EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 14: October 17

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14.1 Announcements

- Guest lecture next Tuesday - Professor Miki Lustig (MRI)
- Reading: Chapter 10 of Lee and Varaiya
- Lecture notes posted have extra material.

14.2 Agenda

- DTFT - Discrete-time Fourier transform
- DFT - Discrete Fourier transform

14.3 Recap of DTFT

The DTFT allows us to find the spectrum of non-periodic signals. It provides us the Fourier representation of aperiodic signals.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega n}d\omega$$

Recall that when $x(n)$ is periodic with period p:

$$x_p(n) = \frac{1}{p}\sum_{k=0}^{p-1} X(k)e^{iw_0 kn}, w_0 = \frac{2\pi}{p}$$

Periodic (Fourier Series): $x(n)$ is a discrete sum of weighted complex exponentials.

$$\{e^{i\omega_0 kn}\}_{k=0}^{p-1}$$

Aperiodic (Discrete-time Fourier transform): x(n) is an "infinite sum" (i.e. integral) of complex exponentials $e^{i\omega n}, \forall\omega \in [-\pi, \pi]$ with weights given by $X_d(\omega)$ for frequency ω.

14.4 Relate the DTFT to Frequency Response

Why is $Y_d(\omega) = X_d(\omega)H(e^{i\omega})$? The synthesis equation of the DTFT shows that:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega n}d\omega$$

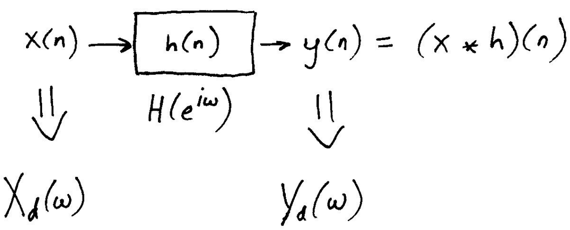

Figure 14.1: LTI system represented in time and frequency

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y_d(\omega)e^{i\omega n}d\omega$$

We know that in LTI systems for an input $x(n) = e^{i\omega_0 n}$ the output is $y(n) = H(e^{i\omega_0})e^{i\omega_0 n}$. That is, for any complex exponential input, the output is simply the input signal scaled by the frequency response. We can think of $X_d(\omega)$ as the set of all input exponentials in the input signal, each of which is scaled appropriately by $H(e^{i\omega})$ By the LTI property:

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)H(e^{i\omega})e^{i\omega n}d\omega$$

$$Y_d(\omega) = X_d(\omega)H(e^{i\omega})$$

Theorem 14.1.

DTFT[OUTPUT] = DTFT[INPUT] × [Frequency response of LTI system]

The DTFT of the output of a system is equal to the DTFT of the input signal multiplied by the frequency response of the system.

Recall, the DTFT$[\delta(n)] = 1, \forall\omega$. Apply our new theorem to this example:

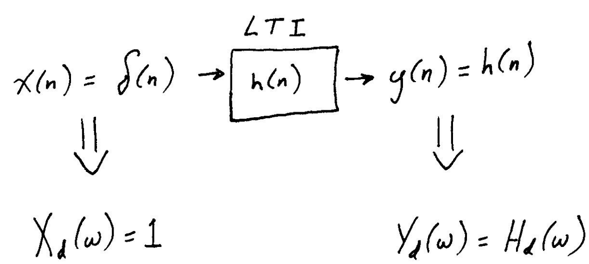

Figure 14.2: LTI system with $x(n) = \delta(n)$

$$\underbrace{X_d(\omega)}_{=1} \times \underbrace{H(e^{i\omega})}_{\text{Freq. resp. of sys.}} = Y_d(\omega) = H_d(\omega) = H(e^{i\omega})$$

Theorem 14.2. *$H(e^{i\omega})$, the frequency response of an LTI system is the DTFT of the impulse response $h(n)$ of the system.*

$$h(n) \xleftrightarrow{DTFT} H(e^{i\omega})$$

Question: How to go from $x(n)$ to $X_d(\omega)$?

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$X_d(\omega) = ?$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega n}d\omega$$

Given the linearity of the integral operator in the DTFT:

$$x_1(n) \longleftrightarrow X_{1,d}(\omega)$$

$$x_2(n) \longleftrightarrow X_{2,d}(\omega)$$

$$x_1(n) + x_2(n) \longleftrightarrow X_{1,d}(\omega) + X_{2,d}(\omega)$$

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \longleftrightarrow \alpha_1 X_{1,d}(\omega) + \alpha_2 X_{2,d}(\omega)$$

The above follows because the synthesis equation is linear. $x(n)$ is just a weighted sum of shifted impulses, therefore we need to find the DTFT$[\delta(n-k)]$.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega n}d\omega$$

What is $x(n-10)$?

$$x(n-10) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega(n-10)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{-i\omega 10}e^{i\omega n}d\omega$$

What is DTFT$[x(n-10)]$?

$$\text{DTFT}[x(n-10)] = X_d(\omega)e^{-i\omega 10}$$

Let's look at these Fourier transform pairs. The time domain representation will be shown in red, and the frequency domain representation will be shown in blue.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{i\omega n}d\omega$$

$$x(n-10) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{-i\omega 10}e^{i\omega n}d\omega$$

Fact 14.1.

$$x(n-k) \xleftrightarrow{DTFT} X_d(\omega)e^{-i\omega k}$$

$$DTFT[x(n)] = \sum_{k=-\infty}^{\infty} x(k)DTFT[\delta(n-k)] = \sum_{k=-\infty}^{\infty} x(k)e^{-i\omega k}$$

The forward, or analysis, DTFT is defined as:

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

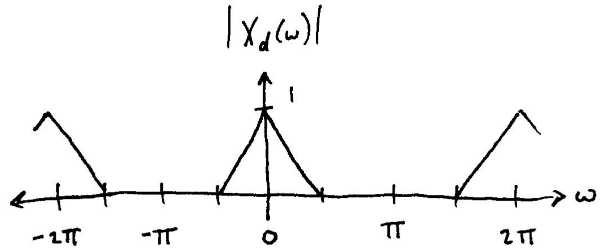

Figure 14.3: Frequency response of $x(n)$

Previously, we have seen the synthesis DTFT:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{i\omega n} d\omega$$

Example 14.1. *Find the DTFT[$x(n)e^{iw_0 n}$], $w_0 = \frac{\pi}{6}$ if $X_d(\omega)$ is shown in Fig. 14.3.*

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{i\pi/6n} e^{-i\omega n} = \sum_{n=-\infty}^{\infty} x(n) e^{-i(\omega - \pi/6)n} = X_d(\omega - \pi/6)$$

$$Y_d(\omega) = X_d(\omega - \pi/6)$$

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} y(n) e^{-i\omega n}$$

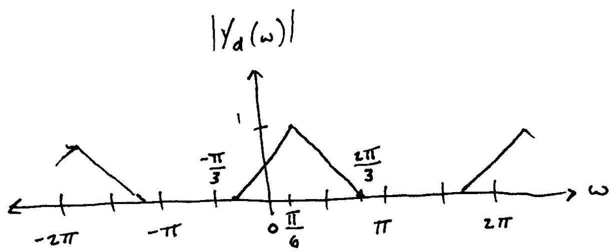

Figure 14.4: Frequency response of $y(n)$

Exercise 14.1. *Find the DTFT[$x(n)\cos(\frac{\pi}{6}n)$] and plot it. Hint:* $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

Not all signals can have a DTFT. The sum must be finite.

Exercise 14.2. *Here are some signals to practice working with the DTFT and DFT:*

- *Find the DTFT of* $x(n) = \alpha^n u(n), \alpha = 0.8$
- *Find the IDTFT of the Ideal low-pass filter.*

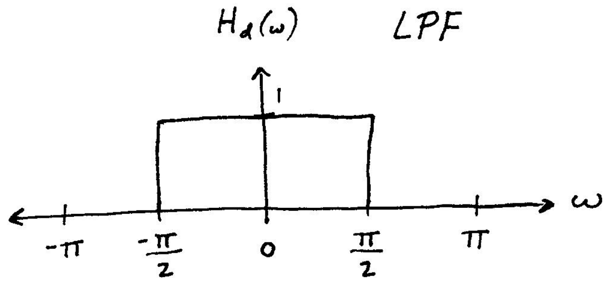

Figure 14.5: Frequency response of ideal low-pass filter with cutoff at $\omega = \pi/2$.

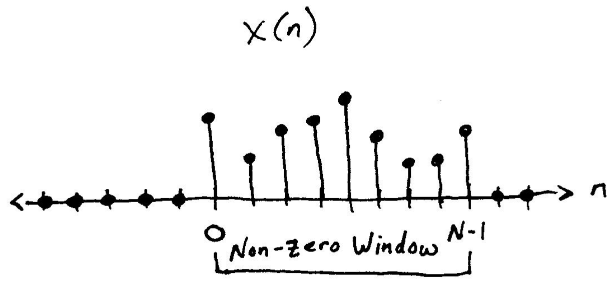

Figure 14.6: Finite length signal $x(n)$ to pass to DFT

14.5 DTFT and DFT

DTFT applies to infinitely long signals that are not practical to implement. DFT is a way of addressing this problem.

We consider only $[0, 1, \ldots, N-1]$ as the domain of $x(n)$. Clearly, this is related to the DFS representation of the periodic extension of $x(n)$.

$$\{x(n)\}_{n=0}^{N-1} \xleftrightarrow{\text{DFT}} \{X(k)\}_{k=0}^{N-1}$$

$$\underline{x}^N \xleftrightarrow{\text{DFT}} \underline{X}^N$$

Analysis:

$$X(k), k = [0, 1, \ldots, N-1] = \sum_{n=0}^{N-1} x(n) e^{-i\omega_0 nk}, \omega_0 = \frac{2\pi}{N}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk}$$

Synthesis:

$$x(n), n = [0, 1, \ldots, N-1] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{i\frac{2\pi}{N}kn}$$

The DFT is used a lot in practice because of its discrete nature in both the time and frequency domains. Restricting the input to a finite length can also be thought of as treating the signal as being periodic. This is only a first glance at the topic of DFT, but will hopefully serve as a reference point for the guest lecture by Professor Lustig on Tuesday.