Homework #3

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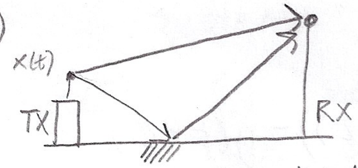
EE20N

1. a)



$y(t) = a_1 x(t-\tau_1) + a_2 x(t-\tau_2)$

Let input $x(t) = e^{i\omega_0 t}$ [complex sinusoid]

$$\Rightarrow y(t) = a_1 e^{i\omega_0 t} e^{-i\omega_0 \tau_1} + a_2 e^{i\omega_0 t} e^{-i\omega_0 \tau_2} = (a_1 e^{-i\omega_0 \tau_1} + a_2 e^{-i\omega_0 \tau_2}) e^{i\omega_0 t}$$
$$= H(\omega)\, x(t)$$

Input / Output: (with obstruction @ $t=2$)

$$x(t) = e^{i\omega_0 t} \longrightarrow \boxed{\text{2 Path System}} \longrightarrow y(t) = \begin{cases} \underbrace{(a_1 e^{-i\omega_0 \tau_1} + a_2 e^{-i\omega_0 \tau_2})}_{H(\omega_0)} e^{i\omega_0 t} & t<2 \\ \underbrace{a_2 e^{-i\omega_0 \tau_2}}_{H^*(\omega_0)} e^{i\omega_0 t} & t \ge 2 \end{cases}$$

$$= \begin{cases} H(\omega_0) e^{i\omega_0 t} = H(\omega_0) x(t) & t<2 \\ H^*(\omega_0) e^{i\omega_0 t} = H^*(\omega_0) x(t) & t \ge 2 \end{cases}$$

Homogeneity:

$$\hat{x}(t) = c x(t) = c e^{i\omega_0 t} \longrightarrow \hat{y}(t) = \begin{cases} H(\omega_0)\hat{x}(t) & t<2 \\ H^*(\omega_0)\hat{x}(t) & t \ge 2 \end{cases}$$

$$\Rightarrow \hat{y}(t) = \begin{cases} H(\omega_0)\, c x(t) & t<2 \\ H^*(\omega_0)\, c x(t) & t \ge 2 \end{cases} = \begin{cases} c y(t) & t<2 \\ c y(t) & t \ge 2 \end{cases} = c y(t)$$

Thus the system is homogeneous

Additivity:

$$\hat{x}(t) = x_1(t) + x_2(t) = e^{i\omega_1 t} + e^{i\omega_2 t} \longrightarrow \hat{y}(t) = \begin{cases} a_1 \hat{x}(t-\tau_1) + a_2 \hat{x}(t-\tau_2) & t<2 \\ a_2 \hat{x}(t-\tau_2) & t \ge 2 \end{cases}$$

$$= \hat{y}(t) = \begin{cases} a_1 e^{i\omega_1 t} e^{-i\omega_1 \tau_1} + a_1 e^{i\omega_2 t} e^{-i\omega_2 \tau_1} + a_2 e^{i\omega_1 t} e^{-i\omega_1 \tau_2} + a_2 e^{i\omega_2 t} e^{-i\omega_2 \tau_2} & t<2 \\ a_2 (e^{i\omega_1 t} e^{-i\omega_1 \tau_2} + e^{i\omega_2 t} e^{-i\omega_2 \tau_2}) & t \ge 2 \end{cases}$$

$$= \begin{cases} e^{i\omega_1 t} (a_1 e^{-i\omega_1 \tau_1} + a_2 e^{-i\omega_1 \tau_2}) + e^{i\omega_2 t} (a_1 e^{-i\omega_2 \tau_1} + a_2 e^{-i\omega_2 \tau_2}) \\ a_2 e^{i\omega_1 t} e^{-i\omega_1 \tau_2} + a_2 e^{i\omega_2 t} e^{-i\omega_2 \tau_2} \end{cases}$$

$$= \begin{cases} H(\omega_1) x_1(t) + H(\omega_2) x_2(t) \\ H^*(\omega_1) x_1(t) + H^*(\omega_2) x_2(t) \end{cases} = y_1(t) + y_2(t)$$

The system is Linear

Time-Invariance:

$$\hat{x}(t) = x(t-\tau_0) = e^{i\omega_0 t} e^{-i\omega_0 \tau_0} \longrightarrow \hat{y}(t) = \begin{cases} a_1 e^{+i\omega_0 t} e^{-i\omega_0 \tau_1} e^{-i\omega_0 \tau_0} + a_2 e^{i\omega_0 t} e^{-i\omega_0 \tau_2} e^{-i\omega_0 \tau_0} \\ a_2 e^{i\omega_0 t} e^{-i\omega_0 \tau_2} e^{-i\omega_0 \tau_0} \end{cases}$$

$$= \begin{cases} (a_1 e^{-i\omega_0 \tau_1} + a_2 e^{-i\omega_0 \tau_2}) e^{-i\omega_0 \tau_0} e^{i\omega_0 t} \\ a_2 e^{i\omega_0 t} e^{-i\omega_0 \tau_2} e^{-i\omega_0 \tau_0} \end{cases} = \begin{cases} H(\omega_0) x(t-\tau_0) & t<2 \\ H^*(\omega_0) x(t-\tau_0) & t \ge 2 \end{cases} \neq \begin{cases} y(t-1) & t-1<2 \\ y(t-1) & t-1 \ge 2 \end{cases}$$

The system is NOT Time Invariant

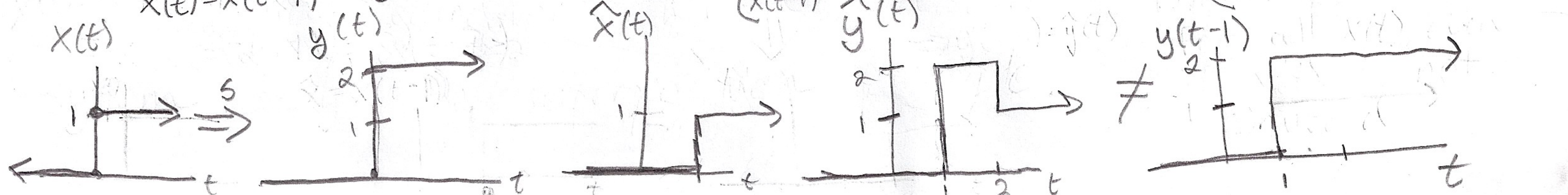
Counter Example: Let $x(t) = u(t)$ [step function]

$a_1 = a_2 = 1 \quad \tau_1 = \tau_2 = 0 \quad \tau_0 = 1$

$y(t) = 2x(t)$

$\hat{x}(t) = x(t-1)$

$$\Rightarrow \hat{y} = 2\hat{x}(t) = 2x(t-1) = \begin{cases} 2x(t-1) & t<2 \\ x(t-1) & t \ge 2 \end{cases} \neq y(t-1) = \begin{cases} 2x(t-1) & t-1<2 \\ x(t-1) & t-1 \ge 2 \end{cases}$$



b) [8.7.6] Timescale: $[R \to R] \to [R \to R]$, $y = \text{Timescale}(x)$:

$\forall t \in R,\ y(t) = x(2t)$

$x(t) \longrightarrow y(t)$

- $\hat{x}(t) = x(t-\tau) \longrightarrow \hat{y}(t) = \hat{x}(2t) = x(2t-\tau) \neq x(2(t-\tau)) = y(t-\tau)$

the system is <u>not</u> Time invariant

c) [8.6.6] A discrete system S is Time Invariant if for every valid I/O pair $x(n) \to y(n)$, the I/O pair $x(n-M) \longrightarrow y(n-M)$ is also a valid pair, where M is the sample delay.

- $D: [Z \to R] \to [Z \to R]$ where if $y = D(x)$

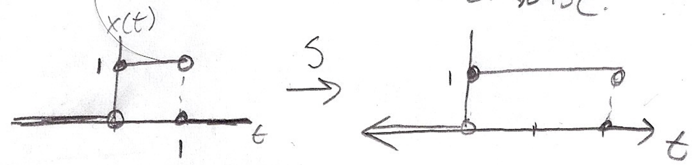
Then for all $n \in Z$, $y(n) = x(n-1)$

$\hat{x}(n) = \underline{x(n-M)} \longrightarrow \hat{y}(n) = \hat{x}(n-1) = x(n-1-M) = \underline{y(n-M)}$

The sytem <u>is</u> <u>Time</u> <u>Invariant</u>

2. LTI system $y = S(x)$:

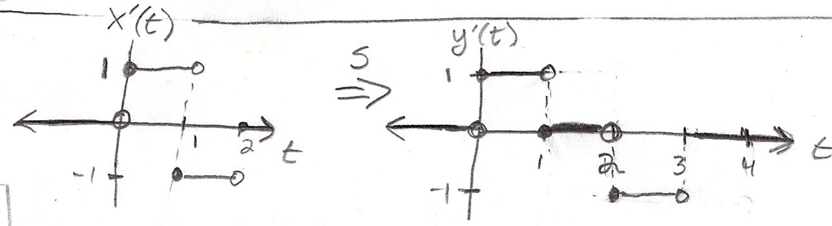
$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \iff y(t) = \begin{cases} 1 & 0 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$



a) $$x'(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

$x'(t) = x(t) - x(t-1)$

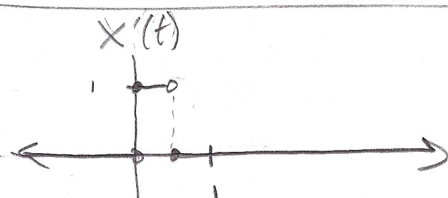
LTI: $\boxed{y'(t) = y(t) - y(t-1)}$



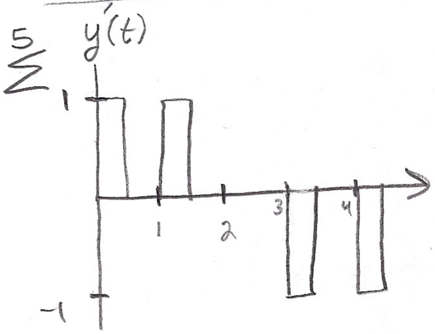
$$y'(t) = \begin{cases} 1 & 0 \le t < 0 \\ -1 & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

b) $$x'(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

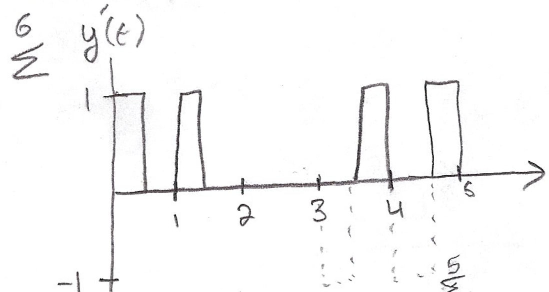
$$x'(t) = x(t) - x(t - \tfrac{1}{2}) + x(t-1) - x(t - \tfrac{3}{2}) + x(t-2) + \dots = \sum_{n=0}^{\infty} x(t - \tfrac{n}{2})(-1)^n$$



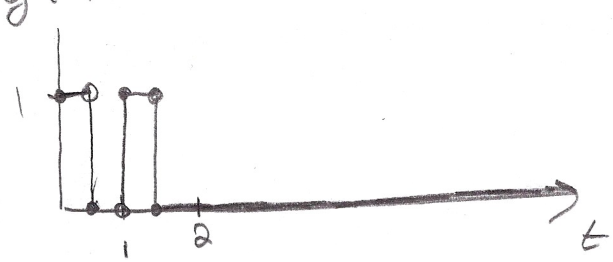
LTI: $$y'(t) = y(t) - y(t - \tfrac{1}{2}) + y(t-1) - y(t - \tfrac{3}{2}) + y(t-2) + \dots = \sum_{n=0}^{\infty} y(t - \tfrac{n}{2})(-1)^n$$



$\Rightarrow$



$\Rightarrow$ $\sum^{\infty}$ $y'(t)$



$$y'(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2},\ 1 \le t < \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

3.a) $x(t) = 1 + \cos(\pi t) + \cos(2\pi t) = e^{i0} + \frac{1}{2}(e^{i\pi t} + e^{-i\pi t}) + \frac{1}{2}(e^{i2\pi t} + e^{-i2\pi t})$

$H(\omega) = \begin{cases} e^{i\omega} & |\omega| < 4 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$ $= \sum_{k=-2}^{2} e^{ik\omega_0 t} = \sum_{k=-2}^{2} e^{ik\pi t}$ $\omega_0 = \pi$

$y(t) = H(\omega)x(t) = H(\omega)\sum_{k=-2}^{2} e^{ik\pi t} = \sum_{k=-2}^{2} e^{ik\pi t} H(k\pi)$ where $\omega = k\pi$

$= e^{i0}e^{i0} + e^{i\pi}\frac{1}{2}(e^{i\pi t} + e^{-i\pi t}) + (0)\frac{1}{2}(e^{i2\pi t} + e^{-i2\pi t})$

$= 1 + (\cos\pi + i\sin\pi)\cos\pi t$

$= 1 + (-1 + 0)\cos\pi t$

$\boxed{y(t) = 1 - \cos\pi t} = 1 - \frac{1}{2}(e^{i\pi t} + e^{-i\pi t})$

b) suppose a general $x(t) = x(t+p)$

$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_0 k t} = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{p}kt}$

$\Rightarrow y(t) = H(\omega)x(t) = H(\omega)\sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{p}kt}$

because $H(\omega) = 0$ for $|\omega| \geq 4$, we know the bounds of k:

$-4 < \frac{2\pi k}{p} < 4 \iff -\frac{2p}{\pi} < k < \frac{2p}{\pi}$

$\Rightarrow -\lfloor\frac{2p}{\pi}\rfloor < k < \lfloor\frac{2p}{\pi}\rfloor$

$y(t) = H(\frac{2\pi k}{p}) \sum_{-\lfloor\frac{2p}{\pi}\rfloor}^{\lfloor\frac{2p}{\pi}\rfloor} c_k e^{i\frac{2\pi}{p}kt}$

$y(t) = \sum_{-\lfloor\frac{2p}{\pi}\rfloor}^{\lfloor\frac{2p}{\pi}\rfloor} c_k e^{i\frac{2\pi k}{p}} e^{i\frac{2\pi}{p}kt} = \boxed{\sum_{-\lfloor\frac{2p}{\pi}\rfloor}^{\lfloor\frac{2p}{\pi}\rfloor} c_k e^{i\frac{2\pi}{p}k(1+t)}}$

cosine form:

$$\boxed{y(t) = \sum_{k=0}^{\lfloor\frac{2p}{\pi}\rfloor} A_k e^{i\frac{2\pi}{p}k(1+t)}}$$