EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 03: September 5

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3.1 Announcement

- Homework 1: Assigned, due Thursday 9/12
- Today's Topic: Frequency domain representations of signals

3.2 Recap

Linear systems pass two tests: homogeneity and linearity.

1. Homogeneity: $x(t) \to y(t) \implies cx(t) \to cy(t)$, so $H(ax) = aH(x)$, for some constant a.
2. Additivity: $H(x_1 + x_2) = H(x_1) + H(x_2)$

3.3 Signals as functions

We now move our discussion back to signals. Last lecture, we mentioned the mathematical definition of a signal as a function of time and/or space. This time, we will elaborate further on this mathematical definition.

Definition 3.1. *A function maps each value in a set (a domain) to a value in another set (a range).*

Example 3.1. *Continuous-time or -space and continuous-valued signal (Domain: $\mathbb{R}$, Range: $\mathbb{R}$), shown in Figure XX.*

Example 3.2. *Continuous-time signal with non-negative domain and continuous-valued (Domain: $[0, \infty) = \mathbb{R}^+$, Range: $\mathbb{R}$), shown in Figure XX.*

Example 3.3. *Finite-duration continuous-time signal (Domain: $[0, T]$, Range: $\mathbb{R}$), shown in Figure XX.*

Example 3.4. *Discrete-time or -space continuous-valued signal (Domain: Integers $= \{\dots, -2, -2, 0, -1, -2, \dots\} = \mathbb{Z}$, Range $= \mathbb{R}$), shown in Figure XX.*

Example 3.5. *Discrete-time continuous-valued signal with non-negative domain (Domain: Natural numbers $= \mathbb{N} = \{0, 1, 2, 3, \dots, \}$, Range: $\mathbb{R}$), shown in Figure XX.*

Note that a system is also a function whose domain and range are signals.

Why do we deal with both continuous-time and discrete-time signals? As engineers, both representations are equally important. For example, speech is a continuous-time signal, but engineers typically perform speech analysis on a computer, which discretizes the speech to store in memory. Later in this course, we will learn how to sample a signal; that is, to transform a signal from continuous-time to discrete-time while preserving signal information.

3.3.1 Does homogeneity imply additivity? Vice versa?

This question was brought up last lecture. To disprove a conjecture, come up with a counterexample. (see Lecture 2 notes for counterexamples in both directions). It turns out that homogeneity does not imply additivity, and additivity does not imply homogeneity.

Example 3.6. *$x(t) \in \mathcal{C}$, and $c \in \mathcal{C}$. Let $y(t) = x^*(t)$. This system is additive, because $(x(t) + y(t))^* = x^*(t) + y^*(t)$. Is this homogeneous? I.e. is $(cx(t))^* = cx^*(t)$? Let $c = i$. The answer is no.*

3.4 Frequency and Sinusoids

Sinusoids are a special class of signals that have a single frequency component. This class of signals is particularly important because they occur naturally in the real world. For instance, our audio and visual sensory networks are frequency-sensitive. The other main reason that sinusoids are important is because they are a good "coordinate" system in which to represent signals; they are an important conceptual tool for decomposing and representing many important classes of signals. The application space is huge. Frequency domain analysis is used in neuroscience, stock market, cognitive radio (allocating spectrum), military, music.

This section's examples are based on these demos (link).

Frequency

Frequency Decomposition

Definition 3.2. *The frequency of a sinusoid signal is the rate (in Hz) at which it oscillates; more specifically, it is the number of cycles it can complete in one second.*

Example 3.7. *Note that the A-440 is playing at 440 Hz, whereas the A one octave higher plays at 880Hz, exactly double the frequency. The scale itself has 12 notes, where the next note up from the current note, x, is determined by $x\sqrt[12]{2}$. (link) plays an A-440 and scales it up to one octave.*

Suppose x is the A-440 signal. Mathematically, the signal $x(t)$ (note the dependency on t, "over time") is represented as follows:

$$x(t) = sin(2\pi * 440t)$$

While **Hz** is the unit representing $\frac{\text{cycles}}{\text{s}} = s^{-1}$, the argument to sin must be in radians. Thus, for one cycle (1 Hz) of a sinusoid, our argument must go from 0 to 2π radians. The 2π in the equation above has units $\frac{\text{radians}}{\text{cycle}}$.

Period

While sinusoids can be defined by their frequencies (e.g., A-440), we can also characterize them by their period, measured in seconds.

Definition 3.3. *The period (T) of a sinusoid is the duration (in seconds) required for a signal to complete one cycle; that is, the smallest time t required for the sinusoid's argument to go from 0 to 2π. So*

$$T = \frac{1}{f_o} = \frac{2\pi}{\omega_o} \tag{3.1}$$

Using our A-440 example, the signal is oscillating at 440 Hz; thus its period is $\frac{1}{440}$ s.

To summarize, a sinusoid $x(t)$ of frequency f Hz has the form $x(t) = \sin(2\pi ft)$. On the other hand, if we define our frequency to be angular frequency ω in units $\frac{radians}{s}$, where $\omega = 2\pi f$, then $x(t) = \sin(\omega t)$.

Generalizing sinusoids

A general sinusoid $x(t)$ can be represented as follows:

$$x(t) = A\sin(2\pi ft + \phi) \tag{3.2}$$

Definition 3.4. *The amplitude A of a sinusoid is the maximum quantity (over the entire range) that the sinusoid can achieve over time. The units of A and $x(t)$ are the same, and depend on the physical signal being represented (e.g., could be volts for a circuit). These units depend on the physical quantity the sinusoid is modeling.*

Definition 3.5. *The phase ϕ of a sinusoid (in radians) is the phase offset at time* 0*.*

Changing amplitude changes the height of the graphed sinusoid over time. Note this contrasts with changing the frequency of the signal, which stretches or compresses the signal in time.

Changing the phase of the signal neither shrinks nor stretches the signal in either direction; instead, it shifts the *entire* signal in time. Let the input to the system be $x(t) = \sin(2\pi ft)$. Let the output of the system be $y(t) = x(t - \tau)$. That is, whatever happens at x happens at y τ seconds later. Then we have:

$$x(t) = \sin(2\pi ft) \rightarrow \text{time delay } \tau \rightarrow y(t) = \sin(2\pi f(t - \tau)) = \sin(2\pi ft - 2\pi f\tau)$$

Here, the time-delay τ has translated into a phase-shift of $2\pi f\tau$. Note that the phase-shift of a signal (i.e., "when it arrives at our ears") does not produce an audible difference. An example is given here (link).

Multiple frequencies

This timbre demo (link) plays harmonics of an A note, where 200 Hz is the fundamental frequency.

Definition 3.6. *Note that the A harmonics are integer multiples of the fundamental frequency. The superposition of two tones is timbre.*

3.5 Periodic Signals

3.6 Periodicity

Definition 3.7. *A signal $x(t)$ is a periodic signal with period p if $\forall t \in R$,*

$$x(t + p) = x(t) \tag{3.3}$$

And we call the smallest p such that (3.3) is true to be the period of the signal $x(t)$.

An example of a periodic signal is $x(t) = \cos(\omega_0 t)$, and its associated period is $p = \frac{2\pi}{\omega_0}$. (To see this, find the smallest T such that $x(t + T) = x(t)$, i.e. $x(\omega_o t + \omega_o T) = x(\omega_o t)$).

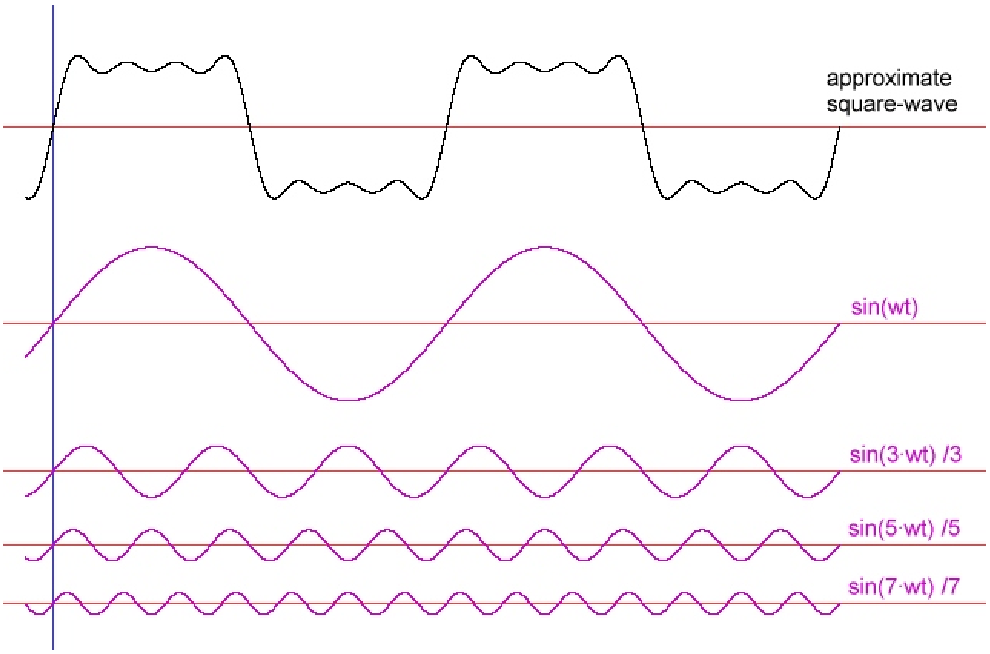

Figure 3.1: Square wave as a sum of sinusoidal signals

3.7 Introduction to Fourier Series

Fourier found that it is possible to approximate the periodic signal by adding many sinusoidal signals of different harmonics of the fundamental frequency. This decomposition is what we call the 'Fourier series' representation. Formally,

Theorem 3.1. *Given "any" periodic signal $x(t)$ with period p, it can be represented as*[1]*,*

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos(k\frac{2\pi}{T}t + \phi_k) \tag{3.4}$$

In other words, "any" periodic signal with period T can be written as a sum of infinitely many sinusoids, each of which has a frequency that is an integer multiple of $\omega_o = \frac{2\pi}{T}$.

This representation (3.4) is powerful because we can reduce the problem of understanding the properties of general periodic signals by only understanding the basic family of sinusoidal signals. For example, given a square wave, we can represent it as a sum of sinusoidal signals as in Fig 3.1.

[1]Remark: Indeed, in the equation (3.4), the equality sign means that the equation is true except possibly in countably many points. This means, there can be some points that the equality does not hold and it is called the Gibbs phenomenon. Moreover, we also assume the periodic function to be smooth enough. Hence the qualifier "any" periodic signal.