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CS70 - zh

Homework 3

Problem 1

$$F(n) = \begin{cases} n/4 & \text{if } n \% 4 = 0 \\ (n+2)/4 & \text{if } n \text{ is even but not divisible by } 4 \\ 3n-1 & \text{if } n \text{ is odd} \end{cases}$$

Prove: $\forall n \in \mathbb{N}$, if n ninjas start on the field, then the battle will end with the ninjas fleeing (By strong Induction

Base Cases:

$n=1$: the battle ends right away since the only ninja flees.

$n=2$: $2 \to 2+2=4 \to 1$

Since 2 is not a multiple of 4 but is even, two ninjas are added. 3 are eaten, leaving 1 ninja to flee

$n=3$: $3 \to 8 \to 2 \to 4 \to 1$

Ninjas lose because $n=3$ works down to 1 after 3 turns

$n=4$: $4 \to 1$ ninjas lose after 1 turn.

Inductive Hypothesis: Assume that for $n=k$, $P(1) \wedge \ldots \wedge P(k)$ is true, where $P(n)$ is the proposition that n starting ninjas will lose the battle. That is, for all $n<k$, n starting ninjas result in a ninja fleeing the field.

Inductive step:

Consider $n=k+1$. There are 3 cases:

- $k+1$ is divisible by 4: (case 1)

$$k+1 = 4l \text{ for some } l \in \mathbb{Z}$$

then the next turn there will be

$$F(k+1) = k+1/4 = 4l/4 = l < k$$

l ninjas left. Since $l<k$, this case is included in the inductive hypothesis so we know the ninjas will eventually reduce to a base case and flee

- $k+1$ is divisible by 2 but not 4 (case 2)

then $k+1 = 2a \neq 4b$ for some $a, b \in \mathbb{Z}$

Now we have to prove that any even number not divisible by 4 becomes divisible by 4 after adding 2. Any even number is divisible by 2 by definition. Only every other even number is divisible by 4. So if all our even numbers are $2a$ for $a \in 1, 2, 3, 4, \ldots$, then only $a \in 2, 4, 6, 8, \ldots$ are numbers divisible by 4. If we start on an even not divisible by 4, (ie $2a$ for $a = 1, 3, 5, \ldots$) and add 2, we are just obtaining the next higher even number, which must be divisible by 4 since every other even is divisible by 4 and we started with one that wasn't. Now we show this case is included in our inductive hypothesis:

$$\frac{((k+1)+2)}{4} = \frac{k+3}{4} = \frac{k}{4} + \frac{3}{4}$$

$\frac{k}{4} + \frac{3}{4} < k$ since clearly $\frac{k}{4} < k - 3/4$ for any valid k in this case $(k = 1, 5, 9, \ldots)$. So the case is included in the inductive hypothesis, and the ninjas eventually get to a base case with $k \leq 4$. Intuitively, case 2 just reduces to case 1 via a variable substitution. let $k+2 = k'$. So $\frac{k'+1}{4}$ is our new ninja count. Since k in our inductive hypothesis is arbitrary, we can say for all $n < k'$ $p(n)$ is true; Thus case 2 really does reduce to case 1, which we have proven ends with the ninjas losing. We can also argue that a number divided by 4 will end up as an even number (in which case we recursively consider case 1 and 2 on a number decreasing toward the base cases) or it is an odd number; in which case we go to case 3.

- Case 3: $k+1$ is odd. This implies $k+1 = 2z+1$ for some $z \in \mathbb{N}$. Now multiplying an odd number by an odd number is an odd number. Adding or subtracting 1 from an odd number gives an even number (even & odd are alternating). We can see this in the math:

$$k+1 = 2z+1 \quad \text{for } z \in \mathbb{N}$$

$$3(k+1) = 3(2z+1) = 6z+3 = 6z+2+1$$

$$= 2(3z+1) + 1 = 2a+1$$

let $a = 3z+1$

Hence $3(k+1)$ is an odd. Now subtract 1:

$3(k+1)-1 = 2a+1-1 = 2a$

So $3(k+1)-1$ is in the end just some even number. And we know all even numbers reduce down to cases 1 and 2, which always resolt in a ninja count smaller than before the dragons eat, so we know case 3 is also included in the inductive hypothesis.

Thus since all 3 cases have been proved by strong induction, we have proved that all battles end with the ninjas fleeing.

Problem 2 $a_1 = 2 \quad a_2 = 3 \quad a_k = a_{k-1} \cdot a_{k-2}$

$a_3 = 6 \quad a_4 = 18 \quad a_5 = 108$

$A_n = \sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{18} + \frac{1}{108} + \cdots$

Compare this to the geometric series

$S_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

we know a geometric series $\sum_{k=0}^{\infty} ar^k$ with $|r| < 1$ will converge. Clearly $1/2 < 1$ means S_n will converge, but we can also prove this:

$$\frac{S_n}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = S_n - 1$$

$$\Rightarrow \frac{S_n}{2} = 1 \quad \Rightarrow S_n = 2$$

So S_n converges to 2. Now compare A_n to S_n:

$$\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) + \left(\frac{1}{6}\right) + \left(\frac{1}{18}\right) + \cdots < (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) + \cdots$$

since $\frac{1}{2} < 1$ and $\frac{1}{3} < \frac{1}{2}$ and $\frac{1}{6} < \frac{1}{4}$, and so on

Thus $A_n < S_n$. Since the direct comparison test says that a series with terms a_n that is bounded by a converging series with terms b_n such that $b_n \geq a_n$ is itself a converging series, we know A_n also converges. Hence by the direct comparison test, we have proved that the series A_n converges.

Proof: $A_n < S_n$

Base case: $n = 1 \quad A_1 = \frac{1}{2} < S_1 = 1.5$

Inductive hypothesis: Assume for $n = k$ $A_k < S_k$

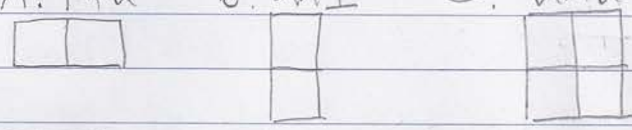
$$\sum_{k=1}^{n} \frac{1}{a_k} < \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$\sum_{k=1}^{n+1} \frac{1}{a_k} = \underbrace{\sum_{k=1}^{n} \frac{1}{a_k}} + \frac{1}{a_{n+1}} < \underbrace{\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k} + \frac{1}{2^{n+1}} = \sum_{k=1}^{n+1} \left(\frac{1}{2}\right)^k$$

$$\frac{1}{a_{n+1}} = \frac{1}{a_n a_{n-1}} < \frac{1}{2^n} < \frac{1}{2^{n+1}} \quad \text{since } a_n a_{n-1} > 2^n$$

so $\sum_{k=1}^{n+1} \frac{1}{a_k} < \sum_{k=1}^{n+1} \left(\frac{1}{2}\right)^k$ by induction, and A_n converges

Problem 3 A: 1×2 B: 2×1 C: 2×2

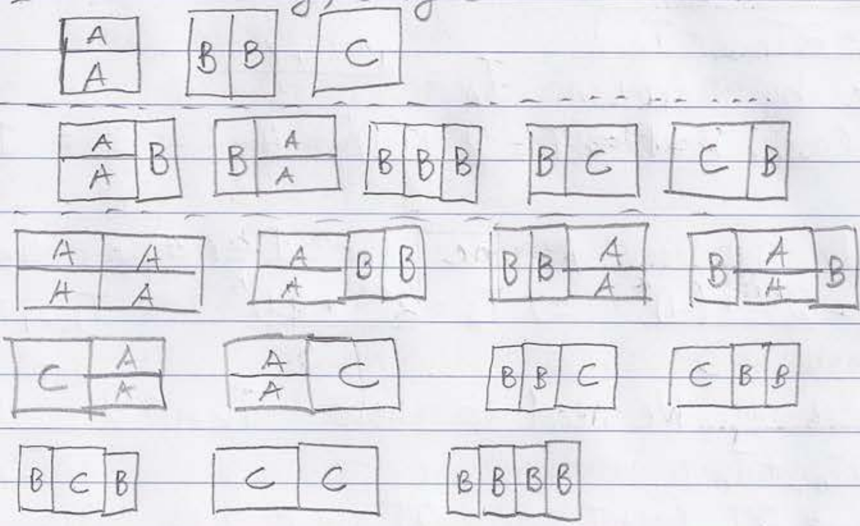


a) $\boxed{n=1}$ $T_1 = 1$. Clearly, only B Fits in a 2×1 board.

$\boxed{n=2}$ $T_2 = 3$

$\boxed{n=3}$ $T_3 = 5$

$\boxed{n=4}$ $T_4 = 11$



$\boxed{n=5}$ AAAAB AABAA BAAAA AABBB BAABB BBAAB BBBAA

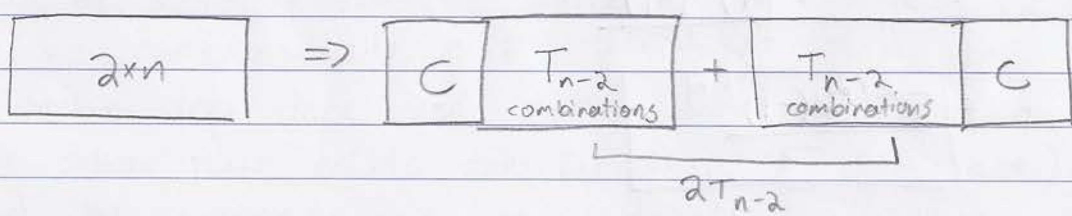
$T_5 = 21$ AABC BAAC BCAA CBAA CAAB AACB

CCB CBC BCC

BBBC CBBB BCBB BBCB

BBBBB

From the patterns above, we notice $\boxed{T_n = 2T_{n-2} + T_{n-1}}$.
This can be "derived" by noticing a recursive construction.
For any $2\times n$ grid, we know we can take a "c" block from either side, so we get $2\times T_{n-2}$ combinations



From the examples above, we can figure out how to get the other combinations in T_n without a c block from looking at the T_{n-1} combinations. In all T_{n-1} combinations, turn the "c" blocks into pairs of 2 A blocks. Then put a "b" block on the appropriate end of each T_{n-1} combo so that there aren't any repeats!

$$\boxed{2 \times n} \Longrightarrow \boxed{B \mid T_{n-1}}$$

So we have an additional T_{n-1} combinations. Hence

$$\boxed{T_n = 2T_{n-2} + T_{n-1}}$$

b) Prove by induction: $T_n = \dfrac{2^{n+1} + (-1)^n}{3}$

Base Case: $n = 1 \Rightarrow T_1 = \dfrac{2^2 + (-1)^1}{3} = \dfrac{4-1}{3} = \dfrac{3}{3} = 1$ ✓

Inductive Hypothesis: Assume for $1 \leq n \leq k$ that

$$T_n = \frac{2^{n+1} + (-1)^n}{3} \Rightarrow T_k = \frac{2^{k+1} + (-1)^k}{3} \Rightarrow T_{k-1} = \frac{2^k + (-1)^{k-1}}{3}$$

Inductive step: We need to show $T_{k+1} = \dfrac{2^{k+2} + (-1)^{k+1}}{3}$

$$T_n = 2T_{n-2} + T_{n-1}$$

$$\Rightarrow T_{k+1} = 2T_{(k+1)-2} + T_{(k+1)-1} = 2T_{k-1} + T_k$$

$$= 2\left(\frac{2^k + (-1)^{k-1}}{3}\right) + \left(\frac{2^{k+1} + (-1)^k}{3}\right)$$

$$= \frac{2 \cdot 2^k + 2(-1)^{k-1} + 2^{k+1} + (-1)^k}{3}$$

$$= \frac{2^{k+1} + 2^{k+1} + 2(-1)^{k-1} + (-1)^k}{3}$$

$$= \frac{2 \cdot 2^{k+1} + (-1)^k \left(2(-1)^{-1} + 1\right)}{3}$$

$$= \frac{2^{k+2} + (-1)^k(-1)}{3}$$

$$T_{k+1} = \boxed{\frac{2^{k+2} + (-1)^{k+1}}{3}}$$

Since we have shown the inductive step from the hypothesis, the induction axiom holds and for all $n \in \mathbb{Z}$, $T_n = \dfrac{2^{n+1} + (-1)^n}{3}$

Problem 4 a) Claim: $\forall n \in \mathbb{N}$, $n^2 \le n$

Proof: Base case $n=1 \Rightarrow 1^2 = 1 \le 1$ ✓

Inductive hypothesis: Assume $k^2 \le k$

Inductive step: need to show $(k+1)^2 \le k+1$

$$k^2 \le (k+1)^2 - 1 \le (k+1) - 1 = k$$

The proof is incorrect, because one of the statements in the inductive step is clearly wrong.

$$(k+1)^2 - 1 = k^2 + 2k + 1 - 1 = k^2 + 2k$$

so $k^2 \le k^2 + 2k$ is true but

$$k^2 + 2k \le k$$

is not true at all for any k since the statement implies

$$k^2 \le -k$$

which is not true for any natural number $k > 1$ clearly. So the proof fails to use induction since there was a false step in the induction step; hence the proof is incorrect

b) claim: $\forall n \in \mathbb{N}$, $7^n = 1$

Proof: Base case $7^0 = 1$. Assume $7^j = 1$ for $0 \le j \le k$. Then

$$7^{k+1} = \frac{7^k 7^k}{7^{k-1}} = \frac{1 \cdot 1}{1} = 1$$

The proof is incorrect, because the inductive step breaks down for certain values of k. Take $k=0$.

Then $7^{0+1} = 7^1 = \frac{7^0 \cdot 7^0}{7^{-1}}$ $\quad 7^{-1} = \frac{1}{7} \neq 1$

But we don't know the value of 7^{-1} since $j \ge 0$, so we can't conclude that $7^1 = \frac{1}{1} = 1$. Thus the inductive step does not work for all k (we have shown there exists one for which it does not), and we cannot use the induction axiom, thereby showing this proof is incorrect.

c) claim: $\forall x, y, n \in N$; if $\max(x,y) = n$ $x \leq y$

Proof: Base case $n=0$. $\text{Max}(x,y) = 0 \Rightarrow x=0, y=0 \Rightarrow x \leq y$.

Assume $\max(x,y) = n$ such that $x \leq y$.

$\max(x,y) = n+1 \Rightarrow \max(x-1, y-1) = n \Rightarrow x-1 \leq y-1 \Rightarrow x \leq y$.

This proof is <u>incorrect</u> because of a logical failure in the inductive step. Even though we proved the base case, consider the inductive step for $x=0$ and $y=0$. Then: $x=0$, $y=0$

$\max(0,0) = 0 = n+1 \Rightarrow n = -1$

$\max(x-1, y-1) = \text{Max}(-1,-1) = -1 \notin N$

The inductive step fails for $x, y = 0$ since the max of -1 and -1 is not a natural number! (natural numbers must be positive). So $\max(x-1, y-1)$ is not gaurenteed to be a natural number, and the proof breaks down.

Problem 5 a) Direct Proof:

- From the lecture notes on stable marriage, we can reference the lemma that a male-propose/women-reject algorithm produces a male optimal and female pesimal pairing. The opposite of this scenario is that a women-propose, men-reject algorithm results in a women optimal and male pesimal stable pairing.
- Since both scenarios produce the same stable pairing, we can transitively conclude that the male-optimal is also the female-optimal pairing (they are the same), and the pairing is also the male and female pesimal. By definition, there can be no other inbetween pairing that is optimal for one side but pesimal for the other. Hence there is really only 1 stable pairing, and all other pairings have rogue couples.

b)

Men	Women
1	A B C
2	C B A
3	A C B

Women	Men
A	1 2 3
B	1 2 3
C	1 2 3

Men Propose:

	A	B	C
day 1:	1 3		2
day 2:	1		2, 3
day 3:	1	3	2

Final pairing: $\{(A,1),(B,3),(C,2)\}$ in $\boxed{\text{3 days}}$

Problem 6 | a) Stable pairing $\{(A,1), (B,2), (C,3)\}$

1	x	$\boxed{A}$	x		A	$\boxed{2}$	$\boxed{1}$	$\boxed{3}$
2	x	x	x		B	x	x	$\boxed{3}$
3	$\boxed{C}$	x	x		C	x	x	$\boxed{3}$

Preferences:

1.

1	B A C	A	2 1 3
2	B A C	B	2 1 3
3	C A B	C	2 1 3

8 total preference lists

2.

1	B A C	A	2 1 3
2	B A C	B	2 1 3
3	C A B	C	1 2 3

3.

1	B A C	A	2 1 3
2	B A C	B	2 1 3
3	C B A	C	2 1 3

4.

1	B A C	A	2 1 3
2	B A C	B	2 1 3
3	C B A	C	1 2 3

5.

1	B A C	A	2 1 3
2	B C A	B	2 1 3
3	C B A	C	2 1 3

6.

1	B A C	A	2 1 3
2	B C A	B	2 1 3
3	C B A	C	1 2 3

7.

1	B A C	A	2 1 3
2	B C A	B	2 1 3
3	C A B	C	2 1 3

8.

1	B A C	A	2 1 3
2	B C A	B	2 1 3
3	C A B	C	1 2 3

b) Since man 3 is the least preffered by all women, we should be able to switch his first and second choice and still get a stable pairing. No matter who he starts off proposing to, man 3 will always be rejected until he proposes to a woman who has not been proposed to (the last one), which always ends the algorithm and leads to a stable pairing.

c) This depends on whether n is even or n is odd. If n is even, there can be at most $n/2$ rogue couples. If n is odd, there can be at most $(n-1)/2$ rogue couples.

d)

Problem 7 a) The students propose to the universities. If the university has an open spot, they give a maybe to the student. Universities should rank their students on a string in order, such that if a student proposes when the spots are full and the university likes that student at least as much as the lowest student on the preference list, then the lowest student's spot is taken and the proposing student is made room in the new preference list.

b) If S proposes to U on day k, then on every day after U has a list of students that have proposed and U likes each student at least as much as S. (the first such day)
Suppose for contradiction that on day $j > k$, someone inferior S^- proposes or nobody proposes. On day $j-1$, U has S' who U likes at least as much as S. Then S' must propose again to U on the next day, which is day J, so now U has at least S' on day j. But S' is better than S^- and no one, so by contradiction we have shown that the improvement lemma holds.

c)

Problem 8 Consider the sequence from problem 2:

$a_1 = 2 \quad a_2 = 3 \quad a_k = a_{k-1} \cdot a_{k-2}$

Prove that a_2 is the only odd number in the sequence. Try simple induction, and compare to strong induction.

- Suppose for the sake of contradiction that $a_2 = 3$ is not the only odd number, that is there is some $n > 2$ such that $a_n = 2z+1$ for some $z \in \mathbb{Z}$.
- Let's try to use simple induction to see what numbers come after a_2: Prove all numbers after a_2 are even.

base cases: $a_1 = 2$ [even] $a_2 = 3$ [only odd so far]

Inductive hypothesis: Assume for $n=k$ that a_k is even.

$$a_k = 2b \quad \text{for } b \in \mathbb{Z}$$

Inductive step: $a_{k+1} = a_k \cdot a_{k-1}$
$= (2b)(a_{k-1}) = ?$

The inductive step tells us nothing about the parity of a_{k-1} since we don't know any terms less than k index. But we do know we can make the variable substitution $b a_{k-1} = c$

$$a_{k+1} = 2c = \text{even}$$

So a_{k+1} is even by definition. We know $2b$ is even, so even if a_{k-1} is odd, a_{k+1} is even. Hence since all other terms after $a_2 = 3$ are even, there is no $n > 2$ such that $a_n = 2z+1$. This contradicts our original assumption, so 3 is the only odd number.

- Strong induction:

Base cases: $n=1 \quad a_1 = 2 \qquad n=2 \quad a_2 = 3$

Hypothesis: Assume for all $3 \leq n \leq k$, a_n is even

Inductive step: $a_{k+1} = a_k \cdot a_{k-1}$

since the inductive step includes k & $k-1$, we know a_k and a_{k-1} are both even.

So $a_k = 2a$ and $a_{k-1} = 2b$ for $a, b \in \mathbb{Z}$

Then $a_{k+1} = 2a \cdot 2b = 4(ab) = 2(2ab)$

By definition, a_{k+1} is even, since an even times an even is even and both a_k and a_{k-1} are even by the inductive hypothesis. So all a_n for $n > 2$ are even. This contradicts the assumption that there is another odd. Hence 3 is the only odd.

Simple induction and strong induction look very similar here, but they both prove the original claims in different ways. Both rely on the inductive hypothesis to show all a_n but a_2 are even, but in very different ways.