Physics 105 (Fall 2013): Solution to HW #1

1. Practice

2. We should moment of inertia about the z-axis can be easily obtained if we know $\int dA\,(x^2+y^2)$ for a single triangular slice. We can use polar coordinate in this case, with the origin at the centroid of the triangle. Putting one of the vertices of the triangle at $\phi=\pi$, then the equation of a side, that is opposite to this vertex, is given by $r\cos\phi = \frac{a}{\sqrt{3}}$, $-\frac{\pi}{3}\le\phi\le\frac{\pi}{3}$. The desired integral is then:
$$3\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\phi \int_0^{\frac{a}{\sqrt{3}}\sec\phi} r^3\,dr = \frac{a^4}{12}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^4\phi d\phi = \frac{\sqrt{3}a^4}{3}.$$
The area of the triangle is $A=\sqrt{3}a^2$, the moment of inertia of the triangle prism is then
$$I_{zz} = \frac{M}{A}\frac{\sqrt{3}a^4}{3} = \frac{M}{3}a^2.$$
The two products of inertia are zero: $I_{xz}=I_{yz}=0$, as there is a reflectional symmetry about the x-y plane.

3. Practice.

4. Practice.

5. Problem 10.13:

   (a) $\dot{L}_z = \Gamma_z$ gives $I\ddot{\phi} = -mga\sin\phi$. For small angle, $I\ddot{\phi} = -mga\phi$ and so the period is $T = 2\pi\sqrt{\frac{I}{mga}}$.

   (b)For a simple pendulum, $T=2\pi\sqrt{\frac{l}{g}}$. The two periods are equal if $l = I/ma$.

   Problem 10.18:

   (a) The angular momentum is $L=\xi b$, coming out from the impulse. The momentum is then $P = m(\frac{L}{I}a) = \frac{m\xi ab}{I}$, $\frac{L}{I}$ being the angular speed.

   (b) The impulse given by the pivot is $\eta = P-\xi = (\frac{mab}{I}-1)\xi$.

   (c) $\eta=0$ when $b=\frac{I}{ma}$.

6. Practice.

7. We already know $I_{zz}=\frac{M}{3}a^2$, $I_{xz}=I_{yz}=0$. Also with reflectional symmetry about the x-z plane, $I_{xy}=0$. To find the other two moments of inertia, we need $\int dm\,x^2$, $\int dm\,y^2$ and $\int dm\,z^2$. The last one is just like that of a rod, so $\int dm\,z^2 = \frac{M}{12}h^2$. The prism has a discrete rotational symmetry, a rotation by $\frac{2\pi}{3}$. With this rotation, $\int dm\,x^2 = \int dm\,(\mathbf{r}\cdot\hat{x})^2$ becomes $\int dm\,[\mathbf{r}\cdot(\cos\frac{2\pi}{3}\hat{x}+\sin\frac{2\pi}{3}y)]^2 = \int dm\,(x\cos\frac{2\pi}{3}+y\sin\frac{2\pi}{3})^2 = \int dm\,(x^2\cos^2\frac{2\pi}{3}+y^2\sin^2\frac{2\pi}{3})$. Note that the cross term vanishes from $I_{xy}=0$. This implies $\int dm\,x^2 = \int dm\,y^2 = \frac{1}{2}I_{zz} = \frac{M}{6}a^2$. Thus, $I_{xx}=I_{yy}=\frac{M}{12}(h^2+2a^2)$.

8. (a) The rotational symmetry about z-axis implies both x-z and y-z plane are planes of reflectional symmetry. So $I_{zx}=I_{zy}=0$ and this is sufficient to prove that z-axis is a principle axis.

   (b) As above, x-z plane is a plane of reflectional symmetry, so $I_{xy}=0$. Thus both x-axis and y-axis are principle axes.

   (c) Since a rotation by $\frac{\pi}{2}$ is a symmetry of the body, $\int dm\,x^2 = \int dm\,y^2$. This shows $I_{xx}=I_{yy}$.

9. (a) Simple computation from definitions shows $I_{xx} = 10ma^2$, $I_{yy} = I_{zz} = 6ma^2$, $I_{xz} = I_{xy} = 0$ and $I_{yz} = I_{zy} = ma^2$.

   (b) The characteristic equation is: $0 = \det(\lambda\mathbf{1} - \mathbf{I}) = (\lambda - 10ma^2)(\lambda - 7ma^2)(\lambda - 5ma^2)$. The three principle moments are $\lambda_1 = 10ma^2$ with the principle axis vector $\mathbf{e_1} = (1, 0, 0)$, $\lambda_2 = 7ma^2$ with the axis pointing towards the mass 2m: $\mathbf{e_2} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $\lambda_3 = 5ma^2$ with $\mathbf{e_3} = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, towards the mass 3m.

10. (a) To verify the matrix is a rotation matrix, we need to verify that the three row vectors are orthonormal, and the determinant is one. Simple algebra shows indeed these two conditions are satisfied.

    (b) Taking the trace of the matrix, $1 + 2\cos\theta = \mathrm{tr}\,\mathbf{R} = 2$. So the rotation angle is $\theta = \frac{\pi}{3}$.

    (c) To find the rotation axis, we need to find the vector that is invariant under rotation, that is the eigenvector eigenvalue 1. It is not hard to see that this vector is $\hat{n} = \pm(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. To determine the sign, we consider the image of the point $(1, 0, 0)$, under rotation it becomes $(\frac{3}{4}, \sqrt{\frac{3}{8}}, \frac{1}{4})$. Since the y-component is positive we determine to sign to be a plus, $\hat{n} = +(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

11. For spinning about the shortest symmetry axis with frequency ω, the small oscillations of the rotation axis will have a frequency: $\Omega^2 = \frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega$. The principle moments are $\lambda_1 = \frac{M}{12}(b^2 + c^2)$, $\lambda_2 = \frac{M}{12}(a^2 + c^2)$ and $\lambda_3 = \frac{M}{12}(a^2 + b^2)$, where $a = 30$cm, $b = 20$cm and $c = 3$cm. Thus we get $\Omega = 174$rpm.
    If instead the rotation is about the longest symmetry axis, we just need to swap $\lambda_1 \leftrightarrow \lambda_3$, and we get $\Omega = 111$rpm.

12. Practice.

13. One component of Euler equations is $\lambda_3\dot{\omega}_3 = \Gamma$, giving $\omega_3 = \omega_{30} + \frac{\Gamma}{\lambda_3}t$. Using the complex combination $\eta = \omega_1 + i\omega_2$, the other two Euler equations reads: $\dot{\eta} = i\frac{\lambda_3-\lambda}{\lambda}\omega_3\eta$. This is separable and can be solved easily: $\eta = \omega_{10}\exp[\frac{\lambda_3-\lambda}{\lambda}(\omega_{30}t + \frac{\Gamma}{2\lambda_3}t^2)]$. Hence $\omega_1 = \omega_{10}\cos[\frac{\lambda_3-\lambda}{\lambda}(\omega_{30}t + \frac{\Gamma}{2\lambda_3}t^2)]$ and $\omega_2 = \omega_{10}\sin[\frac{\lambda_3-\lambda}{\lambda}(\omega_{30}t + \frac{\Gamma}{2\lambda_3}t^2)]$.

14. (a) We start from $\omega = \dot{\phi}\hat{z} + \dot{\theta}\mathbf{e'_2} + \dot{\psi}\mathbf{e_3}$, refer to page 401 of the text for relevant definitions. Now $\mathbf{e'_2} = -\sin\phi\hat{x} + \cos\phi\hat{y}$ and $\mathbf{e_3} = \cos\theta\hat{z} + \sin\theta\mathbf{e'_2} = \cos\theta\hat{z} + \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y})$. This gives $\omega = (-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\hat{x} + (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\hat{y} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{z}$.

    (b) Using $\hat{z} = \cos\theta\mathbf{e_3} - \sin\theta\mathbf{e'_1}$, $\mathbf{e'_1} = \cos\psi\mathbf{e_1} - \sin\psi\mathbf{e_2}$ and $\mathbf{e'_2} = \sin\psi\mathbf{e_1} + \cos\psi\mathbf{e_2}$, we can arrive at $\omega = (-\dot{\psi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\mathbf{e_1} + (-\dot{\psi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{e_2} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{e_3}$.