EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 05: September 12

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5.1 Announcements

1. HW1 Due at 11:59am September 12
2. Reading:
3. Midterm 1 on September 26th

5.2 Continuous-time Fourier Series (Recap.)

5.2.1 Continuous-time Fourier Series Expansion (Cosine basis)

Let us start by wrapping up the example we started in the last lecture. We have a periodic square wave with a period of 3. Recall that "any" p-periodic signal $x(t)$ can be decomposed into a linear combination of cosines with frequencies at integer multiples of the signal's fundamental frequency $\omega_0 = \frac{2\pi}{p}$ as

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k).$$

Note that this equality is "almost everywhere" and is not pointwise $(*)$.

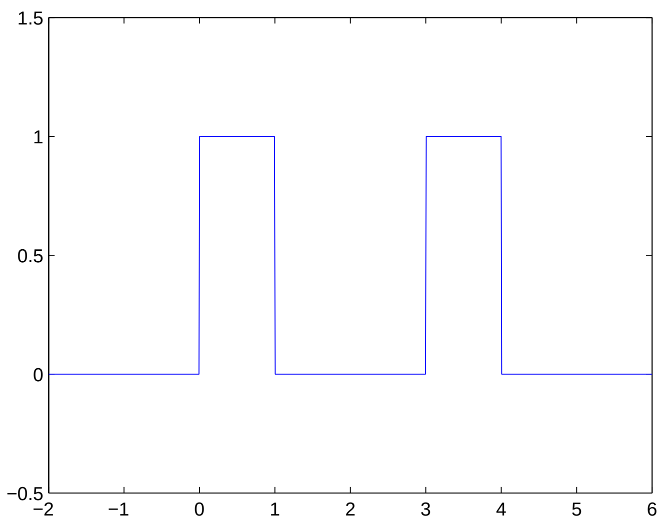

Figure 5.1: squarewave with a period of 3

5.2.2 Continuous-time Fourier Series Expansion (Sine and Cosine basis)

Alternatively, as proved in the last lecture, we can represent the same function in terms of both sines and cosines:

$$x(t) = A_0 + \sum_{k=1}^{\infty}(\alpha_k \cos kw_0 t + \beta_k \sin kw_0 t)$$

where A_o, α_k and β_k are:

$$A_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$\alpha_k = \frac{2}{T}\int_0^T x(t)\cos w_0 ktdt$$

$$\beta_k = \frac{2}{T}\int_0^T x(t)\sin w_0 ktdt$$

As usual, our goal is to find the Fourier coefficients, or the weights, for the sine and cosines. First, let us calulate A_0, or the DC offset of the signal:

$$A_0 = \frac{1}{3}\int_0^3 x(t)dt = \frac{1}{3}\int_0^1 1dt = \frac{1}{3}$$

Next, α_k's for the cosines:

$$\alpha_k = \frac{2}{T}\int_0^T x(t)\cos w_0 ktdt = \frac{2}{3}\int_0^1 1\cos\frac{2\pi k}{3}tdt = \frac{1}{k\pi}\sin(\frac{2\pi}{3}k)$$

What about the β_k's? They are a bit more complicated:

$$\beta_k = \frac{2}{T}\int_0^T x(t)\sin w_0 ktdt = \frac{2}{3}\int_0^1 1\sin\frac{2\pi k}{3}tdt = \frac{2}{\pi k}\sin^2(\frac{\pi k}{3})$$

5.2.3 Interpreting CTFS coefficients

Now that we have calculated the coefficients, let us interpret the results. As previously discussed, A_0 is the average value, or the DC offset, of the signal. We have calculated α_k's and β_k's for the sine and cosine terms with frequencies $\omega_0, \omega_1, \omega_2, ...$ Here, ω_0 is called the fundamental frequency of the signal. And $\omega_1, \omega_2, ...$ are the harmonics (integer multiples) of ω_0. Looking at the coefficients, we observe that the magnitude of coefficients, both α_k's and β_k's, decreases as the frequency gets higher. To be more precise, let us plot the magnitude of α_k's vs k (Figure 5.2): The magnitude of α_k's (depicted with the red stems) decrease with k and is bounded by $\frac{1}{k\pi}$ (green line). As k approaches infinity, the magnitude of α_k's goes to 0. (Decreasing at the rate in the order of $\frac{1}{k}$). Since the signal has big α_k's for the lower values of k, we say that the signal has a low-pass spectrum. (and β_k's are similar)

5.2.4 Applications of CTFS

Fourier series expansion is defined for periodic signals. Another equivalent interpretation of the Fourier series expansion is that we are approximating a signal over a finite duration $[0, p]$ (or over any interval of duration p). When we actually process signal, we are almost always dealing with the signal over a finite duration. So this interpretation is more computationally relevant.

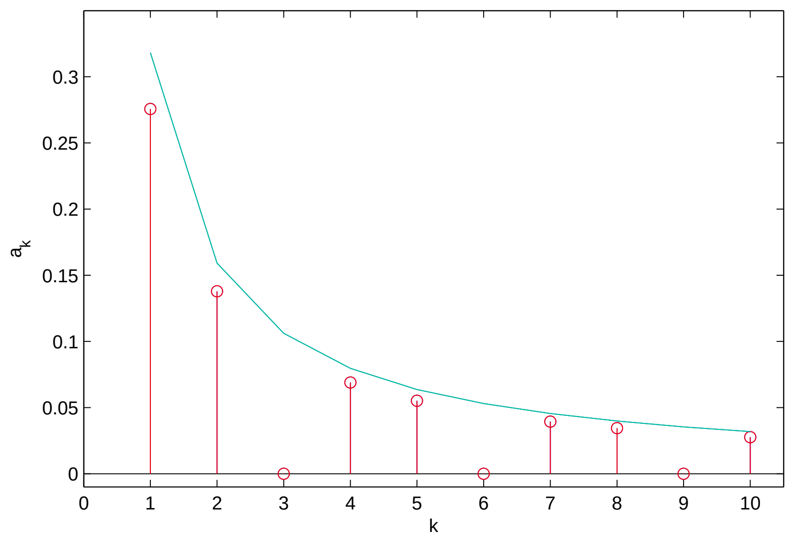

Figure 5.2: magnitude of α_k

Example 5.1. *Speech Recognition. A long duration audio recording of speech can be cut into snippets, say of* 100 *ms long. Each snippet contains the signal over a finite duration. The Fourier series expansion of each snippets yields the frequency content of the phoneme being uttered. This is often the first processing step of a speech recognizer.*

Example 5.2. *Finance. Stock market data can be treated as a signal and approximated with a Fourier series. From looking at the energies of low frequency components (namely A_i for small i), a day trader can judge how strong slow changing trends are in that stock. Similarly, high energy in high frequency components (A_i for large i) indicate a volatile market.*

5.3 Discrete-time Signals

We started our discussion on Discrete-time signals last lecture. The term "discrete-time signals" encompasses signals defined on both time and space. Whereas continuous-time signals have time in the units of *seconds*, discrete-time signals have time in the units of *samples*. Commonly, discrete-time signals are the result of sampling continuous-time signals. However, there are many signals that are inherent discrete-time in nature.

Definition 5.1. *A discrete-time signal $x(n)$ is a function defined on the domain $\mathbb{Z}$.*

Definition 5.2. *The discrete-time sinusoid is the function $x(n) = \cos(2\pi f n)$.*

Because of the interesting properties of the Fourier series in continuous-time, we are motivated to consider a discrete-time analogy. To begin, we try to understand the notion of periodicity and frequency in the discrete-time domain.

Example 5.3. *If $f = 1$, then we have $x(n) = \cos(2\pi n) = 1$, the constant function. So $p = 1$.*

Example 5.4. *If $f = \frac{1}{2}$, then we have $x(n) = \cos(\pi n) = (-1)^n$ which is the sequence $1, -1, 1, -1, \ldots$, so $p = 2$.*

Example 5.5. *If* $f = \frac{1}{3}$*, then we have* $x(n) = \cos(\frac{2\pi}{3}n)$ *which is the sequence* $1, -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, ...,$ *so* $p = 3$*.*

It may be tempting to conclude from the above examples the relation of $p = f^{-1}$, as in the continuous-time case. To show that this is not true, consider the case where $f = 2$. By the above formula, we guess that the period $p = 2^{-1} = \frac{1}{2}$, but a half-sample makes no sense in the context of discrete-time. Furthermore, the resulting signal $x(n) = \cos(2\pi \cdot 2n) = \cos(4\pi \cdot n) = 1$, yielding a period of $p = 1$, so the formula $p = f^{-1}$ is not always true. In fact, sometimes increasing f does not change the period. Let's explore a few more cases:

Example 5.6. *If* $f = \frac{2}{5}$*, then we have* $x(n) = \cos(\frac{4\pi}{5}n)$*,* $p = 5$*.*

What if f is irrational?

Example 5.7. $f = \sqrt{2}$*, then we have* $x(n) = \cos(2\pi\sqrt{2}n)$

Then $x(n)$ will never repeat the same value, and $x(n)$ is not periodic. (period is infinity) In particular, if $x(n)$ is p-periodic, then

$$\begin{aligned} x(n) &= \cos(2\pi f n) \\ &= \cos(2\pi f(n+p)) \qquad p\text{-periodicity implies } x(n) = x(n+p) \\ &= \cos(2\pi f n + 2\pi f p). \end{aligned}$$

This can only be true if $2\pi f p$ is an integer multiple of 2π. Since p must be an integer, this also implies that f is rational. We summarize this result in the following:

Proposition 5.1. *If* $f = \frac{a}{b}$ *where* a *and* b *are relatively prime, then* $p = b$*.*

The main distinction between continuous-time and discrete-time is the restriction on integers in DT. This means that we need to worry about periodicity and frequencies in discrete-time. In real-world engineering, for example, speech signals (in continuous-time) is sampled and processed in computers (discrete-time). The material might very sound mathy but the applications are very engineering.

5.4 Discrete-time Fourier Series

5.4.1 Discrete-time Fourier Series Expansion (Cosine basis)

With the concept of a discrete-time frequency and period, we are now in a position to consider the discrete-time analogy of the Continuous-time Fourier series.

Definition 5.3. *Let* $x(n)$ *be a* p*-periodic discrete-time signal. Then*

$$x(n) = A_0 + \sum_{k=1}^{K} A_k \cos(k\omega_0 n + \phi_k).$$

where

$$\omega_0 = \frac{2\pi}{p} \text{ and } K = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases}.$$

One important observation here is that K is finite in the discrete domain, whereas the sum is infinite in continuous-time. This is related to the fact that increasing f may not yield a different sinusoid.

This is also an important fact when considering how to implement the Fourier series expansion. Note that the sum is finite, so it is pointwise equality in the most literal sense. Contrast this to the case in continuous-time where the Gibbs phenomenon makes the equality in the continuous-time Fourier series only "almost everywhere" (see $*$ in section 4.1). Perhaps even more importantly for the engineer, the series can be stored as a finite array of coefficients $A_0, A_1, \phi_1, A_2, \phi_2, ..., A_K, \phi_K$ after finite computation.

Although it is a convenient fact that any p-periodic discrete signal can be decomposed into a linear combination of its constituent frequencies, how does one actually find this decomposition? Note that in a discrete=time signal, p-periodicity implies that we only need to consider $x(n)$ for $n = 0, 1, 2, ..., p-1$. Also note that if p is odd, in there are a total of $1 + 2K = 1 + (p-1) = p$ unknowns of the form $A_0, A_1, \phi_1, A_2, \phi_2, ..., A_K, \phi_K$. This leaves a set of p equations in p unknowns

$$
\begin{aligned}
x(0) &= A_0 + \sum_{k=1}^{K} A_k \cos(k\omega \cdot 0 + \phi_k) \\
x(1) &= A_0 + \sum_{k=1}^{K} A_k \cos(k\omega \cdot 1 + \phi_k) \\
&\vdots \\
x(p-1) &= A_0 + \sum_{k=1}^{K} A_k \cos(k\omega \cdot (p-1) + \phi_k).
\end{aligned}
$$

5.4.2 Discrete-time Fourier Series Expansion (Sine and Cosine basis)

Unfortunately, ϕ_i for $i = 1, 2, ...K$ is a phase term inside of a sinusoid, so our system of equations is not a system of *linear* equations. However, we can rewrite the Fourier series expansion as

$$
x(n) = A_0 + \sum_{k=1}^{K} A_k \cos(k\omega_0 n + \phi_k) = A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 n) + \beta_k \sin(k\omega_0 n)
$$

with the angle-sum identity to convert our system of equations into the form

$$
\begin{aligned}
x(0) &= A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot 0) + \beta_k \sin(k\omega_0 \cdot 0) \\
x(1) &= A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot 1) + \beta_k \sin(k\omega_0 \cdot 1) \\
&\vdots \\
x(p-1) &= A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot (p-1)) + \beta_k \sin(k\omega_0 \cdot (p-1))
\end{aligned}
$$

5.4.3 Discrete-time Fourier Series Expansion (Matrix representation)

This form is much more preferable, because the sine and cosine parts are constants for each value of n, leaving p linear equations in p variables where the unknowns are now $A_0, \alpha_1, \beta_1, ..., \alpha_K, \beta_K$. Let's do a quick example. (Figure 5.3)

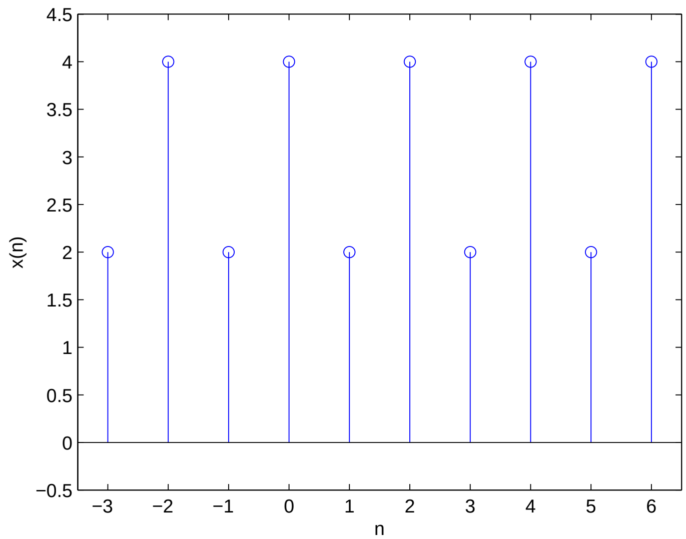

Figure 5.3: a discrete time signal

If you find the Fourier series expansion, you will get:

$$x[n] = A_0 + \alpha_1 \cos\left(\frac{2\pi}{2} n\right) + \beta_1 \sin\left(\frac{2\pi}{2} n\right) = A_0 + \alpha_1 \cdot (-1)^n$$

How many unknowns or degrees of freedom do we have in the signal? The answer is 2 because the period is 2 (which explains why A_0 and α_1 are non-zero but β_1 is zero.)

$$n = 0 : x[0] = A_0 + \alpha_1 = 4$$

$$n = 1 : x[1] = A_0 - \alpha_1 = 2$$

It is now possible to rewrite this problem in matrix form:

$$\begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_0 \\ \alpha_1 \end{pmatrix}$$

In practice, we may choose to keep the average and difference, throwing out other smaller coefficients, in order to speed up calculations.

5.4.4 Discrete-time Fourier Series Expansion (general Matrix representation)

In general we can use the following matrix notation:

$$\begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(p-1) \end{pmatrix} = \begin{pmatrix} 1 & \cos(1\omega_0 \cdot 0) & \sin(1\omega_0 \cdot 0) & ... & \cos(K\omega_0 \cdot 0) & \sin(K\omega_0 \cdot 0) \\ 1 & \cos(1\omega_0 \cdot 1) & \sin(1\omega_0 \cdot 1) & ... & \cos(K\omega_0 \cdot 1) & \sin(K\omega_0 \cdot 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(1\omega_0(p-1)) & \sin(1\omega_0(p-1)) & ... & \cos(K\omega_0(p-1)) & \sin(K\omega_0(p-1)) \end{pmatrix} \begin{pmatrix} A_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_K \\ \beta_K \end{pmatrix}.$$