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Homework 2

Problem 1. $n=0$: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$n=1$: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^1=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

$n=2$: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^2=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}$

$n=3$: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^3=\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&3\\0&1\end{pmatrix}$

$n=4$: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^4=\begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&4\\0&1\end{pmatrix}$

Guess: $\begin{pmatrix}1&1\\0&1\end{pmatrix}^n=\begin{pmatrix}1&n\\0&1\end{pmatrix}$

Proof by Induction:

We have shown that the base case $n=0$ is true.

- Assume the guess formula is true for $n=k$:

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}^k=\begin{pmatrix}1&k\\0&1\end{pmatrix}$$

- Show true for $n=k+1$:

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}^{k+1}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^k\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&k\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}1(1)+k(0) & 1(1)+k(1)\\0(1)+1(0) & 0(1)+1(1)\end{pmatrix}=\boxed{\begin{pmatrix}1&k+1\\0&1\end{pmatrix}}$$

- Hence by the induction hypothesis we have shown that the formula is true for $n=k+1$, so by the induction axiom we have proved that the formula is true for all $n\in\mathbb{N}$.

problem 2. $|A| = n \Rightarrow 2^{n^2}$ Binary Relations

$\underline{n=0}$ BR = set of binary relations

$\{\}$ $|BR| = 2^{0^2} = 2^0 = 1$

$\underline{n=1}$ $A = \{a\}$

$\{\}$, $\{(a,a)\}$ $|BR| = 2^{n^2} = 2^{1^2} = 2^1 = 2$

$\underline{n=2}$ $A = \{a, b\}$

$\{\}$, $\{(a,a), (a,b), (b,a), (b,b)\}$

$\{(a,a)\}$ $\{(a,b)\}$ $\{(b,a)\}$ $\{(b,b)\}$

$\{(a,a), (a,b), (b,a)\}$ $\{(a,a), (a,b), (b,b)\}$ $\{(a,a), (b,a), (b,b)\}$

$\{(a,b), (b,a), (b,b)\}$

$\{(a,a), (a,b)\}$ $\{(a,a), (b,a)\}$ $\{(a,a), (b,b)\}$

$\{(a,b), (b,b)\}$ $\{(b,a), (b,b)\}$

$\{(a,b), (b,a)\}$

$|BR| = 2^{n^2} = 2^{2^2} = 2^4 = 16$

- First we need to show how many subsets are in a set of n elements, by induction. (subsets $= 2^n$)

<u>Base Case</u>: $n = 1$. Then $A = \{a\}$ for $a \in A$.
It has subsets $\{\}$ and $\{a\}$, for a total $2^{1^2} = 2$ subsets.

<u>Inductive hypothesis</u>: Assume that for $n=k$ elements, there are 2^k subsets.

<u>Inductive step</u>: Now consider a set A with $k+1$ elements. $|A| = k+1$. We can easily get all subsets of A from the set with k elements by inserting the $(k+1)$th element into all subsets of the set of k elements, and collecting these with the original subsets of the k-element set. In effect, we've doubled the number of subsets going from k to $k+1$:

$$\begin{aligned} \text{subsets}(k+1) &= 2 \cdot \text{subsets}(k) \\ &= 2 \cdot 2^k \quad \text{[by inductive hypothesis]} \\ &= 2^1 \cdot 2^k \\ &= \boxed{2^{k+1}} \end{aligned}$$

Thus by induction we have proved that a set $|A| = n$ with n elements has 2^n subsets.

For any set with n elements, we have n possibilities for the first element in an ordered pair, and n possibilities for the second element:

([n possibilities], [n possibilies])

giving us $n \times n = n^2$ total possible ordered pairs. We can generate all the binary relations from knowing that a set A of n elements has 2^n subsets and n^2 ordered pair.

We can think of any binary relation as a subset of all possible ordered pairs. Since there are n^2 ordered pairs (that is, there is a set B such that $|B| = n^2$), there must be 2^{n^2} subsets of the set of all ordered pairs.

Hence a set with n elements has 2^{n^2} binary relations, which we have proven by induction (more specifically, we inducted on the subsets of this set, and showed it leads to 2^{n^2} through logic)

Problem 3

Part 1) Show $a \sim b$ over $\mathbb{Z}$ is an equivalence if and only if $a = b$:

reflexivity:

$a \sim a$ implies $a = a$. We can switch the a's on either side of the equality and the relation $a \sim a$ still holds since a is always equal to itself, hence reflexive.

symmetry:

$a \sim b$ implies $a = b$. This means a "equals" b, but we can also say $b = a$, which means the same thing, and can be represented by the relation $b \sim a$.

So $(a \sim b) \Rightarrow (a = b) \equiv (b = a) \Rightarrow (b \sim a)$, and the relation is symmetric.

transitivity:

$a \sim b$ implies $a = b$. $b \sim c$ implies $b = c$. Since equality is transitive, we can write $a = c$, which by definition implies the relation $a \sim c$. So the relation is transitive.

- Since $\sim$ is a reflexive, symmetric, and transitive relation when $\sim$ is $=$ (equality), it is an equivalence.

Part 2) $a \sim b \Longleftrightarrow 3 \mid (a-b)$ is an equivalence relation

reflexivity:

$a \sim a$ implies $(a-a)$ is divisible by 3, of course, $a - a = 0$, which is divisible by three even when you switch a with itself. Hence $a - a = a - a \Rightarrow a \sim a$ shows $\sim$ is reflexive.

symmetry:

$a \sim b \Rightarrow 3 \mid (a-b)$. $b \sim a \Rightarrow 3 \mid (b-a)$. If $a-b$ is divisible by 3, then $(b-a) = 3k$ for some $k \in \mathbb{Z}$.

- then $a - b = -3k$ which is still divisible by 3, so $a \sim b \Rightarrow 3 \mid (a-b) \Rightarrow b \sim a$. Hence $\sim$ is symmetric.

Transitivity: $a \sim b \Rightarrow 3 \mid (a-b)$. $b \sim c \Rightarrow 3 \mid (b-c)$.

so $(a-b) = 3k$ for $k \in \mathbb{Z}$ and $(b-c) = 3\ell$, $\ell \in \mathbb{Z}$.

Then $(a-b) + (b-c) = (a-c) = 3k + 3\ell = 3(k+\ell)$ so $(a-c)$ is divisible by 3, so $a \sim c$ is true, thus $\sim$ is transitive.

Part 3) $a \sim b \iff 3 \mid (a+b)$

Reflexivity:

$a \sim a \Rightarrow 3 \mid (a+a)$. This means $a+a = 3k$ for some $k \in \mathbb{Z}$.

This is always divisible by 3, so $a \sim a$ is a reflexive relation.

Symmetry:

$a \sim b \Rightarrow 3 \mid (a+b)$. So $a+b = 3k$ for $k \in \mathbb{Z}$.

Since addition is commutative, $a+b = b+a = 3k$, so $b+a$ is still divisible by 3, implying $b \sim a$ ($\sim$ is symmetric).

Transitivity:

$a \sim b \Rightarrow 3 \mid (a+b)$, so $(a+b) = 3k$ for $k \in \mathbb{Z}$.

$b \sim c \Rightarrow 3 \mid (b+c)$, so $(b+c) = 3l$ for $l \in \mathbb{Z}$.

$$a = 3k - b \qquad c = 3l - b$$

$$(a+c) = 3k - b + 3l - b = 3(k+l) - 2b$$

$(a+c)$ is divisible by 3 only when b itself is a multiple of 3, so $a \sim b \wedge b \sim c \Rightarrow a \sim c$ is not always true, and $\sim$ is <u>not transitive</u>.

· Since $\sim$ fails transitivity, it is not an equivalence relation.

Part 4) Show if $a \sim b \Rightarrow [a] = [b]$.

Assume $a \sim b$.

We know $[a] = \{c \in A \mid a \sim c\}$.

$[b] = \{c' \in A \mid b \sim c'\}$

any element c in $[a]$ also satisfies $c \sim a$ since $\sim$ is an equivalence and has symmetric properties.

Then $(c \sim a \wedge a \sim b) \Rightarrow c \sim b$, by the transitive property.

Using symmetry again, we know $b \sim c$. So we have shown any $c \in [a]$ satisfies $a \sim c$ and $b \sim c$.

We also know that $c' \in [b]$ if and only if $b \sim c'$.

Since we know $b \sim c$, we can conclude that every c in $[a]$ is also an element of $[b]$.

$$\{c \in A \mid a \sim c\} \subseteq \{c' \in A \mid b \sim c'\}$$

Now $b \sim c' \Rightarrow c' \sim b$, and $b \sim a$ (by symmetry), so transitively $c' \sim a$, and symmetrically $a \sim c'$.

So any $c' \in [b]$ satisfies $a \sim c'$;

Problem 3

Part

4 (continued)

giving us $\{c' \in A \mid b \sim c'\} \subseteq \{c \in A \mid a \sim c\}$.

We have shown that all $\{c\}$ is a subset of all $\{c'\}$, and that all $\{c'\}$ is a subset of all $\{c\}$. Hence we have shown that all elements in $[a]$ are in $[b]$ and all elements in $[b]$ are in $[a]$. Equivalently,

$$\{c \mid a \sim c\} = \{c' \mid b \sim c'\}$$

$$\Rightarrow \boxed{[a] = [b]}$$

So the two equivalence classes are equal sets given $a \sim b$.

Part 5) Prove: If $a \nsim b$, then $[a]$ and $[b]$ do not have a common element.

Proof by Contradiction:

We are trying to prove that

$$a \nsim b \Rightarrow \neg(\exists x \mid x \in [a] \land x \in [b])$$

For contradiction, assume $a \nsim b$ means that there is in fact an element in both equivalence classes; that is:

$$\text{assume } a \nsim b \Rightarrow \exists x \mid x \in [a] \land x \in [b]$$

call this common element z. By definition

$$z \in [a] \land z \in [b] \Rightarrow z \in \{c \mid a \sim c\}$$
$$z \in \{c' \mid b \sim c'\}$$

So $a \sim z$ and $b \sim z$. By symmetric property, any element c' in $[b]$ satisfies $c' \sim b$ as well, so:

$$z \sim b$$

$$\Rightarrow a \sim z \land z \sim b \Rightarrow a \sim b \text{ by transitive property.}$$

But $a \sim b$ contradicts our assumption $a \nsim b$, So our original proposition is in fact true by contradiction. Thus

$$a \nsim b \Rightarrow \neg(\exists x \mid x \in [a] \land x \in [b])$$

is true and there are no common elements in the equivalence classes of a & b when there is no relation between a & b.

Problem 4

Part 1)

```
Binary Search(element x, array A, start a, end b):
    m = (a+b)/2          // round down to integer
    if A[m] equals x:    // checks if mth element is x
        return m
    else if A[m] is less than x:
        if (m+1) is greater than b:   // x not in array, empty range
            return 0
        return BinarySearch(x, A, m+1, b)   // recursive search m+1 to b
    else:     // A[m] is less than x
        if m-1 is less than a:  // x not in array
            return 0
        return BinarySearch(x, A, a, m-1)  // search a to m-1
```

Part 2) Proof by Strong Induction: Binary Search will return 0 when x is not in an Array, and return the correct index if it is.

<u>Base Case</u>: Consider an Array search with a search range of 1 element; that is, the start and end search indices a & b are the same. Then the pseudocode is:

```
BinarySearch(x, A, a, a):
    m = (a+a)/2 = a
    if A[a] is equal to x,
        return a
    else if A[a] is less than x:
        m+1 = a+1 > a:
            return 0
    else: // A[a] is greater than x:
        m-1 = a-1 < a
            return 0
```

If the 1 element sub array has x, the correct index a is returned. In both else cases, incrementing one of the start or end indices makes the search range 0, so it properly returns 0 (x not found in range).

Inductive Hypothesis:

Assume that the binary search works correctly for all ranges $n = b - a \leq k$. If x is in range of length k, the correct index is returned, otherwise 0.

Inductive Step: Consider the array search range length. $k+1 > k = b - a$, where $b > a$

- Again, if x is actually the element in the middle of the $(k+1)$ range search, then the correct index is returned.
- If the element in the middle of the search $A[m]$ is less than x, we see that the algorithm tries to increment the starting index. Since $m > a$, except in the case where $m = a$ (hence $m \geq a$), $m+1 > a$. Then we know the new search range $b-(m+1) < b-a$, so $b-(m+1) < k$ which is included in our inductive hypothesis and will thus correctly perform a binary search.
- If the middle element $A[m]$ is greater than x, the algorithm will try to decrement the ending index. Since $m < b$, $m - a < b - a$ implies that the new search range $(m-1) - a < k$, and $m-1$ is smaller than the search range k and included in the inductive hypothesis, we know using a binary search on a range length $(m-1)a$ will properly return
- Since a range $k+1$ length reduces down to a range length smaller than k, and all ranges less than k work correctly, by inductive axiom we have proved that the Binary search will work for all search ranges $n = b - a$ for any $a, b \in \mathbb{N}$

Problem 5) Prove $\sum_{i=0}^{n} i r^i = \frac{n \cdot r^{n+1}}{r-1} - \frac{r(r^n-1)}{(r-1)^2}$

<u>Base Case:</u> $n=0$: $\sum_{i=0}^{0} i r^i = 0 \cdot r^0 = 0 = \frac{0 \cdot r^{0+1}}{r-1} - \frac{r(\cancel{r^0-1})^{\to 0}}{(r-1)^2} = 0$ ✓

<u>Inductive Hypothesis:</u>

Assume that the formula holds for $n=k$, that is

$$S_k = \sum_{i=0}^{k} i \cdot r^i = \frac{k \cdot r^{k+1}}{r-1} - \frac{r(r^k-1)}{(r-1)^2}$$

<u>Inductive Step:</u>

$$S_{k+1} = S_k + (k+1) r^{k+1}$$

$$= \frac{k \cdot r^{k+1}}{r-1} - \frac{r(r^k-1)}{(r-1)^2} + (k+1) r^{k+1} \frac{(r-1)}{(r-1)}$$

$$= \frac{k r^{k+1} + (r-1)(k+1)(r^{k+1})}{(r-1)} - \frac{r(r^k-1)}{(r-1)^2}$$

$$= \frac{r^{k+1}(\cancel{k} + kr \cancel{-k} + r - 1)}{(r-1)} - \frac{r(r^k-1)}{(r-1)^2}$$

$$= \frac{r^{k+1}(rk+r) - r^{k+1}}{r-1} - \frac{r(r^k-1)}{(r-1)^2}$$

$$= \frac{r \cdot r^{k+1}(k+1)}{(r-1)} - \frac{r^{n+1}}{r-1} - \frac{r(r^k-1)}{(r-1)^2}$$

$$= \frac{(k+1) r^{k+2}}{(r-1)} - \frac{(r^{n+1})(r-1)}{(r-1)^2} - \frac{r(r^k-1)}{(r-1)^2}$$

$$= \frac{(k+1) r^{k+2}}{(r-1)} - \left[\frac{r(r^{k+1}) - \cancel{r^{k+1}} + \cancel{r^{k+1}} - r}{(r-1)^2}\right]$$

$$\boxed{S_{k+1} = \frac{(k+1) r^{k+2}}{(r-1)} - \frac{r(r^{k+1}-1)}{(r-1)^2}}$$

Since we were able to go from the inductive hypothesis to showing the formula works for $k+1$, we have proved by induction that for all $n \in N$

$$S_n = \sum_{i=0}^{n} i r^i = \frac{n r^{n+1}}{r-1} - \frac{r(r^k-1)}{(r-1)^2}$$ [except for $r=1$]

Problem 6

Proof by cases and strong induction

Case 1: Consider the case when the warriors start off with $x \geq 1000$ warriors on the field. Then the dragon will be scared and leave the field. Even if x is divisible by 3, we assume the warriors automatically win because the dragons leave as soon as the warriors have greater than or equal to 1000.

Case 2: Consider the case when the number of warriors, x, is between

$$0 \leq x \leq 999.$$

- The base cases are when $x=0$ and $x=1$. When $x=0$, after 1 hour, the warriors will send 1 soldier, who will run away, thus ending the battle. When $x=1$, this first soldier will instantly run away from the field, ending the battle.
- Now assume that for in all cases $x \leq k$, the battle ends with one side fleeing. That is for $0 \leq x \leq k$, the battle ends with a runaway.
- Then for $x = k+1$, we know that this is either a multiple of 3 or it is not. If $k+1$ is a multiple of 3, then

$$k+1 = 3L \text{ for some } L \in \mathbb{Z}$$

Since $k+1$ is divisible by 3, the dragons will eat $2/3$ of them and only leave $1/3$, so there will only be

$$1/3(k+1) = \tfrac{1}{3} 3L = L \text{ warriors left.}$$

since $k = 3L-1 > L$ for all $L > 1$, it follows that L must be less than k, so by the inductive hypothesis we know that these L warriors will end in a battle where someone flees the field.
- If $k+1$ is not a multiple of 3, then nothing happens for 1 hour until another warrior is added.

(continued on back)

Warriors will be added hourly until their numbers are divisible by 3:

$$x = k + n = 3m \quad \text{for} \quad m, n \in \mathbb{Z}$$

One third will be spared, leaving

$$\tfrac{1}{3}(k+n) = \tfrac{1}{3}3m = m$$

take the smallest value of k for which this is true;

$k = 2$, $n = 1$, $m = 1$

We see $m < k$. Since as k increase the division will dominate the addition of n (k and $k+n$ are at most 2 numbers apart), it follows that $\frac{1}{3}(k+n) = m < k$ is always true, so the warriors will always be reduced down to a case included by our strong inductive hypothesis. Thus for all x in $[1, 999]$ the battle will end with a runaway by induction, and by cases we have shown that all battles will end.

Since the base cases $x = 1$ ends in loss for the warriors, and all $x < 1000$ will reduce down to $x = 1$, it follows that the warriors can only win if they start with at least 1000 warriors.

Problem 7)

n	1	2	3	4	5	6	7	8	9	n
F_n	0	1	1	2	3	5	8	13	21	$F_{n-1}+F_{n-2}$

Prove that $\forall n \in \mathbb{N}^+$, n can be written as a sum of distinct Fibonacci numbers such that no two have consecutive indices.

Proof by Strong Induction:

Base Cases: $n=1$, $n=2$, $n=3$

We know that 1, 2, 3 themselves are fibonacci numbers, so they can be written as the trivial sums F_3, F_4, F_5, respectively.

Inductive Hypothesis:

Assume that for all $n \leq k$, n can be written as a sum of nonconsecutive Fibonacci numbers:

$$\forall n \leq k, \quad n = F_x + F_y + \cdots \qquad |x-y| \neq 1$$

Inductive Step: Need to show $k+1$ is also a sum of fib numbers.

Consider the largest Fibonacci number less than or equal to $k+1$:

$$F_a \leq k+1$$

We know the difference between F_a and $(k+1)$ should be less than k:

$$(k+1) - F_a \leq k$$

$$-k \qquad\qquad -k$$

$$\Rightarrow 1 - F_a \leq 0 \Rightarrow F_a \geq 1$$

So we know there is a largest Fibonacci number bounded by

$$1 \leq F_a \leq k$$

If F_a is k, then we know $k+1 = F_a + 1$ is a sum of Fibonacci numbers since $1 = F_2 = F_3$ is a Fibonacci number and if F_a consecutively follows F_2 or F_3 we can contract this sum into 1 single fibonacci number since by definition a fib number is the sum of the Fib numbers before it.

If F_a is 1, then we know that $k+1$ must be 1, which is just a base case.

When F_a is in the range $1 < F_a < k$, then we already know upon defining F_a that its difference with $(k+1)$, namely $(k+1) - F_a = X \leq k$ for some $x \in Z$. Since $x \leq k$, it is included in our inductive hypothesis and we know it can be written as a sum of Fibonacci numbers.

$$x = F_x + F_y + F_z + \cdots$$

then $(k+1) = F_a + x = F_a + F_y + F_z + \cdots$

Which is definitely a sum of Fibonacci numbers. If any of the indices of the sum are consecutive, we can always contract pairs in the sum into a single Fibonacci number, until we have distinct, non-consecutive Fibonacci numbers. Hence we have shown that from $n \leq k$ we can get $n = k+1$ as a sum of Fib numbers, and by induction axiom we have proved that for any n, n is a sum of non-consecutive Fibonacci numbers.

Problem 8 - Write Your own problem

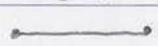
Problem: Consider n points in a plane. How many line segments can be draw between any 2 points? Equivalently, given a polygon with n points, how many edges plus diagonals can you draw? Now consider n points on a sphere. How many line segments along a great circle can you draw between these n points?

Solution: Start with examples in the euclidean (2D or 3D) space: n points, ℓ line segments

$n=0$, $\ell=0$ (no points, no lines)

$n=1$, $\ell=0$ (line requires 2 points)

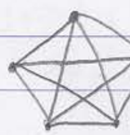
$n=2$, $\ell=1$: $e=1$ $d=0$



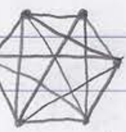
$n=3$, $\ell=3$ $e=3$ $d=0$

$n=4$, $\ell=6$ $e=4$ $d=2$

$n=5$, $\ell=10$: $e=5$ $d=5$



$n=6$, $\ell=15$: $e=6$ $d=9$



I started with counting diagonals & edges, because it was most intuitive. However, there seemed to be no connection between the number of points and how many diagonals there are (edges $=n$). It doesn't seem like there is an easy recursive definition either, so I looked at just n and its relation to n. It looks like twice the number of lines segments is equal to $n(n-1)$, which makes an easy formula:

$$\ell = \frac{n(n-1)}{2}$$

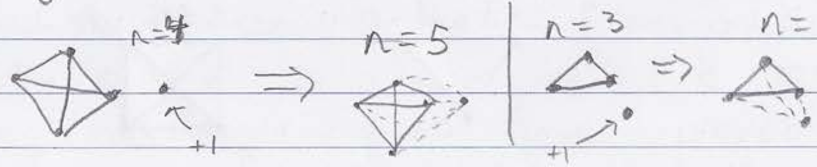
Note that the polygons draw could have been drawn in more concave or convex shapes, but the results would be the same.

Proof by Induction

Base cases: $n=0 \Rightarrow \ell = 0(0-1) = 0$] Base cases work
$n=1 \Rightarrow \ell = 1(1-1) = 0$

Induction Hypothesis:
Assume true that given $n=k$ points, you can draw $\ell = \frac{k(k-1)}{2}$ line segments

Inductive Step: start with k points, which we know has $\frac{1}{2}k(k-1)$ lines. To figure out how many line $k+1$ points makes, just add 1 point to the original k points with $\frac{1}{2}k(k-1)$. Since a new line segment needs 2 points, we see that adding 1 point gives us k more lines:



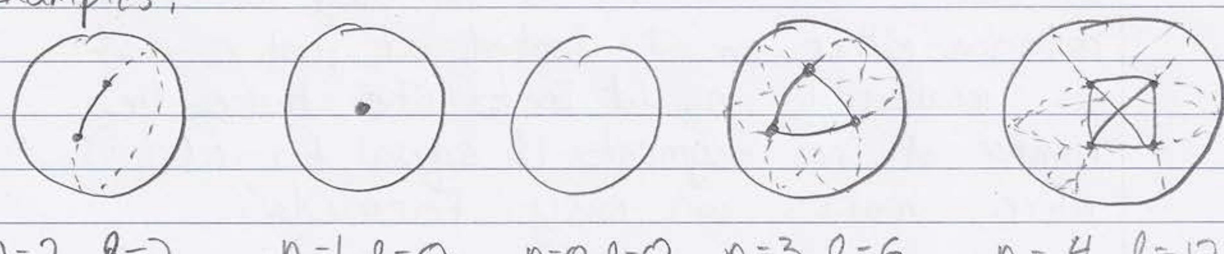
$$\ell_{k+1} = \ell_k + k = \frac{k(k-1)}{2} + k = \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$$

$$\ell_{k+1} = \frac{(k+1)[(k+1)-1]}{2}$$

Hence by the induction axiom we have proved that the formula $\ell = \frac{1}{2}n(n-1)$ holds, for all $n \in \mathbb{N}$

Spherical Space

examples:



$n=2$ $\ell=2$ $n=1$ $\ell=0$ $n=0$ $\ell=0$ $n=3$ $\ell=6$ $n=4$ $\ell=12$

It looks like you just double the number of lines compared to the euclidean problem, or simply:

$$\boxed{\ell_n = \frac{n(n-1)}{2} \times 2 = n(n-1)}$$