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EE20N

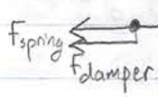
Homework 4

1. 

$F_{spring}(t) = Kx(t)$ $\quad F_{mass}(t) = M\frac{d^2x(t)}{dt2}$

$f_{damper}(t) = f_v \frac{dx(t)}{dt}$

a) Ignoring kinectic friction (and $\Sigma F_y = 0$):



$Ma = M\frac{d^2x(t)}{dt} = F(t) - kx(t) - F_v\frac{dx(t)}{dt}$

$\Rightarrow \boxed{M\frac{d^2x(t)}{dt} + F_v\frac{dx(t)}{dt} + kx(t) = F(t) = M\ddot{x} + F_v\dot{x} + kx}$

b) Since we are asked to find the frequency response, the system is LTI, and x(t) is a complex exponent:

input $= F(t) = e^{i\omega t}$

output $= x(t) = H(\omega)e^{i\omega t}$

$\Rightarrow F(t) = e^{i\omega t} = M\ddot{x} + F_v\dot{x} + kx = Mi^2\omega^2 H(\omega)e^{i\omega t} + F_v i\omega H(\omega)e^{i\omega t} + kH(\omega)e^{i\omega t}$

$\Rightarrow$ divide by $f(t) \Rightarrow 1 = Mi^2\omega^2 H(\omega) + F_v i\omega H(\omega) + kH(\omega)$

$\Rightarrow 1 = H(\omega)(Mi^2\omega^2 + F_v i\omega + k)$

$\Rightarrow \boxed{H(\omega) = \frac{1}{Mi^2\omega^2 + F_v i\omega + K}}$

c) $\omega = 0$

$\Rightarrow |H(\omega=0)| = \left|\frac{1}{0+0+k}\right| = \boxed{\frac{1}{k}}$

when the driving force is constant, the displacement only depends on the spring element. This intuitively makes sense: our Force is constant, so our dampening forces becomes irrelevant and the position only depends on the spring constant.

d) $\lim_{\omega\to\infty} |H(\omega)| = \left|\frac{1}{\infty}\right| = \boxed{0}$

As $\omega \to \infty$, the frequency response becomes more dependent on the mass element, because it is scaled by a factor of ω^2 (F_v only ω). This matches our real life intuition. If we push and pull on a mass very rapidly — infinitely rapidly — then it looks like it won't move anywhere.

e) $M = .1 \quad F_v = 1.1 \quad k = 1$

$$F(t) = 1 + \cos(t) + \cos(10t) + \cos(100t) + \cos(1000t)$$

$$x(t) = 1\left|\frac{1}{M(0)+F_v(0)+k}\right| e^0 + \left|\frac{1}{Mi^2+F_v i+k}\right| e^{it} e^{\angle H(1)}$$

$$+ \left|\frac{1}{100Mi^2+10F_v i+k}\right| e^{i10t} e^{\angle H(10)} + \left|\frac{1}{10000Mi^2+100F_v i+k}\right| e^{i100t} e^{\angle H(100)}$$

$$+ \left|\frac{1}{1000000Mi^2+1000F_v i+k}\right| e^{i1000t} e^{\angle H(1000)}$$

$$= \frac{1}{k} + \frac{1}{\sqrt{(k-M)^2+F_v^2}} e^{it} e^{-\tan^{-1}\left(\frac{F_v}{k-M}\right)} + \frac{e^{i10t} e^{-\tan^{-1}\left(\frac{10F_v}{k-100M}\right)}}{\sqrt{(k-100M)^2+(10F_v)^2}}$$

$$+ \frac{e^{i100t} e^{-\tan^{-1}\left(\frac{100F_v}{k-10000M}\right)}}{\sqrt{(k-10000M)^2+(100F_v)^2}} + \frac{e^{i1000t} e^{-\tan^{-1}\left(\frac{1000F_v}{k-1000000M}\right)}}{\sqrt{(k-1000000M)^2+(1000F_v)^2}}$$

$$= 1 + \frac{e^{it} e^{-.885}}{1.42} + \frac{e^{i10t} e^{+.885}}{14.2} + \frac{e^{i100t} e^{.1096 7}}{1005} + \frac{e^{i1000t} e^{.011}}{100005}$$

$$= \boxed{1 + \frac{\cos(t-.885)}{1.42} + \frac{\cos(10t+.885)}{14.2} + \frac{\cos(100t+.1097)}{10005} + \frac{\cos(1000t+.011)}{100005}}$$

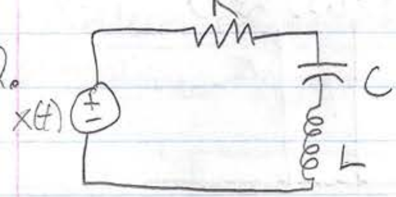
f) This is a low pass filter that attenuates high frequencies and passes low frequencies. We can see this from the limits of $H(\omega)$. As $\omega \to \infty$, $H(\omega) \to 0$, so the higher frequencies are filled

b) CT: $LCG(\omega)(i\omega)^2 \cancel{e^{i\omega t}} + RCi\omega \cancel{e^{i\omega t}} G(\omega) + G(\omega)\cancel{e^{i\omega t}} = LC(i\omega)^2 \cancel{e^{i\omega t}} + \cancel{e^{i\omega t}}$

$$G(\omega) = \frac{LC(i\omega)^2 + 1}{LC(i\omega)^2 + RCi\omega + 1}$$

compared to the discrete-time frequency response, the CTFR has no depency on $e^{i\omega}$ and is <u>not periodic</u>

2.



$$LC\frac{d^2y}{dt} + RC\frac{dy}{dt} + y = LC\frac{d^2x}{dt^2} + x$$

$$LCy[n-2] + RCy[n-1] + y[n] = LCx[n-2] + x[n]$$

b) Since LTI, we can input $x(t) = e^{i\omega n}$, so $y(t) = G(\omega)e^{i\omega n}$

$$\Rightarrow LCG(\omega)e^{i\omega n}e^{-i\omega 2} + RCG(\omega)e^{i\omega n}e^{-i\omega} + G(\omega)e^{i\omega n} = LCe^{i\omega n}e^{-i\omega 2} + e^{i\omega n}$$

$$\Rightarrow G(\omega)\left[LCe^{-i\omega 2} + RCe^{-i\omega} + 1\right] = LCe^{-i\omega 2} + 1$$

$$\Rightarrow G(\omega) = \frac{LCe^{-i\omega 2} + 1}{LCe^{-i\omega 2} + RCe^{-i\omega} + 1}$$

$$\Rightarrow G(\omega) = \frac{e^{-i\omega 2} + \frac{1}{LC}}{e^{-i\omega 2} + \frac{R}{L}e^{-i\omega} + \frac{1}{LC}}$$

$$2\zeta\omega_0 = 2\zeta\frac{1}{\sqrt{LC}} = \frac{R}{L}$$

$$\Rightarrow \zeta = \frac{R\sqrt{C/L}}{2}$$

$$\Rightarrow \boxed{G(\omega) = \frac{(e^{-i\omega})^2 + (\omega_0)^2}{(e^{-i\omega})^2 + 2\zeta\omega_0(e^{-i\omega}) + (\omega_0)^2}}$$

compared to CTFR, depends on $e^{i\omega}$ and periodic $\rightarrow G(e^{i\omega})$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ $\quad \zeta = \frac{R}{2L}\sqrt{LC} = \frac{R}{2L\omega_0}$

c) $$G(\omega) = \frac{(e^{-i\omega} - z_1)(e^{-i\omega} - z_2)}{(e^{-i\omega} - P_1)(e^{-i\omega} - P_2)}$$

$z_1 = e^{i\frac{\pi}{2}}$ $\quad P_1 = .99e^{i\frac{\pi}{2}}$

$z_2 = e^{-i\frac{\pi}{2}}$ $\quad P_2 = .99e^{-i\pi/2}$

$$G(\omega) = \frac{(e^{-i\omega} - e^{i\frac{\pi}{2}})(e^{-i\omega} - e^{-i\frac{\pi}{2}})}{(e^{-i\omega} - .99e^{i\frac{\pi}{2}})(e^{-i\omega} - .99e^{-i\pi/2})}$$

$$G(\omega) = \frac{(\cos\omega - i\sin\omega - \cancel{\cos\tfrac{\pi}{2}}^{0} - i\sin\tfrac{\pi}{2})(\cos\omega - i\sin\omega - \cancel{\cos\tfrac{\pi}{2}}^{0} + i\sin\tfrac{\pi}{2})}{(\cos\omega - i\sin\omega - \cancel{.99\cos\tfrac{\pi}{2}}^{0} - i.99\sin\tfrac{\pi}{2})(\cos\omega - i\sin\omega - \cancel{.99\cos\tfrac{\pi}{2}}^{0} + i.99i\sin\tfrac{\pi}{2})}$$

$$= \frac{(\cos\omega - i(\sin\omega + 1))(\cos\omega - i(\sin\omega - 1))}{(\cos\omega - i(\sin\omega + .99))(\cos\omega - i(\sin\omega - .99))}$$

$$|G(\omega)| = \frac{\sqrt{\cos^2\omega + (\sin\omega + 1)^2}\ \sqrt{\cos^2\omega + (\sin\omega - 1)^2}}{\sqrt{\cos^2\omega + (\sin\omega + .99)^2}\ \sqrt{\cos^2\omega + (\sin\omega - .99)^2}}$$

$$\angle G(\omega) = \tan^{-1}\left(-\frac{\sin\omega + 1}{\cos\omega}\right) + \tan^{-1}\left(-\frac{\sin\omega - 1}{\cos\omega}\right) - \tan^{-1}\left(-\frac{\sin\omega + .99}{\cos\omega}\right) - \tan^{-1}\left(-\frac{\sin\omega - .99}{\cos\omega}\right)$$

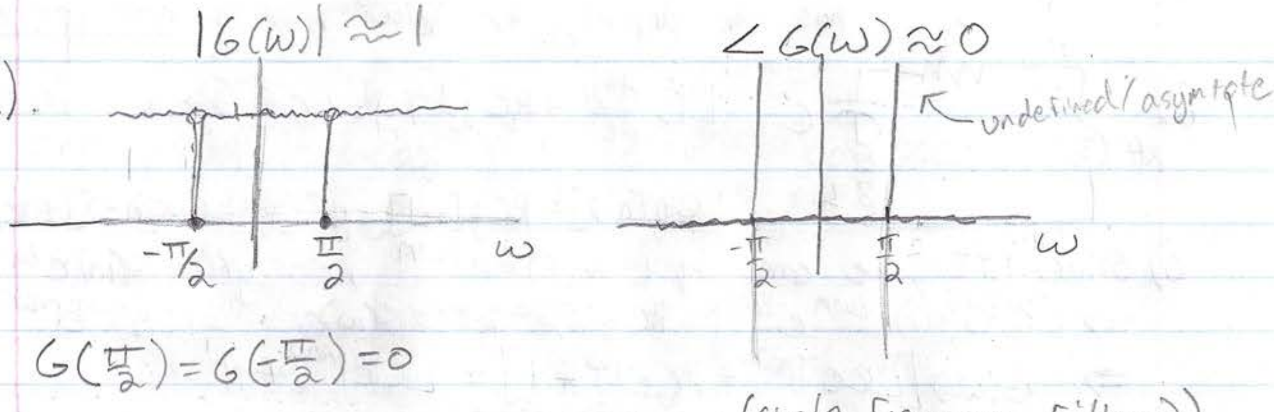
$$x[n] = \cos(0\pi n) + \cos(\tfrac{\pi}{8}n) + \cos(\tfrac{\pi}{4}n) + \cos(\tfrac{\pi}{2}n) + \cos(\pi n)$$

$$y[n] = G(0)\cos(0 + \angle G(0)) + G(\tfrac{\pi}{8})\cos(\tfrac{\pi}{8}n + \angle G(\tfrac{\pi}{8})) + G(\tfrac{\pi}{4})\cos(\tfrac{\pi}{4}n + \angle G(\tfrac{\pi}{4}))$$
$$+ G(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2}n + \angle G(\tfrac{\pi}{2})) + G(\pi)\cos(\pi n + \angle G(\pi))$$

$$y[n] = 1.01005\cos(0) + 1.01004\cos(\tfrac{\pi}{8}n + .004) + 1.0099\cos(\tfrac{\pi}{4}n + .01)$$
$$\cancel{(0)\cos(\tfrac{\pi}{2}n)} + 1.01\cos(\pi n + 0)$$

$$\boxed{y[n] = 1.01005 + 1.01004\cos(\tfrac{\pi}{8}n + .004) + 1.0099\cos(\tfrac{\pi}{4}n + .01) + 1.01\cos(\pi n)}$$

$$\approx 1 + \cos(\tfrac{\pi}{8}n) + \cos(\tfrac{\pi}{4}n) + \cos(\pi n)$$

d) 

$G(\frac{\pi}{2}) = G(-\frac{\pi}{2}) = 0$

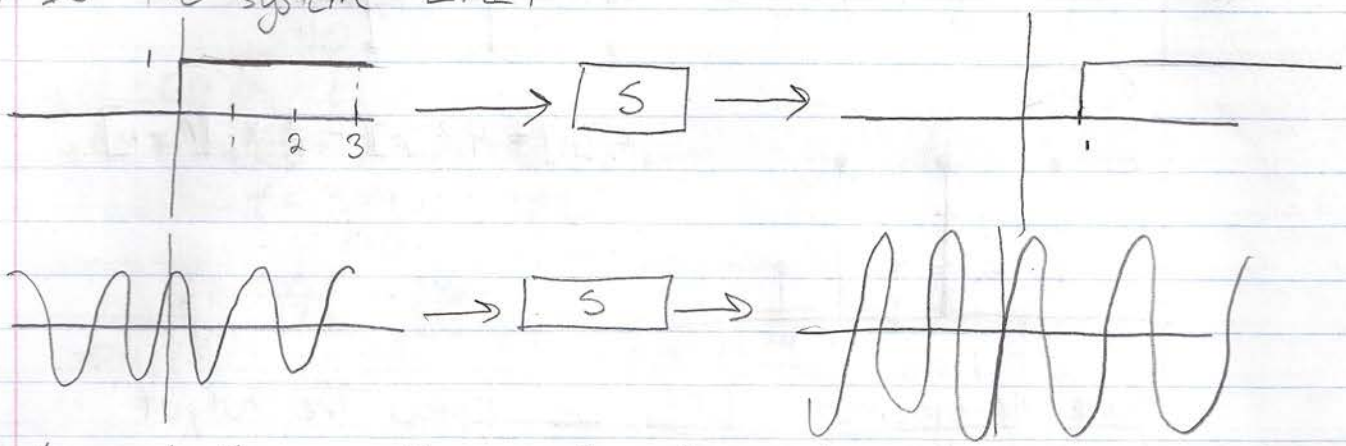
(single Frequency Filter?) or bandstop for $\omega \approx \frac{\pi}{2} + 2\pi k$

This is neither a highpass nor lowpass nor band pass Filter. It's attenuation/Gain across all frequencies are negligible (because $G(\omega) \approx 1$), though it Fluctuated around 1 rapidly. Furthermore, the system kills Frequencies of $|\omega| = \frac{\pi}{2}$, and all $\omega = \frac{\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$. The phase is also fairly negligible, since it very approximately zero.

- If we increase or decrease α, $|G(\omega)|$ will not be approximately 1, but rather have much greater fluctuations and oscillations in magnitude. We can see that with $\alpha = .99$, $|G(\omega)| \approx 1$ since $.99 \approx 1$, and the only differences between the numerator and denominator are small. changing α will amplify the differences in $G(\omega)$'s numerator & denominator.

Midterm Problem Redos:

Q3a) Is the system LTI?



Answer: There isn't enough information to make a conclusion.

We cannot determine if the entire system is LTI from only 2 input and output pairs. LTI is a property of a system as a whole, not just the behavior of two specific inputs, so we cannot conclude that the system is LTI without explicitly knowing what S does to any input $x(t)$. Though it looks well behaved, theres no way to tell if the system follows additivity, homogeneity, and time invariance.

Q4c) $y(t) = 2x(t-1)$

$$y(t) = 2A_0 + \sum_1^\infty 2A_k \cos(k\omega_0(t-1)) + \cancel{\sum_1^\infty 2B_k \sin(k\omega_0(t-1))} \rightarrow 0$$

$$\boxed{C_0 = 2A_0}$$

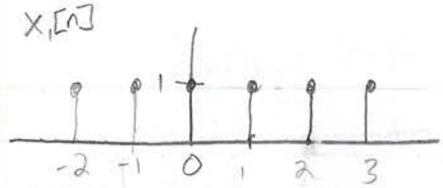
$$y(t) = 2A_0 + \sum^\infty 2A_k \left(\cos(k\omega_0 t)\cos(k\omega_0) + \sin(k\omega_0 t)\sin(k\omega_0)\right)$$

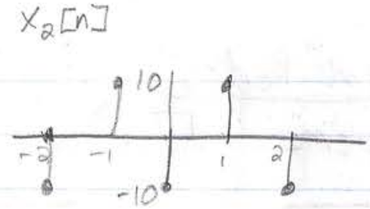
$$y(t) = 2A_0 + \sum_1^\infty 2A_k \cos(k\omega_0) \cos k\omega_0 t + \sum_1 2A_k \sin k\omega_0 \sin k\omega_0 t$$

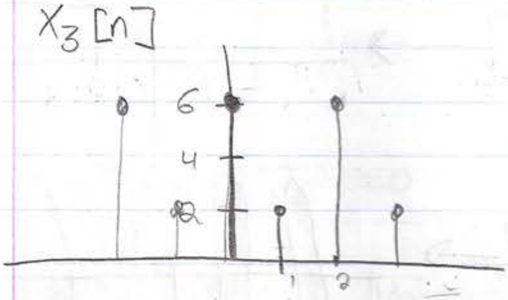
$$\boxed{C_k = 2A_k \cos(k\omega_0)}$$

$$\boxed{D_k = 2A_k \sin(k\omega_0)}$$

Q5 a)





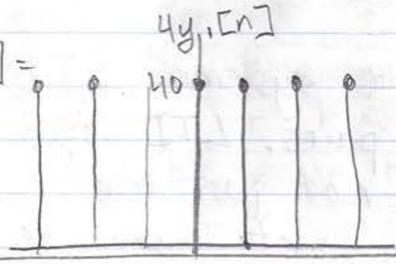


$$x_3[n] = 4x_1[n] + \frac{1}{5}x_2[n-1]$$

Since the system is LTI, we know the output:

$$\boxed{y_3[n] = 4y_1[n] + \frac{1}{5}y_2[n-1]}$$

$y_3[n] =$



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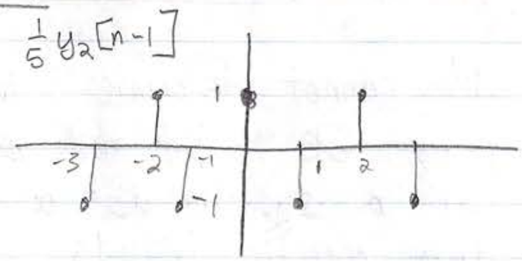
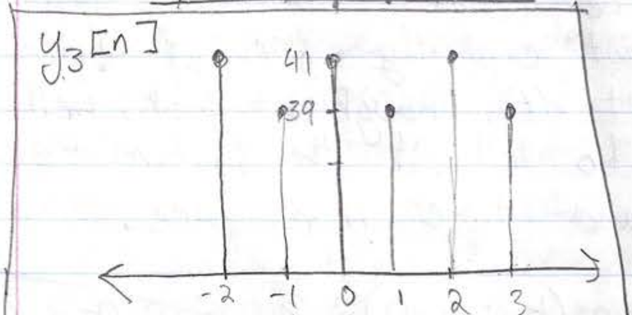




b) $y[n] = a_1 x[n-1] + a_2 x[n-2]$

$y_1[0] = 10 = a_1 x_1[-1] + a_2 x_1[-2] = a_1(1) + a_2(1)$

$y_2[0] = -5 = a_1 x_2[-1] + a_2 x_2[-2] = a_1(10) + a_2(-10)$

$$\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 10 & -10 & -5 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 10 \\ 1 & -1 & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 10 \\ 2 & 0 & 9.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 10 \\ 1 & 0 & \frac{19}{4} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & \frac{21}{4} \\ 1 & 0 & \frac{19}{4} \end{bmatrix}$$

$$\boxed{a_1 = \frac{19}{4} \quad a_2 = \frac{21}{4}}$$

Q6a) $\tau(t) = \dfrac{374 + 34t}{340}$

$y(t + \tau(t)) = x(t)$

$y\left(t + \frac{374+34t}{340}\right) = y\left(\frac{340t}{340} + \frac{374+34t}{340}\right) = y\left(\frac{340+34}{340}t + \frac{374}{340}\right)$

$t = \dfrac{374}{340}t + \dfrac{374}{340}$

$t = \frac{340t}{374} - \frac{374}{374}$

$\Rightarrow \boxed{y(t) = x\left(\frac{340}{374}t - 1\right)}$ $\boxed{a = \frac{374}{340} \quad b = 1}$

c) $x(t) = \cos(2\pi \times 10^6 t)$

$y(t) = x\left(\frac{340}{374}t - 1\right) = \cos\left(2\pi \times 10^6 \left(\frac{340}{374}t - 1\right)\right)$

$= \cos\left(2\pi \times 10^6 \left(\frac{340}{374}\right)t - 2\pi \times 10^6\right)$

$\boxed{\omega' = \left(\frac{340}{374}\right)(2\pi \times 10^6) \qquad f = \frac{340}{374} \times 10^6}$

This makes sense, the doppler effect should decrease the frequency when the wave source is moving away.

Q7. $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)$

$H\{x(t)\} = y(t) = x(2t)$
$x(2t) = A_0 + \sum_{k=1}^{\infty} \left[A_k \cos(k\omega_0 2t) + B_k \sin(k\omega_0 2t)\right]$

The only difference between $x(t)$ and $y(t)$ is that they have different fundamental frequencies and different periods:

$x(t)$: $\omega_0 = \omega_0$ Period $= T$ | $x(t+T) = x(t)$

$y(t)$: $\omega' = 2\omega_0$ T': Period $= \frac{T}{2}$

So $y(t)$ has half the period and twice the frequency. Otherwise $x(t)$ & $y(t)$ have the same "shape" and therefore each harmonic has the same weight.

$\langle x(t) \rangle_T = \langle y(t) \rangle_T$ = time average

Since they have the same average, $\boxed{A_0' = A_0}$

$$\boxed{A_k' = A_k \quad B_k' = B_k}$$

These results can be confirmed through integration.

Q9. a) $y(t) = .5y(t-1) + .3x(t)$

we know that since F is LTI, $y(t) = F(\omega)x(t)$

where $x(t) = e^{i\omega t} \rightarrow y(t) = F(\omega)e^{i\omega t}$

$$\Rightarrow F(\omega)\cancel{e^{i\omega t}} = .5F(\omega)\cancel{e^{i\omega t}}e^{-i\omega} + .3\cancel{e^{i\omega t}}$$

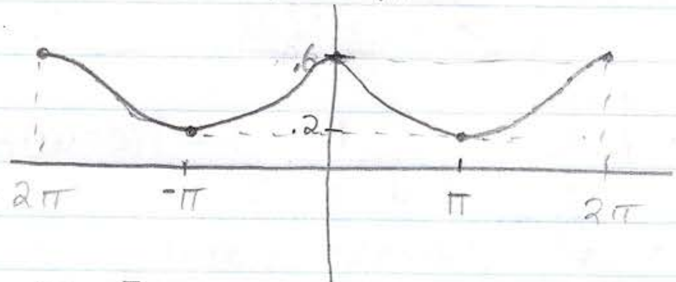
$$F(\omega)(1-.5e^{-i\omega}) = .3$$

$$\boxed{F(\omega) = \frac{.3}{1-.5e^{-i\omega}}}$$

$$\boxed{|F(\omega)| = \left|\frac{.3}{1-.5\cos\omega + .5i\sin\omega}\right| = \frac{.3}{\sqrt{(.5\sin\omega)^2 + (1-.5\cos\omega)^2}}}$$

$$\angle F(\omega) = -\tan^{-1}\left(\frac{.5\sin\omega}{1-.5\cos\omega}\right)$$

$|F(\omega)|$



b) $x(t) = 5\sqrt{2} + \frac{5}{3}\cos(\pi t - \frac{7\pi}{12})$

$$y(t) = |F(0)|\, 5\sqrt{2}\, e^{0} + |F(\pi)|\, \frac{5}{3}\cos\left(\pi t - \frac{7\pi}{12} - \tan^{-1}\left(\frac{.5\sin\pi}{1-.5\cos\pi}\right)\right)$$

$$= 3\sqrt{2} + \frac{1}{3}\cos\left(\pi t - \frac{7\pi}{12} - 0\right)$$

$$\boxed{y(t) = 3\sqrt{2} + \frac{1}{3}\cos\left(\pi t - \frac{7\pi}{12}\right)}$$