H7C HW6: Tipler Ch2: 21, 41, 42, 47, 50, 51

Note that $c=1$, though students may or may not do this.

21) $m_\pi \simeq 139.6$ MeV $\qquad m_p = 938$ MeV

Let's look at the threshold condition in the CM frame, where all final particles may have zero (three-) momenta. Conservation of four-momentum says

$$\begin{pmatrix} E \\ p \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E \\ -p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m_\pi \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2E = 2m_p + m_\pi \qquad \Rightarrow E = \frac{2m_p + m_\pi}{2}$$

Now we just need to transform to the lab frame, where the protons have four-momentum

$$\begin{pmatrix} E' \\ p' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E' + m_p \\ p' \\ 0 \\ 0 \end{pmatrix}$$

The $(\text{invariant mass})^2$ is

$$\underset{\uparrow \atop \text{CM frame}}{(2E)^2} = \underbrace{(E' + m_p)^2 - (p')^2}_{\text{lab frame}}$$

$$\Rightarrow 4E^2 = (E')^2 + 2E'm_p + m_p^2 - \left[(E')^2 - m_p^2\right]$$

$$= 2E'm_p + 2m_p^2$$

$$\Rightarrow E' = \frac{2E^2 - m_p^2}{m_p} = \frac{2(m_p + m_\pi/2)^2 - m_p^2}{m_p}$$

$$= \boxed{\frac{m_p^2 + 2m_p m_\pi + \frac{m_\pi^2}{2}}{m_p}}$$

41) a) The diameter is the same, $d = 10\,\mu m$.

The length is contracted in the lab frame: $L = L_R/\gamma$

$$\Rightarrow L_R = \gamma L = \frac{E}{m_e} L = \frac{50\text{ GeV}}{511\text{ keV}}(1\text{ cm}) = 9.8 \times 10^4\text{ cm} \approx 10\text{ m}$$

b) Presumably T&L mean "in the bundle's own reference frame". Well, the bundle has $L_R \approx 10\text{m}$ and "observes" the accelerator to have length $L'_{ac} = L_{ac}/\gamma$

$$\Rightarrow L_R = \frac{L_{ac}}{\gamma} \Rightarrow L_{ac} = \gamma L_R = \gamma^2 L = \left(\frac{50\text{ GeV}}{511\text{ keV}}\right)^2 (1\text{cm})$$

$$\approx 10^{10}\text{ cm} = 10^4\text{ Km}$$

c) For two successive Lorentz transformations with the same velocity, the net effect is a Lorentz transformation with

$$\gamma_{net} = \gamma^2 + \gamma^2\beta^2 = \gamma^2\left(1 + 1 - \tfrac{1}{\gamma^2}\right) = 2\gamma^2 - 1$$

so the desired length is $\frac{1}{\gamma_{net}}(10\text{m})$

$$= (2\gamma^2 - 1)^{-1}(10\text{m}) = \left(2\left(\frac{50\text{GeV}}{511\text{keV}}\right)^2 - 1\right)^{-1}(10\text{m})$$

$$= \boxed{5.2 \times 10^{-10}\text{ m}}$$

Notice this is not the same as "contracting the contracted length". To see what γ_{net} is, I composed Lorentz transformations by multiplying their matrix representations:

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2 + \gamma^2\beta^2 & -2\gamma^2\beta \\ -2\gamma^2\beta & \gamma^2 + \gamma^2\beta^2 \end{pmatrix}$$

but you could do this in other ways, too.

d) Unlike the "length of an object", energy and momentum transform "simply" (technically, as a four-vector) and we can just calculate

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} E \\ -p \end{pmatrix} = \begin{pmatrix} \gamma(E + \beta p) \\ \gamma(-\beta E - p) \end{pmatrix}$$

Since $E = 50\text{ GeV} >> 511\text{ keV} = m_e$, $E \approx p$ and so

$$E' \approx p' \approx \gamma(1+\beta)E$$

$$\approx 2\gamma E \qquad (\gamma >> 1)$$

$$= 2\left(\frac{50\text{ GeV}}{511\text{ keV}}\right)(50\text{ GeV}) = 9.8\times10^{6}\text{ GeV}$$

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42) $E_k = E - m = m \Rightarrow E = 2m$

a) $E^2 = p^2 + m^2$

$$\Rightarrow p = \sqrt{E^2 - m^2} = \sqrt{(2m)^2 - m^2}$$

$$= \sqrt{3}\, m$$

$$\approx 1.6 \text{ GeV}$$

b) $v = \frac{p}{E} = \frac{\sqrt{3}\, m}{2m} = \frac{\sqrt{3}}{2} = .87$ (that is, $.87c = 2.6\times10^{8}\,\text{m/s}$)

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47) For any mass, there is a frame in which an isolated photon has less energy than that mass. To an observer in such a frame, conservation of energy makes it obvious that the photon cannot produce a pair.
(Students: The question didn't ask, but you should understand clearly why this argument fails in the presence of other particles.)

50) a) In the CM (or, more precisely, zero-momentum) frame, the protons have four momenta

$$\begin{pmatrix} E \\ p \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} E \\ -p \\ 0 \\ 0 \end{pmatrix}$$

We want $2E_K \equiv 2(E - m_p) = 2m_p \Rightarrow E = 2m_p$

$$\Rightarrow \gamma = 2 = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \frac{1}{4} = 1 - \beta^2$$

$$\Rightarrow \beta = \sqrt{3/4} = \frac{\sqrt{3}}{2} = .87$$

$$\Rightarrow u = \beta c = 2.6 \times 10^8 \text{ m/s}$$

b) In the lab frame, the protons have four-momenta

$$\begin{pmatrix} E' \\ p' \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The (invariant mass)^2 of the combined system is

$$(2E)^2 = (E' + m_p)^2 - (p')^2 = (E')^2 + 2m_pE' + m_p^2 - (E')^2 + m_p^2$$

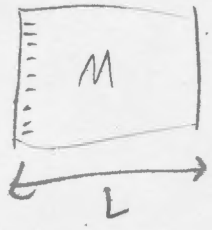
$$= 2m_pE' + 2m_p^2$$

$$\Rightarrow E' = \frac{2E^2 - m_p^2}{m_p} = \frac{8m_p^2 - m_p^2}{m_p} = 7m_p$$

$$\Rightarrow \gamma = 7 \Rightarrow \beta = \sqrt{1 - \tfrac{1}{49}} = .99 \qquad \Rightarrow u = \beta c \simeq 3 \times 10^8 \text{ m/s}$$

c) $(E')_K = E' - m_p = 6m_p$ (that is, I showed it in part b)

51)



a) The light carries $p_\ell = E/c$ right, so the box picks up

$$p_b = -p_\ell = -\frac{E}{c}$$

$$\Rightarrow v = \frac{-E}{Mc}$$ (the minus meaning "to the left")

b) $$|\Delta x| = |v\Delta t| = \frac{E}{Mc}\frac{L}{c} = \frac{EL}{Mc^2}$$

c) The center of mass moves by

$$\frac{M\Delta x - m_\ell L}{M + m_\ell} = \frac{\frac{EL}{c^2} - m_\ell L}{M + m_\ell}$$

$$\approx \frac{L}{M}\left(\frac{E}{c^2} - m_\ell\right)$$ assuming $M >> m_\ell$

$$\Rightarrow m_\ell = E/c^2$$ to keep the CM stationary.

Note: I find this problem somewhat misleading. Light carries energy and momentum but photons themselves are massless (or have an absurdly tiny mass that we haven't been able to measure).