1. (a) The frequency response is given by

$$
\begin{aligned}
h(n) &= \sum_{k=-M}^{M} \delta(n-k) \\
H(\omega)e^{i\omega n} &= \sum_{k=-M}^{M} e^{i\omega(n-k)} \\
&= e^{i\omega n} \sum_{k=-M}^{M} e^{-i\omega k} \\
H(\omega) &= \sum_{k=-M}^{M} e^{-i\omega k} \\
&= e^{-i\omega M} + e^{-i\omega(M-1)} + \ldots + e^{i\omega(M-1)} + e^{i\omega M} \\
&= e^{-i\omega M} \frac{1 - e^{i\omega(2M+1)}}{1 - e^{i\omega}} \\
&= \frac{e^{-i\omega M} - e^{i\omega(M+1)}}{1 - e^{i\omega}} \\
&= \frac{e^{-i\omega(M+\frac{1}{2})} - e^{i\omega(M+\frac{1}{2})}}{e^{-i\frac{\omega}{2}} - e^{i\frac{\omega}{2}}} \\
&= \frac{-2i\sin(\omega(M+\frac{1}{2}))}{-2i\sin(\frac{\omega}{2})} \\
&= \frac{\sin(\omega(M+\frac{1}{2}))}{\sin(\frac{\omega}{2})}
\end{aligned}
$$

(b) The plot of $X(\omega)$ is shown in Fig. 1.

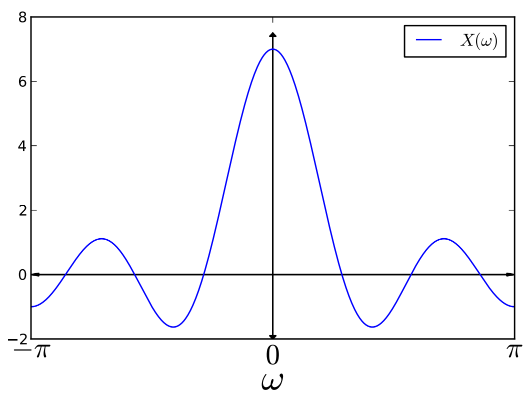


Figure 1: Plot of $X(\omega)$ for Q.1

2. Note that $G(\omega) = \sum_{k=-\infty}^{\infty} g(k)e^{-i\omega k}$.

(a) $h(n) = \begin{cases} g(\frac{n}{N}) & n \ \ mod \ \ N = 0 \\ 0 & \text{else.} \end{cases}$.

$$
\begin{aligned}
H(\omega) &= \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} \\
&= \sum_{r=-\infty}^{\infty} h(rN)e^{-i\omega rN} \\
&= \sum_{r=-\infty}^{\infty} g(r)e^{-i\omega rN} \\
&= \sum_{r=-\infty}^{\infty} g(r)e^{-i(\omega N)r} \\
&= G(\omega N)
\end{aligned}
$$

(b) $w(n) = g(n)e^{i\alpha n}$.

$$
\begin{aligned}
W(\omega) &= \sum_{k=-\infty}^{\infty} w(k)e^{-i\omega k} \\
&= \sum_{k=-\infty}^{\infty} g(k)e^{i\alpha k}e^{-i\omega k} \\
&= \sum_{k=-\infty}^{\infty} g(k)e^{-i(\omega-\alpha)k} \\
&= G(\omega - \alpha)
\end{aligned}
$$

(c) $z(n) = g(n)\cos(\alpha n)$.

$$
\begin{aligned}
Z(\omega) &= \sum_{k=-\infty}^{\infty} z(k)e^{-i\omega k} \\
&= \sum_{k=-\infty}^{\infty} g(k)\cos(\alpha k)e^{-i\omega k} \\
&= \sum_{k=-\infty}^{\infty} g(k)\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}e^{-i\omega k} \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} g(k)e^{i\alpha k}e^{-i\omega k} + \frac{1}{2}\sum_{k=-\infty}^{\infty} g(k)e^{-i\alpha k}e^{-i\omega k} \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} g(k)e^{-i(\omega-\alpha)k} + \frac{1}{2}\sum_{k=-\infty}^{\infty} g(k)e^{-i(\omega+\alpha)k} \\
&= \frac{G(\omega+\alpha) + G(\omega-\alpha)}{2}
\end{aligned}
$$

3. (a)

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} \\
&= \sum_{n=-\infty}^{\infty} [\delta(n+1) + \delta(n) + \delta(n-1)]e^{-i\omega n} \\
&= e^{i\omega} + 1 + e^{-i\omega} \\
&= 1 + 2\cos(\omega)
\end{aligned}
$$

Since $\cos(\omega)$ is periodic with period $2\pi$, it is clear that $X(\omega)$ is periodic with period $2\pi$. The plot of $X(\omega)$ vs. $\omega$ is shown in Fig. 2.

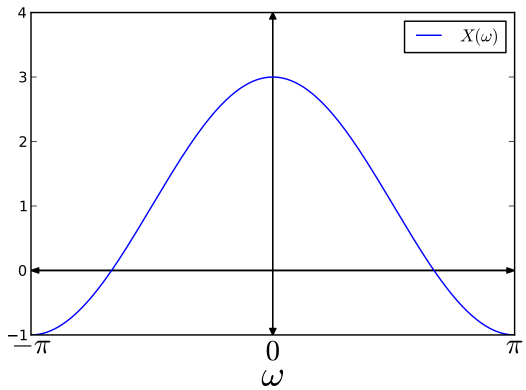


Figure 2: Plot of $X(\omega)$ for Q.3

(b) $H(\omega) = e^{-i\omega}$

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= e^{-i\omega}(1 + 2\cos(\omega))
\end{aligned}
$$

(c) As $Y(\omega) = e^{-i\omega}(1 + 2\cos(\omega)) = 1 + e^{-i\omega} + e^{-i\cdot 2\omega}$, we have $y(n) = \delta(n) + \delta(n-1) + \delta(n-2)$.

(d) From the sketch of $x(n)$ and $y(n)$ in Fig. 3, it looks like the system $K$ delays the input $x(n)$ by one time unit to get the output $y(n)$, i.e. $y(n) = x(n-1)$. This is easily verified by observing that the impulse response $h(n) = \delta(n-1)$ has exactly the specified frequency response $H(\omega) = e^{-i\omega}$.

4. (a) The impulse train has a spacing of $p$ samples in between each delta in the summation.

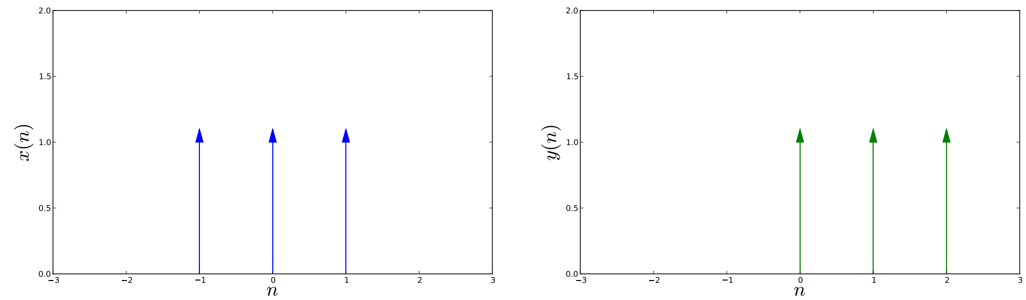


Figure 3: Sketch of $x(n)$ and $y(n)$ for Q.3

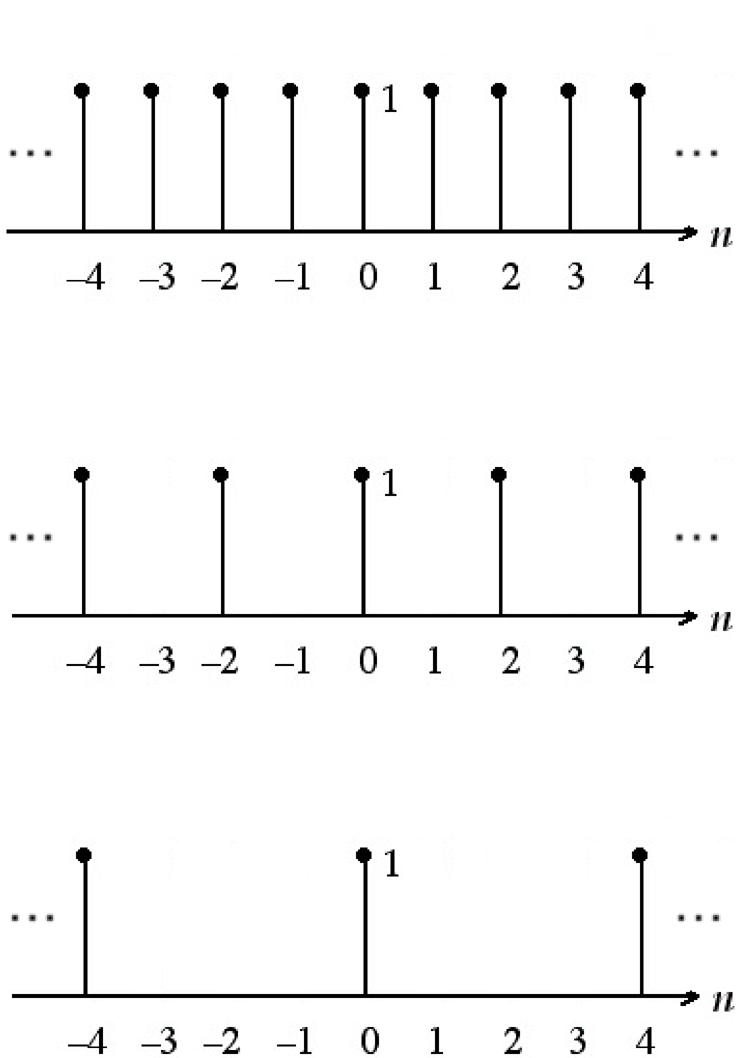


Figure 4: Plots of the impulse train for $p = 1, 2, 4$

(b)

$$
\begin{aligned}
x(n) \star y(n) = 1 \\
&= \sum_{m=-\infty}^{\infty} \delta(n - mp) \star y(n) \\
&= \sum_{m=-\infty}^{\infty} y(n - mp)
\end{aligned}
$$

Now since $y(n)$ is nonzero in $0 \leq n < p$, we can reason that $y(n - mp)$ is nonzero in $mp \leq < (m+1)p$. Now expanding the summation,

$$= ... + y(n + 2p) + y(n + 1p) + y(n) + y(n - p) + y(n - 2p) + ...$$

Where each term is non-zero for some interval of length $p$, and all other terms are 0 for that interval. Since $y(n)$ takes on a constant value of 1 when it is nonzero, this must mean that $x(n) \star y(n) = 1$.