EE20N: Structure and Interpretation of Systems and Signals **Fall 2013**

Lecture 13: October 15

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13.1 Announcements

- Practice problem posted (with answers)

13.2 Recap of IIR Filters

Example 13.1.

$$y(n) = \alpha y(n-1) + (1-\alpha)x(n)$$

Questions: 1) What is the block diagram of the above filter? 2) What is the impulse response? 3) What is the frequency response?

Answer 1: The parameter α is the relative weight we give to the previous output value relative to the current input value in determining the filter's current output. How does on implement the above filter? We use feedback! In block diagram form, this looks as follows:

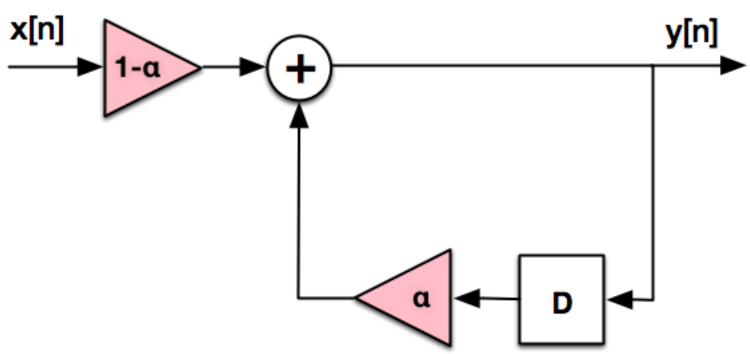


Figure 13.1: Diagram for Example 12.2

Triangles denote multiplicative gains, while a square with a D inside denotes a unit delay. This takes the input and outputs the value of the input at the previous time-step. In circuits language, a delay block could be implemented by something like a shift register.

Answer 2: To find the impulse response, we can't just plug in $\delta[n]$ again because of the feedback term. So instead, we will build up the impulse response from time 0. Assume initial rest, i.e. $y(n) = 0$ for $n < 0$. Now we can see that

$$h[0] = (1 - \alpha)$$
$$h[1] = \alpha(1 - \alpha)$$
$$h[2] = \alpha^2(1 - \alpha)$$
$$\dots$$

and so forth. In fact, we can prove by induction that the impulse response is given by:

$$h[n] = \begin{cases} 0 & : n < 0 \\ \alpha^n(1 - \alpha) & : n \geq 0 \end{cases}$$

Alternatively, you may use the unit step, in order to simplify the expression:

$$u[n] = \begin{cases} 0 & : n < 0 \\ 1 & : n \geq 0 \end{cases}$$

$$h[n] = \alpha^n(1 - \alpha)u[n]$$

If $|\alpha|$ is greater than 1 then $h(n)$ will infinity as n approaches infinity. If $|\alpha| < 1$ (as we should assume) we have more of a focus on the present and care less and less about the past but we don't want to forget about it completely.

Answer 3:

In order to find the frequency response of the system, we will use the frequency response formula we derived in the last lecture:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-iwn}$$

Plugging in our equation for h(n), we get the following:

$$H(\omega) = \sum_{n=-\infty}^{\infty} \alpha^n(1 - \alpha)u[n]e^{-iwn}$$

Since we know that h(n) has a value of 0 for all $n < 0$, we can simplify the expression (change the lower bound of sum):

$$H(\omega) = \sum_{n=0}^{\infty} \alpha^n(1 - \alpha)e^{-iwn}$$

The next step is to use the geometric sum formula: $\sum_{k=0}^{\infty} \beta r^k = \frac{\beta}{1-r}$ for $|r| < 1$. If $|r| \geq 1$, the summation diverges. In our example, we will assume $r = |\alpha e^{-iw}| = |\alpha| < 1$:

$$H(\omega) = \sum_{n=0}^{\infty} (1 - \alpha)(\alpha e^{-iw})^n = \frac{1 - \alpha}{1 - \alpha e^{-iw}}$$

Now in order to plot the magnitude $|H(\omega)|$ we use the following method, we know that:

$$|H(\omega)| = \frac{|1-\alpha|}{|1-\alpha e^{-i\omega}|}$$

The numerator is always $|1-\alpha| = 1-\alpha$ the interesting part is the denominator. In order to determine its magnitude as a function of w we first draw 1 and $\alpha e^{-i\omega}$ on the complex plane. But for what value of ω? We draw it for different values and attempt to determine the magnitude of the resultant subtraction of the two vectors. We then divide $|1-\alpha|$ by what we found and plot the magnitude $|H(\omega)|$

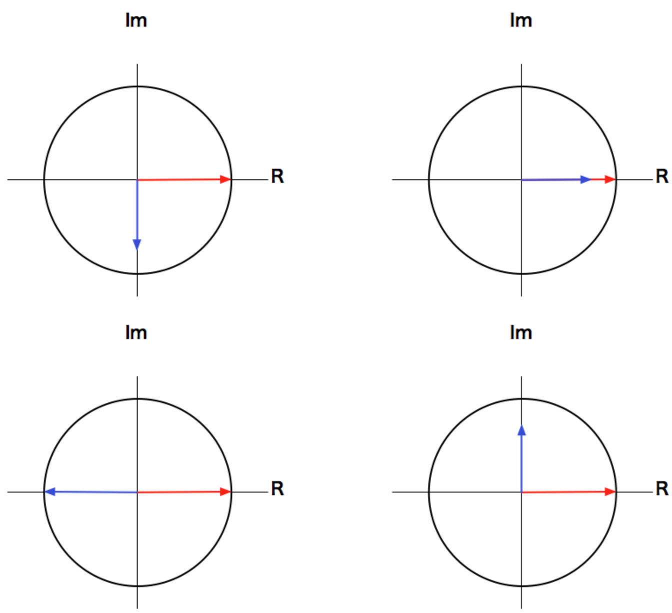


Figure 13.2: Plots on the complex plan and the unit circle of the vectors 1 (red) and $\alpha e^{-i\omega}$ (blue): (i) $\omega = \frac{\pi}{2}$, (ii) $\omega = 0 \rightarrow |H(\omega)| = 1-\alpha$, (iii) $\omega = \pi, -\pi \rightarrow |H(\omega)| = 1+\alpha$, (iv) $\omega = -\frac{\pi}{2}$

Is this a high-pass or a low-pass filter?

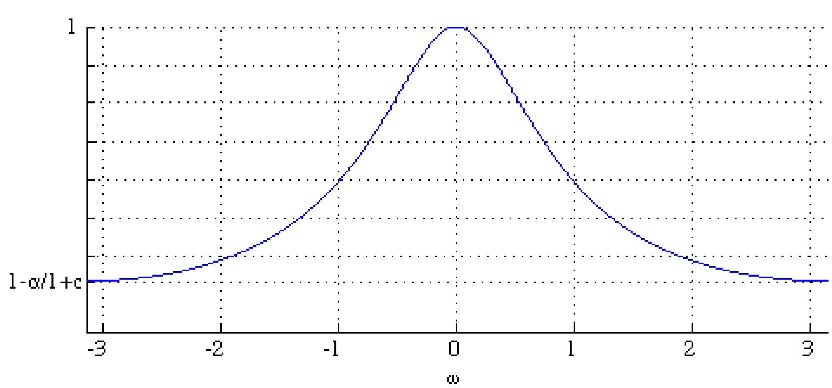


Figure 13.3: $|H(\omega)|$ for $[-\pi, \pi]$

13.3 Discrete Time Fourier Transform

13.3.1 Warm-up: Fourier series is not enough!

Suppose we have an LTI system with impulse response $h(n)$ and frequency response $H(\omega)$, the following input-output relationships are true.

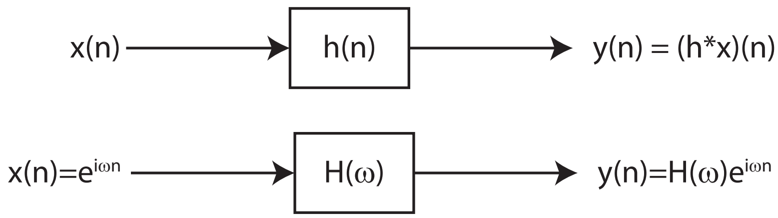


Figure 13.4

In time domain, the output, $y(n)$, of the filter H, which is the convolution between $x(n)$ and $h(n)$ is true for all input $x(n)$. Therefore, the impulse response, $h(n)$, completely characterizes the behavior of the system for any type of input $x(n)$.

However, in frequency domain, $y(n) = H(\omega)x(n)$ is true *if and only if* you can express $x(n)$ as a linear combination of e^{iwn}.

Clearly, we have an imbalance right now in terms of our ability to analyze LTI systems in the time domain (using $h(n)$) and the frequency domain (using $H(\omega)$)

In other words, so far, in frequency domain, we can analyze the signals that are:

- periodic (using Fourier series expansion into e^{iwn}s)
- over a finite duration (view them as one period of periodic signals.)

How do we analyze signals that are neither periodic nor over finite duration?

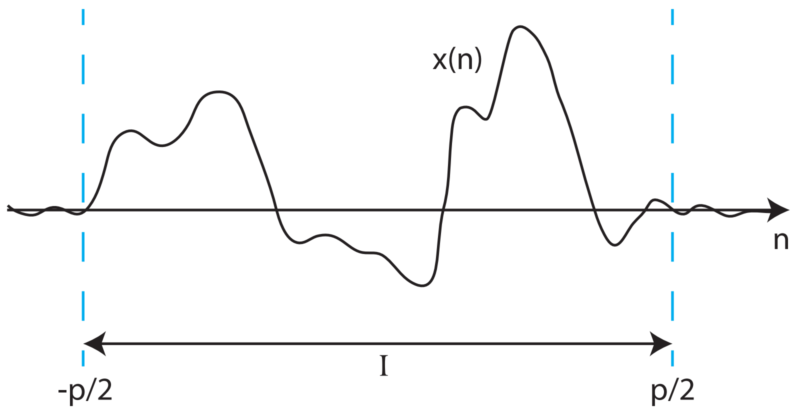
13.3.2 Derivation of Discrete Time Fourier Transform

Solution: Let's try to find the analogous "Fourier series" for signals that are neither periodic nor over finite duration! As it turns out, such a thing exists, and it's called the Fourier transform.

Fourier transform can be thought of as a generalization for Fourier series to more general signals.[1]

Let's consider the following signal $x(n)$:

Figure 13.5: an aperiodic, infinite duration signal



Note that we are in discrete time ($x(n)$) but we are plotting continuous time graphs for the simplicity of drawing the picture.

How should we go about finding a Fourier series equivalent for $x(n)$? First of all, what is the period, p, of an aperiodic signal? $p = \infty$ Well, while there is no Fourier series for $x(n)$ because it's aperiodic and not over finite duration, we know that for a chunk of $x(n)$ there exists a Fourier series. Because a chuck of $x(n)$ is a signal over a finite duration.

We chose an interval $I = [-\frac{p}{2},, \frac{p}{2}]$, with duration p. On this finite interval, we can find the Fourier series:

$$x(n) = \sum_{k=0}^{p-1} X(k)e^{ik\omega_0 n} \text{ for } n \in I$$

where $\omega_o = \frac{2\pi}{p}$

Note that this is only for $n \in I$. It's NOT true for n outside I!

What can we do? Well, we can make p large so that the interval I is large. But there is no choice of finite p that will allow you to have the Fourier series relationship be true $\forall n$.

Luckily for us, this signal has a nice property that it decays as n goes to either infinity. So if we make p larger and larger, $x(n)$ for $n \in I$ is a closer and closer approximation of the original $x(n)$ and the error (the difference between the approximation and the original signal) is very small.

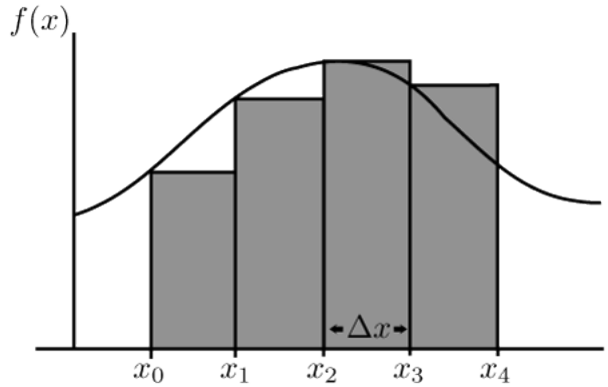
If we let $p \to \infty$, then $I \to (-\infty, \infty)$. Note that we also have the fundamental frequency $\omega_0 \to 0$.

[1]But not all signals. There are signal for which even the Fourier transform won't be enough! Stay tuned for EE120 to learn about how to deal with them.

Therefore, we must take the limit as p approaches ∞:

$$x(n) = \lim_{\left\{\begin{matrix} p \to \infty \\ \Delta\omega \to 0 \\ p\Delta\omega \to 2\pi \end{matrix}\right\}} \sum_{k=0}^{p-1} X(k) e^{ik\Delta\omega n}$$

This looks very horrendous. However, this will make more sense once we take this detour. Recall Riemann sum approximation of integral from basic calculus:



$$\int_{x_0=0}^{x_4=4\Delta x} f(x)dx \approx \lim_{\Delta x \to 0} \sum_{k=0}^{4-1} f(k\Delta x)\Delta x$$

Basically, we will take the limit as $\Delta\omega$ approaches 0 for our signal $x(n)$. Manipulating the expression for $x(n)$ slightly will give us the following approxmiation of $x(n)$:

$$x(n) = \lim_{\Delta\omega \to 0} \frac{1}{p\Delta\omega} \sum_{k\Delta\omega=0}^{(p-1)\Delta\omega} X(k\Delta\omega) e^{ik\Delta\omega n} \Delta\omega$$

What are the limits of the integral?

Remember that we are in discrete time. The frequency is bounded. Look at the Fourier series representation, we see that the largest k is $p-1$, so the largest frequency is $(p-1)\omega_0 = \frac{p-1}{p}2\pi$. If $p \to \infty$, this largest frequency goes to 2π. So we only need to integrate over a period of 2π. Let's pick the interval $[\pi, -\pi]$.[2]

What's the integrand of the integral?

As $p \to \infty$, $\omega_0 \to 0$. So the distance between $k\omega_0$ and $(k+1)\omega_0$ gets smaller and smaller - in fact, that distance goes to zero as well, until they are just become continuous with each other. This suggests that we are simply integrating over ω. Instead of having discrete X_k's, you will have a continuous function of ω, $X_d(\omega)$.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{i\omega n} d\omega \tag{13.1}$$

[2]There's nothing to prevent you from using any other intervals of length 2π. In fact, you can choose whichever interval of 2π that you want to integrate over.

The $X(\omega)$ in Equation 13.1 is called the **Fourier Transform** of $x(n)$.

Equation 13.1 is true for "any" signal $x(n)$. Any in quotation marks because there are signals for which we cannot define Fourier transforms like this. If a signal never "dies off" like our example $x(n)$ does, then this idea won't work because no matter how large p is, there will still be things outside that can't catch. But don't worry about dealing with such signals in this class!

We added a factor of $\frac{1}{2\pi}$ in the front. You can think of it as a normalization factor - with it, the Fourier transform of the impulse, $\delta(n)$, will look very nice. And since $\delta(n)$ is such an important and fundamental object, we'd like to have it that way. [3]

13.4 The Fourier Transform of the Impulse

While we can't find the Fourier series of $\delta(n)$ because it doesn't exist, we can find the Fourier transform of $\delta(n)$.

Claim: The Fourier transform of $\delta(n)$ is simply the constant 1 function:

$$X_d(\omega) = 1, \forall \omega$$

Let's check:

$$\delta(n) \stackrel{?}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot e^{i\omega n} d\omega$$

When $n = 0$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot e^{i\omega 0} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\omega = \frac{2\pi}{2\pi} = 1 = \delta(0)$$

And for other n:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot e^{i\omega n} d\omega = 0 = \delta(n) \text{ for } n \neq 0$$

The function $e^{i\omega n}$ has period $\frac{2\pi}{n}$ (corresponding to one rotation on the unit circle in the real-imaginary plane). If we integrate it over 2π, we are integrating over an integer multiple of the period, so it will integrate to 0.

Therefore, the Fourier transform of $\delta(n)$ is $X_d(\omega) = 1$. And here you see why we had a factor of $\frac{1}{2\pi}$ in our formula - it's to make the Fourier transform of $\delta(n)$ to be normalized to 1.

[3] Actually, there's a couple different conventions people use for the Fourier transform formulas, and some versions don't put that 2π there. They will instead put it in another formula that you will soon learn. But it remains that a factor of 2π will show up somewhere in this Fourier transform business.

13.5 Relate our new friend to our old friends

13.5.1 FTs of the input and output signals of an LTI system

So we learned about the Fourier transform. How does it fit in with our existing knowledge such as impulse response and frequency response?

Recall that so far, we know that for an LTI system with impulse response $h(n)$ and frequency response $H(\omega)$, the relationships shown in Fig. 13.6 are true.

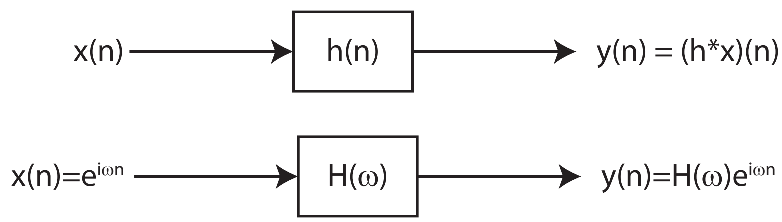


Figure 13.6

Now let's consider $X_d(\omega)$ and $Y_d(\omega)$, the Fourier transforms of $x(n)$ and $y(n)$, respectively. How do they relate to each other through your system (Fig. 13.7)?

$$x(n) \overset{FT}{\leftrightarrow} X_d(\omega)$$

is called the Fourier Transform pair. This pair is unique and one-to-one (invertible)

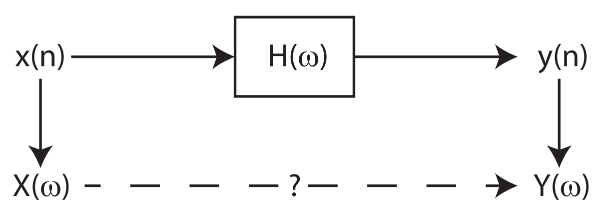


Figure 13.7

Because of:

$$x(n) = e^{i\omega n} \rightarrow \boxed{\text{H}(\omega)} \rightarrow y(n) = H(\omega)e^{i\omega n}$$

The following is true due to linearity of our system:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(w)e^{i\omega n}d\omega \rightarrow \boxed{\text{H}(\omega)} \rightarrow y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)X_d(w)e^{i\omega n}d\omega$$

And we also have:

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y_d(\omega)e^{i\omega n}d\omega$$

Therefore, it must be that:

$$Y_d(\omega) = H(\omega)X_d(\omega) \tag{13.2}$$

Equation 13.2 gives the relationship between the Fourier transforms of the input and output signals for an LTI system: *the Fourier transform of the output signal is simply the frequency response of the system multiplied by the Fourier transform of the input signal.*

13.5.2 The impulse response and the frequency response

Now let's consider the impulse response. Recall that the Fourier transform of $\delta(n)$ is $X(\omega) = 1$.

We've already figured out that the relationships in Fig. 13.8 are true.

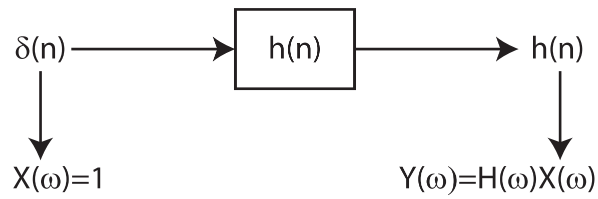


Figure 13.8

Therefore:

$$\mathrm{FT}\{h(n)\} = H(\omega)X_d(\omega) = H(\omega)$$

$H(\omega)$, the frequency response of an LTI system, is the Fourier transform of $h(n)$, the impulse response of the system.

13.6 Guitar Lab

With what we've talked about in lecture today, hopefully next week's lab - the guitar lab - makes some more sense to you.

In the lab, to synthesize the guitar sound, we pass an impulse, $\delta(n)$, through a comb filter. Why do we use $\delta(n)$?

It's because $\delta(n)$ contains all frequencies. The Fourier transform of $\delta(n)$ is $X(\omega) = 1$. Therefore, if we pass $\delta(n)$ through the comb filter, the output signal's spectrum is simply the frequency response of the comb filter. This way we get all the harmonics we need for a guitar sound.

A subtlety of this lab is that, other than the harmonics we need - say the $440Hz$ and its higher harmonics ($880Hz$ and so on) - we also get a peak at $0Hz$ due to the shape of the comb filter. Why does this $0Hz$ peak not bother us?

It's because we can't hear $0Hz$! Human years won't pick up low frequencies (approximately those $< 20Hz$) so even though the $0Hz$ is there, we can't tell.