Lecture 09: September 26th

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9.1 Announcements

1. MTI is today! Please arrive 10 minutes early as we will be starting at 6pm sharp!!
2. There will be no class next Thursday, October 3rd
3. HW4 will be assigned tomorrow and due next Thursday at midnight

9.2 Clarifications

Given a frequency response $H(\omega)$ that can be represented as

$$H(\omega) = \frac{A(\omega)}{B(\omega)}$$

the $\angle H(\omega) = \angle A(\omega) - \angle B(\omega)$. In other words,

$$H(\omega) = \frac{A(\omega)}{B(\omega)} = \frac{|A(\omega)|e^{i\angle A(\omega)}}{|B(\omega)|e^{i\angle B(\omega)}} = \frac{|A(\omega)|}{|B(\omega)|}e^{i(\angle A(\omega) - \angle B(\omega))}$$

The frequency response of a CT LTI system is $H(\omega)$, and the frequency response of a DT LTI system is $H(e^{i\omega})$. DT frequency responses are 2π-periodic, which is why we write the frequency response as a function of $e^{i\omega}$, which is itself 2π-periodic. Note that this is still a function of ω, but this notation is used to serve as a reminder of its periodicity.

9.3 More Examples of Frequency Response

Example 9.1. *Differentiator system*

Now we will look at a system that takes the derivative of the input, or in other words,

$$y(t) = \frac{d}{dt}x(t).$$

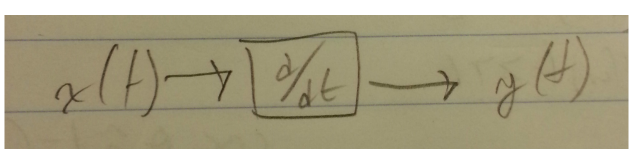

Figure 9.1: The differentiator system

First, we should verify that the system is LTI, because frequency responses only make sense for LTI systems! This exercise is left to the reader, but you should find that the differentiator is in fact LTI.

Now, to find the frequency response, we consider the case when $x(t) = e^{i\omega t}$ and $y(t) = H(\omega)e^{i\omega t}$. After setting $x(t) = e^{i\omega t}$ and $y(t) = H(\omega)e^{i\omega t}$, we get the expression

$$H(\omega)e^{i\omega t} = \frac{d}{dt}e^{i\omega t}.$$

Now we have to take the derivative of a complex exponential to solve for $H(\omega)$!
. Fortunately, the derivative is with respect to t, and both i and ω are constant with respect to t. Therefore, we can use the standard differentiation forumula for exponentials, and just treat $i\omega$ as a constant value.

$$H(\omega)e^{i\omega t} = \frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}.$$

(You can also verify this by expanding out the complex exponential in terms of sin and cos.) Solving for $H(\omega)$, we see that the frequency response is:

$$H(\omega) = i\omega$$

which means that the magnitude of the frequency response is:

$$|H(\omega)| = |\omega|.$$

This can be interpreted as a system which is very insensitive to low frequencies, and is sensitive to

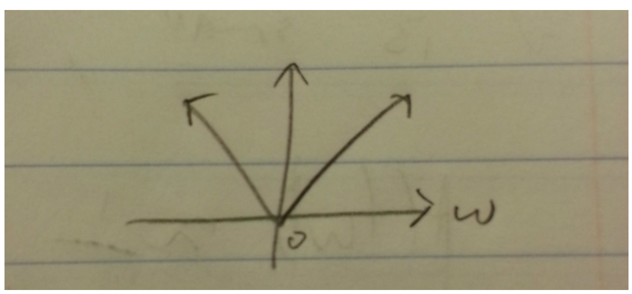

Figure 9.2: The differentiator magnitude frequency response, $|H(\omega)|$

high frequencies. This makes sense because differentiation tells us the rate of change of a function. By looking at the slopes of low and high frequency signals, we can see that the slope of a low frequency signal is also very small, and so the derivative is also very small. Conversely, the slope of a high frequency signal takes on high values, and so the derivative also takes on high values.

Example 9.2. *Averaging system*

Now we will look at a system that takes the average of the input over the past window of duration τ via integration. Intuitively, we should expect the opposite effects of the differentiator system, as integration and differentiation are inverse operations of each other. The system can be expressed as

$$y(t) = \frac{1}{\tau}\int_{t-\tau}^{t} x(s)ds.$$

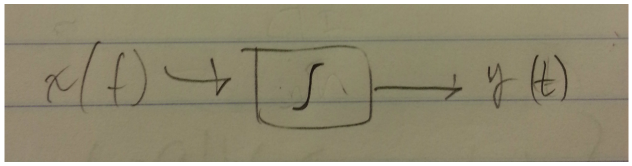

Figure 9.3: The averaging system

After setting $x(t) = e^{i\omega t}$ and $y(t) = H(\omega)e^{i\omega t}$, we have

$$y(t) = \frac{e^{i\omega s}}{i\omega\tau}\Big|_{t-\tau}^{t}$$

solving for $H(\omega)$, we see that the frequency response is:

$$H(\omega) = \frac{1 - e^{-i\omega\tau}}{i\omega\tau}$$

which means that the magnitude of the frequency response is:

$$|H(\omega)| = \frac{1}{\omega\tau}\sqrt{2(1 - cos\omega\tau)}.$$

Note that we can also rewrite this using a different method:
First, we can take our original expression and factor out a constant phase term:

$$H(\omega) = \frac{1 - e^{-i\omega\tau}}{i\omega\tau} = \frac{e^{-i\omega\frac{\tau}{2}}(e^{i\omega\frac{\tau}{2}} - e^{-i\omega\frac{\tau}{2}})}{i\omega\tau}$$

And now we can simplify the difference into a sinusoid:

$$= \frac{e^{-i\omega\frac{\tau}{2}}(e^{i\omega\frac{\tau}{2}} - e^{-i\omega\frac{\tau}{2}})}{i\omega\tau} = \frac{e^{-i\omega\frac{\tau}{2}}(2sin(\omega\frac{\tau}{2}))}{\omega\tau} = \frac{e^{-i\omega\frac{\tau}{2}}sin(\omega\frac{\tau}{2})}{\omega\frac{\tau}{2}}$$

This function $\frac{sin(x)}{x}$ is also known as $sinc(x)$. The sinc function is very useful and we will be using it a lot, so let's start here!

$$= e^{-i\omega\frac{\tau}{2}}sinc(\omega\frac{\tau}{2})$$

We can graph $|H(\omega)|$ systematically by analyzing the behavior of this function at 0 and ∞. Using L'Hopital's rule we can determine that $|H(0)| = 1$ and $lim_{\omega\to\infty}H(\omega) = 0$ since the denominator is a hyperbolic. Looking at the numerator, we can see that there will be zero crossings at multiples of $\frac{2\pi}{\tau}$, which is when the sinusoid is 0. Now we know the behavior at some key points and we can sketch the sinusoid with a decaying amplitude in between like so:

For a small frequency ω, the magnitude of the frequency response $|H(\omega)| = 1$. This can be interpreted as a system which has no effect on low frequencies. Looking at the behavior at high frequencies, we see that as $\omega \to \infty$, $|H(\omega)| \to 0$, which can be interpreted as a system that doesn't preserve high frequencies that are passed through it. This is what we would expect because the system takes a moving average over the last τ seconds. In general, low frequency signals will not change much over an interval of τ seconds, so the moving average will not stray much from the input value. On the other hand, high frequency signals can go through a full period or more in an interval of τ seconds, so the average may include both positive and negative numbers and end up very small, or even 0.

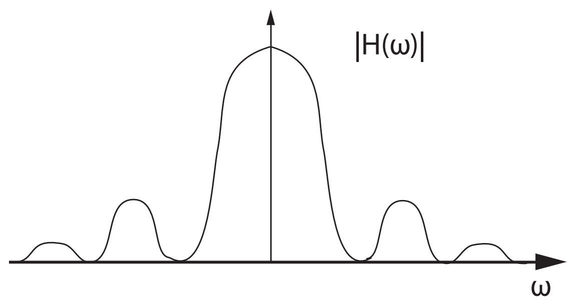

Figure 9.4: Magnitude response of integrator system.

9.4 Complex Conjugate Symmetry

Recall that the frequency response for a 2 path wireless channel is

$$H(\omega) = a_1 e^{i\omega\tau_1} + a_2 e^{i\omega\tau_2}.$$

If we plug in $\omega' = -\omega$ to the frequency response,

$$H(\omega') = H(-\omega) = a_1 e^{-i\omega\tau_1} + a_2 e^{-i\omega\tau_2}$$

which can also be expressed as

$$(a_1 e^{i\omega\tau_1} + a_2 e^{i\omega\tau_2})^* = H(\omega)^*$$

or put in terms of $H(\omega)$,

$$H(-\omega) = H^*(\omega)$$

when $a_1, a_2 \in \mathbb{R}$. This relation is called complex conjugate symmetry. This property generalizes to any LTI system which is *real*, i.e. takes real inputs to real outputs (like real friends). There is a similar relation for phase in this case:

$$\angle H(-\omega) = -\angle H(\omega)$$

Given input $x_1(t) = e^{i\omega t}$, the output from the LTI system with frequency response H is $y_1(t) = H(\omega)e^{i\omega t}$. Given input $x_2(t) = e^{-i\omega t}$, the output is $y_2(t) = H(-\omega)e^{-i\omega t}$. Now rewrite $x_1(t) = x_R(t) + ix_I(t)$ and $y_1(t) = y_R(t) + iy_I(t)$, where the output from $x_R(t)$ is $y_R(t)$ and the output from $x_I(t)$ is $y_I(t)$. (This follows from the fact that the system is real, or in brief, a system observed in the real world. See lecture Note 7.) By linearity, the output from $x_2(t) = x_R(t) - ix_I(t)$ is hence $y_R(t) - iy_I(t) = y^*(t)$. Hence $y_2(t) = y_1^*(t)$, and this implies $H(-\omega) = H^*(\omega)$.

When looking at the magnitude of the frequency response, we see that it is even.

$$|H(-\omega)| = |H^*(\omega)| = |H(\omega)|$$

An analogous relation for the phase of the frequency response is

$$\angle H(\omega) = -\angle H(-\omega)$$

due to the properties of complex conjugates..

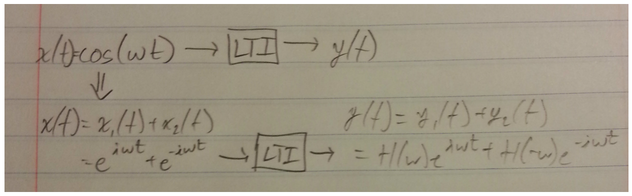

Figure 9.5: Input-output relationship for an LTI system in sinusoid and exponential form

9.5 DT Frequency Response

Now consider a DT LTI system H which can be described as

$$y(n) - 0.8y(n-1) = 0.2x(n)$$

This is expressed as a difference equation, which is the analog of a differential equation in continuous time. Another name for this is a Linear Constant Coefficient Difference Equation or LCCDE. You will be observing more of these in lab next week!

Now convince yourself that the frequency response $H(e^{i\omega}) = \frac{0.2}{1-0.8e^{-i\omega}}$. The graph of this is shown below:

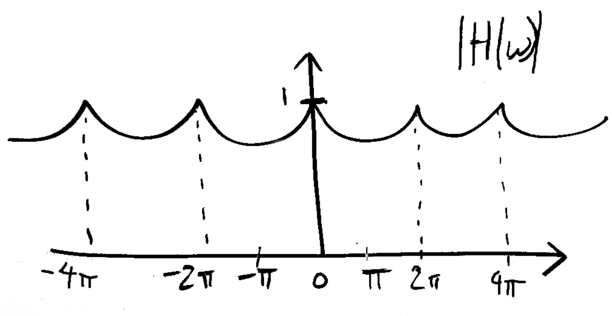

Figure 9.6: Input-output relationship for an LTI system in sinusoid and exponential form

9.6 Impulse Response

The **impulse response** is the time domain counterpart to the frequency response; that is, it is the time domain representation of an LTI system.

For example, the last midterm question asked to design a frequency response to preserve the LTI nature of a system. This frequency domain design was natural because we assumed that the input has specific frequency components. However, in general, to implement such a system in software or hardware, we need a time domain representation.

We will focus on the discrete-time impulse response first.

Definition 9.1. *An impulse, denoted by $\delta(n)$, is defined by:*

$$\delta(n) = \begin{cases} 1 & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

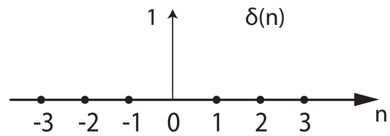

Figure 9.7: The impulse function $\delta(n)$.

A graphical representation is shown in Figure 9.7.

Definition 9.2. *The impulse response of an LTI system is the response of that system to the impulse function, as shown in Figure 9.8.By convention, the impulse response is denoted h. Note that the notation h for impulse response is closely related to the uppercase notation H for frequency response.*

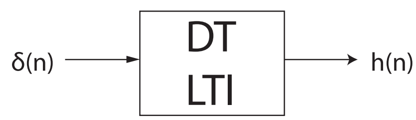

Figure 9.8: The impulse response $h(n)$.

Now consider a 2 channel wireless system (now in discrete time) described by

$$y(n) = \frac{1}{2}x(n-2) + \frac{1}{4}x(n-3)$$

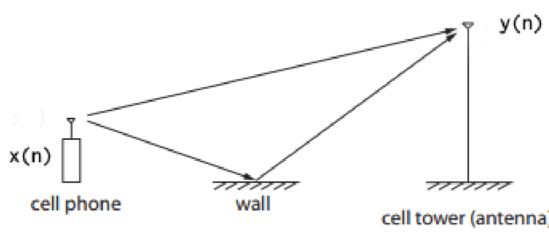

Figure 9.9: A wireless channel with one reflected path.

What is the output when the input $x(n) = \delta(n)$?
By plugging in $x(n) = \delta(n)$, we can see that

$$y(n) = \frac{1}{2}\delta(n-2) + \frac{1}{4}\delta(n-3)$$

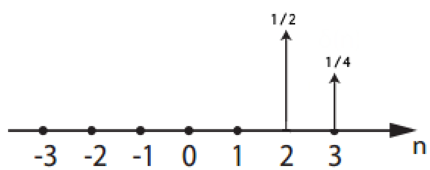

Figure 9.10: Output to the system, also called the "impulse response" of the system