1. Q. 5 in Lee-Varaiya Chapter 10.

   (a) We are given $x(n) = e^{2\pi i f n} = e^{2\pi i m n/p}$. Let $f = m/p$ so that $2\pi f = m \cdot \frac{2\pi}{p} = m\omega_0$. The DFT is given by

$$\begin{aligned}
X'_k &= \sum_{n=0}^{p-1} x(n) e^{-i\omega_0 k n} \\
&= \sum_{n=0}^{p-1} e^{i2\pi f n} e^{-i\omega_0 k n} \\
&= \sum_{n=0}^{p-1} e^{i\omega_0 (m-k) n} \\
&= \begin{cases} p & \text{if } p \text{ divides } m-k \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} p & \text{if } k \in \{\dots, m-2p, m-p, m, m+p, m+2p, \dots\} \\ 0 & \text{else} \end{cases}
\end{aligned}$$

   (b) $x(n) = \cos(2\pi f n) = \frac{1}{2}[e^{i2\pi f n} + e^{-i2\pi f n}] = \frac{1}{2}[y(n) + z(n)]$ where $y(n) = e^{i2\pi f n}$, $z(n) = e^{-i2\pi f n}$. By part a), we have the DFT of y_n and z_n to be given by Y'_k and Z'_k where

$$Y'_k = \begin{cases} p & \text{if } k \in \{\dots, m-2p, m-p, m, m+p, m+2p, \dots\} \\ 0 & \text{else} \end{cases}$$

$$Z'_k = \begin{cases} p & \text{if } k \in \{\dots, -m-2p, -m-p, -m, -m+p, -m+2p, \dots\} \\ 0 & \text{else} \end{cases}$$

   By linearity of DFT, we have $X'_k = \frac{1}{2}[Y'_k + Z'_k]$. Thus,

$$X'_k = \begin{cases} p/2 & \text{if } k \in \{\dots, m-2p, m-p, m, m+p, m+2p, \dots\} \\ p/2 & \text{if } k \in \{\dots, -m-2p, -m-p, -m, -m+p, -m+2p, \dots\} \\ 0 & \text{else} \end{cases}$$

   (c) $x(n) = \sin(2\pi f n) = \frac{1}{2i}[e^{i2\pi f n} - e^{-i2\pi f n}] = \frac{1}{2i}[y(n) - z(n)]$ where $y(n) = e^{i2\pi f n}$, $z(n) = e^{-i2\pi f n}$.
   By linearity of DFT, we have $X'_k = \frac{1}{2i}[Y'_k - Z'_k]$. Thus,

$$X'_k = \begin{cases} p/2i & \text{if } k \in \{\dots, m-2p, m-p, m, m+p, m+2p, \dots\} \\ -p/2i & \text{if } k \in \{\dots, -m-2p, -m-p, -m, -m+p, -m+2p, \dots\} \\ 0 & \text{else} \end{cases}$$

(d) We are given $x(n) = 1 = e^{2\pi imn/p}$, with $m = 0$. By part a), the DFT is given by

$$X'_k = \begin{cases} p & \text{if } k \in \{\ldots, -2p, -p, 0, p, 2p, \ldots\} \\ 0 & \text{else} \end{cases}$$

2. Q. 6 in Lee-Varaiya Chapter 10.

(a) Let $\omega_0 = \frac{2\pi}{p}$.
We have

$$\begin{aligned}
X'_k &= \sum_{n=0}^{p-1} x(n) e^{-i\omega_0 kn} \\
&= \sum_{n=0}^{M} e^{-i\omega_0 kn} + \sum_{n=p-M}^{p-1} e^{-i\omega_0 kn} \\
&= \sum_{n=0}^{M} e^{-i\omega_0 kn} + \sum_{n=-M}^{-1} e^{-i\omega_0 kn} \qquad \text{since } e^{-i\omega_0 kn} = e^{-i\omega_0 k(n-p)} \\
&= \sum_{n=-M}^{M} e^{-i\omega_0 kn} \\
&= \begin{cases} 2M+1 & \text{if } k \text{ divides } p \\ e^{i\omega_0 kM} \frac{1-e^{-i\omega_0 k(2M+1)}}{1-e^{-i\omega_0 k}} & \text{else} \end{cases} \\
&= \begin{cases} 2M+1 & \text{if } k \text{ divides } p \\ \frac{e^{i\omega_0 k(M+0.5)} - e^{-i\omega_0 k(M+0.5)}}{e^{i\omega_0 \frac{k}{2}} - e^{-i\omega_0 \frac{k}{2}}} & \text{else} \end{cases} \\
&= \begin{cases} 2M+1 & \text{if } k \text{ divides } p \\ \frac{2i\sin(\omega_0 k(M+\frac{1}{2}))}{2i\sin(\omega_0 \frac{k}{2})} & \text{else} \end{cases} \\
&= \begin{cases} 2M+1 & \text{if } k \text{ divides } p \\ \frac{\sin(\omega_0 k(M+\frac{1}{2}))}{\sin(\omega_0 \frac{k}{2})} & \text{else} \end{cases}
\end{aligned}$$

(b) The required plots are shown in Fig. 1.

3. Q. 15 in Lee-Varaiya Chapter 10.

Suppose the system is fed input $\delta(n)$. Then, the output of the system would be the impulse response $h(n) = b_0\delta(n) + b_1\delta(n-1) + b_2\delta(n-2) + b_3\delta(n-3)$. The frequency response of this system is given by $H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} = b_0 + b_1 e^{-i\omega} + b_2 e^{-i2\omega} + b_3 e^{-i3\omega}$.

Now, if all the unit delays in this system are replaced by double unit delays, then the impulse response of this system would be given by $h'(n) = b_0\delta(n) + b_1\delta(n-2) +$

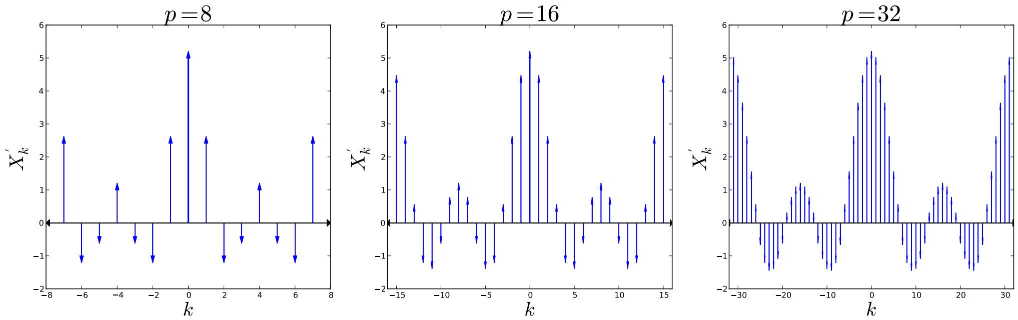


Figure 1: Plot of X_k' for different values of p

$b_2\delta(n-4)+b_3\delta(n-6)$ and similarly, the frequency response would be given by $H'(\omega) = b_0 + b_1 e^{-i2\omega} + b_2 e^{-i4\omega} + b_3 e^{-i6\omega}$.

By inspection, we can see that $H'(\omega) = H(2\omega)$.

4. a) The DTFT of the impulse signal $\delta(n)$ is given by $\Delta(\omega) = \sum_{k=-\infty}^{\infty} \delta(k)e^{-i\omega k} = 1$.

   b) Let us first study the frequency response $H(\omega)$ of the comb filter. Let the input $x(n) = e^{i\omega n}$ and the output be $y(n) = H(\omega)e^{i\omega n}$. Then,

$$H(\omega)e^{i\omega n} = \alpha H(\omega)e^{i\omega n - N} + e^{i\omega n}$$
$$H(\omega) = \frac{1}{1 - \alpha e^{-i\omega N}}$$

   The DTFT of the output $y(n)$ will be given by $Y(\omega) = H(\omega)\Delta(\omega) = \frac{1}{1-\alpha e^{-i\omega N}}$. This has been plotted for particular values of α and N in Fig. 2.

   The peaks of $Y(\omega)$ are at $\omega = \frac{2\pi k}{N}$ for $k = \ldots, -2, -1, 0, 1, 2, \ldots$ and the peak value is $\frac{1}{1-\alpha}$. The troughs of $Y(\omega)$ are at $\omega = \frac{\pi(2k+1)}{N}$ for $k = \ldots, -2, -1, 0, 1, 2, \ldots$ and the trough value is $\frac{1}{1+\alpha}$. The number of peaks in $[-\pi, \pi]$ is exactly N (where we count a peak at π and $-\pi$, if any as only one peak).

   Increasing the parameter N will increase the density of peaks in the magnitude of $|Y(\omega)|$. Increasing α in $(0, 1)$ will increase the magnitude of the peak and decrease the magnitude of the trough, thus increasing α leads to more variation in the magnitude of $|Y(\omega)|$.

   c) We want to have a fundamental frequency of 440 Hz and two higher components at twice and thrice that frequency. The frequency response has its first non-trivial peak at $\frac{2\pi}{N}$. This corresponds to a discrete signal with N samples as period.

   If our sampling rate is f_s Hz, then the signal goes through one cycle in time $\frac{1}{440}$ seconds which must be the same as $\frac{N}{f_s}$. So, $f_s = 440N$. Now, we must choose N so that all frequencies that are four times of the fundamental frequency or higher are absent. So, we should choose $N = 6$ or 7. This is shown in Fig. 3. Thus, an acceptable choice of parameters is either of

   - $N = 6, f_s = 6 \times 440 = 2640$ Hz

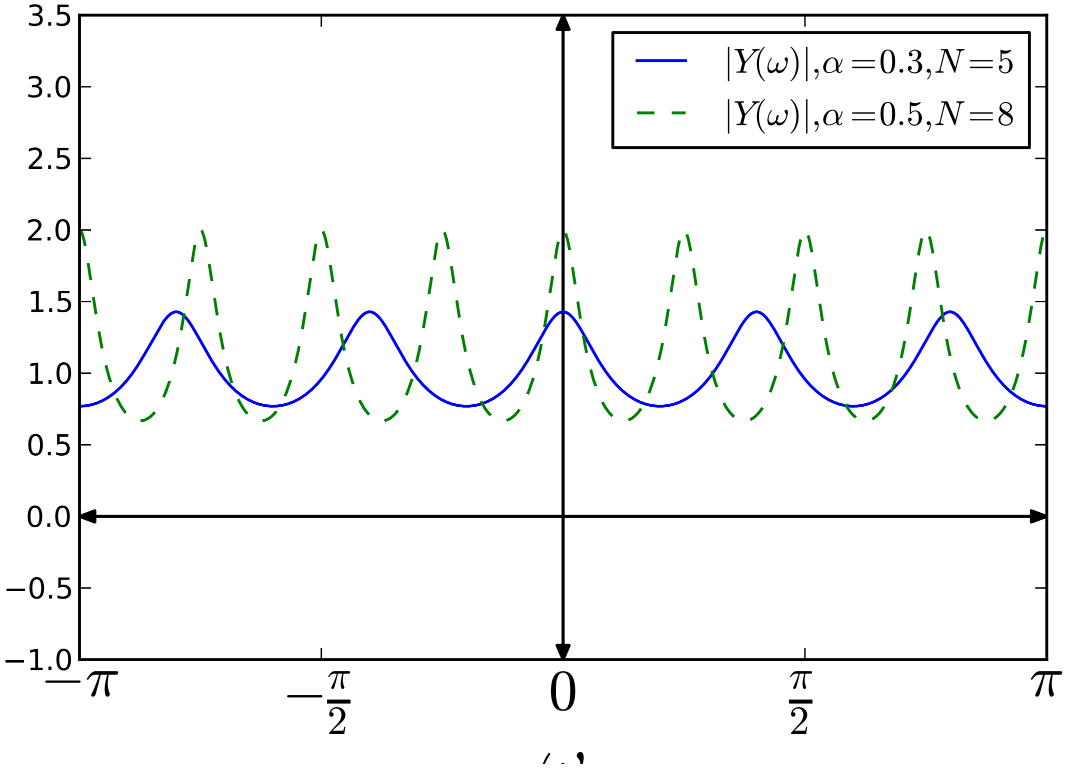


Figure 2: Plot of $|Y(\omega)|$ for $(\alpha, N) = (0.3, 5), (0.5, 8)$

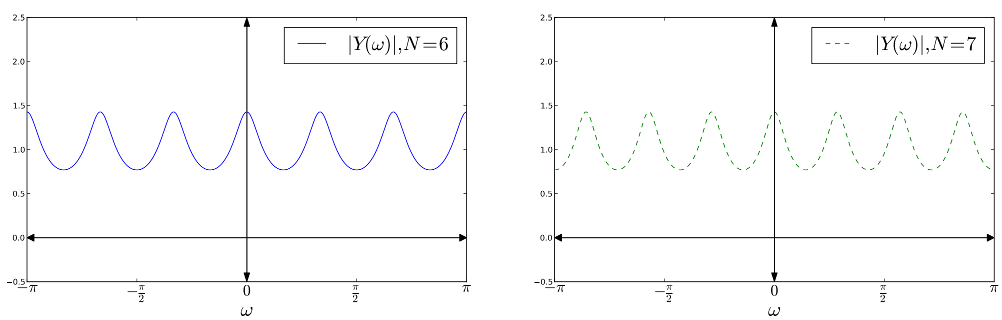


Figure 3: Plot of $|Y(\omega)|$ for $N = 6, 7$.

- $N = 7, f_s = 7 \times 440 = 3080$ Hz

d) Now, we choose α such that the output continuous time signal has a sufficiently small magnitude at 20 Hz.

- If we had chosen $N = 6, f_s = 2640Hz$, then, $f = 20Hz$ will correspond to $\omega = 2\pi\frac{20}{f_s} = \frac{\pi}{66}$. We want to have $20\log_{10}\frac{|Y(\frac{\pi}{66})|}{|Y(0)|} = -10$. This gives

$$\frac{1}{|1 - \alpha e^{-i\frac{\pi}{66}6}|} = \frac{1}{\sqrt{10}}\frac{1}{1-\alpha}.$$

Rearranging,

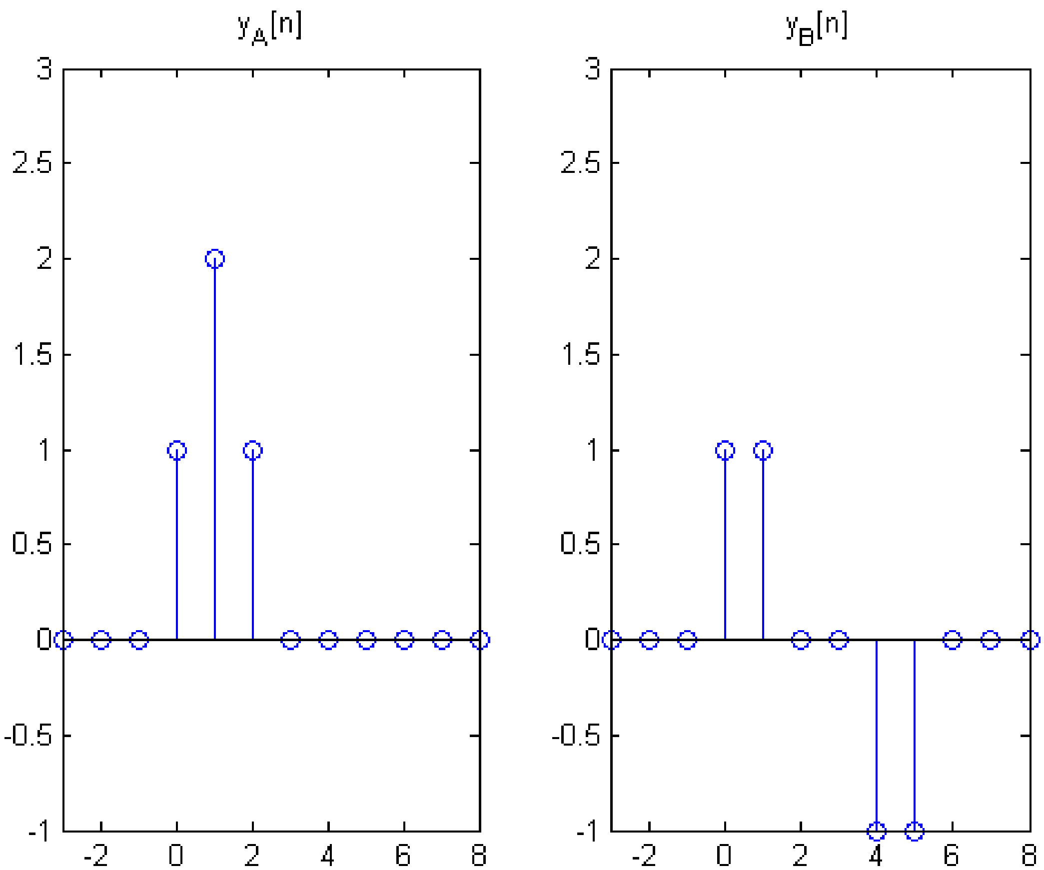
$$
\begin{aligned}
10(1-\alpha)^2 &= |1-\alpha e^{-i\frac{\pi}{11}}|^2 \\
10(1-\alpha)^2 &= (1-\alpha\cos(\pi/11))^2 + (\alpha\sin(\pi/11))^2 \\
10 - 20\alpha + 10\alpha^2 &= 1 - 2\cos(\pi/11)\alpha + \alpha^2 \\
9\alpha^2 - (20 - 2\times 0.9595)\alpha + 9 &= 0 \\
\alpha^2 - 2.009\alpha + 1 &= 0 \\
\alpha &= 0.91
\end{aligned}
$$

- If we had chosen $N = 7, f_s = 3080Hz$, then, $f = 20Hz$ will correspond to $\omega = 2\pi\frac{20}{f_s} = \frac{\pi}{77}$. We want to have $20\log_{10}\frac{|Y(\frac{\pi}{77})|}{|Y(0)|} = -10$. This gives

$$
\frac{1}{|1-\alpha e^{-i\frac{\pi}{77}7}|} = \frac{1}{\sqrt{10}}\frac{1}{1-\alpha},
$$

which is the same equality as above. Thus, we get $\alpha = 0.91$.

5. a) The plots of $y_A[n]$ and $y_B[n]$ are shown below.



b) Using the definition of convolution we have:

$$\begin{aligned} y_A[n] &= \sum_{k=-\infty}^{\infty} h_A[n-k]x[k] \\ &= \sum_{k=-\infty}^{\infty} h_A[n-k]\delta[k] \\ &= h_A[n] \\ &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

$$\begin{aligned} y_B[n] &= \sum_{k=-\infty}^{\infty} h_B[n-k]y_A[k] \\ &= \sum_{k=-\infty}^{\infty} h_B[n-k](\delta[k] + 2\delta[k-1] + \delta[k-2]) \\ &= h_B[n] + 2h_B[n-1] + h_B[n-2] \\ &= \delta[n] + \delta[n-1] - \delta[n-4] - \delta[n-5] \end{aligned}$$

c) Using the definition of DTFT we have:

$$\begin{aligned} H_A(e^{i\omega}) &= 1 + 2e^{-i\omega} + e^{-i2\omega} \\ H_B(e^{i\omega}) &= 1 - e^{-i\omega} + e^{-i2\omega} - e^{-i3\omega} \end{aligned}$$

d) Using the definition of DTFT and the multiplication property we have:

$$\begin{aligned} X(e^{i\omega}) &= 1 \\ Y_A(e^{i\omega}) &= X(e^{i\omega})H_A(e^{i\omega}) \\ &= 1 + 2e^{-i\omega} + e^{-i2\omega} \\ Y_B(e^{i\omega}) &= Y_A(e^{i\omega})H_B(e^{i\omega}) \\ &= (1 + 2e^{-i\omega} + e^{-i2\omega})(1 - e^{-i\omega} + e^{-i2\omega} - e^{-i3\omega}) \\ &= 1 + e^{-i\omega} - e^{-i4\omega} - e^{-i5\omega} \end{aligned}$$

e) Impulse responses convolve; frequency responses multiply:

$$\begin{aligned} y_B[n] &= (x \star h_A \star h_B)[n] \\ Y_B(e^{i\omega}) &= X(e^{i\omega})H_A(e^{i\omega})H_B(e^{i\omega}) \end{aligned}$$