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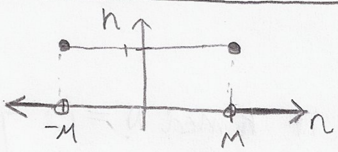
Homework #6

1. $h[n] = \begin{cases} 1 & |n| \leq M \\ 0 & \text{else} \end{cases}$



$$H(\omega) = \sum_{-\infty}^{\infty} h[k]e^{-i\omega k} = \sum_{-M}^{M} h[k]e^{-i\omega k} = \sum_{-M}^{M}(1)e^{-i\omega k}$$

$$= e^{i\omega M} + e^{-i\omega(-M+1)} + e^{-i\omega(-M+2)} + \cdots + 1 + \cdots + e^{-i\omega(M-1)} + e^{-i\omega(M)}$$

$$= e^{i\omega M} + e^{i\omega M}e^{-i\omega} + e^{i\omega M}e^{-i2\omega} + \cdots + 1 + \cdots + e^{-i\omega M}e^{i\omega} + e^{-i\omega M}$$

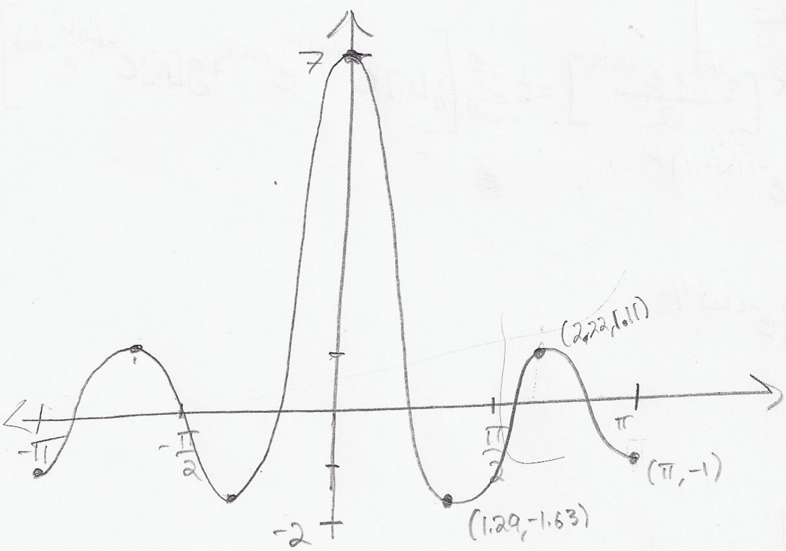
$$= e^{i\omega M}(1 + e^{-i\omega} + e^{-2i\omega} + \cdots + e^{-i\omega M} + \cdots + e^{-i\omega(2M-1)} + e^{-i\omega 2M})$$

$$= e^{i\omega M}\sum_{0}^{2M} e^{-i\omega k} \quad \text{since} \quad \sum_{0}^{N} ar^k = \frac{a(1-r^{n+1})}{(1-r)}:$$

$$H(\omega) = \frac{e^{i\omega M}(1-e^{-i\omega(2M+1)})}{1-e^{-i\omega}} = \frac{e^{i\omega M}e^{-i\omega(M+\frac{1}{2})}(e^{i\omega(M+\frac{1}{2})} - e^{-i\omega(M+\frac{1}{2})})}{e^{-i\omega/2}(e^{i\omega/2} - e^{-i\omega/2})}$$

$$= \cancel{\frac{e^{i\omega(M-M-\frac{1}{2})}}{e^{-i\omega/2}}}\frac{\sin(\omega(M+\frac{1}{2}))}{\sin(\omega/2)} \Rightarrow \boxed{H(\omega) = \frac{\sin(\omega(M+\frac{1}{2}))}{\sin(\omega/2)}}$$

b) $M=3 \Rightarrow H_3(\omega) = \frac{\sin(3.5\omega)}{\sin(.5\omega)}$



$\boxed{2}$ impulse response $g(n)$ with frequency response $G(\omega)$

a) H: $h[n] = \begin{cases} g[n/N] & n \bmod N = 0 \\ 0 & \text{else} \end{cases}$

$$H(\omega) = \sum_{-\infty}^{\infty} h[k] e^{-i\omega k} = \sum_{-\infty}^{\infty} \begin{bmatrix} g[k/N] e^{-i\omega k} \ldots \text{ if } k \bmod N = 0 \\ 0 \ldots \text{else} \end{bmatrix}$$

$$= \cdots + g[-1]e^{i\omega N} + g[0]e^{0} + g[1]e^{-i\omega N} + g[2]e^{-i\omega 2N} + \cdots$$

$$= \sum_{-\infty}^{\infty} g[k] e^{-i\omega k N} = \sum_{-\infty}^{\infty} g[k] e^{-i(\omega N)k}$$

since we know that $\sum_{-\infty}^{\infty} g[k] e^{-i\omega k} = G(\omega)$, we can just make the substitution $\omega' = \omega N$ and thus we see that

$$\sum_{-\infty}^{\infty} g[k] e^{-i\omega' k} = G(\omega') = G(\omega N), \text{ so } \boxed{H(\omega) = G(\omega N)}$$

b) W: $w(n) = g(n) e^{i\alpha n}$

$$W(\omega) = \sum_{-\infty}^{\infty} w(k) e^{-i\omega k} = \sum_{-\infty}^{\infty} g(k) e^{i\alpha k} e^{-i\omega k} = \sum_{-\infty}^{\infty} g(k) e^{-i(\omega - \alpha)k}$$

using a similar substitution, $\omega' = \omega - \alpha$:

$$W(\omega) = \sum_{-\infty}^{\infty} g(k) e^{-i\omega' k} = G(\omega') = G(\omega - \alpha) \Rightarrow \boxed{W(\omega) = G(\omega - \alpha)}$$

c) Z: $z(n) = g(n)\cos(\alpha n) = g(n)\left[\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right]$

$$Z(\omega) = \sum_{-\infty}^{\infty} z[k] e^{-i\omega k} = \sum_{-\infty}^{\infty} g[n] e^{-i\omega k}\left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right] = \frac{1}{2}\sum_{-\infty}^{\infty}\left[g[n]e^{-i\omega k}e^{i\alpha k} + g[n]e^{-i\omega k}e^{-i\alpha k}\right]$$

$$Z(\omega) = \frac{1}{2}\sum_{-\infty}^{\infty} g[k] e^{-i(\omega-\alpha)k} + \frac{1}{2}\sum_{-\infty}^{\infty} g[k] e^{-i(\omega+\alpha)k}$$

$$\Rightarrow \omega' = \omega - \alpha \qquad \omega'' = \omega + \alpha$$

$$Z(\omega) = \frac{1}{2}\sum_{-\infty}^{\infty} g[k] e^{-i\omega' k} + \frac{1}{2}\sum_{-\infty}^{\infty} g[k] e^{-i\omega'' k}$$

$$= \frac{G(\omega') + G(\omega'')}{2}$$

$$\Rightarrow \boxed{Z(\omega) = \frac{G(\omega-\alpha) + G(\omega+\alpha)}{2}}$$

3 $x[n] = \delta(n+1) + \delta(n) + \delta(n-1) \implies X$

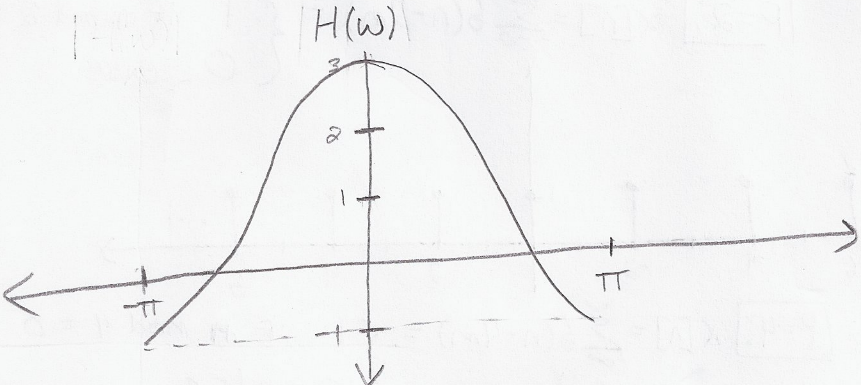
$H \implies H(w) = e^{-iw}$

a) $X(w) = \sum_{-\infty}^{\infty} x[n] e^{-iwn} = \sum_{-1}^{1} x[n] e^{-iwn} = x[-1]e^{iw} + x[0]e^{0} + x[1]e^{-iw}$

$= (\delta(-1+1) + \delta(-1) + \delta(-2))e^{iw} + (\delta(1) + \delta(0) + \delta(-1)) + (\delta(2) + \delta(1) + \delta(0))e^{-iw}$

$X(w) = \boxed{e^{iw} + 1 + e^{-iw}}$

$X(w) = \cos(w) + \cancel{i\sin(w)} + 1 + \cos(w) \cancel{- i\sin(w)} = \boxed{1 + 2\cos w}$

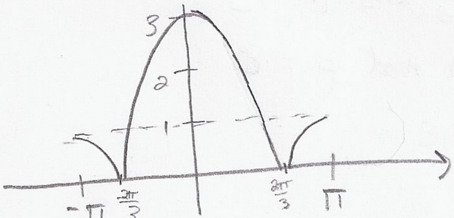


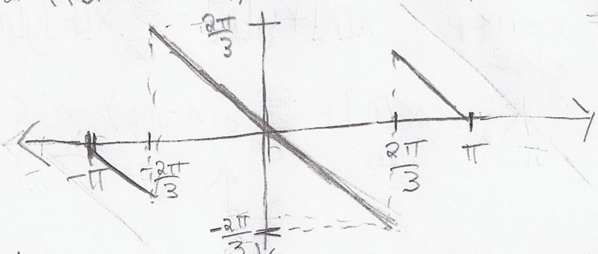
b) $Y(w) = X(w)H(w) = e^{-iw}(e^{iw} + 1 + e^{-iw}) = (e^{-iw}e^{iw} + e^{-iw} + e^{-iw}e^{-iw})$

$\boxed{Y(w) = (1 + e^{-iw} + e^{-i2w})}$ $= (1 + 2\cos(w))e^{-iw}$

$|Y(w)| = |(1 + \cos w - i\sin w + \cos 2w - i\sin 2w)| = \sqrt{(1 + \cos w + \cos 2w)^2 + (\sin w + \sin 2w)^2}$

$|Y(w)| = |1 + 2\cos(w)|$ $\quad \angle Y(w) = \tan^{-1}(-(\sin w + \sin 2w)/(\cos w + \cos 2w + 1)) = \angle(1 + 2\cos(w)) + \angle e^{-iw}$





c) Using Inverse Fourier Transform:

$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 + e^{-iw} + e^{-i2w}) e^{iwn}\, dw$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iwn} + e^{iw(n-1)} + e^{iw(n-2)}\, dw$

$= \frac{1}{2\pi}\left(\frac{e^{iwn}}{in} + \frac{e^{iw(n-1)}}{i(n-1)} + \frac{e^{iw(n-2)}}{i(n-2)}\right)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{i\pi n}}{in} + \frac{e^{i\pi(n-1)}}{i(n-1)} + \frac{e^{i\pi(n-2)}}{i(n-2)} - \frac{e^{-i\pi n}}{in} - \frac{e^{-i\pi(n-1)}}{i(n-1)} - \frac{e^{-i\pi(n-2)}}{i(n-2)}\right]$

$\to \frac{1}{2\pi}\left(\frac{2\sin \pi n}{n} + \frac{2\sin(\pi(n-1))}{n-1} + \frac{2\sin(\pi(n-2))}{n-2}\right)$

$= \frac{\sin(\pi(n))}{\pi n} + \frac{\sin(\pi(n-1))}{\pi(n-1)} + \frac{\sin(\pi(n-2))}{(n-2)\pi}$

$= \boxed{\text{sinc}(n) + \text{sinc}(n-1) + \text{sinc}(n-2)}$

$= \delta[n] + \delta[n-1] + \delta[n-2]$

In discrete time, $\text{sinc}[n] = \delta[n]$,

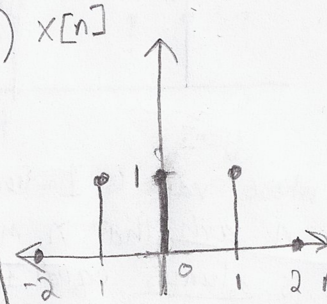
so $\boxed{y[n] = \delta[n] + \delta[n-1] + \delta[n-2] = x[n-1]}$

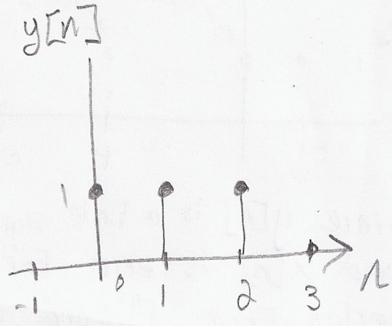
We also know that $y[n] = H(w)e^{iwn} = e^{-iw}e^{iwn}$

$\implies y[n] = e^{iw[n-1]} = x[n-1]$, which confirms our result.

$\boxed{\text{System } H \text{ introduces a sample shift of } 1}$

d)

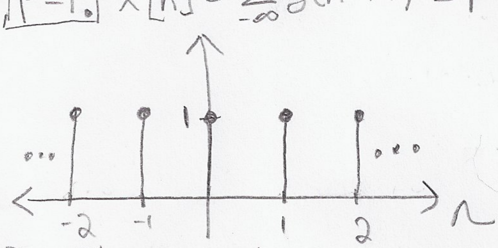




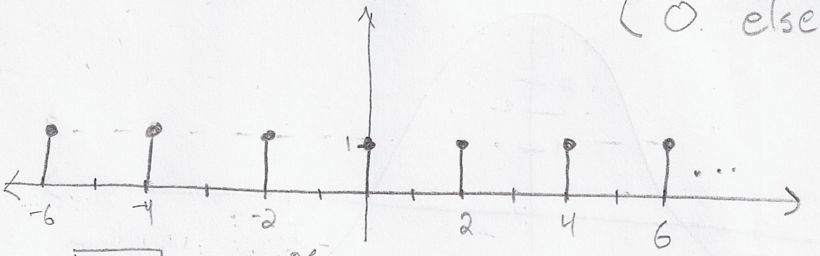
4) $x(n) = \sum_{n=-\infty}^{\infty} \delta(n-mp)$ $p>0$ is the integer period

a) $x[n]$ is just the sum of shifted / delayed impulses. We can also think of $x[n]$ as a periodic signal whose value everywhere is 1 with a period of p. We can see this since the summation over m just shifts the impulse by interger multiples of p, that is shifts the impulse by whole periods of p. As p gets larger, the discrete time between impulses increases, which is intuitive since we are just increasing the period (time).

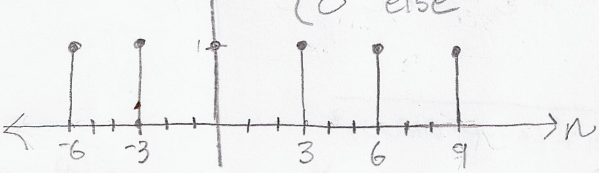
$\boxed{P=1:}$ $x[n] = \sum_{-\infty}^{\infty} \delta(n-m) = 1$



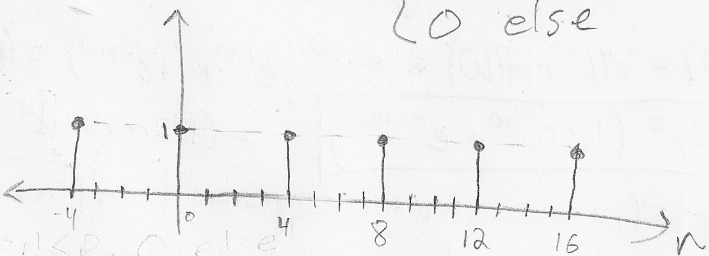
$\boxed{P=2:}$ $x[n] = \sum_{-\infty}^{\infty} \delta(n-2m) = \begin{cases} 1 & \text{if } n \bmod 2 = 0 \\ 0 & \text{else} \end{cases}$



$\boxed{P=3:}$ $x[n] = \sum_{-\infty}^{\infty} \delta(n-3m)$ $= \begin{cases} 1 & \text{if } n \bmod 3 = 0 \\ 0 & \text{else} \end{cases}$



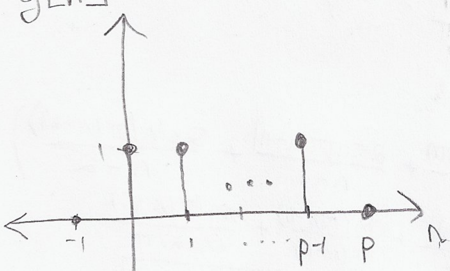
$\boxed{P=4:}$ $x[n] = \sum_{-\infty}^{\infty} \delta(n-4m) = \begin{cases} 1 & \text{if } n \bmod 4 = 0 \\ 0 & \text{else} \end{cases}$

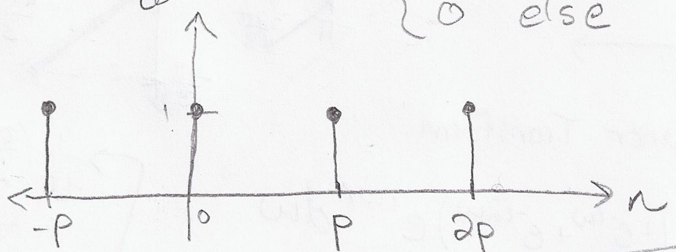


b) Show $x[n] * y[n] = 1$ if $y[n] = \begin{cases} 1 & \text{if } 0 \le n < p \\ 0 & \text{else} \end{cases}$

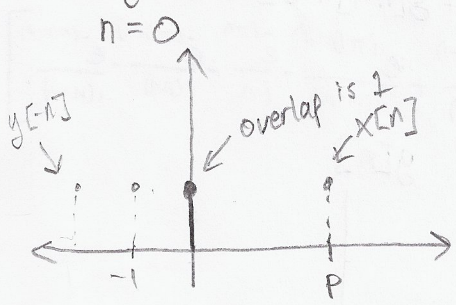
$x[n] * y[n] = \sum_{-\infty}^{\infty} x[k]y[n-k] = \sum_{-\infty}^{\infty} y[k]x[n-k]$

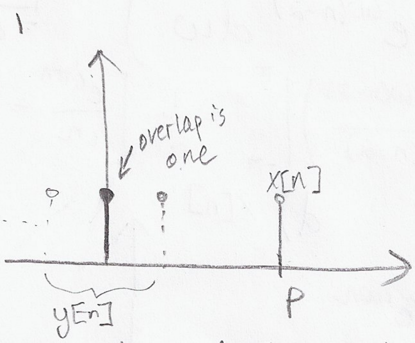
$x[n] = \sum_{-\infty}^{\infty} \delta(n-mp) = \begin{cases} 1 & \text{if } n \bmod p = 0 \\ 0 & \text{else} \end{cases}$



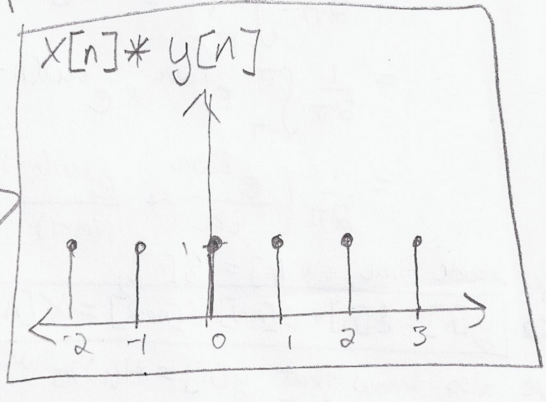


Using flip and drag:





$\Rightarrow$



Since $y[n]$ is a finite signal whose value is 1 from 0 to $p-1$, and $x[n]$ is zero for all n such that $n \bmod p \neq 0$, the overlap from flip and drag is always equal to 1. So the convolution is the constant signal $\boxed{x * y = 1}$