EE20N: Structure and Interpretation of Systems and Signals **Spring 2013**

Lecture 08: September 24

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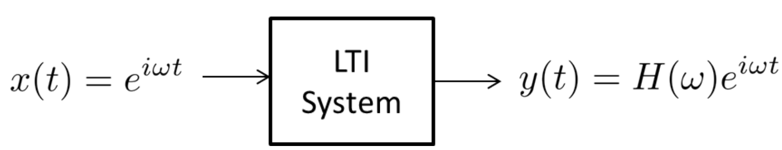
8.0.1 Announcements

- MT1 on Thursday from 6-7.30. Even SIDs go to 10 Evans, and odd SIDs go to 100 GPB. If you don't have a SID, go to either. You get 1 hand-written cheat sheet. NO CALCULATORS, BOOKS, OR ELECTRONICS.
- Homework 3 is due Thursday at noon, but subsequent homeworks will be due on Thursday at 11:59 pm.
- We will have class this Thursday, 9/26
- No class next Thursday, Oct. 3

8.1 Frequency Response

Remark 8.0. *Recall from last lecture that we will use the term LTI to refer to both continuous-time (CT) and discrete-time (DT) systems. In reality, the units of discrete-time signals are in samples, not seconds, so we should be using the term "Linear Shift-Invariant". But we are not pedants, so we'll just use LTI to describe CT and DT systems.*

Now, we know that LTI systems have associated frequency response behavior. So in CT, this means



This is a nice property because if you input a signal of a particular frequency, the ouput is the same frequency, times some scaling factor. This is useful because we can decompose general signals into a sum of components with different frequencies using Fourier decompositions. Therefore, an LTI system is fully defined by its frequency response.

Remark 8.0. *Recall that we can write $H(w)$ as $|H(w)| * e^{i\angle H(w)}$, i.e., the product of the magnitude of the frequency response and the phase of the frequency response.*

Example 8.1. *Low-pass filter*

Suppose we have a system whose frequency response magnitude looks like this:

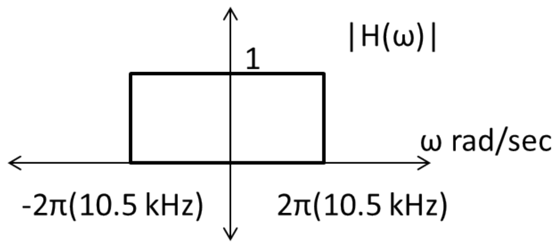


Figure 8.1: The magnitude of our system's frequency response, $|H(\omega)|$

and whose phase is 0 for all ω, i.e., $\angle H(\omega) = 0$. Now suppose we input the following signal $x(t)$:

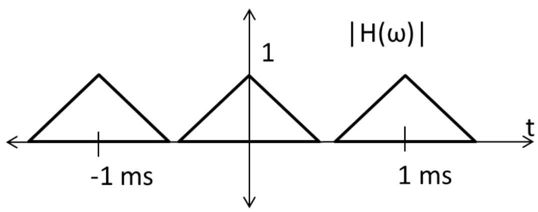


Figure 8.2: The input to our system, $x(t)$

We observe that the period here is $T = 1 \times 10^-3$ seconds, so we have $\omega_o = 2\pi/T = 2\pi \times 10^3$ rad/sec.

Question: What does the Fourier series for $x(t)$ look like? (You already know the general form below. Which terms are nonzero?)

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ki\omega_o t}$$

Answer: The frequencies present in this signal are all integer multiples of ω_o.

Question: What is the output of the LTI system given the Fourier expansion of $x(t)$?
Answer: We know that $y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_o) X_k e^{ikw_o t}$. However, in our example, $H(k\omega_o)$ is 1 on the range $[-10.5kHz, 10.5kHz]$ and 0 elsewhere.

To see what is happening visually, we can plot our exponential coefficients X_k as a function of k.

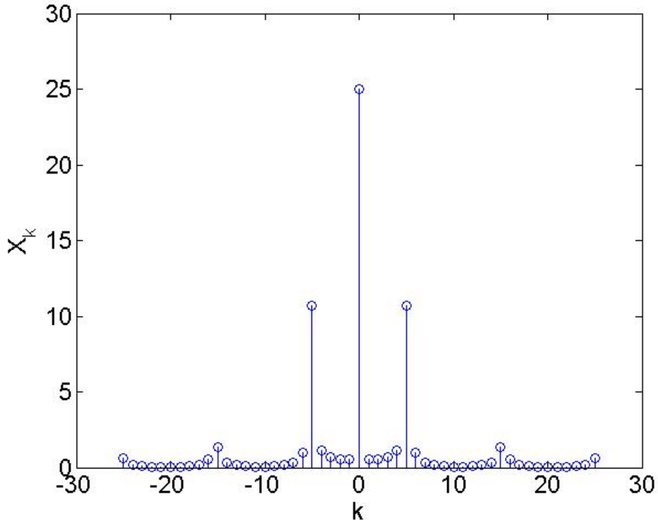


Figure 8.3: The Fourier coefficients for our triangle wave input $x(t)$. (I.e. the X_k coefficients plotted as a function of k.)

But each of those C_k coefficients is associated with a complex sinusoid of frequency $\omega_o k$; basically, the coefficient X_k tells you how much of frequency $\omega_o k$ is present in the overall signal $x(t)$. So let's plot the X_k coefficients for our signal as a function of the corresponding frequency, and then overlay our system's frequency response.

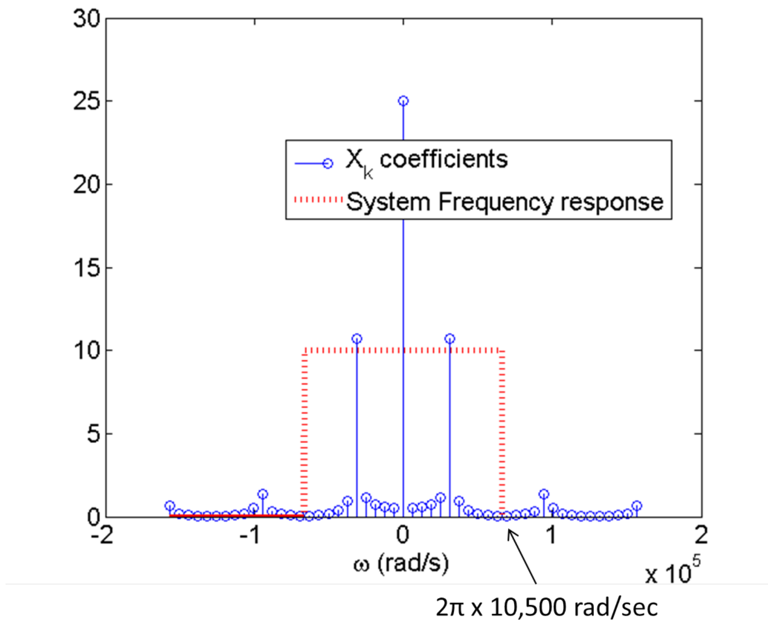


Figure 8.4: The Fourier coefficients for our triangle wave input $x(t)$ as a function of angular frequency ω. Each tickmark on the x-axis represents a frequency of the form $k\omega_o$. The system frequency response has been amplified (the rectangle has height 10 in the figure, instead of height 1), just to show it on the same axes as the coefficients.

When our signal passes through the system, we know that the C_k coefficients are multiplied by $H(\omega_o k)$. Therefore, our system will zero out all frequencies present in $x(t)$ that are greater than 10.5 kHz in magnitude. In particular, this means that if $k\omega_o > 2\pi \times 10.5 \times 10^3$ rad/sec, then that term of the Fourier series will be zeroed out. To find the maximum k for which this holds, we find $\lfloor 2\pi \times 10.5 \times 10^3/\omega_o \rfloor = \lfloor 10.5 \rfloor = 10$, where $\lfloor\ \rfloor$ denotes the floor function. Therefore, we can write our output $y(t)$ as

$$y(t) = \sum_{k=-10}^{10} X_k e^{ikw_o t}$$

This LTI system is called a *low-pass filter* because it removes (i.e., it filters) all the high frequencies from a signal that gets passed through it.

8.2 Backing up our claims about LTI systems

Why is it that for LTI systems, output frequency is a scaled version of the input frequency for a complex sinusoid? To understand why this is the case, let's look at the key properties of linearity and time-invariance.

Claim 8.1. *Assume $x(t) = e^{i\omega t}$ is passed through an LTI system, and the output is $y(t)$. We claim that $y(t) = H(\omega)e^{i\omega t}$.*

Proof: We are given that the system is LTI. So let's suppose the input to the system is $x(t-\tau)$. Then by LTI-ness, the output should be $y(t-\tau)$. Now, we know that $x(t-\tau) = e^{i\omega(t-\tau)}$. This can be written as $e^{-i\omega\tau}e^{i\omega t}$, so let's invoke the homogeneity property of linearity. Homogeneity implies that the output for $x(t-\tau)$ when passed through the LTI system is $e^{-i\omega\tau}y(t)$. So $y(t-\tau) = e^{-i\omega\tau}y(t)$. Then $y(t) = e^{i\omega\tau}y(t-\tau)$. This relationship holds for ALL values of τ. So let's pick $\tau = t$. Then we get $y(t) = y(0)e^{i\omega t}$. But $y(0)$ is just a constant! Let's call it $H(\omega)$. So we've shown that if the input is $x(t) = e^{i\omega t}$, then the output is a constant times the same complex exponential. So $y(0)$ is precisely what we call the frequency response. NOTE: $y(0)$ depends on ω because $y(t)$ is the output at time 0 when you input a sinusoid of frequency ω. Therefore, if you change the input to a different frequency ω_2, then you would expect $y(0)$ to change (in general).

Example 8.2. *RC Circuit*

Suppose we have a circuit with an input, a resistor of resistance R, and a capacitor of resistance C:

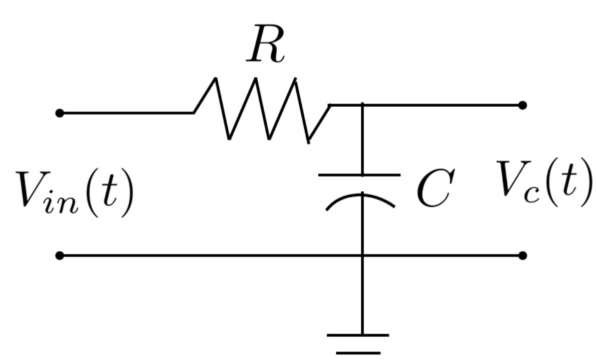


Figure 8.5: Our input is the voltage $V_{in}(t)$, which is a function of time, and we wish to measure our output $V_c(t)$, which is the voltage across the capacitor.

To find the output of this system, we first take a look at some important equations. If you don't know these equations, it's fine. We just want to make sure that you can use them to find a frequency response for the system!

1. $i_c(t) = C\frac{dVc(t)}{dt}$ describes the current through the capacitor.
2. $V_{in}(t) = i_c(t)R + V_c(t)$ results from Kirchoff's Voltage Law.

Now, we wish to solve for $V_c(t)$. Substituting for $i_c(t)$, we get $V_{in}(t) = V_c(t) + RC \cdot dV_c(t)/dt$. Now let's treat the circuit as a black box and find the frequency response. To match our earlier notation, we can call $V_{in}(t) = x(t)$, and $y(t) = V_c(t)$. Rewriting our equation, we get:

$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

Question: Is this system LTI?
Answer: Yes. Verify this for yourselves.

Question: If the system is LTI, it must have a frequency response. So what is the frequency response $H(\omega)$?
Answer: Since the system is LTI, we know that if the input is $e^{i\omega t}$, the output must be $H(\omega)e^{i\omega t}$. So let's start by inputting that, and trying to solve for $H(\omega)$. Remember that $d/dte^{i\omega t} = i\omega e^{i\omega t}$. This gives $e^{i\omega t} = RC \cdot H(\omega)(i\omega)e^{i\omega t} + H(\omega)e^{i\omega t}$. This is a really nice expression, because we can divide through by $e^{i\omega t}$ and solve for $H(\omega)$. We get

$$1 = H(\omega)[RC \cdot i\omega + 1] \implies \tag{8.1}$$

$$H(\omega) = 1/(1 + i\omega \cdot RC) \tag{8.2}$$

This is precisely the frequency response of the RC circuit.

To find and plot the magnitude of this frequency response, we just need to divide the magnitudes of the numerator and denominator of $H(\omega)$. The numerator clearly has magnitude 1, and the denominator has magnitude $\sqrt{1 + \omega^2 R^2 C^2}$. For $RC = 0.1$, this looks something like the following:

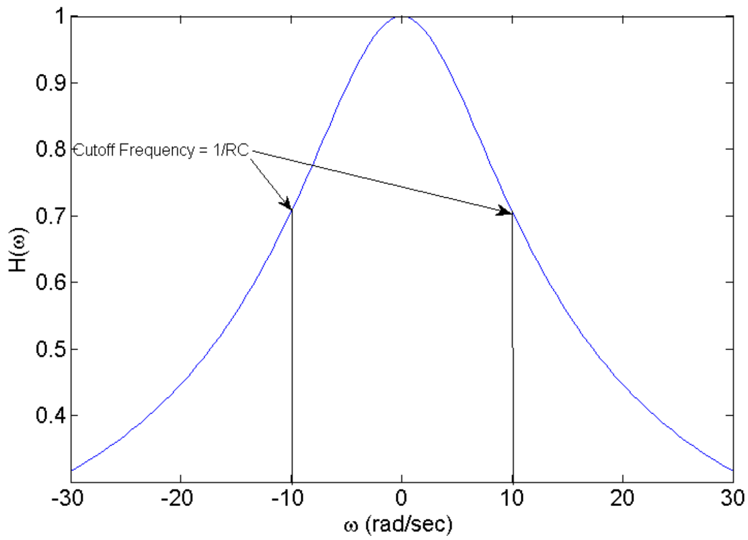


Figure 8.6: Frequency response of the RC circuit. Note the nominal cutoff at frequency $\omega = 1/(RC)$. This is the frequency where the frequency response is 3 dB less than the peak response value (i.e., 1).

As the frequency increases, the frequency response tends to zero. Therefore, this system is also a low-pass filter! It lets only the low frequencies through. Note that at $\omega = 1/RC$, the frequency response magnitude is 3 decibels below the peak. This frequency is called the "nominal cutoff" for the filter.

8.3 Parting thoughts

When we are dealing with LTI systems, they can be either continuous or discrete time. Similar frequency response behavior happens for DT-LTI systems and CT ones. But there's an important difference!

In DT, since n is an integer, $e^{i\omega n} = e^{i(\omega+2pi)n}$. That is, the complex exponential is periodic, with period $2\pi/\omega$. Thus for an LTI-DT system, $H(\omega)$ is also periodic with period 2π. To emphasize this fact, we write $H(\omega)$ in the discrete-time case as $H(e^{i\omega})$. So our notation is as follows:

$$
\begin{aligned}
x(t) &\rightarrow LTI(CT) \rightarrow y(t) = H(\omega)e^{i\omega t} \\
x[n] &\rightarrow LTI(DT) \rightarrow y(t) = H(e^{i\omega})e^{i\omega n}
\end{aligned}
$$