EE20N: Structure and Interpretation of Systems and Signals	Fall 2013

Lecture 11: October 8

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Announcements

1. Homework now due on Friday at 11:59 PM. Self grades are still due on Monday at 11:59.
2. Survey out on bspace. Please respond.
3. Kannan's office hours are changed to Wednesday 10-11 AM in 269 Cory.
4. Midterm 2 date and time change: Thursday 11/7/13 in class.

Agenda

1. Recap: convolution, causality, and commutivity
2. Relationship between impulse response and frequency response
3. More convolution examples
4. Flip and Slide
5. Intro to Filtering

11.1 Convolution, causality, and commutivity recap

Recall the system composed of LTI_A and LTI_B in cascade. 11.1.

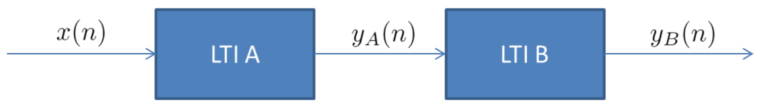


Figure 11.1: Cascade of LTI system

This system satisfies the following properties:

1. The system is LTI.
2. The system is causal if LTI_A and LTI_B are both causal.
3. The system remains the same if LTI_B and LTI_A switch places.

Property 1 follows from the general proof for linearity and time invariance.
Property 2 can be verified through the property of causal LTI systems. If h_A is the impulse response of LTI_A and h_B is the impulse response of LTI_B, then it is clear that the impulse response of the cascaded system is $h_B * h_A$ and that

$$h_A(n) = h_B(n) = 0 \quad \text{for all } n < 0.$$

Every delta of h_A that is fed into LTI_B will generate a delta in $h_B * h_A$ at the same time or later. Thus

$$(h_B * h_A)(n) = 0 \quad \text{for all } n < 0.$$

Property 3 can be shown in two ways. Let h_1 be the impulse response of LTI_A cascaded before LTI_B and h_2 be the reverse system. By commutativity of convolution,

$$h_1 = h_A * h_B = h_{B*A} = h_2.$$

LTI systems are uniquely determined by their impulse response, so the two systems must be equal. Alternatively, let $H_A(\omega)$ and $H_B(\omega)$ be the frequency responses of LTI_A and LTI_B, respectively. If the input is $x(n) = e^{i\omega n}$, then $y_A(n) = H_A(\omega)e^{i\omega n}$. Because $H_A(\omega)$ is a complex number for fixed ω, passing $y_A(n)$ through LTI_B is no different than passing a scaled complex exponential as input. Thus our output is

$$y(n) = H_B(\omega)H_A(\omega)x(n) = H_B(\omega)H_A(\omega)e^{i\omega n},$$

and the overall frequency response is $H_B(\omega)H_A(\omega)$. Commutativity of multiplication indicates that the order does not matter.

11.2 Impulse Response→Freq. Response

Because both $h(n)$ and $H(\omega)$ capture all the information about LTI systems, we can naturally obtain $H(\omega)$ from $h(n)$.

With this in mind, let H be a LTI system with frequency response $H(\omega)$ and impulse response $h(n)$. On the frequency side, if the input is $x(n) = e^{i\omega n}$, then the output is $y(n) = H(\omega)e^{i\omega n}$. At the same time on the time domain,

$$\begin{aligned}
y(n) &= (h * x)(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{i\omega(n-k)} \\
&= \left(\sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} \right) e^{i\omega n}.
\end{aligned}$$

Since k is merely a dummy variable for the summation, our result is a function only in ω,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k},$$

where $h(n)$ is the impulse response of a LTI system. For a simple sanity check, note that there is no n term in the expression for $H(\omega)$; the frequency response of a system is completely independent of time.

Here's an example. We have a three point averager represented by the system

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)].$$

From this sytem, we have

$$h(n) = \frac{1}{3}[\delta(n) + \delta(n-1) + \delta(n-2)]$$

$$H(\omega) = \sum_{k=0}^{N-1} \frac{1}{3} e^{-i\omega k}$$

We now note that this expression is a geometric series, and so we recall our geometric series formula

$$\sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha}.$$

With this, we can simplify our frequency response to

$$\frac{1}{3}(\frac{1-e^{-3i\omega}}{1-e^{-i\omega}})$$

11.3 Convolution Exercises

We have an input $x[n] = u[n]$, where $u[n]$ is the unit step defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Let's say that we send the input $x[n]$ into a system with impulse respone $h[n] = 2\delta[n] + \delta[n-1]$. What is the output $y[n]$?

We know that $y[n] = x[n] * h[n]$. We can split $x[n]$ into a series of delayed imuplses and convolve each with $h[n]$. We can iteratively solve for the result using the convolution formula.

$$y(n) = (h * x)(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(0) = h(0) * x(0) + h(1) * x(-1) = 2 + 0 = 2$$

$$y(1) = h(0) * x(1) + h(1) * x(0) = 2 + 1 = 3$$

$$y(2) = h(0) * x(2) + h(1) * x(1) = 2 + 1 = 3$$

and so on. Using this method, we'll have very many calculations to do. Instead, let's use $h[n]$ as the input. We now have a resulting

$$\begin{aligned} y(n) &= (x * h)(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= 2x[n] + x[n-1] \\ &= 2u[n] + u[n-1] \end{aligned}$$

11.4 Flip And Slide

We have another method to visually understand how to solve the convolution. This method is flip and slide. Let's say you have

$$y(n) = (x * h)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

The "Flip" part comes from the fact that the k dummy time value in $h[n-k]$ is time reversed due to the negative sign. The "Slide" part comes from the n in $h[n-k]$. As n is incremented, $h[n-k]$ appears to slide along the time axis. There is a very good applet that visually demonstrates this procedure. It is available at `www.jhu.edu/signals/discreteconv2/` . We will perform an example flip and slide convolution in the figures at the end of this set of notes.

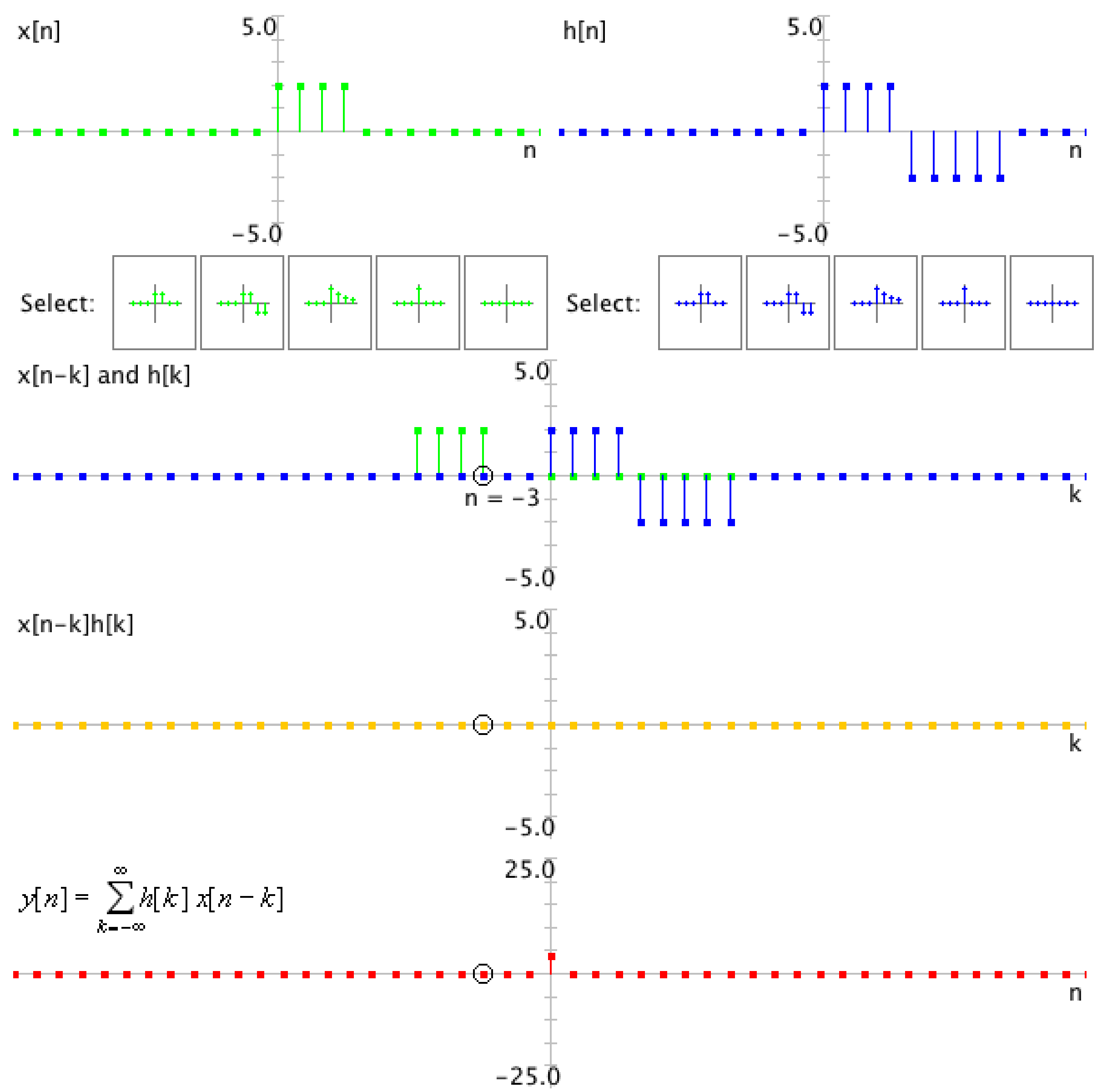
11.5 Intro to Filtering

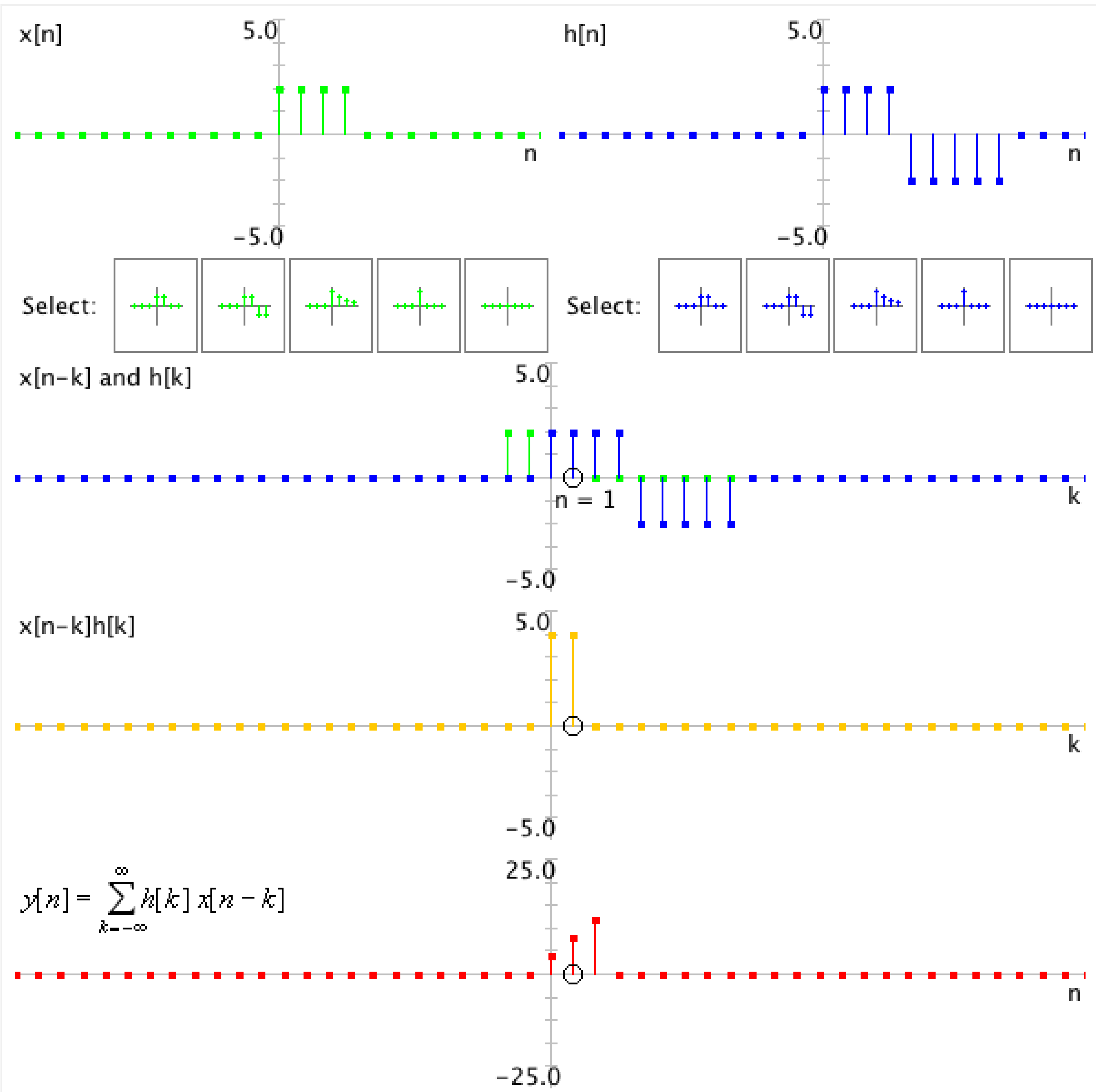
This section included two demos available at `http://ptolemy.eecs.berkeley.edu/eecs20/week10/index.html`.

First play the origin music sample under Filtering sounds, and then play the sample after it has gone through low, band, and high pass filters and observe the sound sample's Fourier coefficients[1].

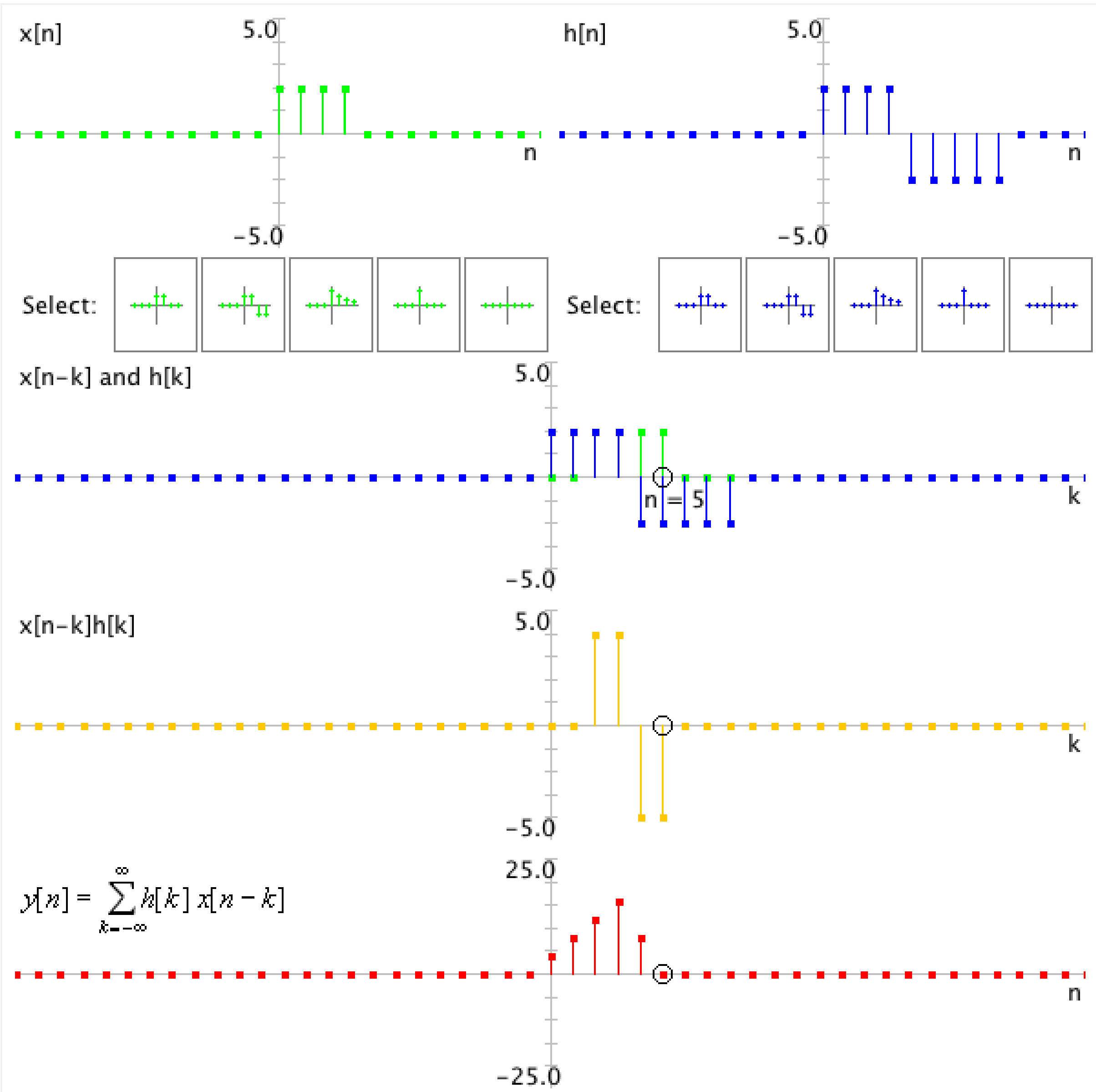
Similarly, take a look at the images under Filtering images. A low pass filter blurs the image since edges are encoded in higher frequencies. A high pass filter darkens the solid colors because solid colors which are encoded as lower frequencies are attenuated.

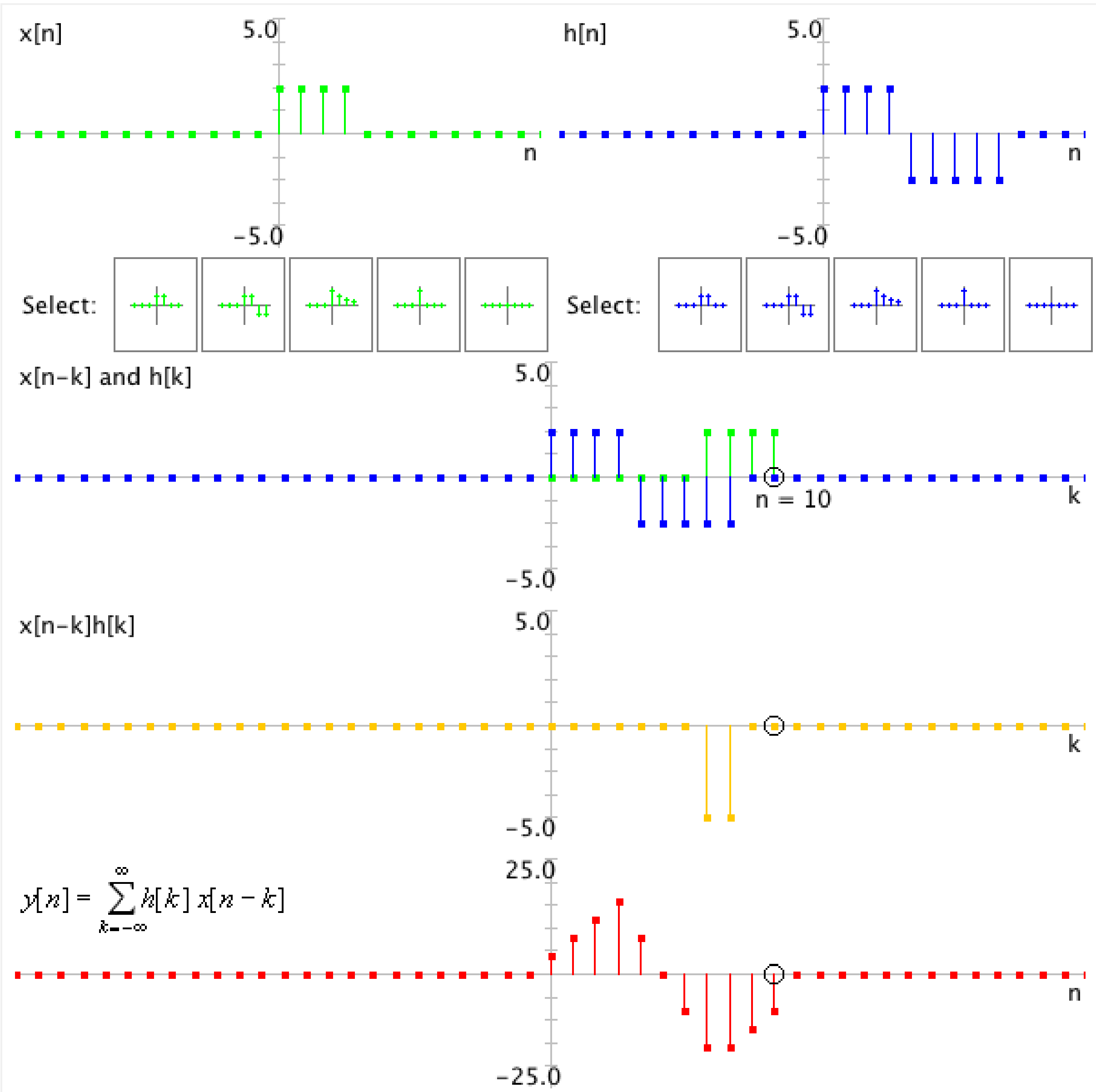
[1]Note that the Fourier coefficients are changing in time because a new set of coefficients is computed for each time slice of 16 milliseconds.

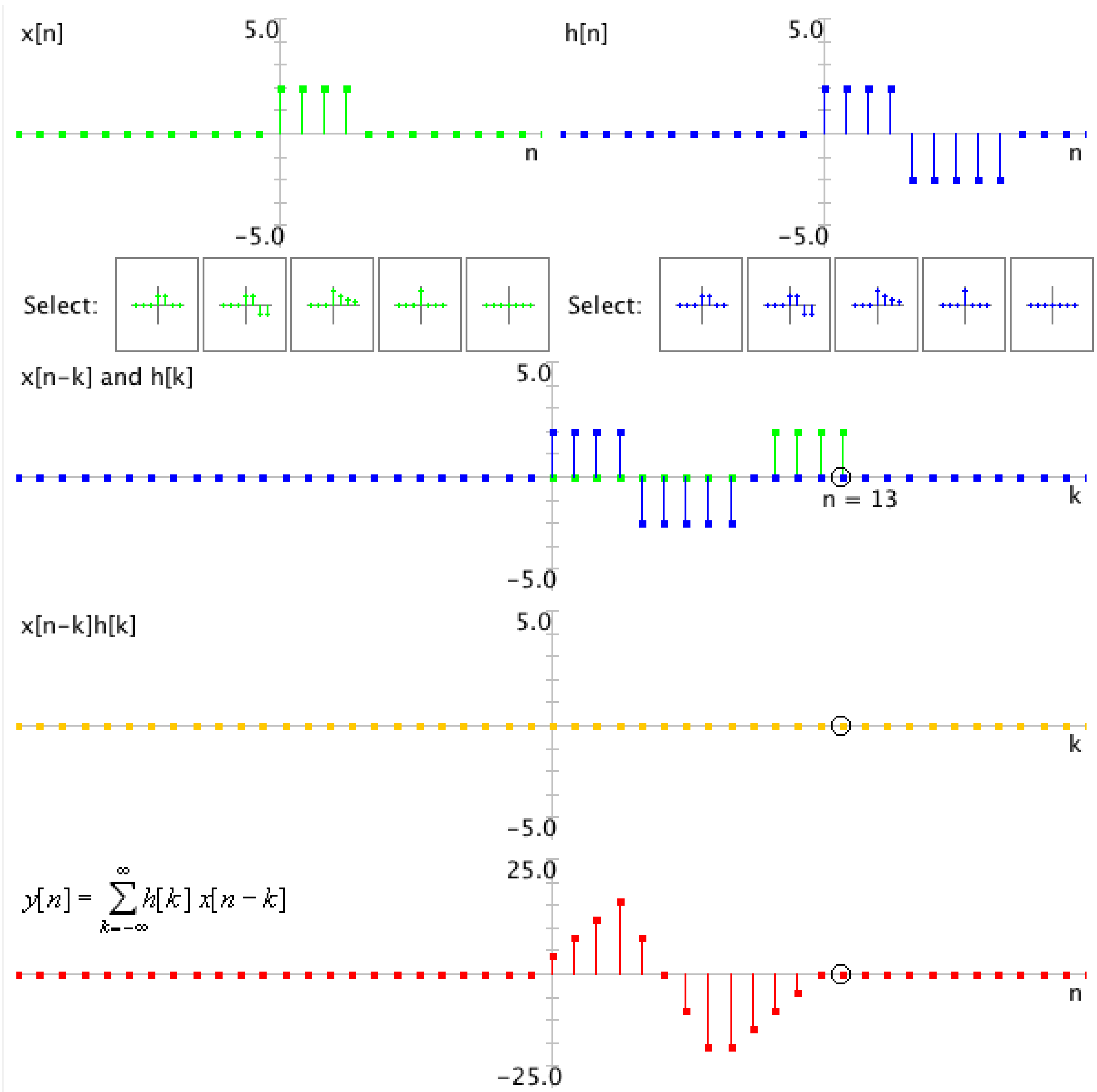

x[n]
5.0
n
-5.0
h[n]
5.0
n
-5.0
Select:
Select:
x[n-k] and h[k]
5.0
n = -3
k
-5.0
x[n-k]h[k]
5.0
k
-5.0
$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$
25.0
n
-25.0



x[n]
5.0
-5.0
n
h[n]
5.0
-5.0
n
Select:
Select:
x[n-k] and h[k]
5.0
-5.0
n = 1
k
x[n-k]h[k]
5.0
-5.0
k
$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$
25.0
-25.0
n


x[n]
5.0
n
-5.0
h[n]
5.0
n
-5.0
Select:
Select:
x[n-k] and h[k]
5.0
n = 5
k
-5.0
x[n-k]h[k]
5.0
k
-5.0
$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$
25.0
n
-25.0


x[n]
5.0
n
-5.0
h[n]
5.0
n
-5.0
Select:
Select:
x[n-k] and h[k]
5.0
n = 10
k
-5.0
x[n-k]h[k]
5.0
k
-5.0
$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$
25.0
n
-25.0



x[n]
5.0
-5.0
n
h[n]
5.0
-5.0
n
Select:
Select:
x[n−k] and h[k]
5.0
-5.0
n = 13
k
x[n−k]h[k]
5.0
-5.0
k
$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$
25.0
-25.0
n