history. We will refer back to this type of experiment often.

C191 - Lecture 5 - Two-qubit gates, circuit model, teleportation

I. MAKING TWO QUBIT GATES

Suppose we have the Hamiltonian on two qubits,

$$H_1 = J\sigma_z \otimes \sigma_z = J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And single qubit Hamiltonian,

$$H_0 = \frac{1}{2}\gamma B(\sigma_z \otimes I + I \otimes \sigma_z) = \frac{1}{2}\gamma B \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right)$$

We use these Hamiltonians to produce a CPHASE gate:

$$\text{CPHASE} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

By choosing B such that $\frac{1}{2}\gamma B = -J$, we have a total Hamiltonian,

$$H = J \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Now, we can always add an operator proportional to the identity to the Hamiltonian, which simply moves the zero on the energy scale. The effect on the unitary operator generated by the Hamiltonian is to multiply it by a complex phase - check this! So adding $JI \otimes I$ gives us

$$H' = J \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

A system evolving under this Hamiltonian for a time t experiences a unitary operator,

$$U = \exp(-iH't/\hbar) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-4iJt/\hbar} \end{pmatrix}$$

Choosing t such that $4Jt/\hbar = \pi$, we generate our CPHASE gate.

II. QUANTUM CIRCUITS

Important quantum circuits and their matrix representations

A. Single qubit operators

Pauli X	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
Pauli Y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	
Pauli Z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	
Hadamard	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	
S	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	
T	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	

B. Two qubit operators

CNOT $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ 

CPHASE $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ 

SWAP $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 

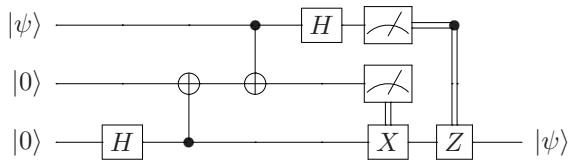
C. Three qubit operators

Toffoli (Controlled-CNOT) If first qubit is $|1\rangle$, perform a CNOT on the second and third qubits 

Fredkin (Controlled-SWAP) If first qubit is $|1\rangle$, swap the second and third qubits 

III. TELEPORTATION

Let's work out the details of the following quantum circuit:



The initial state is

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle$$

After the first Hadamard, the state becomes:

$$|\psi\rangle \otimes |0\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

The CNOT puts the last qubits into a maximally entangled state:

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

Assuming the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, the state at this point is,

$$(\alpha |0\rangle + \beta |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

We'll expand this out and drop the tensor product signs:

$$\frac{1}{\sqrt{2}} (\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle)$$

After the second CNOT, the state becomes:

$$\frac{1}{\sqrt{2}} (\alpha |000\rangle + \alpha |011\rangle + \beta |110\rangle + \beta |101\rangle)$$

And after the Hadamard, the state becomes,

$$\frac{1}{2} (\alpha |000\rangle + \alpha |100\rangle + \alpha |011\rangle + \alpha |111\rangle + \beta |010\rangle - \beta |110\rangle + \beta |001\rangle - \beta |101\rangle)$$

The measurement results, the resulting state of the third qubit, the correction operator, and the resulting state are summarized in the following table: So this circuit uses two bits of classical information and one shared Bell pair to

Qubit 1	Qubit 2	State of qubit 3	Correction step	Final state
0	0	$\alpha \|0\rangle + \beta \|1\rangle$		$\alpha \|0\rangle + \beta \|1\rangle$
0	1	$\beta \|0\rangle + \alpha \|1\rangle$	X	$\alpha \|0\rangle + \beta \|1\rangle$
1	0	$\alpha \|0\rangle - \beta \|1\rangle$	Z	$\alpha \|0\rangle + \beta \|1\rangle$
1	1	$-\beta \|0\rangle + \alpha \|1\rangle$	ZX	$\alpha \|0\rangle + \beta \|1\rangle$

move a state from qubit 1 to qubit 3. Neat!