Physics 105 (Fall 2013): Solution to HW #9

1. The constraint is: $\dot{x} = R\dot{\phi}$, which can be integrated to give $f = x - R\phi = 0$. The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx$. Using Lagrange equation with constraint we have: $m\ddot{x} - mg = \lambda\frac{\partial f}{\partial x} = \lambda$ and $I\ddot{\phi} = \lambda\frac{\partial f}{\partial \phi} = -R\lambda$. One then get: $\ddot{x} = R\ddot{\phi} = \frac{mg}{m+I/R^2}$ and $\lambda = -\frac{mgI/R^2}{m+I/R^2}$ which is the negative of the tension, while $\lambda\frac{\partial f}{\partial \phi}$ is the torque on the wheel.

2. If $u = r^{-1}$, then the equation of orbit is: $u''(\phi) + u = -\frac{\mu F}{l^2 u^2}$. Now $u = (\kappa\phi^2)^{-1}$, $u'' = \frac{6}{\kappa\phi^{-4}} = 6\kappa u^2$. So $F(r) = -\frac{l^2}{\mu}(6\kappa u^4 + u^3) = -\frac{l^2}{\mu}(6\kappa/r^4 + 1/r^3)$.

3. The equation of the orbit is: $r = \frac{c}{1+\epsilon\cos\phi}$, where $c = \frac{l^2}{\gamma\mu}$, with γ being the strength of the interaction($F(r) = -\gamma/r^2$) and the mass of the satellite being μ. We know $\dot{r} = -\frac{l}{\mu}\frac{d}{d\phi}(r^{-1})$ from chain rule and angular momentum conservation, this gives $\dot{r} = \frac{l\epsilon}{\mu c}\sin\phi$ and the maximum is $\dot{r}_{max} = \frac{l\epsilon}{\mu c}$. From Kepler's second law, $\tau = \pi\alpha b/\frac{l}{2\mu} = 2\pi\alpha b\mu/l$, here $A = \pi\alpha b$ is the area of the ellipse and $\frac{dA}{dt} = \frac{l}{2\mu} = \frac{1}{2}r^2\dot{\phi}$. Finally using $b = \frac{c}{\sqrt{1-\epsilon^2}}$ we arrive at $\dot{r}_{max} = \frac{2\pi\alpha\epsilon}{\tau\sqrt{1-\epsilon^2}}$.

4. (a) Let θ be the angle between the vertical and the line joining axis of the can and that of the drum, also let ϕ be the rotation angle of the can. The no slipping condition is $(R-r)\dot{\theta} + r\dot{\phi} = R\Omega$, which can be integrated to give the constraint $f = (R-r)\theta + r\phi - R\Omega t = 0$. The Lagrangian is then $L = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 + mg(R-r)\cos\theta - \lambda f$.

   (b) It is better to use the differential constraint to eliminate $\dot{\phi}$, instead of including a Lagrange multiplier. Then $L = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{I}{2r^2}[(R-r)\dot{\theta} - R\Omega]^2 + mg(R-r)\cos\theta$. The equation of motion is: $(m + I/R^2)(R-r)\ddot{\theta} = -mg\sin\theta$. There is a equilibrium at $\theta = 0$, it's stable since the potential is a minimum there.

   (c) For small deviations, $(m+I/R^2)(R-r)\ddot{\theta} = -mg\theta$, this is a harmonic oscillation of frequency $\omega = \sqrt{\frac{mg}{(m+I/R^2)(R-r)}}$.

5. (a) $\mathbf{A}\cdot\mathbf{L} = 0$ since both $\mathbf{r}\times\mathbf{L}$ and $\hat{r}$ are perpendicular to $\mathbf{L}$.
   $\mathbf{A}\cdot\mathbf{p} = -\mu\kappa\hat{r}\cdot\mathbf{P} = -\mu^2\kappa\dot{r} = -\mu^2\kappa\frac{l\epsilon}{\mu c}\sin\phi$ from the result of question 3. Now $c = \frac{l^2}{\kappa\mu}$ and $\epsilon = \sqrt{1 + \frac{2El^2}{\mu\kappa^2}}$, thus $\mathbf{A}\cdot\mathbf{p} = -\frac{\mu^2\kappa^2}{l}\sqrt{1 + \frac{2El^2}{\mu\kappa^2}}\sin\phi$.
   $\mathbf{A}\cdot\mathbf{A} = p^2l^2 - 2\mu\kappa\hat{r}\cdot(\mathbf{p}\times\mathbf{L}) + \mu^2\kappa^2$. From $\hat{r}\cdot(\mathbf{p}\times\mathbf{L}) = \mathbf{L}\cdot(\hat{r}\times\mathbf{p}) = \frac{l^2}{r}$ and $E = \frac{p^2}{2\mu} - \frac{\kappa}{r}$, we get $\mathbf{A}\cdot\mathbf{A} = 2\mu El^2 + \mu^2\kappa^2$.
   $\mathbf{A}\cdot\mathbf{r} = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - \mu\kappa r = l^2 - \mu\kappa r$.

   (b) $\dot{\mathbf{A}} = \dot{\mathbf{p}}\times\mathbf{L} - \mu\kappa\frac{d}{dt}\hat{r}$, and $\dot{\mathbf{p}} = -\mu\kappa\hat{r}/r^2$. Now $\hat{r}\times(\mathbf{r}\times\mathbf{p}) = \mu\dot{r}\mathbf{r} - r\mathbf{p} = -\mu r^2[\dot{\mathbf{r}}(\frac{1}{r}) + (\frac{-\dot{r}}{r^2})\mathbf{r}] = -\mu r^2\frac{d}{dt}(\frac{\mathbf{r}}{r})$ from BAC-CAB rule. This shows $\dot{\mathbf{A}} = 0$.

6. Practice.

7. Practice.

8. (a) Using spherical coordinate with fixed $\theta = \alpha$, the Lagrangian is $L = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2) - mgr\cos\alpha$. The conserved angular momentum is $L = mr^2\sin^2\alpha\dot{\phi}$ and this gives the effective potential: $U_{eff} = \frac{L^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha$. For a stable circular orbit, one has to minimize the effective potential: $U' = mg\cos\alpha - \frac{L^2}{mr^3\sin^2\alpha} = 0$. This will give the condition $r = (\frac{L^2}{m^2\sin^2\alpha g\cos\alpha})^{1/3}$ or $g\cos\alpha = r\sin^2\alpha\dot{\phi}^2$.

   (b) One expand about the minimum of the potential, giving $U_{eff} \approx U(r_{cir}) + \frac{1}{2}(\frac{3L^2}{mr_{cir}^4\sin^2\alpha})\Delta r^2$. The oscillation frequency for small deviation is then $\omega = \sqrt{\frac{3L^2}{mr_{cir}^4\sin^2\alpha}}$.

(c) We have to solve the equation: $E = U_{eff} = \frac{L^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha$, since $\dot{r} = 0$ in this case. There are two positive roots for this cubic equation with a physical value of energy E, so this will give the maximum and minimum r, and hence the corresponding vertical height.

9. Practice.

10. Practice.

11. Practice.

12. (a) The force is $F = -knr^{n-1}$. So $kn > 0$ means the force is attractive. The sketches for the effective potentials are omitted. The key point is limiting behavior for large or small r, for $n > -2$ the centripetal term dominates at small distance while the true potential dominates for large r, and vice versa.

    (b) A fixed radius corresponds to the critical point of the effective potential. Differentiating the potential one get $r_0 = (\frac{l^2}{nkm})^{1/n+2}$. From simple sketches or from second derivative tests ($U''_{eff}(r_0) = (n+2)\left(\frac{l}{mr_0^2}\right)^2$), one can see that $n > -2$ will give a stable orbit.

    (c) The frequency of the circular orbit is $\omega_{osc} = \frac{l}{mr_0^2}$. For small deviation, one can show that the oscillation frequency, from expanding the effective potential, is $\omega = \sqrt{\frac{U''_{eff}(r_0)}{m}} = \sqrt{n+2}\omega$. So the period has the desired relation. The orbit will be closed if $\sqrt{n+2}$ is rational, as in this case it is possible to have both orbital and radial motion return to the starting position together. The sketches are omitted, they are similar to the wave functions in Bohr's quantization of Hydrogen atoms.

13. Practice.

14. Practice.

15. Practice.

16. Practice.

17. Practice.

18. (a) The equation of motion using this Lagrangian is $\frac{d}{dt}(m\dot{\mathbf{R}} + \frac{q}{c}\mathbf{A}) + q\nabla\phi - \frac{q}{c}\nabla(\dot{\mathbf{R}} \cdot \mathbf{A}) = 0$. From chain rule, $\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial t} + (\dot{\mathbf{R}} \cdot \nabla)\mathbf{A}$. By explicitly writing in components, one can show that $\nabla(\dot{\mathbf{R}} \cdot \mathbf{A}) - (\dot{\mathbf{R}} \cdot \nabla)\mathbf{A} = \dot{\mathbf{R}} \times (\nabla \times \mathbf{A})$. Note that the derivative does not acts on the velocity vector. Hence, we have

$$m\ddot{\mathbf{R}} = q(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) + \frac{q}{c}\dot{\mathbf{R}} \times (\nabla \times \mathbf{A}) = q(\mathbf{E} + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{B}).$$

    (b) As above, the canonical momentum is $\mathbf{P} = m\ddot{\mathbf{R}} + \frac{q}{c}\mathbf{A}$.

    (c) If we perform a gauge transformation on the electromagnetic potential: $\phi' = \phi - \frac{1}{c}\frac{\partial \psi(\mathbf{R},t)}{\partial t}$ and $\mathbf{A}' = \mathbf{A} + \nabla\psi(\mathbf{R}, t)$, the Lagrangian is transformed as $L' = L + \frac{q}{c}(\dot{\mathbf{R}} \cdot \nabla\psi + \frac{\partial \psi}{\partial t}) = L + \frac{d}{dt}(\frac{q}{c}\psi(\mathbf{R}, t))$. This is a gauge transformation of Lagrangian, a shift in *total* time derivative of any function of position and time only, but not the velocity. One can explicitly verify that this total time derivative term will not contribute to the equation of motion.