Problem 3 $\phi(n) = |\{i : 1 \le i \le n,\ gcd(n,i) = 1\}|$

For m, n such that $gcd(m,n) = 1$, $\phi(mn) = \phi(m)\cdot\phi(n)$

1. Let p be a prime number.

$\phi(p) = |\{i : 1 \le i \le p,\ gcd(p,i) = 1\}|$

Since all integers less than p are coprime with p, and there are $p-1$ numbers less than p and greater than or equal to 1,

$\boxed{\phi(p) = p-1}$

3. Let p be a prime number and $a \in \mathbb{Z}^+$, $a < p$.
What is $a^{\phi(p)} \bmod p$? From part 1, $\phi(p) = p-1$

$a^{\phi(p)} \bmod p = a^{p-1} \bmod p = \boxed{1 \bmod p}$

2. Let p be prime and k be a positive integer.
What is $\phi(p^k)$?
This is just all the numbers up to p^k minus all the multiples of p. For p^k, there are p^{k-1} multiples of p less than p^k.
So $\boxed{\phi(p^k) = p^k - p^{k-1}}$

4. b with prime factors $p_1, p_2, p_3, \cdots, p_k$. $b = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$gcd(a,b) = 1$ given, show $a^{\phi(b)} \equiv 1 \bmod p_i$.

Since a is coprime with b, it is also coprime with all factors of b, and all $p_i^{\alpha_i}$ are coprime with $p_j^{\alpha_j}$ for $j \ne i$. Using the theorem mentioned earlier:

$$a^{\phi(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})} = a^{\phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2})\cdots\phi(p_k^{\alpha_k})}$$
$$= a^{(p_1^{\alpha_1} - p_1^{\alpha_1 - 1})(p_2^{\alpha_2} - p_2^{\alpha_2 - 1})\cdots(p_k^{\alpha_k} - p_k^{\alpha_k - 1})}$$
$$= a^{(p_1^{\alpha_1 - 1})(p_1 - 1)\cdots(p_k^{\alpha_k - 1})(p_k - 1)}$$
$$= a$$

For any i, this can be rewritten $a^{k(p_i - 1)}$ for $k \in \mathbb{Z}$

$$a^{k(p_i-1)} = (a^{p_i - 1})^k \equiv 1^k \bmod p_i \equiv \boxed{1 \bmod p_i} \text{ using FLT}$$