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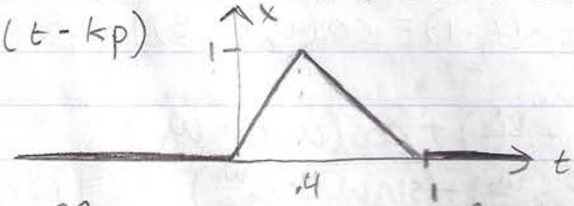
EE20

Homework 2:

1. $x: \mathbb{R} \to \mathbb{R}$ $\quad x(t) = 0 \ \forall t \in [0, 1]$ and $x(.4) = 1$

$y(t) = \sum_{k=-\infty}^{\infty} x(t - kp)$

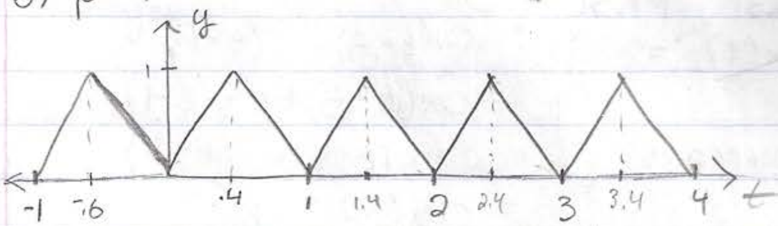


a) $y(t+p) = \sum_{k=-\infty}^{\infty} x(t+p-kp) = \sum_{k=-\infty}^{\infty} x(t+p(1-k))$

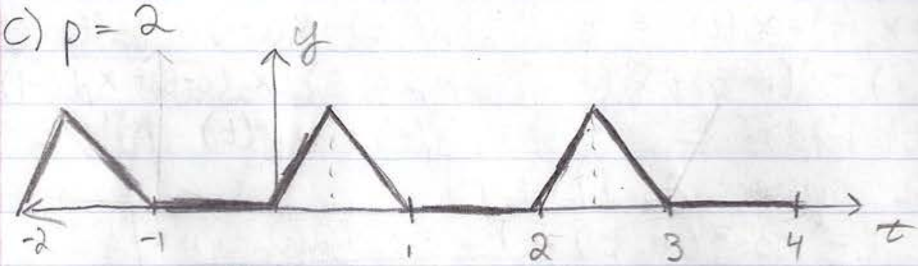
$= \sum_{k=-\infty+1}^{\infty+1} x(t-kp) = \sum_{k=-\infty}^{\infty} x(t-kp) = y(t)$

$\Rightarrow y(t+p) = y(t)$, so $y(t)$ is periodic

b) $p = 1$

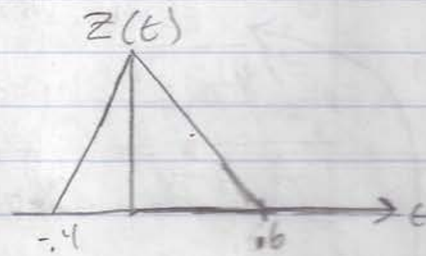


c) $p = 2$



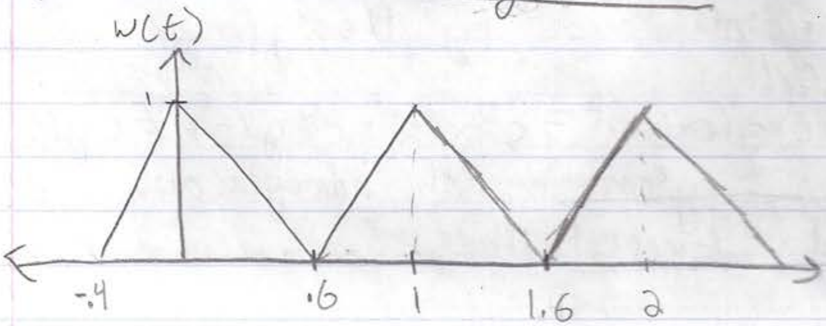
F) $\forall t \in \mathbb{R}$ $\quad z(t) = x(t+.4)$

$\Rightarrow \forall t \in \mathbb{R}$ $\quad w(t) = \sum_{k=-\infty}^{\infty} z(t - kp)$



$\boxed{w(t) = \sum_{-\infty}^{\infty} z(t-kp) = \sum_{-\infty}^{\infty} x(t - kp + .4)}$

$p = 1$ $\quad \boxed{w(t) = y(t+.4)}$



2. $x(t) = \cos w_0 t$

a) $y(t) = x(t-.5) + x(t-1) = \cos(w_0(t-.5)) + \cos(w_0(t-1))$

$\boxed{T = \frac{2\pi}{w_0}}$

$y(t) = \cos(w_0 t - \frac{w_0}{2}) + \cos(w_0 t - w_0)$

$= (\cos w_0 t)(\cos \frac{w_0}{2}) + (\sin w_0 t)(\sin \frac{w_0}{2})$ [cosine sum angle formula]

$+ (\cos w_0 t)(\cos w_0) + (\sin w_0 t)(\sin w_0)$

$= \boxed{(\cos w_0 t)(\cos w_0 + \cos \frac{w_0}{2}) + (\sin w_0 t)(\sin w_0 + \sin \frac{w_0}{2})}$ [Fourier]

* since there are no $\cos k w_0 t$ or $\sin k w_0 t$ terms for $k > 1$, there is only the fundamental frequency, and no new harmonics introduced

$\alpha_1 = \cos w_0 + \cos \frac{w_0}{2} \quad \beta_1 = \sin w_0 + \sin \frac{w_0}{2} \quad A_0 = 0$

$\alpha_{k>1},\ \beta_{k>1} = 0$

Homogeneity: $\hat{x}(t) = cx(t) \Rightarrow \hat{y}(t) = \hat{x}(t-.5) + \hat{x}(t-1)$

$= cx(t-.5) + cx(t-1)$

• The system is homogeneous $\quad = c(x(t-.5) + x(t-1))$

$\hat{y}(t) = cy(t)$ ✓

additivity: $\hat{x} = x_1(t) + x_2(t) \Rightarrow y_1(t) = x_1(t-.5) + x_1(t-1) \quad y_2(t) = x_2(t-.5) + x_2(t-1)$

$\Rightarrow \hat{y}(t) = \hat{x}(t-.5) + \hat{x}(t-1) = x_1(t-.5) + x_2(t-.5) + x_1(t-1) + x_2(t-1)$

• The sytem is additive $\quad \hat{y}(t) = y_1(t) + y_2(t)$

$\Rightarrow$ Thus, the system is $\boxed{\text{Linear}}$

b) $y(t) = x^2(t) = \cos^2(w_0 t) = \frac{1 + \cos(2w_0 t)}{2} = \boxed{\frac{1}{2} + \frac{1}{2}\cos(2w_0 t)}$

$F = \frac{2w_0}{2\pi} = \frac{w_0}{\pi}$

$\Rightarrow \boxed{w = 2w_0} \Leftrightarrow \boxed{T = \frac{2\pi}{w} = \frac{2\pi}{2w_0} = \frac{\pi}{w_0}}$

Fourier expansion

$\boxed{A_0 = \frac{1}{2} \quad \underset{k=2}{\alpha_2 = \frac{1}{2}} \quad \alpha_{k\neq 2},\ \beta_k = 0}$

* The first harmonic, corresponding to $k=2$, with $w = 2w_0$, is a new frequency introduced by the sytem.

Homogeneity: $\hat{x}(t) = cx(t)$

$\hat{y}(t) = \hat{x}^2(t) = (cx(t))^2 = c^2 x(t) = c^2 y(t) \neq cy(t)$

* Because the sytem is not homogeneous, it is $\boxed{\text{not Linear}}$ either.

2c) $y(t) = x(t)\sin(2w_0 t) = \cos w_0 t \sin(2w_0 t)$

$= \frac{1}{2}[\sin(3w_0 t) - \sin(-w_0 t)]$

$= \frac{1}{2}[\sin(3w_0 t) + \sin(w_0 t)]$

$\Rightarrow \boxed{y(t) = \frac{1}{2}\sin 3w_0 t + \frac{1}{2}\sin w_0 t}$ [Fourier expansion]

$A_0 = 0 \quad B_1 = +\frac{1}{2} \quad B_3 = +\frac{1}{2} \quad B_{k\neq 1, k\neq 3},\ \alpha_k = 0$

$F = 3w_0/2\pi$

* the Frequency $\boxed{w = 3w_0}$, corresponding to the harmonic $k=3$, is a new frequency introduced by this sytem.

Homogeneity: $\hat{x} = cx(t)$

$\hat{y}(t) = \hat{x}(t)\sin(2w_0 t) = cx(t)\sin(2w_0 t) = c\,y(t)$ [homogeneous]

Additivity: $\hat{x}(t) = x_1(t) + x_2(t) \qquad y_1(t) = x_1(t)\sin(2w_0 t) \quad y_2(t) = x_2(t)\sin(2w_0 t)$

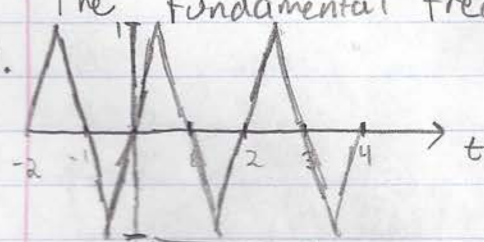
$\hat{y}(t) = \hat{x}(t)\sin(2w_0 t) = x_1(t)\sin 2w_0 t + x_2(t)\sin(2w_0 t)$

$= y_1(t) + y_2(t)$ [Additive]

$\boxed{\text{The system is Linear}}$

The fundamental frequency is still w_0, so $\boxed{T = \frac{2\pi}{w_0}}$

3.



$T = 2 \Rightarrow \boxed{w_0 = \frac{2\pi}{2} = \pi}$

$A_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2}\int_0^2 x(t)\,dt$

$= \frac{1}{2}\left(\int_0^{.5} 2t\,dt + \int_{.5}^{1.5}(-2t+2)\,dt + \int_{1.5}^{2}(2t-4)\,dt\right) = 0$

$\Rightarrow \boxed{A_0 = 0}$

[No DC term, $\overline{x(t)}$ is zero]

$A_k = \frac{2}{T}\int_0^T x(t)\cos(k\pi t)\,dt = \int_0^{.5} 2t\cos(k\pi t)\,dt + \int_{.5}^{1.5}(-2t+2)\cos(k\pi t)\,dt + \int_{1.5}^{2}(2t-4)\cos(k\pi t)\,dt$

$= \left[2t\left(\frac{1}{k\pi}\right)\sin(k\pi t) + 2\left(\frac{1}{k^2\pi^2}\right)\cos(k\pi t)\right]_0^{.5} + \left[(-2t+2)\left(\frac{1}{k\pi}\right)\sin k\pi t - 2\left(\frac{1}{k^2\pi^2}\right)\cos(k\pi t)\right]_{.5}^{1.5} + \left[(2t-4)\left(\frac{1}{k\pi}\right)\sin k\pi t + 2\left(\frac{1}{k^2\pi^2}\right)\cos(k\pi t)\right]_{1.5}^{2}$

$= \cancel{\frac{1}{k\pi}\sin\frac{k\pi}{2}} + \frac{2}{k^2\pi^2}\cos\frac{k\pi}{2} - \cancel{\frac{2}{k^2\pi^2}} - \cancel{\frac{1}{k\pi}\sin\frac{3}{2}k\pi} - \frac{2}{k^2\pi^2}\cos\frac{3}{2}k\pi - \cancel{\frac{1}{k\pi}\sin\frac{k\pi}{2}} + \frac{2}{k^2\pi^2}\cos\frac{k\pi}{2} + \cancel{\frac{2}{k^2\pi^2}} + \cancel{\frac{1}{k\pi}\sin\frac{3}{2}k\pi} - \frac{2}{k^2\pi^2}\cos\frac{3}{2}k\pi$

$= 4\left(\frac{1}{k^2\pi^2}\right)\cos\frac{k\pi}{2} - 4\left(\frac{1}{k^2\pi^2}\right)\cos\left(\frac{3}{2}k\pi\right)$

$= \boxed{0} \qquad \Rightarrow \boxed{A_k = 0 \text{ for all } k}$

(3 continued) $B_k = \frac{2}{T}\int_0^T x(t)\sin(k\pi t)\,dt = \int_0^{.5} 2t\sin(k\pi t)\,dt + \int_{.5}^{1.5}(-2t+2)\sin(k\pi t)\,dt + \int_{1.5}^{2}(2t-4)\sin(k\pi t)\,dt$

$$= \left[-2t\frac{1}{k\pi}\cos k\pi t + 2\frac{1}{k^2\pi^2}\sin k\pi t\right]_0^{.5} + \left[-(-2t+2)\frac{1}{k\pi}\cos k\pi t - \frac{2}{k^2\pi^2}\sin k\pi t\right]_{.5}^{1.5} + \left[-(2t-4)\frac{1}{k\pi}\cos k\pi t + \frac{2}{k^2\pi^2}\sin k\pi t\right]_{1.5}^{2}$$

$$= -\cancel{\frac{1}{k\pi}\cos\frac{k\pi}{2}} + \frac{2}{k^2\pi^2}\sin\frac{k\pi}{2} - \cancelto{0}{\frac{2}{k^2\pi^2}\sin(0)} + \cancel{\frac{1}{k\pi}\cos\frac{3}{2}k\pi} - \frac{2}{k^2\pi^2}\sin\frac{3}{2}k\pi + \cancel{\frac{1}{k\pi}\cos\frac{k\pi}{2}} + \frac{2}{k^2\pi^2}\sin\frac{k\pi}{2}$$
$$+ \cancelto{0}{\frac{2}{k^2\pi^2}\sin 2\pi k} - \cancel{\frac{1}{k\pi}\cos\frac{3}{2}\pi k} - \frac{2}{k^2\pi^2}\sin\frac{3}{2}\pi k$$

$$= 4\frac{1}{k^2\pi^2}\sin\frac{k\pi}{2} - 4\frac{1}{k^2\pi^2}\sin\frac{3}{2}\pi k$$

$$\boxed{B_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{8}{k^2\pi^2} & \text{if } k = 1, 5, 9, 13 \\ -\frac{8}{k^2\pi^2} & \text{if } k = 3, 7, 11, 15 \end{cases}}$$

$$\Rightarrow \underbrace{\boxed{x(t) = \sum_{k=1}^{\infty} B_k \sin(k\pi t)}}_{\text{Fourier series}} \quad \text{where } B_k$$

4. $x[n] = A_0 + \sum_{k=1}^{K} A_k \cos(k\omega_0 n + \phi_k)$ $\quad K = \begin{cases} \frac{p-1}{2} & p \text{ odd} \\ \frac{p}{2} & p \text{ even} \end{cases}$

a) $p = 5 \Leftrightarrow K = \frac{p-1}{2} = \frac{5-1}{2} = 2$

$\omega_0 = \frac{m}{p} 2\pi \qquad m > K \qquad m' \leq K$

case 1. when $m > p = 5$: $m = pt + m'$ where $t > 0$ & $t \in \mathbb{Z}$

$$\cos\left(\frac{2\pi}{p}(pt+m') + \phi_1\right) = \cos\left(\frac{2\pi}{p} m' + \phi_2\right)$$

$$\cos\left(\frac{2\pi}{5}(5t+m') + \phi_1\right) = \cos\left(\frac{2\pi}{5} m' + \phi_2\right)$$

$$\cos\left(2\pi t + \frac{2\pi m'}{5} + \phi_1\right) = \cos\left(\frac{2\pi}{5} m' + \phi_2\right)$$

$$\Rightarrow 2\pi t + \cancel{\frac{2\pi m'}{5}} + \phi_1 = \cancel{\frac{2\pi}{5} m'} + \phi_2$$

$$\phi_2 = \phi_1 + 2\pi t \Rightarrow \Delta\phi = 2\pi t$$

Since the phases differ by $2\pi t$, they are the <u>exact same sinusoid.</u>

Case 2: $K < m \leq p \Leftrightarrow m = 3, 4, 5$

$m = 3 \quad \cos\left(\frac{6\pi}{5}\right) = \cos\left(\frac{2\pi}{5} m'\right) \quad \Rightarrow m' = 2$

$m = 4 \quad \cos\left(\frac{8\pi}{5}\right) = \cos\left(\frac{2\pi}{5} m'\right) \quad \Rightarrow m' = 1$

$m = 5 \quad \cos(2\pi) = \cos\left(\frac{2\pi}{5} m'\right) \quad \Rightarrow m' = 0$

b) case 1: $m > 6 = p \quad \Rightarrow K = 3 \qquad m > K \qquad m' \leq K$

$m = pt + m' \quad \cos\left(\frac{\pi}{3}(6t+m') + \phi_1\right) = \cos\left(\frac{\pi}{3} m' + \phi_2\right)$

$\Rightarrow \phi_1 + 2\pi t = \phi_2$ * Thus they are the same

case 2: $m \leq p \Rightarrow m = 4, 5, 6$

$m = 4 \quad \cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{\pi}{3} m'\right) \Rightarrow m' = 2$

$m = 5 \quad \cos\left(\frac{5}{3}\pi\right) = \cos\left(\frac{\pi}{3} m'\right) \Rightarrow m' = 1$

$m = 6 \quad \cos(2\pi) = \cos\left(\frac{\pi}{3} m'\right) \Rightarrow m' = 0$

c) We will always be able to find a sinusoid of frequency $\frac{m}{p}$ with an equivalent sinusoid of frequency $\frac{m'}{p}$. This is because for any p, whether odd or even, a sinusoid of $\frac{m}{p}$ where $m > p$ will always have a corresponding m' for which the phase ~~of~~ that sinusoid differs by $2\pi n$. Also, for every integer between k and p, there is a corresponding m' since $p - K \leq K + 1$ for any p odd or even. Since this is true for all p, all discrete signals have a minimum frequency which is a rational number with p on the bottom. Any other frequency which is rational is really just a larger period of the same signal.

$P=5 \Rightarrow K=2$

$$x[n] = A_0 + \sum_{k=1}^{2}\left(\alpha_k \cos\left(\frac{2\pi k}{5}n\right) + \beta_k \sin\left(\frac{2\pi k}{5}n\right)\right)$$

5. a)

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & \cos\left(\frac{2\pi}{5}\right) & \sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{4\pi}{5}\right) & \sin\left(\frac{4\pi}{5}\right) \\ 1 & \cos\left(\frac{4\pi}{5}\right) & \sin\left(\frac{4\pi}{5}\right) & \cos\left(\frac{8\pi}{5}\right) & \sin\left(\frac{8\pi}{5}\right) \\ 1 & \cos\left(\frac{6\pi}{5}\right) & \sin\left(\frac{6\pi}{5}\right) & \cos\left(\frac{12\pi}{5}\right) & \sin\left(\frac{12\pi}{5}\right) \\ 1 & \cos\left(\frac{8\pi}{5}\right) & \sin\left(\frac{8\pi}{5}\right) & \cos\left(\frac{16\pi}{5}\right) & \sin\left(\frac{16\pi}{5}\right) \end{bmatrix}}_{\text{Discrete Time Fourier Series Transform Matrix}} \begin{bmatrix} A_0 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{bmatrix}$$

b) $P=3 \Rightarrow k=1$

$$x[n] = A_0 + \sum_{k=1} \alpha_k \cos\left(\frac{2\pi k}{3}n\right) + \beta_k \sin\left(\frac{2\pi k}{3}n\right)$$

$$= A_0 + \alpha_1 \cos\left(\frac{2\pi}{3}n\right) + \beta_1 \sin\left(\frac{2\pi}{3}n\right)$$

$x(0) = 3 = A_0 + \alpha_1$

$x(1) = 2 = A_0 + \alpha_1 \cos\left(\frac{2\pi}{3}\right) + \beta_1 \sin\left(\frac{2\pi}{3}\right) = A_0 + -\frac{\alpha_1}{2} + \frac{\sqrt{3}}{2}\beta_1$

$x(2) = 1 = A_0 + \alpha_1 \cos\left(\frac{4\pi}{3}\right) + \beta_1 \sin\left(\frac{4\pi}{3}\right) = A_0 + -\frac{\alpha_1}{2} + -\frac{\sqrt{3}}{2}\beta_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 2 \\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 2 \\ 0 & 0 & 1 & \frac{\sqrt{3}}{3} \end{bmatrix} \quad \beta_1 = \frac{\sqrt{3}}{3}$$

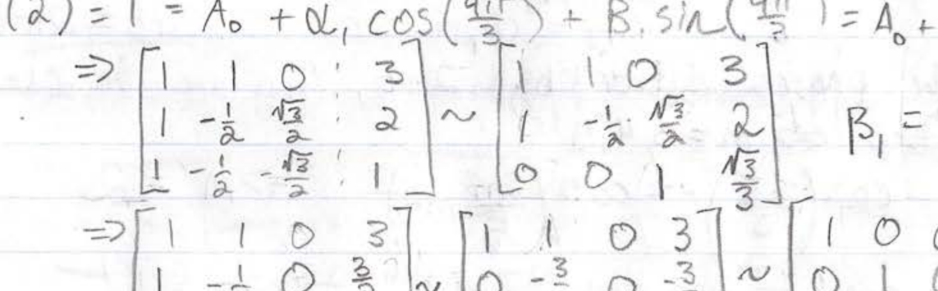
$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{\sqrt{3}}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{\sqrt{3}}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{\sqrt{3}}{3} \end{bmatrix} \quad \begin{aligned} A_0 &= 2 \\ \alpha_1 &= 1 \end{aligned}$$

$\Rightarrow x(0) = 3 = A_0 + \alpha_1 = 1 + 2 = 3$ ✓

$\Rightarrow x(1) = 2 = A_0 - \frac{\alpha_1}{2} + \frac{\sqrt{3}}{2}\beta_1 = 2 - \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{3} = 2$ ✓

$\Rightarrow x(2) = 1 = A_0 - \frac{\alpha_1}{2} - \frac{\sqrt{3}}{2}\beta_1 = 2 - \frac{1}{2} - \frac{1}{2} = 1$

$$\boxed{x[n] = 2 + \cos\left(\frac{2\pi}{3}n\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{2\pi}{3}n\right)}$$



6. $x(t) = x(t+T) \qquad W_0 = \frac{2\pi}{T}$

$$X(t) = A_0 + \hat{\sum_{k=1}} (\alpha_k \cos k w_0 t + \beta_k \sin k w_0 t)$$

$$\int_0^T \cos(m\tfrac{2\pi}{T}t)\cos(n\tfrac{2\pi}{T}t)\,dt = \begin{cases} 0 & m \neq n \\ T & m = n = 0 \\ \frac{T}{2} & m = n \neq 0 \end{cases}$$

$$\int_0^T \sin(m\tfrac{2\pi}{T}t)\sin(n\tfrac{2\pi}{T}t)\,dt = \begin{cases} 0 & m \neq n \text{ or } m = n = 0 \\ T/2 & m = n \neq 0 \end{cases}$$

$$\int_0^T \cos(m\tfrac{2\pi}{T}t)\sin(n\tfrac{2\pi}{T}t)\,dt = 0 \quad \text{for all } m, n$$

$$\Rightarrow \int_0^T x(t)\,dt = A_0 \int_0^T dt + \sum_{k=1}^{\infty} \alpha_k \int_0^T \cos k\tfrac{2\pi}{T}t\,dt + \beta_k \int_0^T \sin k\tfrac{2\pi}{T}t\,dt$$

$$= A_0 T + \sum_{k=1}^{\infty} \alpha_k \frac{T}{2k\pi} \sin(k\tfrac{2\pi}{T}t)\Big|_0^T - \beta_k \frac{T}{2\pi k}\cos(\tfrac{2\pi k}{T}t)\Big|_0^T$$

$$= A_0 T + \sum_{k=1}^{\infty} 0\alpha_k - \cancel{(\beta_k - \beta_k)\frac{T}{2\pi k}} \rightarrow 0$$

$$\Rightarrow \int_0^T x(t)\,dt = A_0 T \quad \Rightarrow \boxed{A_0 = \frac{1}{T}\int_0^T x(t)\,dt}$$

$$\Rightarrow x(t)\cos(k\tfrac{2\pi}{T}t) = A_0 \cos(k\tfrac{2\pi}{T}t) + \sum_{k=1}^{\infty} \alpha_k \cos(k\tfrac{2\pi}{T}t)\cos(k\tfrac{2\pi}{T}t) + \beta_k \cos(k\tfrac{2\pi}{T}t)\sin(k\tfrac{2\pi}{T}t)$$

$$\int_0^T x(t)\cos(k\tfrac{2\pi}{T}t)\,dt = A_0 \cancel{\int_0^T \cos(k\tfrac{2\pi}{T}t)\,dt} \rightarrow 0 + \sum_{k=1}^{\infty} \alpha_k \int_0^T \cos^2(k\tfrac{2\pi}{T}t) + \beta_k \cancel{\int_0^T \cos(\tfrac{2\pi k}{T}t)\sin(\tfrac{2\pi k}{T}t)\,dt} \rightarrow 0$$

$$\int_0^T x(t)\cos(k w_0 t)\,dt = \alpha_k \int_0^T \cos^2(k\tfrac{2\pi}{T}t)\,dt = \alpha_k \frac{T}{2}$$

$$\boxed{\Rightarrow \alpha_k = \frac{2}{T}\int_0^T x(t)\cos(k w_0 t)\,dt}$$

$$\Rightarrow x(t)\sin(k\tfrac{2\pi}{T}t) = A_0 \sin(k\tfrac{2\pi}{T}t) + \sum_{k=1}^{\infty} \alpha_k \sin(k\tfrac{2\pi}{T}t)\cos(k\tfrac{2\pi}{T}t) + \beta_k \sin^2(k\tfrac{2\pi}{T}t)$$

$$\Rightarrow \int_0^T x(t)\sin(k W_0 t)\,dt = A_0 \cancel{\int_0^T \sin(k\tfrac{2\pi}{T}t)\,dt} \rightarrow 0 + \sum_{k=1}^{\infty} \alpha_k \cancel{\int_0^T \sin(\tfrac{2\pi k}{T}t)\cos(\tfrac{2\pi k}{T}t)} + \beta_k \int_0^T \sin^2(k\tfrac{2\pi}{T}t)\,dt$$

$$= \beta_k \frac{T}{2}$$

$$\Rightarrow \boxed{\beta_k = \frac{2}{T}\int_0^T x(t)\sin(k w_0 t)\,dt}$$