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Homework 11

[LV 11.1] 1. Continuous time signal:

$$x(t) = \cos(10\pi t) + \cos(20\pi t) + \cos(30\pi t)$$

a) Fundamental Frequency:

$$\boxed{\omega_0 = 10\pi \text{ rad/s}}$$

$$F_0 = \frac{\omega_0}{2\pi} = \frac{10\pi \text{ rad/s}}{2\pi \text{ rad}} = \boxed{5 \text{ Hz}}$$

b) Fourier Series in the cosine basis:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

comparing this summation to the three terms in x(t):

$$x(t) = \cos 10\pi t + \cos 20\pi t + \cos 30\pi t = A_0 + \sum A_k \cos(k\omega_0 t + \phi_k)$$

and $\omega_0 = 10\pi$

$$\Rightarrow \boxed{\begin{array}{l} A_0 = 0 \quad A_1 = 1 \quad A_2 = 1 \quad A_3 = 1 \\ A_k = 0 \text{ for } k > 3 \\ \phi_k = 0 \text{ for all } k \end{array}}$$

c) y(t) is x(t) sampled at $10 \text{ Hz} = F_s$

for a sampler that samples every $T = \frac{1}{F_s}$ seconds, the sampled signal is:

$$s(n) = \cos(2\pi F n T)$$
$$= \cos(\omega_k n \tfrac{1}{F_s}) \qquad \omega_k = k\omega_0$$

$$\Rightarrow y[n] = \cos(10\pi\, n / 10\text{Hz}) + \cos\left(\frac{20\pi n}{10}\right) + \cos\left(\frac{30\pi n}{10}\right)$$

$$y[n] = \cos(\pi n) + \cos(2\pi n) + \cos(3\pi n)$$

since $\cos(2\pi n) = 1$ for $n \in \mathbb{Z}$

$\cos(\pi n) = \cos(3\pi n)$

$$\Rightarrow \boxed{y[n] = 1 + 2\cos(\pi n)}$$

$$\boxed{\omega_0 = \pi} \text{ rads/sample}$$

d) $y[n] = 1 + 2\cos(\pi n) = A_0 + \sum_{k=1}^{K} A_k \cos(\pi k n + \phi_k)$

$\omega_0 = \pi \Rightarrow p = \frac{2\pi}{\omega_0} = 2$

$K = \frac{p}{2} = \frac{2}{2} = 1 \implies \quad = A_0 + A_1 \cos(\pi n + \phi_1)$

by comparison, we see

$$\boxed{A_0 = 1, \quad A_1 = 2, \quad \forall k > 1 \; A_k = 0 \qquad \omega_0 = \pi \qquad \phi_k = 0 \text{ for all } k}$$

e) Find $w = IdealInterpolator_T(sampler_T(x))$ for $T = .1$ seconds

$z = ImpulseGen_T = \sum_{k=-\infty}^{\infty} y(k)\delta(t - kT)$ weighted Dirac delta function with weight $y(k)$

$IdealInterpolator_T = Sinc_T \circ ImpulseGen_T$

$w = IdealInterpolator_T(sampler_T(x)) = IdealInterpolator_T(y[n])$

$FT(z(t)) = Z(\omega) = Y(\omega T)$

let $FT(Sinc_T) = S(\omega) = \begin{cases} T & -\frac{\pi}{T} < \omega < \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$

$W(\omega) = Z(\omega) S(\omega) = Y(\omega T) S(\omega)$

going from the sampled signal $y[n]$ to the reconstructed signal $w(t)$, we make the substitution:

$T = .1 \qquad n = \frac{t}{T}$ since $t = nT$

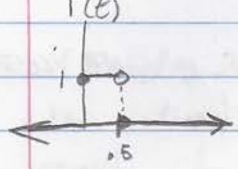
$w(t) = y[n = \frac{t}{T}] = 1 + 2\cos(\pi \frac{t}{.1}) = \boxed{1 + 2\cos(10\pi t) = w(t)}$

f) Since 10Hz < 15Hz (the maximum frequency in $x(t)$), there is aliasing expected. We see that 10Hz cosine was aliased to a constant DC term equal to 1, and since 15 Hz at $F_s = 10Hz$ looks the same as 5Hz, the $\cos 10\pi t$ and $\cos 30\pi t$ were reconstructed as just $2\cos 10\pi t$.

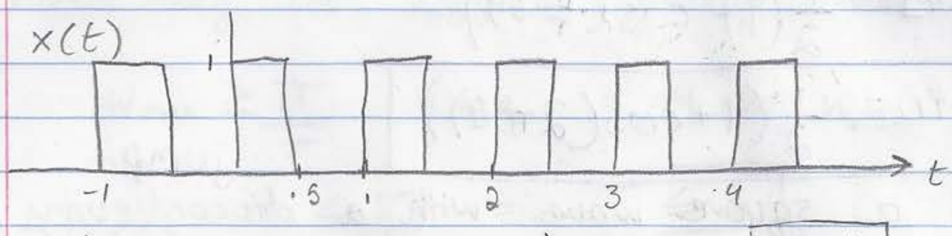
g) By the Nyquist Sampling Theory, $F_s > 2F_m$

$\Rightarrow \boxed{F_{s,min} = 2F_m = 2(15Hz) = 30Hz}$

[LV 11.4] 2. $x(t) = \sum_{-\infty}^{\infty} r(t-k)$ where $r(t) = \begin{cases} 1 & 0 \le t < .5 \\ 0 & \text{otherwise} \end{cases}$



The summation over k makes $x(t)$ just the sum of shifted $r(t)$'s



a) $x(t)$ is periodic, with period of $\boxed{P=1}$

b) Let the sampling period be $T=1$

$y = \text{Sampler}_T(x)$

$t = nT$

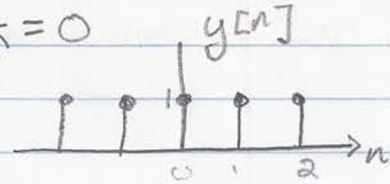
$y[n] = x(nT) = \sum_{k=-\infty}^{\infty} r(nT-k) = \sum_{k=-\infty}^{\infty} r(n-k)$

since the sampler is just taking $x(t)$ every 1 second, we can easily see that $y[n]$ is just a weight 1 impulse train. This can be verified from the sumation:

$y[n] = \sum_{-\infty}^{\infty} r(n-k) = 1 + 0 + 0 + 0 + \cdots = 1$

since the only nonzero term is when $n-k=0$

$\boxed{y[n] = 1 \quad \forall n}$



c) $T = .5s$

$y[n] = x(n\frac{1}{2}) = \sum_{-\infty}^{\infty} r(\frac{n}{2} - k) = \text{sampler}_T(x)$

For n is an odd number, the sum $\sum_{-\infty}^{\infty} r(\frac{n}{2}-k)$ is always zero, even in the case that $\frac{n}{2} - k = .5$, since $r(.5) = 0$. For n even, the summation terms are all zero except for when $n-k=0$, since $r(0)=1$, $y[n\text{-even}]=1$.

$$\boxed{y[n] = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}}$$

$z = \text{Ideal Interpolator}_T(\text{sampler}_T(x))$ where $T = .5$ s

we can rewrite $y[n]$:

$$y[n] = \frac{1}{2}(1 + \cos(\pi n)) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad \text{since } \cos\pi n = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \end{cases}$$

$$z(t) = y\left[\tfrac{t}{T}\right] = \frac{1}{2}\left(1 + \cos\left(\tfrac{\pi}{T}t\right)\right)$$

$$\boxed{z(t) = \frac{1}{2}(1 + \cos(2\pi t))}$$

d) $x(t)$ is a square wave with a jump discontinuity, meaning that it has a sharp edge at $x = n(.5)$ with infinitely many frequency components (remember Gibbs phenomenon), so the spectrum of X is not bandlimited, in otherwords, it has a maximum frequency of infinity, or no highest frequency. Nyquist sampling theorem says we need to sample at twice the highest frequency, but since no such sampling frequency exists, there is $\boxed{\text{no } T}$ for which we can reconstruct the original square wave $x(t)$.

[LV 11.5] 3. $x(t) = \sum_{k=0}^{4} \cos(k w_0 t)$ $w_0 = \pi/4$ rads/s

a) $x = \text{IdealInt}_T(\text{sampler}_T(x))$ means the reconstructed signal is the same as the original signal, so T must obey the Nyquist sampling theorem:

$$w_{max} \geq \frac{\pi}{T} \quad \text{since} \quad k=4 \iff w_{max} = 4w_0 = 4\left(\frac{\pi}{4}\right) = \pi$$

$$w_{max} = \pi \leq \frac{\pi}{T} \implies \boxed{T \leq 1}$$

b) $T = 4s$

$$y[n] = \text{sampler}_T(x) = x(nT) = \sum_{k=0}^{4} \cos(k w_0 4n)$$

$$= \sum_{k=0}^{4} \cos\left(k \frac{\pi}{4} 4n\right) = \sum_{k=0}^{4} \cos(k\pi n)$$

$$= \cos(0) + \cos(\pi n) + \cos(2\pi n) + \cos(3\pi n) + \cos(4\pi n)$$

$$= 1 + \cos(\pi n) + 1 + \cos(\pi n) + 1$$

$$= 3 + 2\cos(\pi n)$$

2π, 4π get aliased to DC

3π gets aliased to 1π

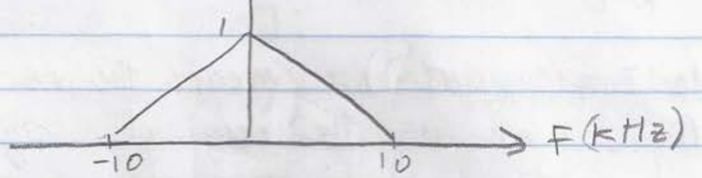
$$\cos(\pi n) = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$\boxed{y[n] = \begin{cases} 5 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}}$$

c) $w = \text{IdealInterpolator}_{T=4}(\text{sampler}_4(x))$

$$w(t) = y[t/T] = 3 + 2\cos\left(\pi \frac{t}{T}\right) = \boxed{3 + 2\cos\left(\pi \frac{t}{4}\right)}$$

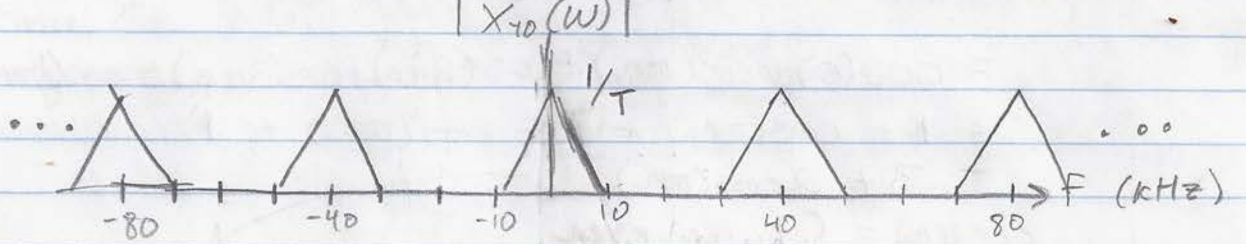
[LV 11.6] 4. $x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega) = X(2\pi F)$



sampled at $f_s = 40\text{kHz}$

$x_{40}[n] \overset{DTFT}{\Longleftrightarrow} X_{40}(\omega)$

a) $|X_{40}(\omega)|$ is just the convolution of $X(\omega)$ with the impulse train since $x_{40}[n]$ is just the multiplication of $x(t)$ with the impulse train. In the frequency domain, the impulse train has impulses at $F = 40n$ (kHz) for $n \in \mathbb{Z}$, which means $X_{40}(\omega)$ is just copies of $X(\omega)$ centered at $F = 40n$ kHz

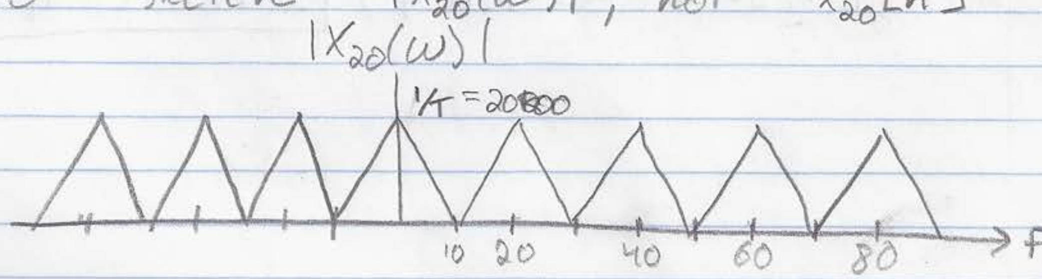


$X_{40}(\omega)$ is also scaled by a factor $1/T = f_s = 40000$ since the impulse in the frequency domain is scaled by $\frac{1}{T}$

b) $F_s = 20000$ samples/sec $= 20\text{kHz}$ $\quad T = 1/f_s$

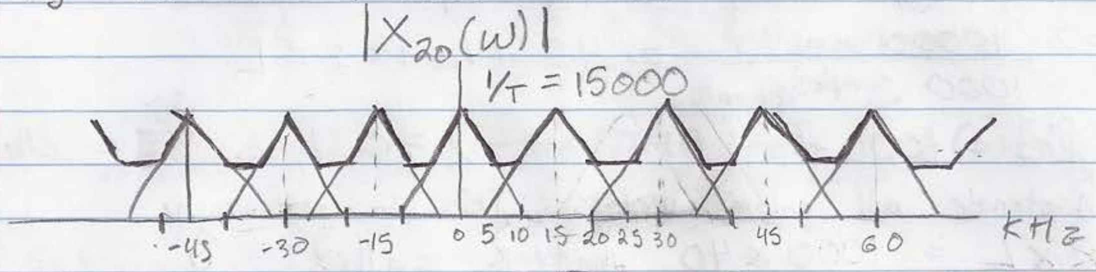
$x_{20}[n] = \text{sampled}_T(x) \overset{DTFT}{\Longleftrightarrow} X_{20}(\omega)$

since $x(t)$ was not given, I assume the problem wants us to sketch $|X_{20}(\omega)|$, not $x_{20}[n]$



$X_{20}(\omega)$ is just twice as many peaks as $X_{40}(\omega)$. Both X_{20} and X_{40} can be used to reconstruct $x(t)$ with a low pass filter with cutoff $F_{cut} = 10\text{kHz}$ by the sampling theorem

c) $F_s = 15000$ samples/sec $= 15kHz$



since the convolution of $X(\omega)$ with the impulse train with spacing of 15 kHz produces overlap in $X_{20}(\omega)$, we will has aliasing. This occurs because we did not sample at at least twice the highest frequency in $x(t)$'s spectrum $X(\omega)$. the reconstructed $x_{20}(t)$ would only have correct frequency components up to 5kHz, while the top 5kHz from 5-10 kHz will be corrupted.

5. 2 second audio signal
sampled at 20kHz $\Rightarrow x[n]$
first half of $x[n]$ is a sinusoid with frequency 200 Hz
second half of $x[n]$ is a 400Hz sinusoid.

a) since $x[n]$ is 2 seconds long at a sampling rate 20kHz $\frac{\text{samples}}{\text{sec}}$, we have 40000 samples total, so the DFT is N=40000 point.

$$\omega_0 = \frac{2\pi}{N} \Rightarrow \Omega_d = \frac{2\pi F_c}{F_s}$$

$$k = \frac{\Omega_d}{\omega_0} = \frac{2\pi F_c N}{F_s 2\pi} = \frac{F_c}{F_s} N$$

$X[k]$ has large values for only 2 of the 40000 k indices, each of the two corresponding to one of the frequencies $F_c = 200Hz, 400 Hz$

$$k = \frac{200}{20000}(40000) = 400 \qquad k = \frac{400}{20000} 40000 = 800$$

$$\boxed{k = 400, 800, 39600, 39200}$$

we also have negative frequency components which get modulo -40000
$-400 \bmod 40000 = 39600$ $\quad -800 \bmod 40000 = 39200$

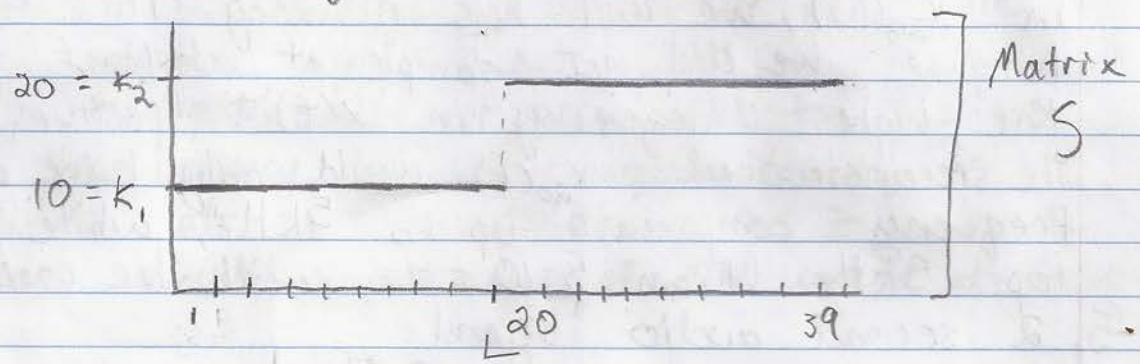
b) Divide $x[n]$ into 1000 sample chunks

$\Rightarrow \dfrac{40000 \text{ samples}}{1000 \text{ samples/chunk}} = 40 \text{ chunks} = L$

$X_\ell[k]$ = 1000-point DFT of $\ell \in 0, 1, \ldots, 39$ chunk

concatenate all column vectors $X_\ell[k]$ to get a $K \times L = 1000 \times 40$ matrix called S

- This matrix is just a spectrograph and looks something like



using the results from part a)

$$K_1 = \frac{F_c}{f_s} N = \frac{200 \text{ Hz}}{20000} 1000 = 10$$

$$K_2 = \frac{400}{20000} 1000 = 20$$

$$K_3 = \frac{-200}{20000} 1000 \bmod 1000 = 990$$

$$K_4 = \frac{-400}{20000} 1000 \bmod 1000 = 980$$

- For chunks $\ell = 0, \ldots, 19$ (the first half), S has large values for $k_1 = 10$, corresponding the frequency 200 Hz, and $k_3 = 990$ corresponding to -200 Hz
- For chunks $\ell = 20, \ldots, 39$ (second half), S has large values for $k_2 = 20$, corresponding to 400 Hz, and $k_3 = 980$ corresponding to -400 Hz
- S has large values for the matrix components

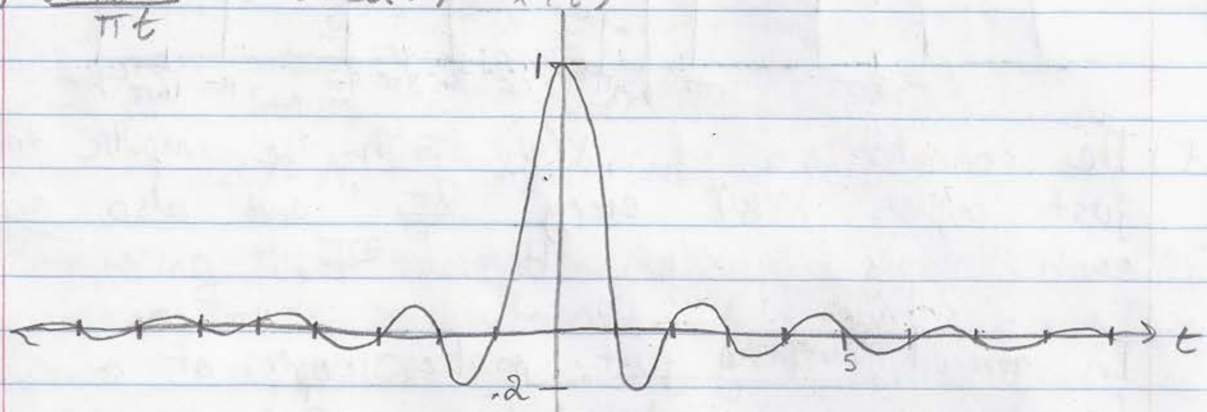
$[k_1 = 10, 0:19]$ $\quad$ $[K_2 = 20, 20:39]$

$[k_3 = 990, 0:19]$ $\quad$ $[k_4 = 980, 20:39]$

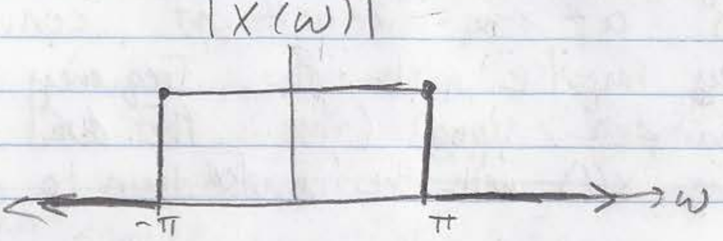
6. $x(t) = \dfrac{\sin(\pi t)}{\pi t} \xleftrightarrow{CTFT} X(\omega) = \begin{cases} 1 & \text{when } -\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$

$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \longleftrightarrow S(\omega) = \dfrac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\tfrac{2\pi}{T})$

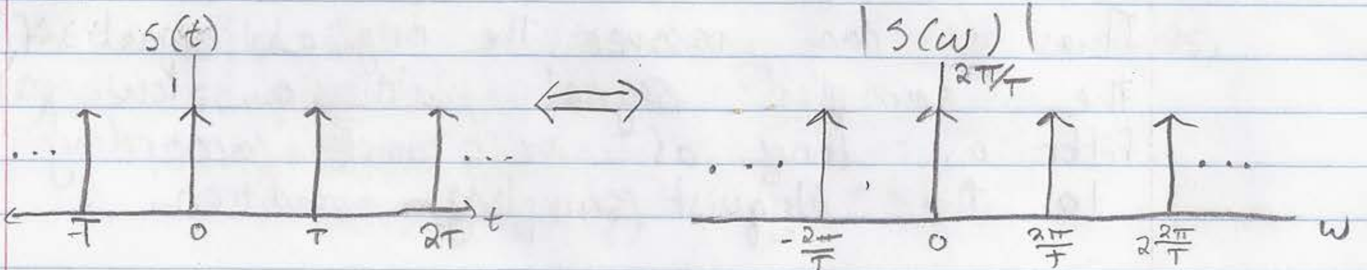
a) $\dfrac{\sin(\pi t)}{\pi t} = \text{sinc}(t) = x(t)$



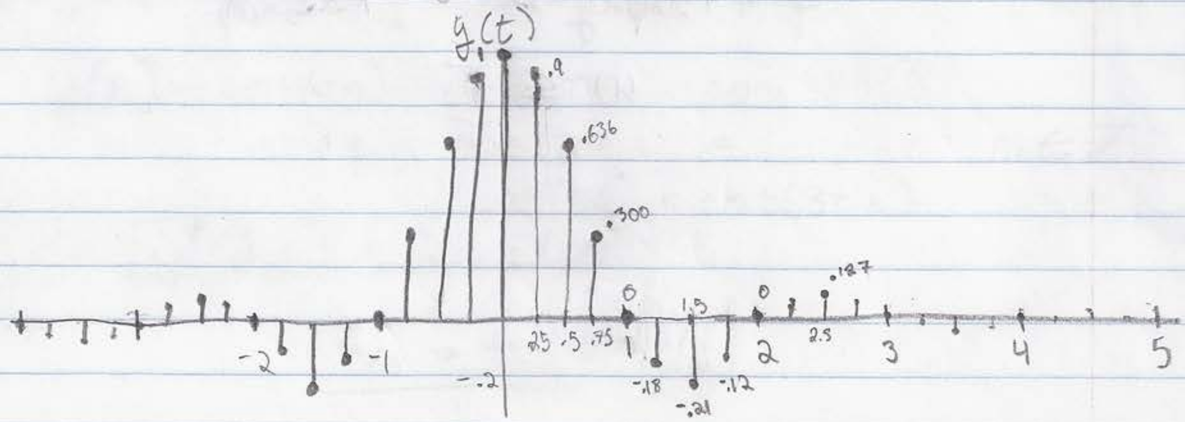
CTFT



b)



c) $y(t) = x(t)s(t)$ For $T = .25$



d) For general T, we know $y(t)=x(t)s(t)$ is multiplication in time domain which is just frequency domain convolution.

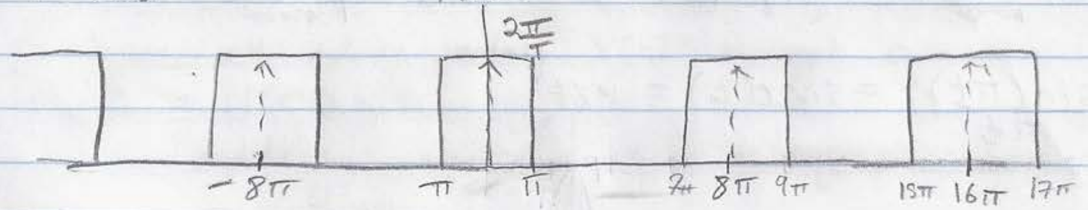
$$\boxed{Y(\omega)=X(\omega)*S(\omega)} = \int_{-\infty}^{\infty} X(l)\, S(\omega-l)\,dl = \int_{-\infty}^{\infty} X(l)\, \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-l-\tfrac{2\pi}{T}n)\,dl$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} X(l)\,\delta(\omega-l-\tfrac{2\pi}{T}n)\,dl$$ nonzero when $\omega-l-\frac{2\pi}{T}n=0$

$$\Rightarrow l=\omega-\tfrac{2\pi}{T}n$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X(l)\,\delta(0) = \boxed{\frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X(\omega-\tfrac{2\pi}{T}n)}$$ this is just scaled, shifted copies of $X(\omega)$

e) $T=.25$



f) The convolution of $X(\omega)$ with the impulse train $S(\omega)$ just copies $X(\omega)$ every $\frac{2\pi}{T}$ and also scales each copy of $X(\omega)$ by $\frac{2\pi}{T}$.

In general, sampling just makes copies of a frequency domain representation, spaced every $2\pi/T$ intervals as a result of convolting the frequency impulse with the frequency representation of the sampled signal (this is the dual operation of multiplying $x(t)$ with the impulse train to get a sampled signal).

Thus we can recover the original signal from the sampled signal with a low pass filter as long as we sample according to the Nyquist sampling condition!

$$\frac{1}{T}=F_{sampling} > 2F_{max,x(t)}$$

$$\omega \le \frac{\pi}{T}$$