EE20N: Structure and Interpretation of Systems and Signals	**Fall 2013**

Lecture 04: September 10

Lecturer: Kannan Ramchandran — *Scribe: Rick Garcia*

4.1 Announcement

- Homework 1 due on Thursday (9/12/13) by noon.
- Homework party in Woz from 3-5pm on Tuesday (9/10/13).
- Please bring original account sheets to lab this week.

4.2 Continuous Time Fourier Series

Joseph Fourier (1768-1830) found in the 18$^{\text{th}}$ century that is is possible to approximate "any" periodic signal as a bunch of sinusoids whose frequencies are integer multiples of the fundamental frequency of the periodic signal: $\omega_0 = \frac{2\pi}{p}$ as

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \tag{4.1}$$

Note that this equality is "almost everywhere" and is not pointwise.

Example 4.1. *Consider the following triangle wave signal with period T, fundamental frequency: $w_o = 2\pi/T$ rad/sec:*

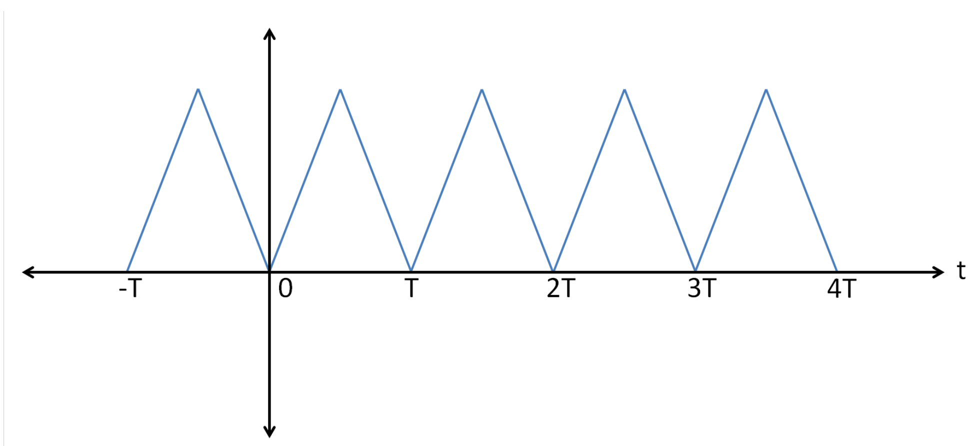

Figure 4.1: Triangle wave as a sum of sinusoidal signals

This signal can be represented by the summation of multiple sinusoids. Each with different amplitudes.

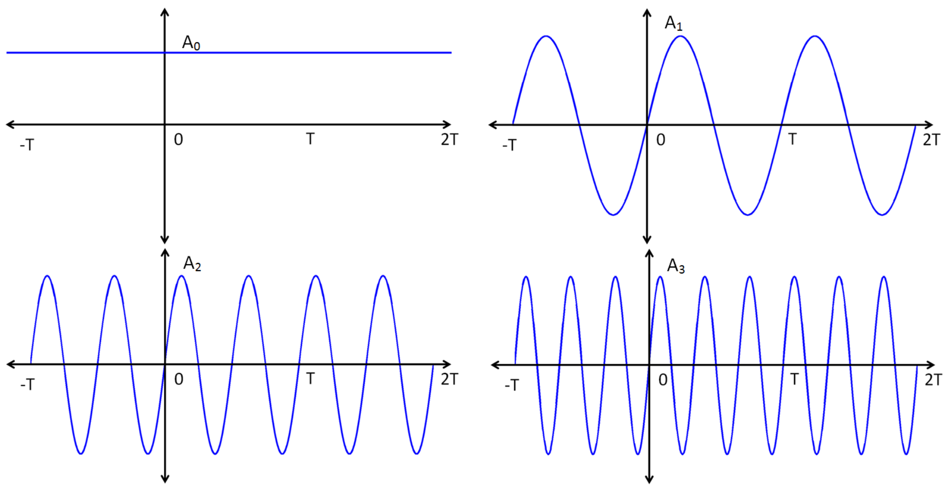

Figure 4.2: Fourier series harmonics.

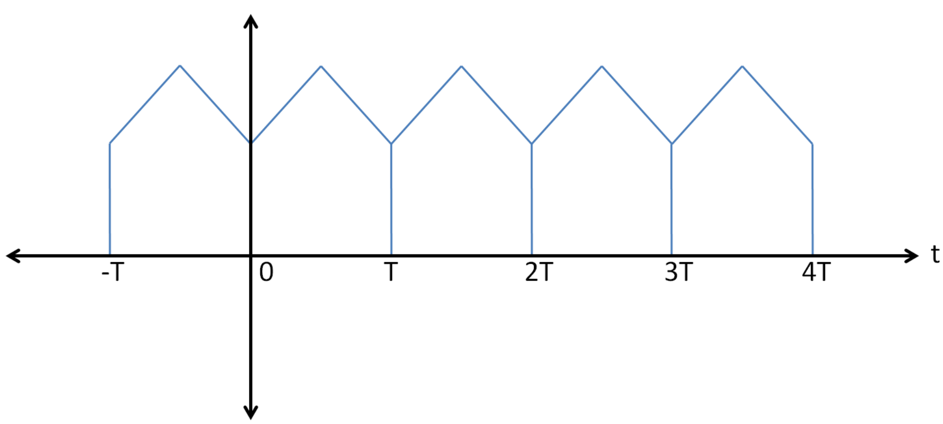

Figure 4.3: Sample “house” signal.

The Fourier series coefficients are unique for each unique periodic signal. A signal such as the one shown in 4.3 would have a different set of coefficients than the ones for 4.1.

The Fourier series representation is true except possibly in countably many points if there is a discontinuity in the signal. This is covered in detail in Chapter 7 of the textbook. For example, in a square wave, the Fourier series reconstruction would be correct except at the points where the signal transitions between the two values of the signal. The ringing effects at these points are known as the Gibbs Phenomenon. The overshoot at this regions can be around $8-9\%$ of the amplitude of the signal. The Gibbs Phenomenon will be explored further in lab. Here are some applets to play with the Gibbs phenomenon further:

http://ptolemy.eecs.berkeley.edu/eecs20/week10/negativefreqs.html
http://robertus.staff.shef.ac.uk/ama349/fourier/
http://robertus.staff.shef.ac.uk/ama349/fourier/
http://homepages.gac.edu/ huber/fourier/

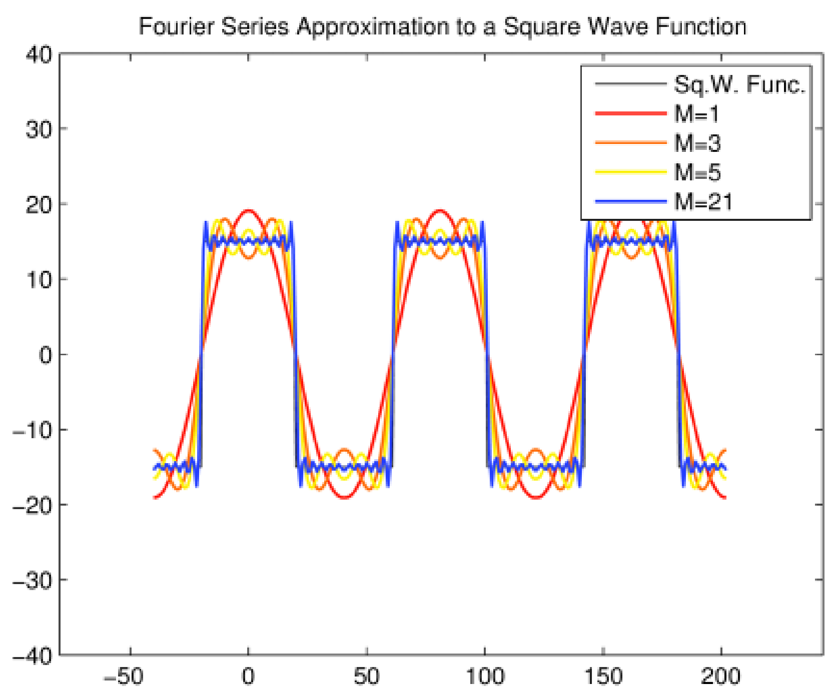

Figure 4.4: The approximation of the square wave improves with more Fourier series coefficients. Source: http://sualtinbasak.wordpress.com/

Note: Finite signals can be thought of as periodic signal, so these methods apply. Other than the "coolness" of the Fourier series expansion of periodic signals, there are many applications.

Example 4.2. *Speech Recognition. A long duration audio recording of speech can be cut into snippets and approximated by phonemes to recognize speech.*

Example 4.3. *Finance. Stock market data can be treated as a signal and approximated with a Fourier series. From looking at the energies of low frequency components (namely A_i for small i), a day trader can judge how strong slow changing trends are in that stock. Similarly, high energy in high frequency components (A_i for large i) indicate a volatile market.*

When working with the Fourier series, the trig identity $\cos\alpha + \beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ can be useful. Eq 4.1 can also be represented as:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k(\cos kw_0 t\cos\phi_k - \sin kw_0 t\sin\phi_k)$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (\alpha_k \cos kw_0 t + \beta_k \sin kw_0 t)$$

The A_0 term in the Fourier series expansion can be determined by integrating the signal over a single period.

$$A_0 = \frac{1}{T}\int_0^T x(t)dt$$

Similarly, the terms for α_k and β_k can be determined.

$$\alpha_k = \frac{2}{T}\int_0^T x(t)\cos kw_0 t dt$$

$$\beta_k = \frac{2}{T}\int_0^T x(t)\sin kw_0 t dt$$

Example 4.4. *Consider the signal in 4.5. Find the value of A_0.*

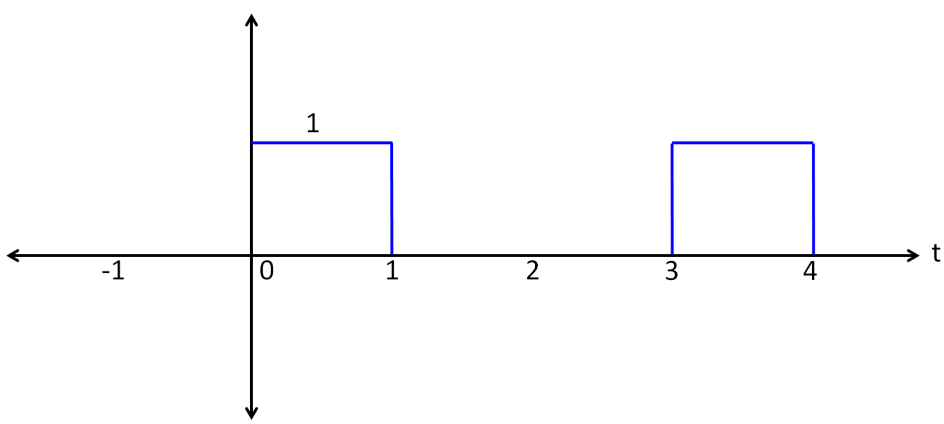

Figure 4.5: Determine the coefficients A_0, α_1, and β_k

$$A_0 = \frac{1}{3}\int_0^3 x(t)dt = \frac{1}{3}\int_0^1 1dt = \frac{1}{3}$$

For practice, work out the values of α_k and β_k. You should find:

$$\alpha_k = \frac{1}{k\pi}\sin(\frac{2\pi}{3}k)$$

4.3 Discrete Time Signals

The term "discrete time signals" encompasses signals in both time and space. Whereas continuous signals in time have units of *seconds*, discrete signals in time have units of *samples*. Commonly, discrete time signals are the result of sampling continuous time signals.

Definition 4.1. *A discrete time signal $x(n)$ is a function where $n \in \mathbb{Z}$.*

Definition 4.2. *The discrete time sinusoid is the function $x(n) = \cos(2\pi f n)$.*

Because of the interesting properties of the Fourier series in continuous time, we are motivated to consider a discrete time analogy. To begin, we try to understand the notion of periodicity and frequency in the discrete time domain. Consider sampling at different rates of f from the continuous time signal $x(t) = \cos(t)$.

Example 4.5. *If $f = 1$, then we have $x(n) = \cos(2\pi n) = 1$, the constant function. So $p = 1$.*

Example 4.6. *If $f = \frac{1}{2}$, then we have $x(n) = \cos(\pi n)$ which is the sequence $1, -1, 1, -1, ...$, so $p = 2$.*

Example 4.7. *If $f = \frac{1}{3}$, then we have $x(n) = \cos(\frac{2\pi}{3} n)$ which is the sequence $1, -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, ...$, so $p = 3$.*

It may be tempting to conclude from the above examples the relation of $p = f^{-1}$. To show that this is not true, consider the case where $f = 2$. By the above formula, we guess that the period $p = 2^{-1} = \frac{1}{2}$, but a half-sample makes no sense in the context of discrete time. Furthermore, the resulting samples of the continuous time signal are $x(n) = \cos(2\pi \cdot 2n) = \cos(4\pi \cdot n) = 1$, yielding a period of $p = 1$, so the formula $p = f^{-1}$ is not always true. In fact, *increasing f after a certain point does not change the period* $(**)$.

What if f is irrational? Then $x(n)$ will never repeat the same value, and $x(n)$ is not periodic. In particular, if $x(n)$ is p-periodic, then

$$\begin{aligned} x(n) &= \cos(2\pi f n) \\ &= \cos(2\pi f(n+p)) \qquad p\text{-periodicity implies } x(n) = x(n+p) \\ &= \cos(2\pi f n + 2\pi f p). \end{aligned}$$

This can only be true if $2\pi f p$ is an integer multiple of 2π. Since p must be an integer, this also implies that f is rational. We summarize this result in the following:

Definition 4.3. *If $f = \frac{a}{b}$ where a and b are relatively prime, then $p = b$.*