Physics 105 (Fall 2013): Solution to HW #1

1. An alternative solution is OK, if a reasonable argument is given and the right order of magnitude is obtained.

   (a) We have 300 million people in the US, and say one-sixth of them are students. Suppose 30% of them are high-school students, thus there are about $3 \times 10^8 \times 0.3/6 = 15$ million high-school students. Let's suppose teacher-student ratio is 1 to 15, hence we need about 1 million high-school teacher.

   (b) Considering both industrial and domestic usage, it is estimated that the daily consumption of gasoline is 1 gallon per person. Thus we have $3 \times 10^8 \times 365$ or about 100 billion gallons of gasoline consumed per year!!

   (c) There are about 1 million people in San Francisco, with in average 3 persons in a household. Suppose there is a piano in every 10 households and each piano is tuned twice a year, then there are $\frac{10^6 \times 2}{3 \times 10 \times 365}$ or roughly 200 tunings per day. Estimate that a piano tuner can tune 4 pianos per day, then there are 50 piano tuners in San Francisco.

2. (a) We have length, time and mass as the three fundamental dimensions in mechanics.

   (b) A dimension is intrinsic to a physical quantity while a unit is a *scale* for measuring a physically quantity *numerically.*
   It's possible that a dimensionless quantity having a unit, for instance an angle has a unit. A dimensionful quantity needs a unit *only when* considered numerically. Of course, a pure number has neither dimensions or units.
   When a equation is considered, both sides must have the same dimension but not necessarily the same unit as conversion between units is possible.

   (c) The argument of a transcendental function is dimensionless since otherwise its series expansion is inconsistent in dimensions. But it can possess a unit, say the argument of trigonometric functions can be either in degrees or in radians.

3. (a) Notice that $|\mathbf{B} \times \mathbf{C}|$ is the area of the parallelogram spanned by the two vectors. The volume of a parallelepiped is base area times height. Here the height is $|\mathbf{A} \cdot \hat{\mathbf{n}}|$ where $\hat{\mathbf{n}}$ is the unit vector normal to the parallelogram above, which is the along the direction of $\mathbf{B} \times \mathbf{C}$. Thus the volume is $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

   (b) Using the so-called BAC-CAB rule for vector triple product,

   $$(\hat{\mathbf{n}} \times \mathbf{A}) \times \hat{\mathbf{n}} = \hat{\mathbf{n}} \times (\mathbf{A} \times \hat{\mathbf{n}}) = \mathbf{A}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) - \hat{\mathbf{n}}(\mathbf{A} \cdot \hat{\mathbf{n}}).$$

   Rearranging we get the desired result.
   Alternatively one may set the coordinate axes such that $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ and $\mathbf{A}$ lies on x-y plane. The identity will be trivial.

   (c) $|\mathbf{A}|$ is a constant $\Leftrightarrow \frac{d}{dt}(\mathbf{A} \cdot \mathbf{A}) = 0 \Leftrightarrow \frac{d\mathbf{A}}{dt} \cdot \mathbf{A} = 0$.

   (d) $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}| \Leftrightarrow (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \Leftrightarrow \mathbf{A} \cdot \mathbf{B} = 0$.

   (e) Suppose the diagonals are represented by vectors $\mathbf{A}$ and $\mathbf{B}$. Observe that $\frac{1}{2}(\mathbf{A} + \mathbf{B})$ corresponds to two opposing sides of the rhombus, while $\frac{1}{2}(\mathbf{A} - \mathbf{B})$ corresponds to the remaining two sides. As the sides are of equal length, from (d) $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

4. (a) $a_r = \ddot{r} - r\dot{\theta}^2 = 2\beta\lambda - 4\beta\lambda^3 t^4$, it is zero when $t = \frac{1}{\sqrt[4]{2\lambda^2}}$.

   (b) $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 10\beta\lambda^2 t^2$.

   The speed of the particle is $v = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = 2\beta\lambda t\sqrt{1 + \lambda^2 t^4}$ and the tangential acceleration is $\frac{dv}{dt} = 2\beta\lambda\frac{1+3\lambda^2 t^4}{\sqrt{1+\lambda^2 t^4}}$.
   The angular acceleration is $\ddot{\theta} = 2\lambda$.

   (c) The have equal magnitude when $1 - 2\lambda^2 t^4 = \pm 5\lambda t^2$, that is when $2\theta^2 \mp 5\theta - 1 = 0$. Solving gives $\theta = \frac{\sqrt{33} \pm 5}{4}$.

(d) $\vec{F} = m\vec{a} = 2m\beta\lambda(1-2\lambda^2 t^4)\hat{r} + 10m\beta\lambda^2 t^2\hat{\theta}$.

(e) $v(t) = 2\beta\lambda t\sqrt{1+\lambda^2 t^4}$ as shown in (b).

5. (a) Either using algebraic or geometric method, we can obtain

$$\dot{\hat{r}} = \dot{\phi}\hat{\phi}, \dot{\hat{\phi}} = -\dot{\phi}\hat{r} \text{ and } \dot{\hat{z}} = 0.$$

Then $\mathbf{v} = \frac{d}{dt}(r\hat{r} + z\hat{z}) = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$, $\mathbf{a} = \frac{d}{dt}\mathbf{v} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$.

(b) $\mathbf{j} = (r^{(3)} - 3\dot{r}\dot{\phi}^3 - 3r\dot{\phi}\ddot{\phi})\hat{r} + (3\ddot{r}\dot{\phi} + 3\dot{r}\ddot{\phi} + r\phi^{(3)} - r\dot{\phi}^3)\hat{\phi} + z^{(3)}\hat{z}$.

6. (a) Here a purely algebraic method is shown, one may also employ differential calculus for this problem. The trajectory of the arrow is given by $y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}$, and where it lands should satisfy $y = -x\tan\phi$. We get from these the x-coordinate of its landing position:

$$\begin{aligned}
x &= \frac{2v_0^2}{g}(\tan\theta + \tan\phi)\cos^2\theta \\
&= \frac{2v_0^2}{g}\frac{\cos\phi\sin\theta\cos\theta + \sin\phi\cos^2\theta}{\cos\phi} \\
&= \frac{2v_o^2}{g\cos\phi}\sin(\phi+\theta)\cos\theta \\
&= \frac{v_0^2}{g\cos\phi}[\sin(\phi+2\theta) + \sin\phi]
\end{aligned}$$

Thus for maximum distance, $\theta = \frac{\pi}{4} - \frac{\phi}{2}$.

(b) The horizontal component of velocity is a constant: $v_x = v_0\cos(\frac{\pi}{4} - \frac{\phi}{2}) = v_0\sqrt{\frac{1+\sin\phi}{2}}$. From one dimensional kinematics the vertical component is

$$\begin{aligned}
v_y = v_0\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right) - g\frac{x_f}{v_x} &= v_0\sqrt{\frac{1-\sin\phi}{2}} - v_0\frac{1+\sin\phi}{\cos\phi}\sqrt{\frac{2}{1+\sin\phi}} \\
= -v_0\sqrt{\frac{1-\sin\phi}{2}}\left(\frac{2}{\cos\phi}\sqrt{\frac{1+\sin\phi}{1-\sin\phi}} - 1\right) &= -v_0\frac{1+\sin\phi}{\sqrt{2(1-\sin\phi)}}.
\end{aligned}$$

7. (a) Consider the train as a whole and apply Newton's second law, the common acceleration of the cars is given by $F = NMa$, $a = \frac{F}{NM}$.

(b) Consider car i and all the cars behind it as a single object, then the force we want to find is the only external force act on the object. So $F_{i,i-1} = (N+1-i)Ma = \frac{N+1-i}{N}F$.

(c) From third law we have $F_{i+1,i} = -F_{i,i+1} = -\frac{N-i}{N}F$.

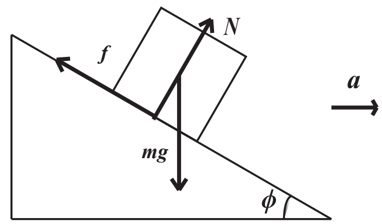
8. As this problem is already discussed during sections only a brief solution is given. Consider a fraction of the rope warping around the capstan, with length being $Rd\theta$. Force cancellation along the tangential direction gives: $f = -\frac{dT}{d\theta}d\theta$, cancellation along radial direction gives: $N = Td\theta$. Then using $f = \mu N$ we come up with $\frac{dT}{d\theta} = -\mu T$, $T = T_0 e^{-\mu\theta}$.

9. (a) Tension of the string is minimum when the mass is at its highest position, thus we only need to ensure tension is non-negative there. By energy conservation, the speed at that point is given by $v^2 = v_0^2 - 2gL$, the centripetal force is $\frac{Mv^2}{L} = T + Mg$, $T \geq 0$ implies $v_0 \geq \sqrt{3gL}$.

(b) Tangential acceleration of the mass is due to circumferential component of the weight. Thus $a_\theta = -g\cos\theta$.

(c) By energy conservation, $v^2(\theta) + 2gL\sin\theta = v_0^2$, so $v(\theta) = \sqrt{v_0^2 - 2gL\sin\theta}$.

(d) Apply Newton's law along the radial direction, $T + Mg\sin\theta = \frac{Mv^2}{L}$, $T = \frac{Mv_0^2}{L} - 3Mg\sin\theta$.



10. (a) Force equilibrium gives $f = mg\sin\phi$ and $N = mg\cos\theta$. Thus $f \le \mu N \Rightarrow \tan\phi \le \mu$.

    (b)(c) We can deal with these two parts together. As in the diagram, if we apply Newton's law along the direction parallel to the slope we get: $f = mg\sin\phi - ma\cos\phi$, and along the direction normal to the slope we get: $N = mg\cos\phi + ma\sin\phi$. Static equilibrium requires $f \le \mu N$ then $a \ge \frac{tan\phi-\mu}{1+\mu\tan\phi}g$, here we should also have $f \ge 0 \Leftrightarrow a \le g\tan\phi$. For the case $0 \ge f \ge -\mu N$ and we get $a(\cos\phi - \mu\sin\phi) \le g(\sin\phi + \mu\cos\phi)$. Here we have to separate into two cases:
    Case 1: $\mu < \cot\phi$, then $a \le \frac{tan\phi+\mu}{1-\mu\tan\phi}g$.
    Case 2: $\mu \ge \cot\phi$, then the inequality is always true given $f \le 0 \Leftrightarrow a \ge g\tan\phi$.
    To sum up, $a_{\min} = \frac{tan\phi-\mu}{1+\mu\tan\phi}g$, $a_{\max} = \frac{tan\phi+\mu}{1-\mu\tan\phi}g$ if $\mu < \cot\phi$ and $a_{\max} = +\infty$ if $\mu \ge \cot\phi$.

11. (a) Either using geometry or derive from the metric tensor in Cartesian coordinate, we can obtain in spherical coordinate:
    $$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$
    This is singular when $r = 0$ or $\sin\theta = 0$, that is at the z-axis. This is because the coordinate system itself is singular there, more precisely ϕ is ill-defined at the z-axis and so does θ at the origin. Then $d\phi$ or $d\theta$ will be ambiguous.

    (b) $\mathbf{r} = r\hat{r}$.

    (c) The derivative of $\hat{r}$ is: $\dot{\hat{r}} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}$. Hence $\mathbf{v} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$. Alternatively, as spherical coordinate is a orthogonal coordinate system, it is not hard to derive the expression geometrically.