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CS70-ZH

Homework 10

a) Let "R" represent traveling right and "U" represent traveling down. Clearly, the shortest path consists of $n=9$ "U"'s and $n=9$ "R" — each path is just a $2n=18$ length string. All the shortest paths are just the number of $2n$ strings choosing n positions. In general, for an $n \times n$, there are $\binom{2n}{n}$ shortest paths.

$\boxed{\binom{18}{9} = 48620}$

b) Label the grid like a normal (x,y) graph, so Tom is at $(0,0)$ and Jerry is at $(9,9)$.

Through x:

ways to get to $(3,3) = \binom{6}{3}$

ways to get from $(3,3)$ to $(9,9) = \binom{12}{6}$

ways from $(3,3)$ to $(9,9)$ through x: $\binom{12}{6}\frac{1}{2}$ since x is traversed for all paths that initially go right, but none that go up first

$\Rightarrow \binom{6}{3}\binom{12}{6}\frac{1}{2} = \boxed{9240}$

through y:

$\binom{9}{5}$ ways to get to $(5,4)$

$\binom{8}{4}$ ways to get from $(5,5) \rightarrow (9,9)$

$\Rightarrow \binom{9}{5}\binom{8}{4} = \boxed{8820}$

through X & Y:

$\binom{6}{3}$ ways to get to $(3,3)$

$\binom{2}{1}$ ways to get from $(4,3)$ to $(5,4)$

$\binom{8}{4}$ ways from $(5,5)$ to $(9,9)$

$\Rightarrow \binom{6}{3} \cdot \binom{2}{1} \cdot \binom{8}{4} = \boxed{2800}$

Neither X nor Y; $48620 - 2800 = \boxed{45820}$

c) through Z: $\binom{11}{4} \cdot \binom{7}{2} = \boxed{6930}$

through W: $\binom{15}{7} \cdot \binom{3}{1} = \boxed{19305}$

through Z & W: $\binom{11}{4}\binom{4}{1}\binom{3}{1} = \boxed{3960}$

neither: $48620 - 3960 = \boxed{44660}$

d) $$\frac{\binom{6}{3}\binom{2}{1}\binom{5}{2}\binom{3}{1}}{\binom{18}{9}} = \boxed{.02468}$$

$\boxed{\text{Problem 2}}$ $n = 10$ $k = 4$ $\rightarrow$ 14 packets

a) All: $(.9)^{14} \approx \boxed{.2288}$

All but 1: $\binom{14}{1}(.1)(.9)^{13} = \boxed{.3558}$

All but 2: $\binom{14}{2}(.1^2)(.9^{12}) \approx \boxed{.257}$

b) She can only tolerate $k=4$ errors:

$\binom{14}{4}(.9)^{10}(.1)^4 + \binom{14}{3}(.9)^{11}(.1)^3 + \binom{14}{2}(.1^2)(.9^{12})$
$+ \binom{14}{1}(.9^{13})(.1) + .9^{14} = \boxed{.99077}$

c) · IF no packets are erased, there are $\binom{14}{0} = 1$ such ways, each with $P = .1^0(.9^{14}) \approx .228$

· For 1 pair erased, there are $\binom{13}{1}$ ways to do that (each choose 1 is really picking 2 packets, so $12 + 2 = 14$), each with $P = .1(.9)^{12} \rightarrow \binom{13}{1}(.1^1)(.9^{12}) \approx .367$

· For 2 pairs erased, there are $\binom{12}{2}$ ways with $P = (.1)^2(.9)^{10} \rightarrow \binom{12}{2}(.1^2)(.9^{10}) \approx .23$

Adding these all together, we get $\boxed{.82605}$

Problem 3

a) n pairs $\rightarrow$ $2n$ total socks

$\boxed{\binom{2n}{k} \text{ distinct subsets}}$

b) k socks, no pair.
For n pairs, there are $\binom{n}{k}$ ways to pick k socks from n pairs. For example, let $n=3$ and $k=2$. So we have socks $aa'bb'cc'$. $\binom{3}{2}=6$ ways to choose 2 pairs. Suppose we have chosen from the a pair and b pair $\rightarrow$ so we could have a or a' and b or b'. If we think of a primed sock as a 1 bit and an unprimed as 0, clearly there are $2^k = 2^2 = 4$ binary strings ab $a'b$ $b'a$ $b'a'$. So for each $\binom{n}{k}$ way we have 2^k strings

$\Rightarrow \boxed{\binom{n}{k}2^k}$ = # of subsets without pairs

c) $\boxed{\dfrac{\left[\binom{2n}{k} - \binom{n}{k}2^k\right]}{\binom{2n}{k}}} = \dfrac{\text{total} - \text{without pairs}}{\text{total}}$

if $k > n$, $P = 1$

d) n socks $\rightarrow$ n^k strings total.

$(nPk) = \dfrac{n!}{(n-k)!}$ ways to pick k socks without repeat

$= (n)(n-1)\ldots(n-k+1)$

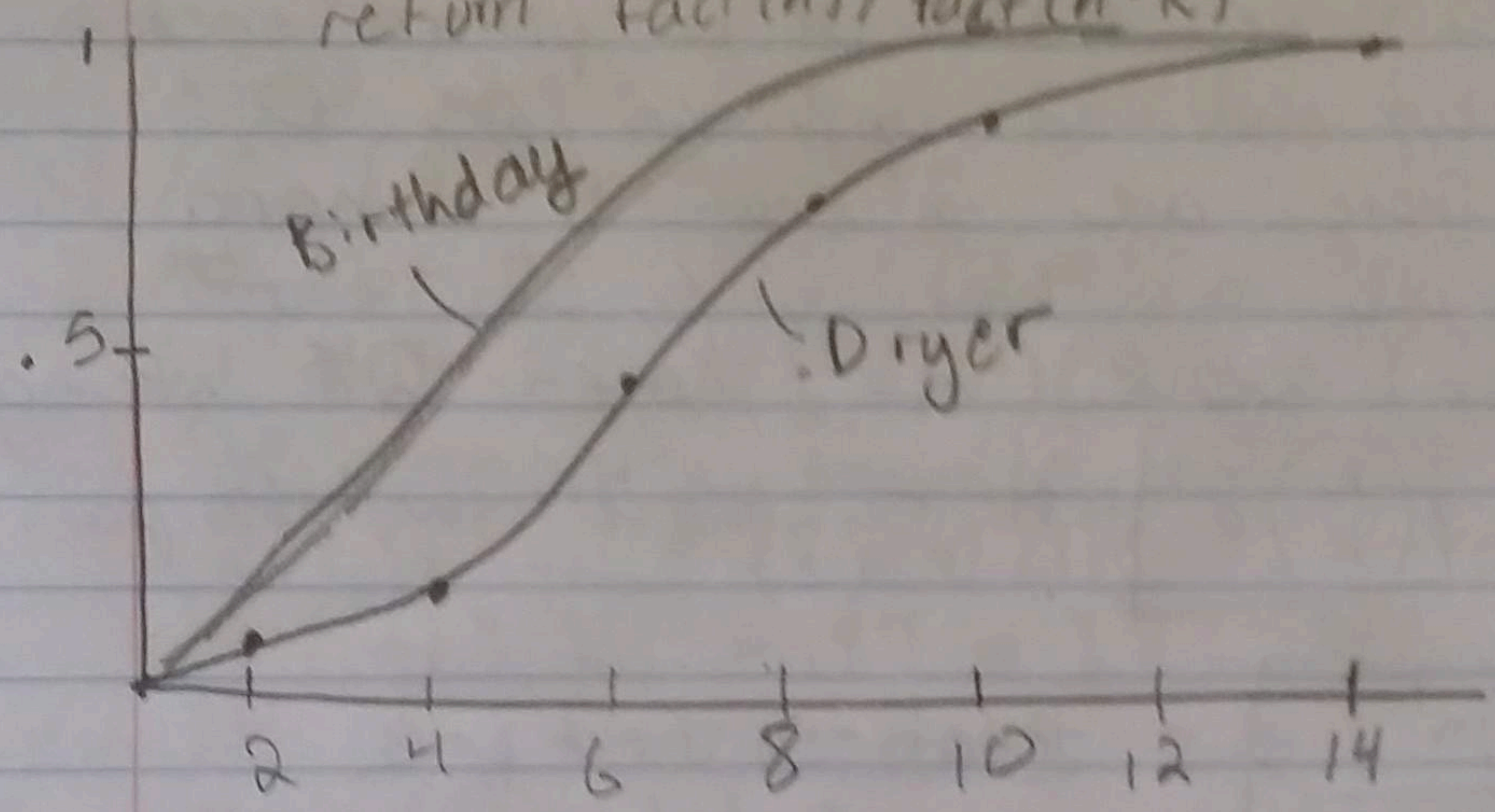
If $k>n$, then the probability is $\boxed{1}$

if $k \leq n$:

$\boxed{\dfrac{n^k - (nPk)}{n^k}}$

e) n=15. Using the python skeleton code, I defined the following functions:

```
def dryer(n,k):
    return (comb(2n,k)-comb(n,k)·2^k)/comb(2n,k)
def birthday(n,k):
    return (math.pow(n,k)-permute(n,k))/math.pow(n,k)
def fact(x):
    if x==0||x==1:
        return 1
    else return x·fact(x-1)
def permute(n,k):
    return fact(n)/fact(n-k)
```



Both experiments have s curves, but overall part d) had higher probabilities since we had replacements. It makes sense that both go to 1 since both have certain probability after k>n.

Problem 4

a) $t=0 \rightarrow x=0$
$t=1 \rightarrow x=-1,0,1$
$t=2 \rightarrow x=-2,-1,0,1,2$
he can be at $\boxed{x \in [-t, t],\ x \in N}$

b) $\underline{t=1}$
$P(-1)=.3$
$P(0)=.2$
$P(1)=.5$

c) $\underline{t=2}$
$P(-2)=(.3)^2=.09$
$P(-1)=(.2)^1(.3)+(.3)(.2)=(.2)(.3)\binom{2}{1}=.12$
$P(0)=.2^2+(.5)(.3)\binom{2}{1}=.34$
$P(1)=\binom{2}{1}(.2)(.5)=.2$ $\rightarrow$ 2 ways $+1-1$, or $-1+1$
$P(2)=.5^2=.25$
$\Sigma P=1$ ✓

d) $\underline{t=3}$
$P(-3)=.3^3=.027$
$P(-2)=(.3)^2(.2)\binom{3}{1}=.054$
$P(-1)=(.3^2)(.5)\binom{3}{1}+(.3)(.2)^2\binom{3}{1}=.171$
$P(0)=(.2)(.3)(.5)(3P3)+.2^3=.188$
$P(1)=(.5)(.2)^2\binom{3}{1}+(.5^2)(.3)\binom{3}{1}=.285$
$P(2)=(.5)^2(.2)(3)=.15$
$P(3)=.5^3=.125$ $\Sigma P=1$ ✓

e) Let $P(x,t)$ be the probability that at t the man is at x.
$P(-(t+1),t+1)=P(-t,t)(.3)$
$P(t+1,t+1)=P(t,t)(.5)$
$P(t,t+1)=P(t,t)(.2)$
$P(-t,t+1)=P(-t,t)(.2)$
In general:
$$\boxed{\begin{array}{l} P(-x_0,t+1)=P(-x_0,t)(.2)+P(-x_0+1,t)(.3) \\ P(x_0,t+1)=P(x_0,t)(.2)+P(x_0-1,t)(.5) \end{array}}$$

Problem 5

a) $n=1$: $P(1)=\{\{(1,1)\}\}$

$$\sum_{p\in P(1)} \prod_{(s,r)\in p} \frac{1}{r!\,x^r} = \frac{1}{1!\,1^1} = \boxed{1} \checkmark$$

$n=2$ $\quad P(2)=\{\{(2,1)\},\{(1,2)\}\}$

$$\sum_{p\in P(2)} \prod \frac{1}{r!\,x^r} = \frac{1}{1!\,2^1} + \frac{1}{2!\,1^2} = \frac{1}{2}+\frac{1}{2} = \boxed{1} \checkmark$$

$n=3$ $\quad P(3)=\{\{(3,1)\},\{(2,1),(1,1)\},\{(1,3)\}\}$

$$\sum_{p\in P(2)} \prod \frac{1}{r!\,x^r} = \frac{1}{1!\,3} + \left(\frac{1}{1!\,2}\times\frac{1}{1}\right) + \frac{1}{3!\,1^3} = \boxed{1} \checkmark$$

$n=4$ $\quad P(4)=\{\{(4,1)\},\{(3,1),(1,1)\},\{(2,2)\},\{(2,1)(1,2)$

$$\sum\prod = \frac{1}{4} + \left(\frac{1}{3}\times 1\right) + \frac{1}{2!\,2^2} + \left(\frac{1}{2}\times\frac{1}{2!}\right) + \frac{1}{4!\,1^4}$$

$$= \frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{4}+\frac{1}{24} = \boxed{1} \checkmark$$

$n=5$ $\quad P(5)=\{\{(5,1)\},\{(4,1),(1,1)\},\{(3,1),(2,1)\},$
$\{(3,1),(1,2)\},\{(2,2),(1,1)\},$
$\{(2,1),(1,3)\},\{(1,5)\}\}$

$$\sum\prod = \frac{1}{5}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{120} = \boxed{1} \checkmark$$

b) In σ_n, there are elements $[1,\dots,n]$. If we choose ℓ of them, how many cycles? Suppose $\ell=3$ and we choose 1 2 3. Clearly, there are $2P2$ cycles since unique cycles can be made by keeping the 1 in front and counting the permutations of 2 and 3. In genera there are $(\ell-1)P(\ell-1)$ permutations of $\ell-1$ elements after the first. So in total,

$$\boxed{\binom{n}{\ell}\cdot\left[(\ell-1)P(\ell-1)\right] \text{ cycles}}$$

c) $$\frac{\frac{n!\,(l_1-1)!}{(n-l_1)!\,l_1!} \times \frac{(n-l_1)!\,(l_2-1)!}{(n-l_1-l_2)!\,l_2!} \cdots (n-l_1-l_2)!}{r_1!\,r_2!\cdots r_k!}$$

$$= \prod_{i=1}^{k} \frac{(n-l_i)!\,(l_i-1)!}{r_i!\,(n-l_i)!\,l_i!} = \prod \frac{(l_i^{-1})(l_i-1)!}{r_i!}$$

the $(n-l_i)!$ term and like terms will cancel, as will the factorials $\frac{(l_i-1)!}{l_i!} = l_i$

$$\Rightarrow \; = n!\left(\frac{1}{l_1 r_1! \cdots l_k r_k!}\right)$$

but $l_i = x_i^{r_i}$

$$\Rightarrow \; \boxed{= n! \prod_{i=1}^{k} \frac{1}{x_i^{r_i}\, r_i!}}$$

d) To recount all elements in σ_n, we can just add up for all $m \leq n$ the number of permutations with m cycles.

$$|\sigma_n| = n! = \sum_{m\leq n}\left(n! \prod_{i=1}^{k} \frac{1}{x_i^{r_i} r_i!}\right)$$

$$\Rightarrow n! = n! \sum_{p\in P(n)} \prod_{i=1}^{k} \frac{1}{x_i^{r_i} r_i!}$$

divide both sides by $n!$

$$\Rightarrow \boxed{1 = \sum_{p\in P(n)} \prod_{(x,r)\in p} \frac{1}{r!\,x^r} = 1}$$

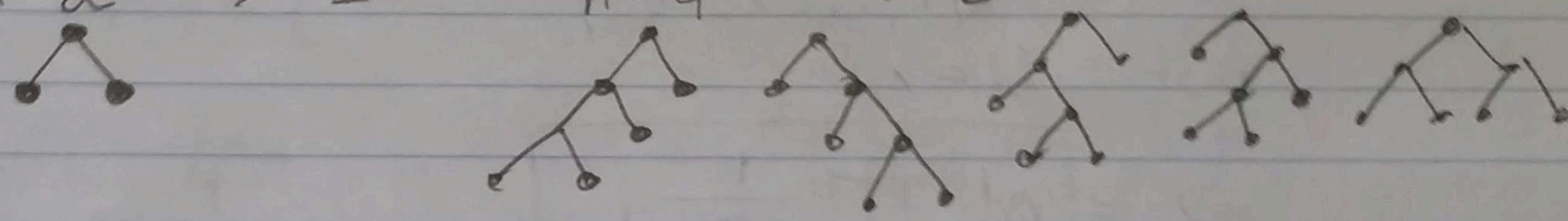
since every (x_i, r_i) corresponds to one partition of m cycles in σ_n.

Problem 7

A binary tree is full if every node has either 2 children or no children. How many distinct full trees are there for n leaves?
For a polygon with n sides, how many different triangulations are there?
Is there a relation? Draw a figure that represents the relation for $n=6$ if you can.

Binary trees:

$n=2 \rightarrow 1$ $\qquad$ $n=4 \rightarrow 5$



Full trees only make sense for even n.
For n leaves, there are $\boxed{C_{n-1}}$ distinct trees, where

$C_n = \binom{2n}{n}\frac{1}{n+1}$ is the nth catalan number.

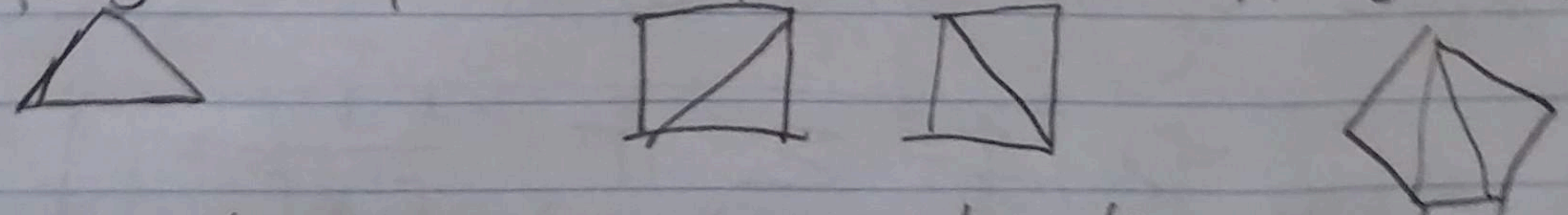
$n=2 \rightarrow C_1 = 1$

$n=4 \rightarrow C_3 = 5$

$n=6 \rightarrow C_5 = 42$

Triangulation:

$n=3 \rightarrow 1$ $\qquad$ $n=4 \rightarrow 2$ $\qquad$ $n=5 \rightarrow 5$



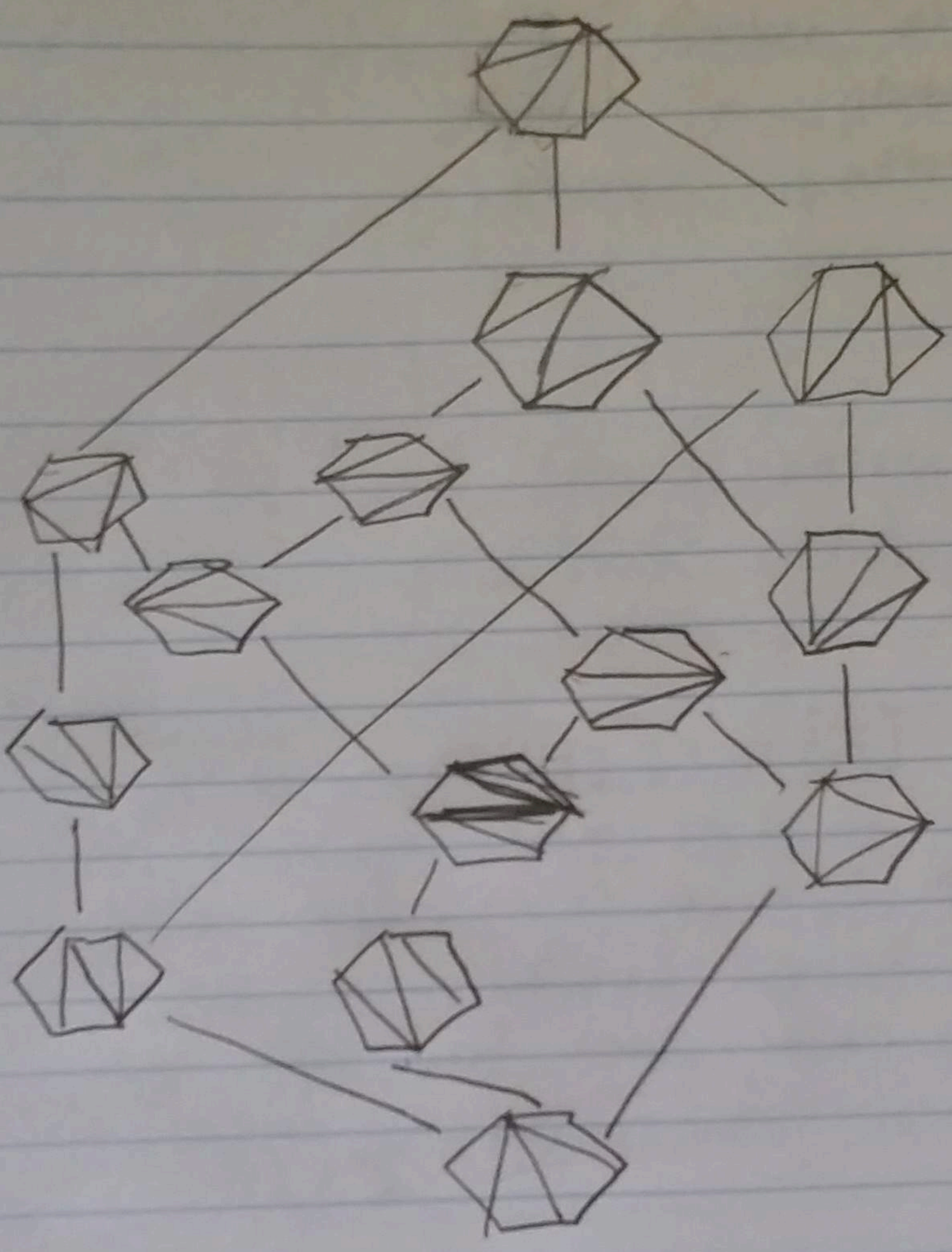
Again these are given by the catalan number $\boxed{C_{n-2}}$

$n=3 \rightarrow C_1 = 1$

$n=4 \rightarrow C_2 = 2$

$n=5 \rightarrow C_3 = 5$

The two problems count the same set, so in a way they are combinatorial proofs of each other, with the catalan numbers being the answer. Here's the <u>tamari lattice</u> for $n = 6$



Problem 6

a) $F(x) = \ln x$