EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 06: September 17

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6.1 Announcements

1. Homework 2 due on Thursday
2. Homework party today from 3-5 PM in Wozniak. Also, we are adding an extra HW "mini-party" Wed 6-7 in 540 Cory.
3. glookup "glitches" are being fixed
4. Please use Piazza for questions. Check bspace for announcements
5. Midterm 1: Thursday 9/26 from 6-7:30. Can bring 1 two-sided 8.5x11 handwritten cheat sheet.
6. Lab 2 has been shortened. You don't have to do the "gaussian" problem. Also you can look up the fourier series coefficients for the triangle wave. If you were in the Monday lab sections, you may get your lab checked off next Monday with no grade reduction for lateness.

6.2 Discrete Time Fourier Series (DTFS)

Recap from last lecture

The DTFS for a general p-periodic DT signal is:

$$x(n) = A_0 + \sum_{k=1}^{K}(\alpha_k \cos kw_0 n + \beta_k \sin kw_0 n)$$

for $n = 0, 1, ...p-1$, and where $\omega_0 = \frac{2\pi}{p}$. We can define a number K such that

$$K = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases}.$$

For example, if $p = 2$, then $K = 1$. If $p = 3$, then $K = 1$.

Example 1

Let's say that p is an odd number. Find the matrix representation of the DTFS.

We have the general equation

$$x(n) = A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 n) + \beta_k \sin(k\omega_0 n).$$

Thus we can plug in distinct time step values for n and use the equation above to find the values for $x[n]$.

$$x(0) = A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot 0) + \beta_k \sin(k\omega_0 \cdot 0)$$

$$x(1) = A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot 1) + \beta_k \sin(k\omega_0 \cdot 1)$$

$$\vdots$$

$$x(p-1) = A_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 \cdot (p-1)) + \beta_k \sin(k\omega_0 \cdot (p-1))$$

These expressions will evaluate to

$$x(0) = A_0 + \alpha_1 + \alpha_2 + ... + \alpha_K$$

$$x(1) = A_0 + \alpha_1 cos(\omega_0) + \beta_1 sin(\omega_0) + \alpha_2 cos(2\omega_0) + ... + \beta_K sin(K\omega_0)$$

$$\vdots$$

$$x(p-1) = A_0 + \alpha_1 cos(\omega_0(p-1)) + \beta_1 sin(\omega_0(p-1)) + \alpha_2 cos(2\omega_0(p-1)) + ... + \beta_K sin(K\omega_0(p-1))$$

This system of equations can be represented by a matrix product as follows.

$$\begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(p-1) \end{pmatrix} = \begin{pmatrix} 1 & \cos(1\omega_0 \cdot 0) & \sin(1\omega_0 \cdot 0) & ... & \cos(K\omega_0 \cdot 0) & \sin(K\omega_0 \cdot 0) \\ 1 & \cos(1\omega_0 \cdot 1) & \sin(1\omega_0 \cdot 1) & ... & \cos(K\omega_0 \cdot 1) & \sin(K\omega_0 \cdot 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(1\omega_0(p-1)) & \sin(1\omega_0(p-1)) & ... & \cos(K\omega_0(p-1)) & \sin(K\omega_0(p-1)) \end{pmatrix} \begin{pmatrix} A_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_K \\ \beta_K \end{pmatrix}$$

Alternate DTFS representation

$$x(t) = \sum_{k=0}^{p-1} D_K e^{ik\omega_0 n}$$

6.3 Linear Time Invariant (LTI) Systems

Time Invariance

We've been talking about linear systems. Now let's talk about another system property, one that's maybe even more important than linearity: **time invariance**.

Intuitively, time invariance (also known as shift invariance) means that the system will behave the same way at any point in time. Mathematically, this means that if output $y(t)$ results from an input $x(t)$, then inputting $x(t-\tau)$ will result in $y(t-\tau)$. In a TI system, the system behavior doesn't vary due to the time delay τ. So, to check for TI we should compare:

1. $y(t-\tau)$

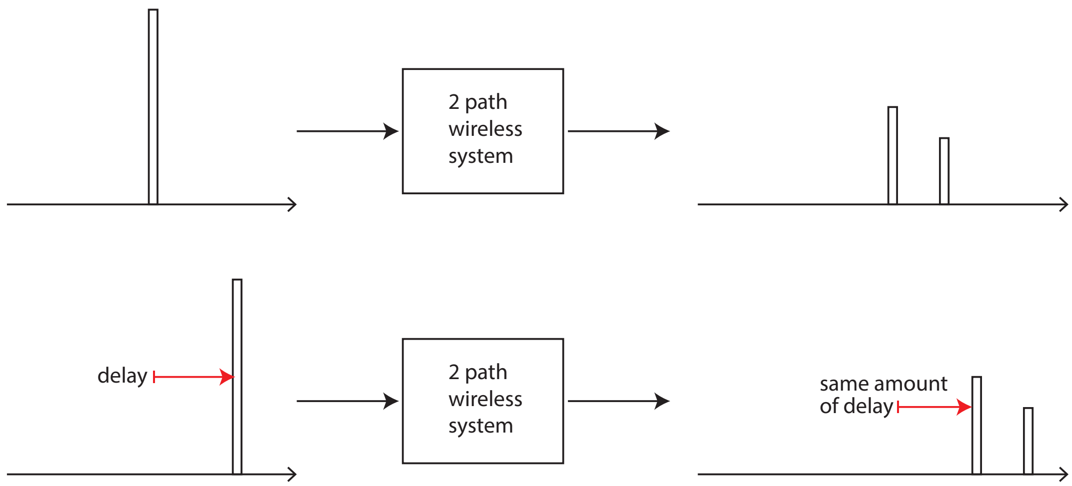
2. The system output when the input is $x(t-\tau)$

If these two operations are equal, then the system is TI.

Example 2

Let's go back to our example, the 2 path wireless channel. This system has a property: if you give it a pulse, you get 2 pulses as output. If you give the same pulse after some time, the output will be the same 2 pulses, with the same amount of delay as the input.

Figure 6.1: a time invariant system



This system is time invariant. As the time suggests, it is invariant in time - and you can see how a property like this is useful. If a system is not time invariant, that means if you give it an input today, it gives you a certain output, but if you give it the same input tomorrow, it may give you something different. How can we design or build systems if they are time variant like that? It would make our life very miserable.

Important: time invariance is a property of systems, NOT of signals.

Time invariance is important by itself, but what's really powerful is if we combine the two properties we've learned so far: linearity and time invariance. If a system is both linear and time invariant, we call it - you guessed it - **linear time invariant**, commonly abbreviated as LTI.

Let the system representation of the wireless channel be

$$y(t) = a_1 x(t-\gamma_1) + a_2 x(t-\gamma_2).$$

You should know by now that this system is linear because it adheres to the rules of superposition and homogeneity. However, how can we check if it's time-invariant (TI)? Let

$$x_\tau(t) = x(t-\tau)$$

$$y_\tau(t) = y(t-\tau)$$

We can find that

$$y_\tau(t) = a_1 x_\tau(t-\gamma_1) + a_2 x_\tau(t-\gamma_2)$$

$$y(t-\tau) = a_1 x(t-\gamma_1-\tau) + a_2 x(t-\gamma_2-\tau)$$

Thus delaying $x(t)$ and $y(t)$ by τ gives the same result.

Example 3

$$y(t) = [x(t)]^2$$

This is an example of a non-linear time invariant system. It goes to show that linearity and time invariance are seperate properties and do not depend on each other.

Example 4

Let's look at the ambulence problem again. You should have found that the system can be represented by

$$y(t) = x(t + \tau(t))$$

For example, let

$$y(t) = x(2t + 1)$$

This system is not time invariant. Here is the test to prove it. First if we delay $y(t)$ by τ, we get

$$y(t - \tau) = x(2t + 1 - 2\tau).$$

Now, if we delay $x(t)$ by τ and pass it through the system, we get

$$x(2t + 1 - \tau) \neq y(t - \tau).$$

LTI systems

Why do we care about LTI systems? Because they have a very special and very useful property.

Theorem 6.1. *For an LTI system, if the input signal is a cosine, $x(t) = \cos(\omega_0 t + \phi)$, the output signal will be a cosine of the same frequency, $y(t) = A(\omega_0)\cos(\omega_0 t + \phi + \theta(\omega_0))$.*

Where $A(\omega_0)$ is the amplitude gain, $\theta(\omega_0)$ is the phase change, and both may depend on the frequency of the cosine ω_0.

$$\cos(\omega_0 t+\varphi) \longrightarrow \boxed{\text{LTI}} \longrightarrow \underbrace{A(\omega_0)}_{\text{amplitude gain}}\cos(\omega_0 t+\varphi+\underbrace{\theta(\omega_0)}_{\text{phase change}})$$

Why is this theorem important/useful?

Recall that:

- "Any" signal can be written as a linear combination of cosines using Fourier series.
- If a system is linear, and we know the output signals for the input signals $x_1(t), x_2(t), \ldots x_k(t)$ are $y_1(t), y_2(t), \ldots y_k(t)$ respectively. Then for an input signal $x(t)$ that is a linear combination of $x_1(t), x_2(t), \ldots x_k(t)$, the output signal is the same linear combination of $y_1(t), y_2(t), \ldots y_k(t)$.

Therefore:

- If a system is LTI, and you know the amplitude gain and phase change for cosines of any frequency, you know the output of the system for any input signal!

Because we can write any input signal as linear combination of cosines, and since we know the output for all these cosines, we know the output of our arbitrary input signal.

In other words, for an LTI system, all you need to know is $A(\omega)$ and $\theta(\omega)$, $\forall\omega$. Then you know everything there is to know about your system.