Physics 105 (Fall 2013): Solution to HW #13

1. The Lagrangian is $L = \frac{m}{2}(c^2 + R^2)\dot{\phi}^2 - mgc\phi$. So the momentum is $p = m(c^2 + R^2)\dot{\phi}$. The Hamiltonian is then: $H = \frac{p^2}{2m(c^2+R^2)} + mgc\phi$. The Hamilton's equations are: $\dot{\phi} = \frac{\partial H}{\partial p} = \frac{p}{m(c^2+R^2)}$ and $\dot{p} = -\frac{\partial H}{\partial \phi} = -mgc$. We get from these $\ddot{z} = c\ddot{\phi} = -\frac{gc^2}{c^2+R^2}$.
   This agrees with the fact that the bead is as if sliding down along an incline with slope $\tan\theta = \frac{c}{R}$. The acceleration along the incline is $a = g\sin\theta$ and indeed the vertical component is $\ddot{z} = -a\sin\theta = -\frac{gc^2}{c^2+R^2}$.
   In the limit $R \to 0$, $\ddot{z} \to -g$. It makes sense as in this case the bead is free falling vertically.

2. We write the Lagrangian in terms of quadratic form: $L = \dot{\mathbf{q}}^T M \dot{\mathbf{q}} + \kappa(q^1)^2$, where $\dot{\mathbf{q}} = (\dot{q^1}, \dot{q^2})^T$ is the column vector for the generalized velocities, M is a symmetric matrix with $M_{11} = 1$, $M_{22} = (a + b(q^1)^2)^{-1}$ and $M_{12} = M_{21} = \mu/2$. The momenta will be then: $\mathbf{p} = (p_1, p_2)^T = 2M\dot{\mathbf{q}}$. Invert this, we get $\dot{\mathbf{q}} = \frac{M^{-1}}{2}\mathbf{p}$. The Hamiltonian will be: $H = \frac{1}{4}\mathbf{p}^T M^{-1}\mathbf{p} - \kappa(q^1)^2 = \frac{a+b(q^1)^2}{4-\mu^2(a+b(q^1)^2)}\left(\frac{p_1^2}{a+b(q_1)^2} - \mu p_1 p_2 + p_2^2\right) - \kappa(q^1)^2$. The first Hamilton's equation is the one above: $\dot{\mathbf{q}} = \frac{M^{-1}}{2}\mathbf{p}$. The second one is: $\dot{p_i} = -\frac{\partial H}{\partial q^i} = \frac{1}{4}\mathbf{p}^T M^{-1}\frac{\partial M}{\partial q^i}M^{-1}\mathbf{p} + \frac{\partial}{\partial q^i}(\kappa(q^1)^2)$. In components it is: $\dot{p_2} = 0$ and $\dot{p_1} = 2\kappa q^1 - \frac{8bq^1}{(4-\mu^2(a+b(q^1)^2))^2}(\frac{\mu}{2}p_1 - p_2)^2$.

3. Practice.

4. .

5. Practice.

6. practice.

7. (a) The momentum is $p = \frac{m}{2}(2\dot{q}\sin^2\omega t + q\omega\sin 2\omega t$, this gives: $\dot{q} = \sin^{-2}\omega t(\frac{p}{m} - \frac{q}{2}\omega\sin 2\omega t)$. The Hamiltonian will be $H = \frac{m}{2\sin^2\omega t}(\frac{p}{m} - \frac{q}{2}\omega\sin 2\omega t)^2 - \frac{m}{2}q^2\omega^2$, which is not a constant of motion as it has explicit time dependence.

   (b) If $Q = q\sin\omega t$, $\dot{Q} = \dot{q}\sin\omega t + q\omega\cos\omega t$. The Lagrangian will be $L = \frac{m}{2}(\dot{Q}^2 + Q^2\omega^2)$. The momentum is $P = m\dot{Q}$ and the Hamiltonian is $H = \frac{P^2}{2m} - \frac{m}{2}Q^2\omega^2$ which is conserved.

8. (a) If $L' = L + \frac{df}{dt} = L + \dot{\mathbf{q}}\cdot\nabla f + \frac{\partial f}{\partial t}$, the new canonical momentum is $\mathbf{P} = \frac{\partial L'}{\partial \dot{\mathbf{q}}} = \mathbf{p} + \nabla f$.

   (b) The Hamiltonian changes into $H' = \dot{\mathbf{q}}\cdot\mathbf{P} - L' = H - \frac{\partial f}{\partial t}$. Note that the new independent variables are: $\mathbf{Q} = \mathbf{q}$ and $\mathbf{p}' = \frac{\partial L'}{\partial \dot{\mathbf{q}}} = \mathbf{p} + \nabla f$.
   We have to show first: $\frac{\partial H'}{\partial \mathbf{P}} = \frac{\partial H}{\partial \mathbf{p}}$ so the $\dot{\mathbf{q}}$ equation is preserved. From chain rule, $\frac{\partial}{\partial \mathbf{p}} = \frac{\partial P^i}{\partial \mathbf{p}}\frac{\partial}{\partial P^i} + \frac{\partial Q^i}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial Q^i} = \frac{\partial}{\partial \mathbf{P}}$ since $\mathbf{Q}$ and $\mathbf{p}$ are independent. Since $\frac{\partial f}{\partial t}$ does not depends on $\mathbf{p}$, we get the required equality.
   To show $\dot{\mathbf{P}} = -\frac{\partial H'}{\partial \mathbf{Q}}$, note that $\frac{\partial}{\partial \mathbf{q}} = \frac{\partial P^i}{\partial \mathbf{q}}\frac{\partial}{\partial P^i} + \frac{\partial Q^i}{\partial \mathbf{q}}\cdot\frac{\partial}{\partial Q^i} = \frac{\partial}{\partial \mathbf{Q}} + \frac{\partial \nabla^i f}{\partial \mathbf{q}}\frac{\partial}{\partial P^i}$. Then

$$-\dot{\mathbf{p}} = \frac{\partial H}{\partial \mathbf{q}} = \left(\frac{\partial}{\partial \mathbf{Q}} + \frac{\partial \nabla^i f}{\partial \mathbf{q}}\frac{\partial}{\partial P^i}\right)\left(H' + \frac{\partial f}{\partial t}\right) = \frac{\partial H'}{\partial \mathbf{Q}} + \frac{\partial \nabla^i f}{\partial \mathbf{q}}\frac{\partial H'}{\partial P^i} + \frac{\partial}{\partial \mathbf{Q}}\frac{\partial f}{\partial t}$$

$$= \frac{\partial H'}{\partial \mathbf{Q}} + \frac{\partial \nabla^i f}{\partial \mathbf{q}}\dot{Q}^i + \frac{\partial}{\partial t}\frac{\partial f}{\partial \mathbf{Q}} = \frac{\partial H'}{\partial \mathbf{Q}} + (\dot{\mathbf{Q}}\cdot\nabla)\nabla f + \frac{\partial}{\partial t}\nabla f = \frac{\partial H'}{\partial \mathbf{Q}} + \frac{d}{dt}\nabla f.$$

   Note that $\frac{\partial f}{\partial t}$ does not depend on $\mathbf{P}$. So $\dot{\mathbf{P}} = -\frac{\partial H'}{\partial \mathbf{Q}}$ as desired.

9. Practice.

10. The Lagrangian here is $L = \frac{m}{2}[(\dot{x}+V)^2 + \dot{y}^2 + \dot{z}^2] - mgz$, this gives $\mathbf{p} = m\dot{\mathbf{x}} + mV\hat{x}$. The Hamiltonian is then: $H = \frac{\mathbf{p}^2}{2m} - Vp_x + mgz$. This is not the same as total mechanical energy in either frame, since in the lab frame, $E = \frac{\mathbf{p}^2}{2m} + mgz$ and in the car's frame, $E = \frac{(\mathbf{p}-mV\hat{x})^2}{2m} + mgz$.

11. The Lagrangian is $L = \frac{m}{2}(\dot{r}^2 + r^2\omega^2)$, with momentum $p_r = m\dot{r}$. The Hamiltonian is $H = \frac{p_r^2}{2m} - \frac{m}{2}\omega^2 r^2$, but the energy in the lab frame is: $E = \frac{p_r^2}{2m} + \frac{m}{2}\omega^2 r^2$, and in the rod's frame, it is: $E = \frac{p_r^2}{2m}$.

12. Practice.

13. Practice.

14. See the last page for the solution.

15. F and G are conserved means $\{F,H\} = \{G,H\} = 0$. We will employ the Jacobi identity: $\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0$, which can be proved using straightforward, albeit tedious, algebra. The proof only requires expanding the brackets using its definition and the identity: $\{AB,C\} = A\{B,C\} + \{A,C\}B$. Now the last two terms are zero, implies $\{\{F,G\},H\} = 0$ and so $\{F,G\}$ is conserved.

16. We only need to check that the Poisson bracket for Q and P satisfies $\{Q,P\} = 1$, which is equivalent to say the transformation is area-preserving in the phase space.
    (a)(b)(c) Yes.
    (d) Yes if $\kappa\beta = 2$.
    (e) Yes if $\alpha = 1/2$ and $\beta = -2$, note that we should swap the role of (q,p) and (Q,P).
    (f) Yes.

17. Practice.

18. Practice.

19. (a) We focus on 1 particle only, let its x-momentum be p_x, which is constant in magnitude. The action variable is $I_x = \frac{1}{2\pi}\oint p_x\,dx = \frac{1}{\pi}|p_x|\,L_x$, using the fact that the phase portrait is a rectangle, so $\frac{p_x^2}{2m} = \frac{\pi^2}{2mL_x^2}I_x^2$. The Hamiltonian is then: $H = \frac{\pi^2}{2m}\sum\left(\frac{I_x^2}{L_x^2} + \frac{I_y^2}{L_y^2} + \frac{I_z^2}{L_z^2}\right)$, the sum is over all N particles.

    (b) The frequency for the x-motion is $\omega_x = \frac{\partial H}{\partial I_x} = \frac{\pi^2}{mL_x^2}I_x = \frac{\pi p_x}{mL_x}$. Adiabatic condition here means the rate of fractional change of L_x is small compared to ω_x, that is $\left|\frac{\dot{L_x}}{L_x}\right| << \omega_x$. In other words, $\left|\dot{L_x}\right| << \frac{p_x}{m}$ or the speed of the wall is slow compared to typical speed of the particle.

    (c) Form adiabatic invariance, $I \sim < pL >$ is invariant. Now in thermodynamic limit, the typical momentum satisfies: $< p > \sim \sqrt{T}$, and $L \sim V^{1/3}$. So $T^{1/2}V^{1/3}$ is invariant, or $T_1 V_1^{2/3} = T_0 V_0^{2/3}$. This agrees with the result $\gamma = 5/3$ from thermodynamics.

20. Practice.

21. Practice.

22. We need $\{Q_i, P_j\} = \delta_{ij}$, which translates to $\frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} = \delta_{ij}$. So $\frac{\partial P_i}{\partial p_k} = \frac{\partial q_k}{\partial Q_i}$ and $P_i = p_k\frac{\partial q_k}{\partial Q_i} + g_i(\mathbf{q},t)$ where g_i are functions subject to a constraint $\{P_i, P_j\} = 0$. The constraint, using Chain rule, implies: $\frac{\partial g_i}{\partial Q_j} = \frac{\partial g_j}{\partial Q_i}$, thus $g_i = \frac{\partial f}{\partial Q_i}$ is a gradient. In conclusion, $P_i = p_k\frac{\partial q_k}{\partial Q_i} + \frac{\partial}{\partial Q_i}f(\mathbf{q},t)$.

23. If $Q = p$ and $P = -q$, then $\frac{\partial H}{\partial Q} = \frac{\partial H}{\partial p} = \dot{q} = -\dot{P}$ and $\frac{\partial H}{\partial P} = -\frac{\partial H}{\partial q} = -\dot{p} = \dot{Q}$. Thus this transformation is canonical.

24. Practice.

25. (a) The Lagrangian is $L = \frac{m}{2}\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}} \cdot \mathbf{A} - q\Psi$ and the momentum is $\mathbf{P} = m\dot{\mathbf{x}} + q\mathbf{A}$. The Hamiltonian is then $H = \frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 + q\Psi$.

    (b) Practice.