Physics 105 (Fall 2013): Solution to HW #7

1. Problem 7.3:
   The Lagrangian is $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$. The two equations of motion is then $m\ddot{x} = -kx$ and $m\ddot{y} = -ky$. They are two simple harmonic motions with the same frequency. In general the trajectory of the particle is an ellipse.

   Problem 7.8:
   (a) The Lagrangian is $L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) - \frac{k}{2}(x_1 - x_2 - l)^2$.

   (b) With the new coordinates, $L = m\dot{X}^2 + \frac{m}{4}\dot{x}^2 - \frac{k}{2}x^2$. The Lagrange equations are $\ddot{X} = 0$ and $\frac{m}{2}\ddot{x} = -kx$.

   (c) $X(t) = At + B$ and $x(t) = C\sin(\sqrt{\frac{2k}{m}}t) + D\cos(\sqrt{\frac{2k}{m}}t)$, where the four constants depends on the initial conditions. So the center of mass is doing a uniform motion while relative to one particle, another is doing a s.h.m. with $\omega^2 = \frac{2k}{m}$.

2. Practice.

3. Note that $\dot{T} = \dot{q}^i \frac{\partial T}{\partial q^i} + \ddot{q}^i \frac{\partial T}{\partial \dot{q}^i} + \frac{\partial T}{\partial t}$ by Chain rule, here summation convention is used. Now using product rule and the fact that partial derivatives commute,

$$\frac{\partial \dot{T}}{\partial \dot{q}^j} = \frac{\partial}{\partial \dot{q}^j}\left(\dot{q}^i \frac{\partial T}{\partial q^i}\right) + \ddot{q}^i \frac{\partial}{\partial \dot{q}^j}\frac{\partial T}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{q}^j}\frac{\partial T}{\partial t} = \left(\frac{\partial \dot{q}^i}{\partial \dot{q}^j}\frac{\partial T}{\partial q^i} + \dot{q}^i \frac{\partial}{\partial q^i}\frac{\partial T}{\partial \dot{q}^j}\right) + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}\frac{\partial T}{\partial \dot{q}^j} + \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}^j}$$

$$= \frac{\partial T}{\partial q^j} + \left(\dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial \dot{q}^j} = \frac{\partial T}{\partial q^j} + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^j}.$$

   Rearranging we get the desired identity.

4. Practice.

5. (a) From Chain rule, $\frac{d}{dt}U(x) = U'(x)\dot{x}$. Thus the equation of motion is

$$\frac{d}{dt}\left(\frac{1}{3}m^2\dot{x}^3 + 2m\dot{x}U(x)\right) = \left(m\dot{x}^2U'(x) - 2U(x)U'(x)\right).$$

   We can rewrite the equation of motion into the following form:

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}U(x) + m\dot{x}^2U'(x) + 2U(x)U'(x) = (m\dot{x}^2 + 2U(x))(m\ddot{x} + U'(x)) = 0.$$

   (b) $\frac{d}{dt}E^2 = 2E\frac{dE}{dt} = (m\dot{x}^2 + 2U(x))[(m\ddot{x} + U'(x))\dot{x}] = 0$ from above.

   (c) The usual Lagrangian will give: $m\ddot{x} + U'(x) = 0$. The system we are considering has one more possible solution: $E = 0$. But this doesn't give anything new, $E = 0$ is just a special case of the Beltrami identity: $E = const$, which is equivalent to the equation of motion for system with 1 degree of freedom.

6. Practice.

7. This is just an easy exercise: $x = A\cos(\omega t) + l\sin(\phi)$, $y = l\cos(\phi)$.
   Conversely $\phi = \tan^{-1}(\frac{x - A\cos(\omega t)}{y})$.

8. Practice.

9. The pulley has kinetic energy $\frac{I}{2}\omega^2 = \frac{I}{2}\frac{\dot{x}^2}{R^2}$. The Lagrangian is then: $L = \frac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 + (m_1 - m_2)gx$. The equation of motion will give:

$$\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2}.$$