EE20N: Structure and Interpretation of Systems and Signals **Spring 2013**

Lecture 12: October 10

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12.1 Announcements

1. HW5 due on Friday at 11:59 pm
2. Please scan your homeworks LEGIBLY

12.2 Introduction

This lecture covers:

1. Convolution review
2. Filtering: FIR and IIR Filters

12.3 Convolution: Different Perspectives

12.3.1 Perspective 1

The first perspective on convolution is to think of it as the contribution of a single input sample to ALL the output samples. The key idea is to decompose your input into a summation of impulses, and then look at how each impulse affects the output. This can be done because the system is LTI, so we know that additivity holds.

Example 12.1. *Consider the 2-path wireless channel below, and suppose you want to input an arbitrary signal $x[n]$ into this system.*

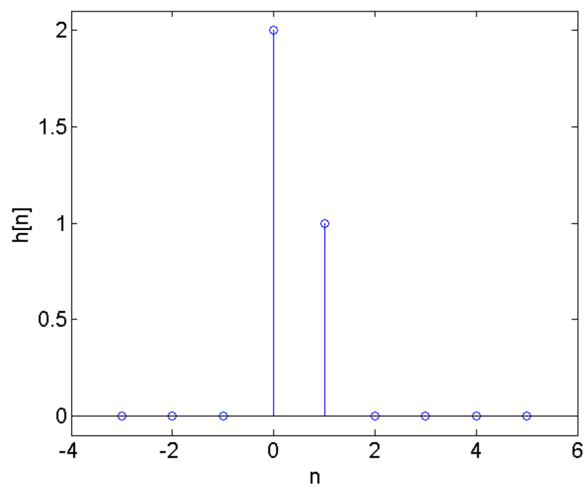

Figure 12.1: Impulse response $h[n]$ for wireless channel with echo (system $\mathcal{H}$).

Our input can take any form, so let's represent it as follows:

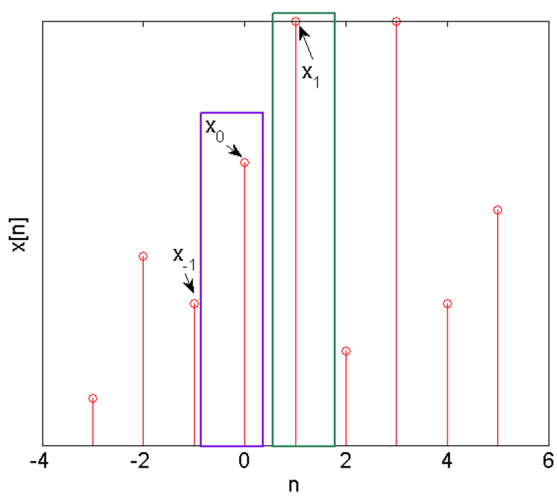

Figure 12.2: Input that we want to send into system $\mathcal{H}$.

Now let's just consider the input highlighted in purple, x_0. We will first determine what happens when we pass only that input (i.e. $x_0\delta[n]$) through the system. Clearly, the output must be $x_0h[n]$, by the definition of impulse response. By linearity, now let's add the the output when we pass only the second impulse through the system (i.e. the contents of the green box). Since that input is simply $x_1\delta[n-1]$, the output due to the green-boxed input must be $x_1h[n-1]$. Continuing in this pattern, we see that the output is simply

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{12.1}$$

which is exactly the formula we have been using. This is the intuition behind the formula.

12.3.2 Perspective 2

Previously we broke our input into a sum of impulses, and passed each through the impulse response. Using the second perspective, we instead look at the world from an output standpoint. That is, we want to find the contribution of ALL input samples to a single, specific output sample. This gives rise to the flip-and-drag method. Recall that our formula for convolution is $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

Example 12.2. *Consider the same wireless channel as before. Which input samples contribute and by how much to output sample n=33?*

Intuitively, we know that the system is just taking $y[n] = 2x[n] + x[n-1]$. Therefore the answer must be $2x[33]$ and $x[32]$. Let's use the flip-and-drag method to confirm this. Since we are looking for $y[33]$, we can set $n = 33$. Then let's plot $h[n-k] = h[33-k]$:

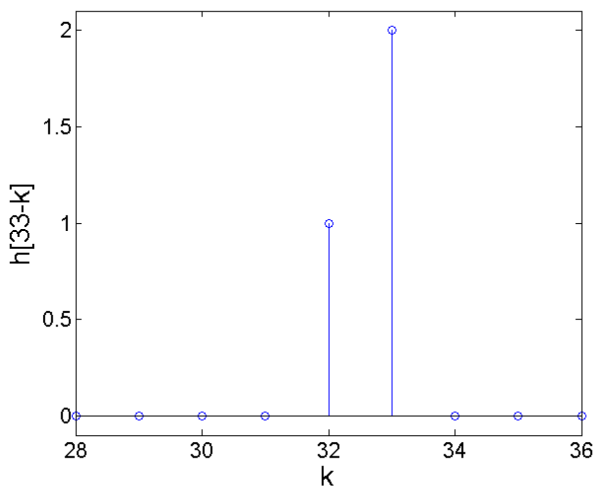

Figure 12.3: Stem plot of $h[n-k]$ when $n = 33$.

So when $n = 33$, we're going to multiply this function $h[n-k]$ by $x[k]$ at every k, and add up the results. Since $h[n-k] = 0$ for all k except 33 and 32, we get $y[33] = x[32]+2x[33]$, which is precisely what we expected from the system definition. Similarly, when $n = 34$, we get $h[34] = x[33]+2x[34]$.

12.4 Filtering

First of all, we encourage you to take a look at the website `http://ptolemy.eecs.berkeley.edu/eecs20/week10/index.html` for some nice demos of filtering in action. Now we're going to look at an example low-pass filter.

Example 12.3.

$$y(n) = \frac{1}{L}\sum_{k=0}^{L-1} x(n-k)$$

This system takes an unweighted average of a signal. We wish to explore the properties of this system as we vary the parameter L, which represents the size of the averager's window. To begin, note that the impulse response of the system can be found by choosing $x(n) = \delta(n)$. This yields

$$h(n) = \frac{1}{L}\sum_{k=0}^{L-1} \delta(n-k) = \begin{cases} \frac{1}{L} & 0 \le n \le L-1 \\ 0 & \text{elsewhere} \end{cases}.$$

To recap, how do we find the frequency response using only the impulse response of a system? We can use the formula derived in class:

$$IR \quad \longleftrightarrow \quad FR \tag{12.2}$$

$$h[n] \quad \longleftrightarrow \quad \sum_{k=-\infty}^{\infty} h[k]e^{-i\omega k} = H(e^{i\omega}) \tag{12.3}$$

Computing this gives

$$H(e^{i\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-i\omega k} \tag{12.4}$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} e^{-i\omega k} \tag{12.5}$$

$$= \frac{1}{L} \cdot \frac{1-e^{-i\omega L}}{1-e^{-i\omega}} \tag{12.6}$$

The last step is computed by taking a partial sum of the geometric series. If $\omega = 0$, then we have an average over L points of $\frac{1}{L}$, so $|H(0)| = H(0) = 1$. The plot reaches zero whenever $1 - e^{-i\omega L} = 0$, or when $\omega = \frac{2\pi C}{L}$ with $C \in \{1, 2, ..., L\}$. The plot consists of a main lobe at the center and secondary lobes at the side. Because we are in discrete time, the frequency response repeats every 2π.

Example 12.4. *Plot the magnitude of the frequency response magnitude when $L = 2$ and $L = 4$.*

Case 1: $L = 2$

$$|H(e^{i\omega})| = \left|\frac{1}{2} \cdot \frac{1-e^{i\omega 2}}{1-e^{-i\omega}}\right| = \left|\frac{1}{2} \cdot \frac{e^{-i\omega}(e^{i\omega} - e^{-i\omega})}{e^{i\omega/2}(e^{i\omega/2} - e^{-i\omega/2}}\right| \tag{12.7}$$

$$= \left|\frac{1}{2}e^{i\omega/2}\frac{2i\sin(\omega)}{2i\sin(\omega/2)}\right| \tag{12.8}$$

$$= \frac{1}{2}\left|\frac{\sin(\omega)}{\sin(\omega/2)}\right| \tag{12.9}$$

$$= \left|\frac{1}{2} \cdot \frac{2\sin(\omega/2)\cos(\omega/2)}{\sin(\omega/2)}\right| \tag{12.10}$$

$$= |\cos(\omega/2)| \tag{12.11}$$

where line 12.10 results from a trig identity. See 12.4.

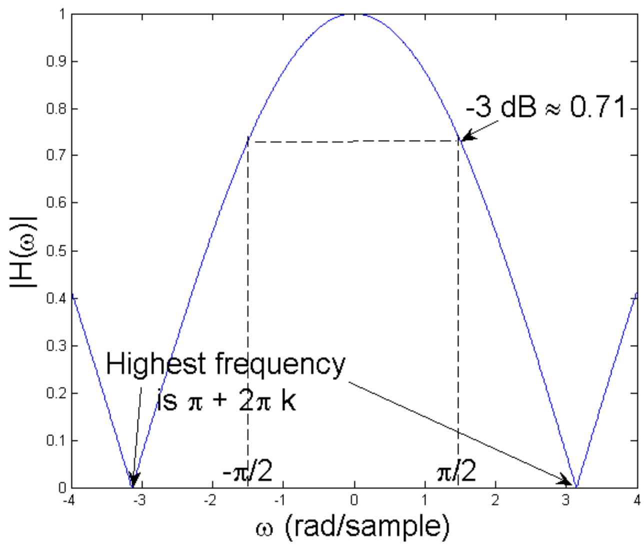

Figure 12.4: Magnitude of averager frequency response with $L = 2$

This is an example of a really crummy low-pass filter. It passes low frequencies better than high frequencies, but it falls off really slowly (i.e. the slope of the main lobe is pretty gradual). Remember that in DT, frequencies are periodic, so $\omega = \pi$ is the highest possible frequency (along with $\pi + 2\pi k$ for any integer k).

To fix this, let's try increasing L. For arbitrary L

$$|H(e^{i\omega})| = \frac{1}{L}\left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|$$

This is called a periodic sinc function. To evaluate it at $\omega = 0$, Remember that

$$\lim_{\omega\to 0}\sin(x) = x. \tag{12.12}$$

Aside: Engineers often give a frequency response in decibels (dB). This means that

$$|H(e^{i\omega})|_{dB} = 20\log_{10}|H(e^{i\omega})| \tag{12.13}$$

So when $\omega = \pi/2$, our frequency response has a magnitude of $\sqrt{2}/2$, which is -3 dB, while when $\omega = 0$, we have a magnitude of 1 dB.

Case 2: L=4 Using this information, we can plot the magnitude of the frequency response when $L = 4$.

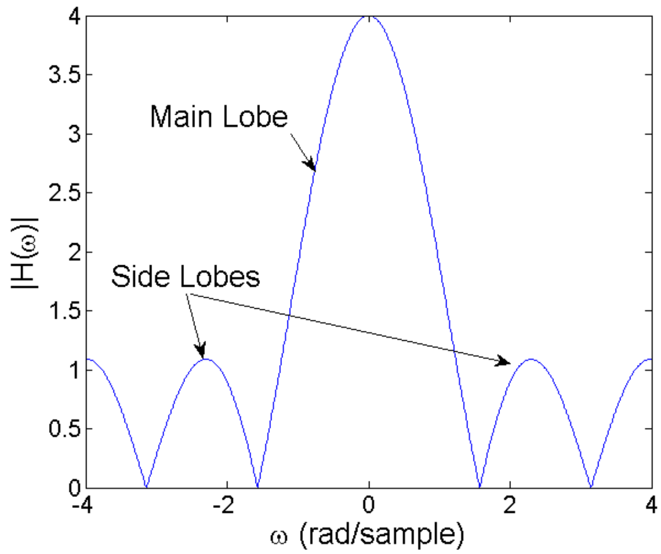

Figure 12.5: The magnitude response of the moving average (low-pass) filter with length $L = 4$.

This is a much better frequency response than the other frequency response. From the figure, we can see that increasing L will result in a tighter main lobe in the frequency response, thus allowing less of the higher frequencies through. This coincides with the intuition that averaging over a larger set of points gives a smoother result.

In the main lobe of the frequency response of the above filter you have a high gain and for the side lobes you have low gain. But this is not the best low pass filter that you can design. In the best low pass filter you'd have a good gain for the passed band of frequency and then you'd have a sharp cut off and kill off all frequencies in the stop band.

The impulse response of the above filter is:

$$h(n) = \begin{cases} \frac{1}{L} \; n = 0 \dots L-1 \\ 0 \text{ otherwise} \end{cases}$$

We refer to these non-zero pulses as 'taps'. This filter's impulse response has L taps.

This type of filter is called a FIR filter, a *finite impulse response* filter. These filters have finite number of taps or nonzero coefficients. The complexity of these filters is determined by the number of the non-zero coefficients. However, sometimes, you'd like filters with *infinite impulse responses*. You can also implement infinite impulse response filters in a easy way. A filter with infinite impulse response is usually described with an input output relationship, for example:

Example 12.5.

$$y(n) = \alpha y(n-1) + (1-\alpha)x(n)$$

Questions: 1) What is the block diagram of the above filter? 2) What is the impulse response?

Answer 1: The parameter α is the relative weight we give to the previous output value relative to the current input value in determining the filter's current output. How does on implement the above filter? We use feedback! In block diagram form, this looks as follows:

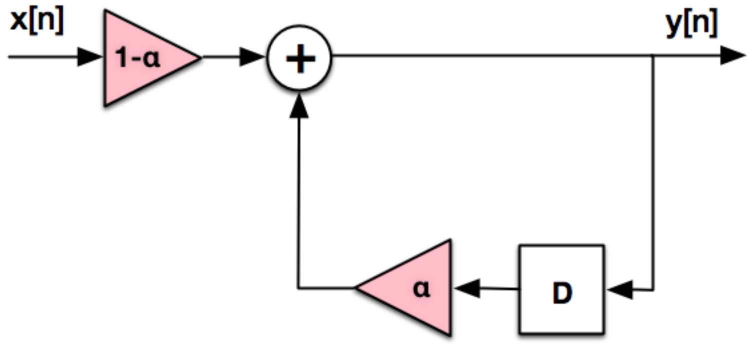

Figure 12.6: Diagram for Example 12.2

Triangles denote multiplicative gains, while a square with a D inside denotes a unit delay. This takes the input and outputs the value of the input at the previous time-step. In circuits language, a delay block could be implemented by something like a shift register.

Answer 2: To find the impulse response, we can't just plug in $\delta[n]$ again because of the feedback term. So instead, we will build up the impulse response from time 0. Assume initial rest, i.e. $y(n) = 0$ for $n < 0$. Now we can see that

$$\begin{aligned} h[0] &= (1-\alpha) \\ h[1] &= \alpha(1-\alpha) \\ h[2] &= \alpha^2(1-\alpha) \\ &\dots \end{aligned}$$

and so forth. In fact, we can prove by induction that the impulse response is given by:

$$h[n] = \begin{cases} 0 \; n < 0 \\ \alpha^n(1-\alpha) \; n \geq 0 \end{cases}$$