EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 07: September 19

Lecturer: Kannan Ramchandran *Scribe: TJ Tsai*

Announcements

- Homework 3 assigned today (short: due Thursday, Sept 26)

Agenda

- Review LTI systems
- Frequency Response

7.1 LTI systems

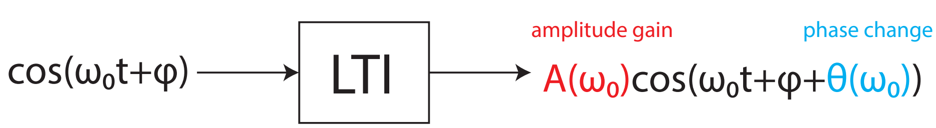
Recall that an LTI system must satisfy two properties:

- Linearity: homogeneity and additivity
- Time Invariance: If the system input $x(t)$ is shifted by τ, the system output $y(t)$ will be shifted by τ

Why do we care about LTI systems? Because they have a very special and very useful property.

Theorem 7.1. *For an LTI system, if the input signal is a sinusoid $x(t) = \cos(\omega_0 t + \phi)$, the output signal will be a sinusoid of the same frequency, $y(t) = A(\omega_0)\cos(\omega_0 t + \phi + \theta(\omega_0))$.*

Here $A(\omega_0)$ is the amplitude gain, $\theta(\omega_0)$ is the phase change, and both may depend on the frequency of the cosine ω_0.



Why is this theorem important/useful?

Recall that:

- "Any" signal can be written as a linear combination of sinusoids of specific frequencies (using Fourier series)
- If a system is linear, and we know the output signals for the input signals $x_1(t), x_2(t), \ldots x_k(t)$ are $y_1(t), y_2(t), \ldots y_k(t)$ respectively. Then for an input signal $x(t)$ that is a linear combination of $x_1(t), x_2(t), \ldots x_k(t)$, the output signal is the same linear combination of $y_1(t), y_2(t), \ldots y_k(t)$.

Therefore:

- If a system is LTI, and you know the amplitude gain and phase change for sinusoids of any frequency, you know the output of the system for any arbitrary input signal!

Because we can write any input signal as linear combination of sinusoids, and since we know the output for all these sinusoids, we know the output of our arbitrary input signal.

In other words, for an LTI system, all you need to know is $A(\omega)$ and $\theta(\omega)$, $\forall\omega$. Then you know everything there is to know about your system. The pair of functions $(A(\omega), \theta(\omega))$ together is called the frequency response of the LTI system. The frequency response tells us what frequencies get attenuated/amplified, and what frequencies get phase shifts from an LTI system. (Note that frequency response only pertains to LTI systems – it does not make sense to refer to the frequency response of a non-LTI system. Why?)

Example 7.1. *Let's revisit the two path wireless channel shown below in Figure 7.1.* $y(t) = a_1x(t-\tau_1) + a_2x(t-\tau_2)$

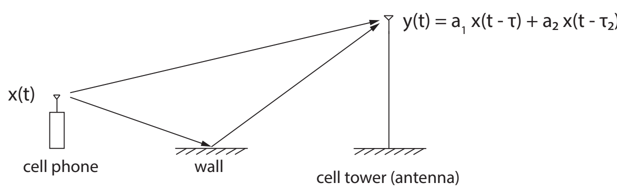


Figure 7.1: Two path wireless channel.

Consider what happens when the input to this system is $x(t) = e^{i\omega_0 t}$*. Such an input is called a complex exponential sinusoid. The output will be*

$$\begin{aligned} y(t) &= a_1e^{i\omega_0(t-\tau_1)} + a_2e^{i\omega_0(t-\tau_2)} \\ &= a_1e^{-i\omega_0\tau_1}e^{i\omega_o t} + a_2e^{-i\omega_o\tau_2}e^{i\omega_o t} \\ &= [a_1e^{-i\omega_0\tau_1} + a_2e^{-i\omega_o\tau_2}]e^{i\omega_o t} \\ &= H(\omega_0)e^{i\omega_o t} \end{aligned}$$

Here, $H(\omega_0)$ *is a scale or gain that is complex valued and frequency-dependent.*

7.2 Frequency Response

Now that we've established some useful properties of LTI systems, we're ready to define the frequency response of an LTI system.

Theorem 7.2. *If you feed a complex exponential sinusoid of frequency* ω_0 *into an LTI system, the output is the same complex exponential sinusoid scaled by some complex number* $H(\omega_0)$ *that depends on the frequency of the input.* $H(\omega)$ *as a function of* ω *is called the frequency response of the LTI system.*

Note the two theorems above are equivalent. You should verify this for yourself as an exercise.

One helpful way to think about frequency response is to visualize things geometrically in the real-imaginary plane. Suppose that the input into an LTI system is a complex exponential sinusoid $e^{i\omega t}$. It's hard to plot $e^{i\omega t}$ as a function of t so we are going to think of it as a vector with its origin in the unit circle and this vector is rotating around the circle at a speed ω. ω is the rate that the vector goes around the circle. This vector is called a phasor.

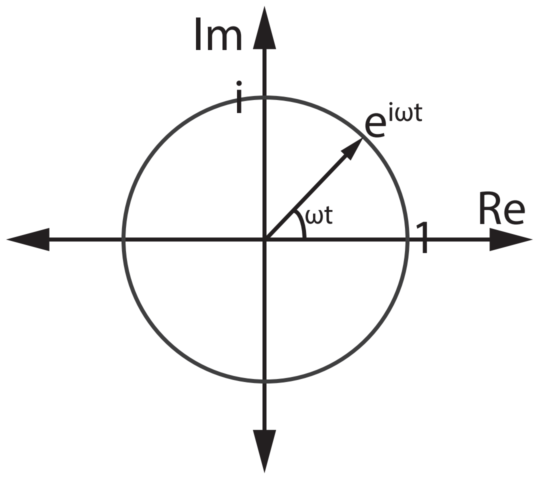


Figure 7.2: $e^{i\omega t}$ represented as a phasor.

So what happens if you pass this phasor through an LTI system? After you pass it through the system, what happens to the phasor? Well, intuitively, it continues to rotate at the same rate, but what happens to the length? It changes length by a factor of $|H(\omega)|$ and the angle shifts by $\angle H(\omega)$ as shown in Figure 7.3.

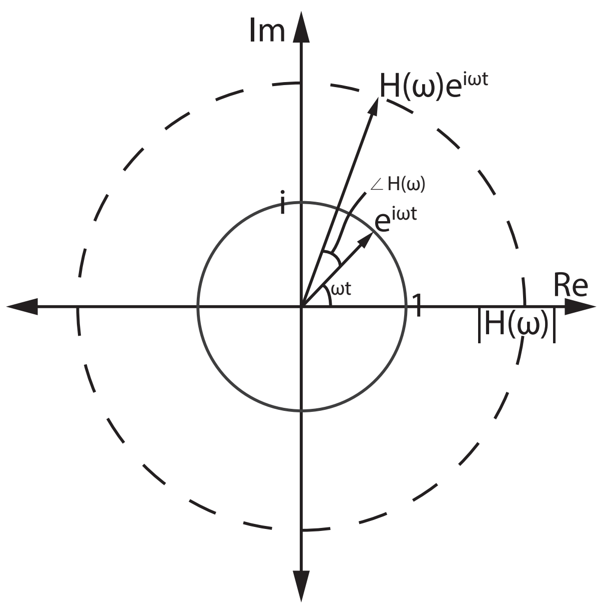


Figure 7.3: $e^{i\omega t}$ and $H(\omega)e^{i\omega t}$ represented as phasors and the relationship between them.

Example 7.2. *If I input a signal $x(t)$ into an LTI system whose frequency response is as shown in figure 7.4 where $x(t) = 2\cos(2\pi 1000t) + 5\sin(2\pi 15000t)$, what is $y(t)$?*

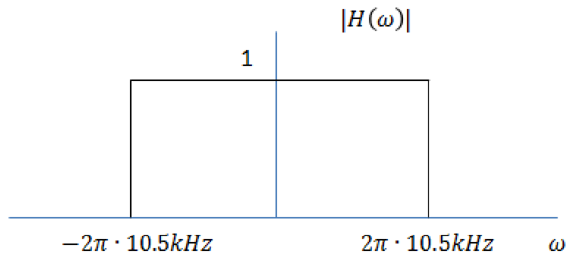


Figure 7.4: Magnitude of frequency response. Assume the phase shift for the frequency response $\angle H(\omega) = 0$ for all ω.

In this case, the input consists of the sum of two complex sinusoids: one has frequency 1kHz and the other has frequency 15kHz. Because the frequency response of the system zeroes out frequencies higher than 10.5kHz, the input complex sinusoid at 15kHz will be removed. The magnitude of the input complex sinusoid at 1kHz will be multiplied by 1 *and the phase will be shifted by 0. So the output will simply be* $y(t) = 2\cos(2\pi 1000t)$*. A system which removes frequencies above a certain cutoff is called a low pass filter (LPF).*

Example 7.3. *Consider the same system whose frequency response is shown in figure 7.4. What is the output* $y(t)$ *if the input* $x(t)$ *is the triangle wave shown in figure 7.5?*

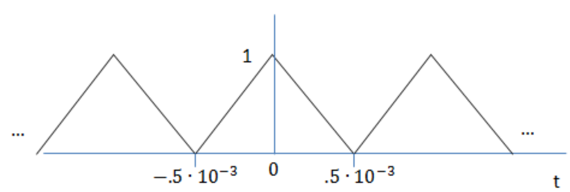


Figure 7.5: Time-domain representation of a triangle wave.

Note that this signal is periodic with period $p = 1ms$*. So, we know that the Fourier series representation of this signal will contain harmonics at integer multiples of the fundamental frequency* $f_0 = 1/p = 1000Hz$*. In fact, this signal will contain an infinite number of harmonics because it has "sharp" edges. However, even though an infinite number of harmonics come in, only harmonics below 10.5kHz will come out, since all the harmonics above this cutoff frequency will be zeroed out by the system. We can represent the input signal in the CTFS expansion:*

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

This is the CTFS expansion using complex exponentials instead of sines and cosines. The output of the system will then be:

$$y(t) = \sum_{k=-10}^{10} X_k e^{ik\omega_0 t}$$