Homework #7

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EE20N

1. [LV 10.5] $f \in Q \Rightarrow f = m/p$ for $m \in Z$ $\Rightarrow 2\pi f = \omega_0 m$ $\quad \omega_0 = \frac{2\pi}{p}$

a) $x[n] = e^{i2\pi f n}$ $\forall n \in Z$, $f \neq 0$

Using 10.9: $\forall k \in Z$ $X'_k = \sum_{n=0}^{p-1} x[n] e^{-ink\omega_0} = \sum_{n=0}^{p-1} e^{i2\pi f n} e^{-ink\omega_0} = \sum_{n=0}^{p-1} e^{i\omega_0 mn} e^{-i\omega_0 nk}$

$= \sum_{n=0}^{p-1} e^{i\omega_0 n(m-k)}$

$k \in \{\dots m-2p, m-p, m, m+p, m+2p, \dots\} = \{m - lp\}$

is equivalent to $(m-k) = lp$ for some integer $l \in Z$,

So $m-k$ is just an integer multiple of p

$\Rightarrow \sum_{n=0}^{p-1} e^{i\omega_0 n(m-k)} = \sum_{n=0}^{p-1} e^{i(\frac{2\pi}{p}) n lp} = \sum_{n=0}^{p-1} e^{i2\pi k}$ where $k = np \in Z$ [integer]

Since $e^{i2\pi k} = 1 \Rightarrow \sum_{n=0}^{p-1} (1) = \boxed{p}$

if $k \notin \{m - lp\}$, then $(m-k)$ is not an integer multiple $\Rightarrow (m-k) = ap$, $\underbrace{a \in \mathbb{R} \setminus Z}_{\text{Non Integer, Real}}$

$\Rightarrow \sum_{n=0}^{p-1} e^{i\omega_0 n(m-k)} = \sum_{n=0}^{p-1} e^{i\frac{2\pi}{p} nap} = \sum_{n=0}^{p-1} (e^{i2\pi a})^n = \frac{1-(e^{i2\pi a})^p}{1-e^{i2\pi a}}$ [geometric sum]

$= \frac{1-(e^{i2\pi})^{ap}}{1-(e^{i2\pi})^a} = \frac{1-1}{1-(e^{i2\pi})^a} = \frac{0}{1-e^{i2\pi a}} = \boxed{0}$ $\Rightarrow \boxed{X'_k = \begin{cases} p & \text{if } k \in \{\dots m-p, m, m+p, \dots\} \\ 0 & \text{else} \end{cases}}$

b) $x[n] = \cos(2\pi f n)$

$X'_k = \sum_{n=0}^{p-1} \cos(2\pi f n) e^{-ink\omega_0} = \sum_{n=0}^{p-1} \frac{1}{2}(e^{i2\pi f n} + e^{-i2\pi f n}) e^{-ink\omega_0} = \frac{1}{2}\sum e^{i2\pi f n} e^{-in2\pi F} + \frac{1}{2}\sum e^{-i2\pi f n} e^{-in2\pi}$

$\Rightarrow \sum e^{-i2\pi f n} e^{-i2\pi f n} = \begin{cases} p & \text{if } k \in \{\dots -m-2p, -m-p, -m, -m+p, -m+2p \dots\} \\ 0 & \text{else} \end{cases} = a_k$

$\Rightarrow \sum e^{i2\pi f n} e^{-2\pi f n} = \begin{cases} p & \text{if } k \in \{m-2p, m-p, m, m+p, \dots\} \\ 0 & \text{else} \end{cases} = b_k$

by superposition:

$$\boxed{X'_k = \frac{1}{2} a_k + \frac{1}{2} b_k = \begin{cases} \frac{p}{2} & \text{if } k \in \{\dots m-p, m, m+p, \dots\} \\ \frac{p}{2} & \text{if } k \in \{\dots -m-p, -m, -m-p \dots\} \\ 0 & \text{else} \end{cases}}$$

c) $x[n] = \sin(i2\pi fn)$

$$X'_k = \sum_{n=0}^{P-1} \frac{1}{2i}\left(e^{i2\pi fn} - e^{-i2\pi fn}\right) e^{-inkw_0}$$

$$= \frac{1}{2i}\sum_{n=0}^{P-1} e^{i2\pi fn} e^{-inkw_0} - \frac{1}{2i}\sum_{n=0}^{P-1} e^{-i2\pi fn} e^{-inkw_0}$$

$$a_k = \sum_{n=0}^{P-1} e^{i2\pi fn} e^{-inkw_0} = \begin{cases} P & \text{if } k \in \{\dots m-p, m, m+p\dots\} \\ 0 & \text{else} \end{cases}$$

$$b_k = \sum_{n=0}^{P-1} e^{-i2\pi fn} e^{-inkw_0} = \begin{cases} P & \text{if } k \in \{-m-p, -m, -m+p \\ 0 & \text{else} \end{cases}$$

by superposition of the DTF

$$\boxed{X'_k = \frac{1}{2i} a_k - \frac{1}{2i} b_k = \begin{cases} P/2i & \text{if } k \in \{m-p, m, m+p, \dots\} \\ -P/2i & \text{if } k \in \{-m-p, -m, -m+p, \dots\} \\ 0 & \text{else} \end{cases}}$$

d) $x[n] = 1 \quad \forall n \in \mathbb{Z}$

$$X'_k = \sum_{n=0}^{P-1} (1) e^{-inkw_0} = \sum_{n=0}^{P-1} e^{-in\frac{k}{P}2\pi}$$

for $l \in \mathbb{Z}$

since $n \in \mathbb{Z}$, $e^{-in\frac{k}{p}2\pi} = 1$ when k is an integer multiple of $p \Rightarrow k = lp$

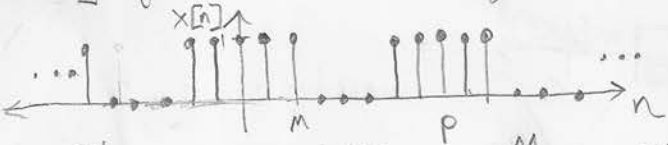
$$\Rightarrow \sum_{n=0}^{P-1} e^{-inl2\pi} = \sum_{n=0}^{P-1} (1) = \boxed{P} \quad \text{if } k \in \{-2p, -p, 0, p, 2p, \dots\}$$

if $k \notin \{lp : l \in \mathbb{Z}\}$: $k \in \{ap : a \in \mathbb{R} \setminus \mathbb{Z}\}$

$$\Rightarrow \sum_{n=0}^{P-1} e^{-i\frac{2\pi apn}{P}} = \sum_{n=0}^{P-1} \left(e^{-i2\pi a}\right)^n = \frac{1 - (e^{-2\pi i})^{ap}}{1 - e^{-2\pi i a}} = \frac{1-1}{1-e^{-2\pi i a}} = \boxed{0}$$

$$\Rightarrow \boxed{X'_k = \begin{cases} P & \text{if } k \in \{lp : l \in \mathbb{Z}\} \\ 0 & \text{else} \end{cases}}$$

2. [LV 10.6] Symmetric discrete square wave: $x[n]$ $\quad w_0 = 2\pi/p \quad p > 2M$



$\forall n \in \{0,1,\cdots,p-1\}$, $x[n] = \begin{cases} 1 & \text{if } n \le M \\ 1 & \text{if } n \ge p-M \\ 0 & \text{otherwise} \end{cases}$

$$X'_k = \sum_{m=0}^{p-1} x[m] e^{-imk\frac{2\pi}{p}} = \sum_{m=0}^{M} e^{-imk\frac{2\pi}{p}} + \sum_{m=p-M}^{p-1} e^{-imk\frac{2\pi}{p}} \qquad \boxed{p-M=0 \Rightarrow p-1=M-1}\ p=M$$

$$= \sum_{m=0}^{M} e^{-imk\frac{2\pi}{p}} + \sum_{m=0}^{M-1} e^{-imk\frac{2\pi}{p}} \qquad \text{if } k = \ell p \text{ for } \ell \in \mathbb{Z},\ e^{-imk2\pi p} = e^{-i\ell m 2\pi} = 1$$

$$= \sum_{m=0}^{M}(1) + \sum_{m=0}^{M-1}(1) = M+1+M = \boxed{2M+1} \quad \text{if } k \text{ is a multiple of } p$$

$$\Rightarrow \sum_{m=0}^{M} e^{-imkw_0} + \sum_{m=p-M}^{p-1} \overbrace{e^{ik\frac{2\pi}{p}p}}^{1} e^{-imkw_0} = \sum_{m=0}^{M} e^{-imkw_0} + e^{ikw_0 M}\sum_{m=0}^{M-1} e^{-imkw_0}$$

$$= e^{-iMkw_0} + (1+e^{ikw_0 M})\sum_{0}^{M-1} e^{-imkw_0} = e^{-iMkw_0} + (1+e^{ikw_0 M})\frac{1-e^{-ikw_0 M}}{1-e^{-ikw_0}}$$

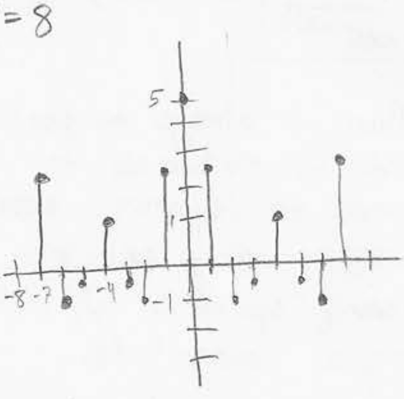
$$= \frac{\cancel{e^{-iMkw_0}} - e^{-ikw_0(M+1)} \cancel{+1} \cancel{-e^{ikw_0 M - ikw_0 M}} + e^{ikw_0 M} \cancel{-e^{-ikw_0 M}}}{1-e^{-ikw_0}} = \frac{e^{ikw_0 M} - e^{-ikw_0(M+1)}}{1-e^{-ikw_0}}$$

$$= \frac{\cancel{e^{-ikw_0/2}}\left(e^{ikw_0(M+\frac{1}{2})} - e^{-ikw_0(M+\frac{1}{2})}\right)}{\cancel{e^{-ikw_0/2}}\left(e^{ikw_0/2} - e^{-ikw_0/2}\right)} = \boxed{\frac{\sin\left(kw_0(M+\frac{1}{2})\right)}{\sin(kw_0/2)}}$$

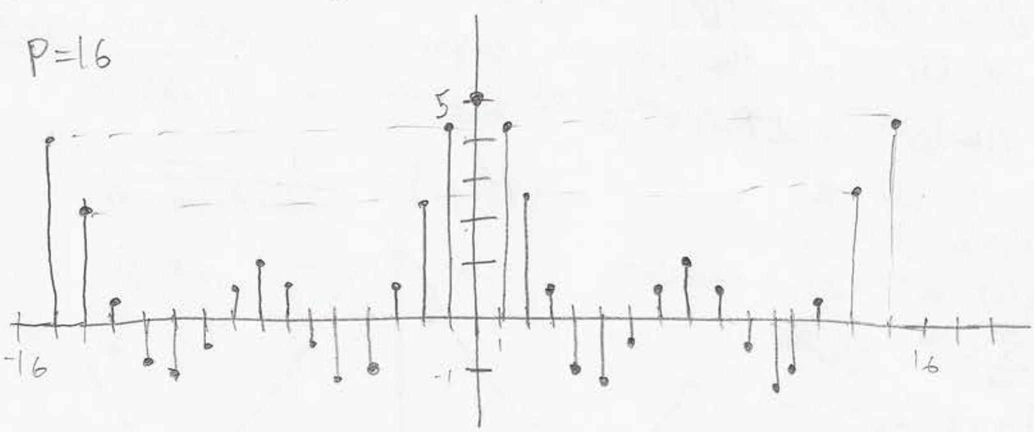
b) MatLab Code:

```
M=2;
p=8;   ## p=16, p=32
m=[-p+1:1:p-1];
k1=[-p+1:1:-1];
k2=[1:1:p-1];
y1=sin(k1*((2*pi)/p)*(M+.5))./sin(k1*.5*2*pi/p);
y2=sin(k2*((2*pi)/p)*(M+.5))./sin(k2*.5*2*pi/p);
y=[y1, 2M+1, y2];  <= we avoid divide by zero error
stem(k,y)
```

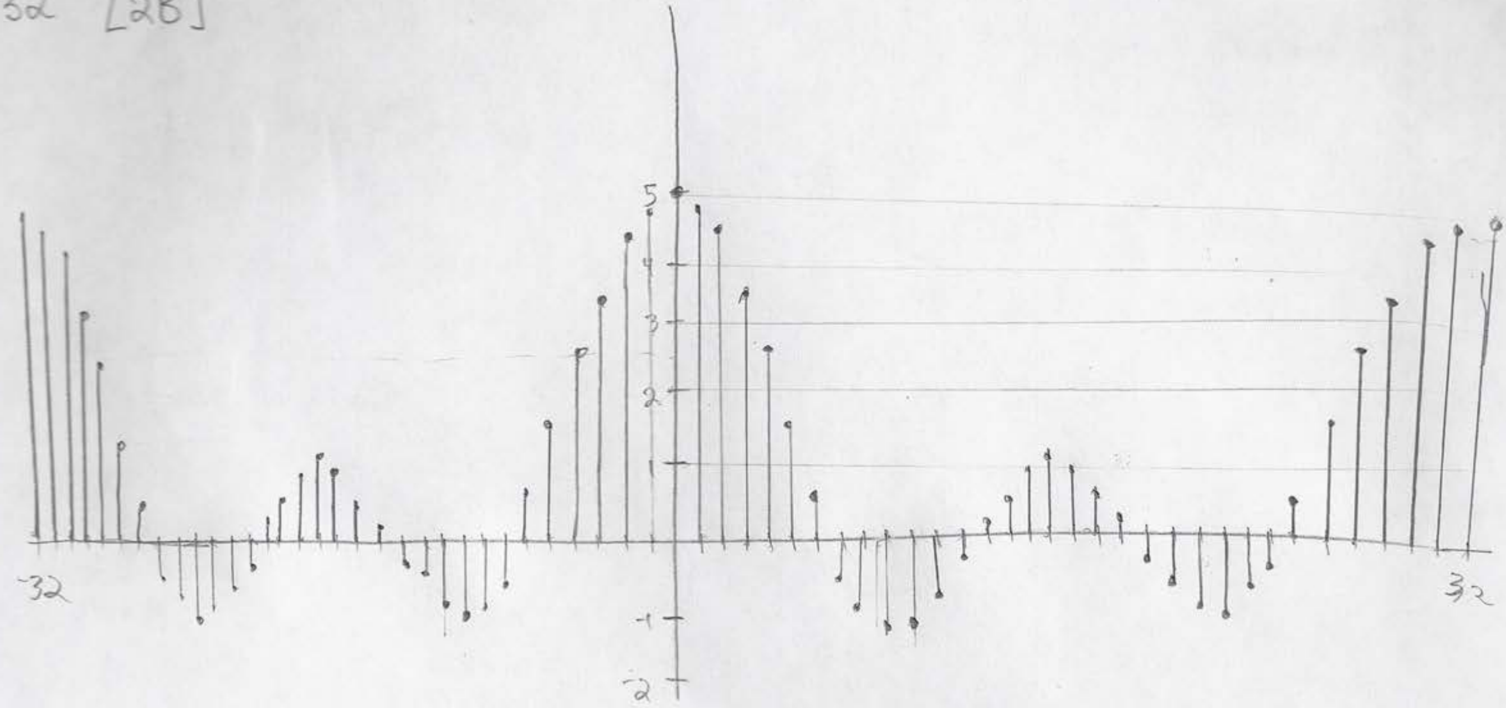
P=8



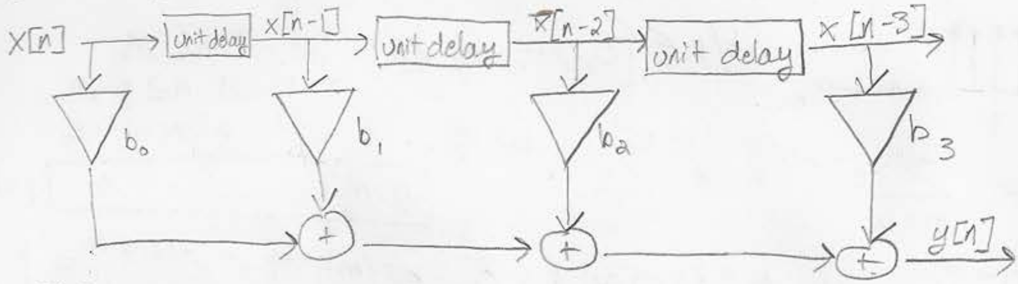
P=16



P=32 [2B]



3. [LV 10.15]



$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3]$$

$$H(\omega) e^{i\omega n} = b_0 e^{i\omega n} + b_1 e^{i\omega n} e^{-i\omega} + b_2 e^{i\omega n} e^{-2i\omega} + b_3 e^{i\omega n} e^{-3i\omega}$$

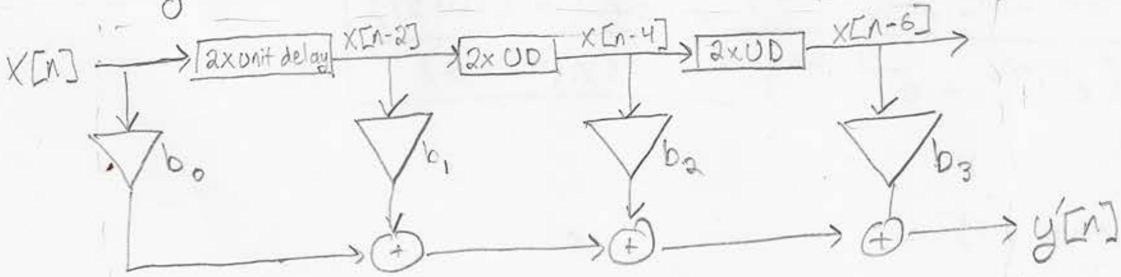
$$\Rightarrow \boxed{H(\omega) = b_0 + b_1 e^{-i\omega} + b_2 e^{-2i\omega} + b_3 e^{-3i\omega}}$$

$$h'[n] = b_0 \delta[n] + b_1 \delta[n-2] + b_2 \delta[n-4] + b_3 \delta[n-6]$$

$$H'(\omega) = \sum_{-\infty}^{\infty} h'[k] e^{-i\omega k} = \sum_{0}^{6} h'[k] e^{-i\omega k}$$

$$= b_0 + b_1 e^{-2i\omega} + b_2 e^{-4i\omega} + b_3 e^{-6i\omega} = b_0 + b_1 e^{-(2\omega)i(1)} + b_2 e^{-(2\omega)i(2)} + b_3 e^{-(2\omega)i(3)}$$

$$= \sum_{0}^{3} b_k e^{-(2\omega)ik} = H(2\omega) \Leftrightarrow \boxed{H'(\omega) = H(2\omega)}$$



4. Comb Filter: $y[n] = \alpha y[n-N] + x[n]$

a) DTFT of $\delta[n]$:

$$D(\omega) = \sum_{-\infty}^{\infty} \delta[n] e^{-i\omega n} = e^{-i\omega(0)} = 1$$

b) $y[n] = \alpha y[n-N] + \delta[n]$

The DTFT of the impulse response is just the frequency response:

$$Y(\omega) \cancel{e^{i\omega n}} = \alpha Y(\omega) \cancel{e^{i\omega n}} e^{-i\omega N} \cancel{+ e^{i\omega n}} \Rightarrow \boxed{Y(\omega) = \frac{1}{1 - \alpha e^{i\omega N}}} = H(\omega)$$

$$|Y(\omega)| = \frac{1}{\sqrt{(1-\alpha\cos\omega N)^2 + (\alpha \sin \omega N)^2}}$$

$y(0) = 1 \quad y(2N) = \alpha^2$

$y(N) = \alpha \quad y(3N) = \alpha^3$

$\Rightarrow y(n) = \alpha^{n/N} u[n]$ if $n \bmod N = 0$

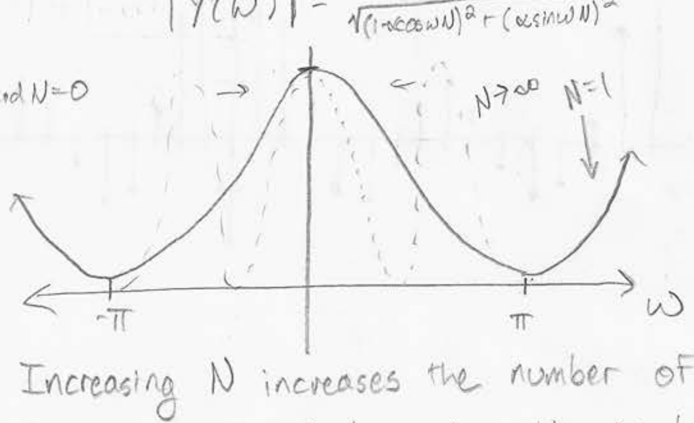
$$Y(\omega) = \sum_{-\infty}^{\infty} \alpha^{n/N} u[n] e^{-i\omega n}$$

$$= 1 + \alpha e^{-i\omega N} + \alpha^2 e^{-i\omega 2N} + \ldots$$

$$= \sum_{k=0}^{\infty} (\alpha e^{-i\omega N})^k$$

$$\boxed{Y(\omega) = \frac{1}{1 - \alpha e^{-i\omega N}}}$$

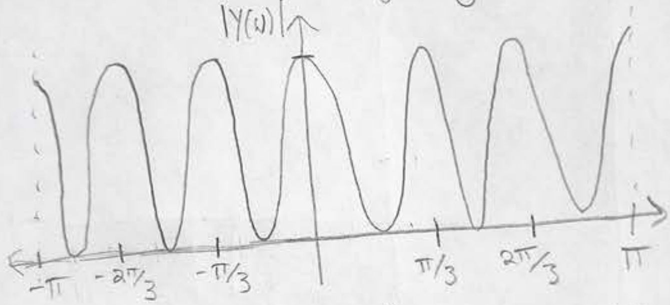
which confirms $Y(\omega) = H(\omega)$



There is always a peak at $\omega = 0$. Subsequent peaks are symmetric about the y axis and are evenly spaced in ω, with equal magnitude.

Increasing N increases the number of peaks per period, and increasing $\alpha \to 1$ makes the peaks narrower.

c) In the period from $\omega = -\pi$ to $\omega = \pi$, we have $N+1$ maximums. For an even choice of N's, $\omega = -\pi$ and $\omega = \pi$ correspond to maximums. Since we want to gain for the frequencies 440, 880, 1320 (3 total), we choose $N = 6$, which has 7 maximums from $-\pi$ to π, including a peak at zero, and 3 on each side of zero for the fundamental frequency and its harmonics.



$W_0 = 2\pi F_c / F_s \qquad F_c = 440\text{hz}$

From the graph, $w_0 = \pi/3$ $\boxed{N = 6}$

$\Rightarrow \frac{\pi}{3} = 2\pi(440\text{hz})/f_s$

$\Rightarrow \boxed{F_s = 6(440) = 2640 \text{ samples/cycle}}$

d) $20\log\left(\frac{|Y(\omega_{20Hz})|}{|Y(0)|}\right) = -10\text{dB} \qquad W_{20Hz} = \frac{2\pi(20Hz)}{2640} = .0476$

$$\frac{1}{\sqrt{10}} = \frac{\sqrt{(1-\alpha\cos(0))^2 + \cancel{(\alpha\sin(0))^2}}}{\sqrt{(1-\alpha\cos((.0476)6))^2 + (\alpha\sin(6(.0476))^2}} \Rightarrow \boxed{\alpha = .90952}$$

the smallest for which we get at least -10dB attenuation at 20Hz is .90952. Of course, we want α to be as close to one as possible for an ideal comb filter.

5) $h_A[n] = \delta[n] + 2\delta(n-1) + \delta(n-2) \qquad h_B[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$

$$x[n] \longrightarrow \boxed{\text{LTI A}} \xrightarrow{\;y_A[n]\;} \boxed{\text{LTI B}} \longrightarrow y_B[n]$$

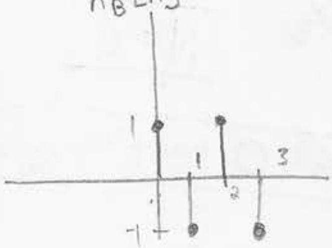
$2(1)+(1)(-1) = 1$
$1(1)+(2)(-1)+(1)(1) = 0$
$-1(1)+2(1)+(-1)(1) = 0$
$-1(2)+1(1) = -1$
$-1(1) = -1$

a) $x[n] = \delta[n]$, so $y_A[n]$ is just $h_A[n]$

$$\boxed{y_A[n] = h_A[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]}$$

$y_B[n] = (y_A * h_B)[n]$
via flip & drag!

$h_B[n]$



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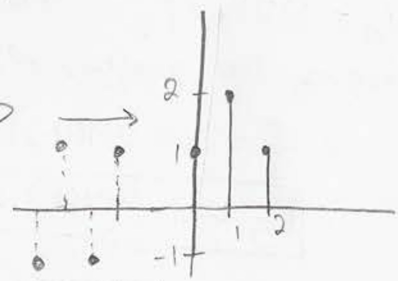

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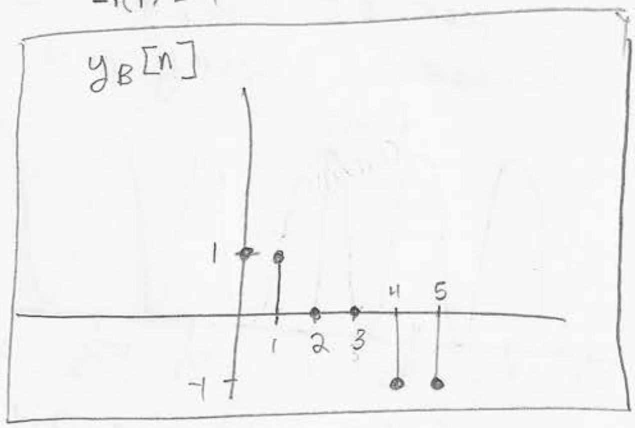

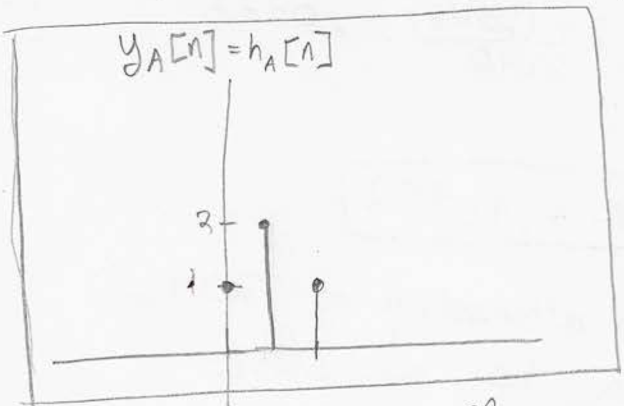

k=0 is only non-zero term in sum

b) $y_A[n] = (\delta * h_A)[n] = \sum_{-\infty}^{\infty} \delta[k] h_A[n-k] = \underbrace{\delta[0] h_A[n-0]}_{k=0 \text{ is only non-zero term in sum}} = \boxed{h_A[n]}$

$y_B[n] = (y_A * h_B)[n] = \sum_{-\infty}^{\infty} y_A[k] h_B[n-k] = \sum_{0}^{2} y_A[k] h_B[n-k]$

$= \sum_{-\infty}^{\infty} h_B[k] y_A[n-k] = \sum_{0}^{3} h_B[k] y_A[n-k]$

$= h_B[0] y_A[n-0] + h_B[1] y_A[n-1] + h_B[2] y_A[n-2] + h_B[3] y_A[n-3]$

$$\boxed{y_B[n] = y_A[n] - y_A[n-1] + y_A[n-2] - y_A[n-3]}$$

$y_B[0] = y_A[0] - y_A[-1] + y_A[-2] - y_A[-3] = 1 - 0 + 0 - 0 = 1$ ✓
$y_B[1] = y_A[1] - y_A[0] + y_A[-1] - y_A[-2] = 2 - 1 + 0 - 0 = 1$ ✓
$y_B[2] = y_A[2] - y_A[1] + y_A[0] - y_A[-1] = 1 - 2 + 1 - 0 = 0$ ✓
$y_B[3] = y_A[3] - y_A[2] + y_A[1] - y_A[0] = 0 - 1 + 2 - 1 = 0$ ✓
$y_B[4] = y_A[4] - y_A[3] + y_A[2] - y_A[1] = 0 - 0 + 1 - 2 = -1$ ✓
$y_B[5] = \cancel{y_A}[5] - y_A[4] + \cancel{y_A[3]} - y_A[2] = -1$ ✓

our convolution verifies the flip and drag method in part A

c) $H_A(\omega) = \sum_{-\infty}^{\infty} h_A[n]e^{-i\omega n} = \sum_{0}^{2} h_A[n]e^{-i\omega n}$

$= h_A[0]e^{-i\omega(0)} + h_A[1]e^{-i\omega(1)} + h_B[2]e^{-i\omega 2}$

$= (1)(1) + 2e^{-i\omega} + e^{-i\omega 2} = \boxed{1 + 2e^{-i\omega} + e^{-2i\omega}}$

$H_B(\omega) = \sum_{-\infty}^{\infty} h_B[n]e^{-i\omega n} = \sum_{0}^{3} h_B[n]e^{-i\omega n}$

$= h_B[0] + h_B[1]e^{-i\omega} + h_B[2]e^{-2i\omega} + h_B[3]e^{-3i\omega}$

$= \boxed{1 - e^{-i\omega} + e^{-2i\omega} - e^{-3i\omega}}$

d) $X(\omega) = \sum_{-\infty}^{\infty} \delta[n]e^{-i\omega n} = \boxed{1}$

$Y_A(\omega) = X(\omega)H_A(\omega) = 1 \cdot H_A(\omega) = H_A(\omega) = \boxed{1 + 2e^{-i\omega} + e^{-2i\omega}}$

$Y_B(\omega) = Y_A(\omega)H_B(\omega) = H_A(\omega)H_B(\omega) = \boxed{(1+2e^{-i\omega}+e^{-2i\omega})(1-e^{-i\omega}+e^{-2i\omega}-e^{-3i\omega})}$

e) $x[n]$, $h_A[n]$, $h_B[n]$, $y_B[n]$ are all discrete time signals. $h_A[n]$ is the impulse response of A, which is the same thing as the output of system A since $x[n] = \delta[n]$. $h_B[n]$ is the output of B when $x[n] = \delta[n]$ is the input (it is the impulse response). Since $y_A[n] = h_A[n]$ is the input to B, $y_B[n]$ is just the convolution of h_A and h_B. So $\boxed{y_B[n] = ((x * h_A) * h_B)[n]}$

- $X(\omega)$, $H_A(\omega)$, $H_B(\omega)$, $Y_B(\omega)$ are all frequency domain signals. In fact, they are just the forward DTFT pairs of the discrete signals $x[n]$, $h_A[n]$, $h_B[n]$, $y_B[n]$. Again, $H_A(\omega) = Y_A(\omega)$ when $X(\omega)$ is the delta impulse (in frequency domain this equals one). Since $Y_A(\omega)$ is the input to B, $Y_B(\omega) = H_B(\omega)Y_A(\omega)$

$$\boxed{Y_B(\omega) = X(\omega)H_A(\omega)H_B(\omega)}$$