

UNIVERSITY OF WATERLOO



PMATH 348
COMPLEX ANALYSIS

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1 Complex Numbers

Definition 1.0.1. A complex number is a vector in \mathbb{R}^2 . The complex plane denoted by \mathbb{C} is the set of complex numbers.

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In \mathbb{C} , we usually write,

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

with $x, y \in \mathbb{R}$. If $z = x + iy, x, y \in \mathbb{R}$, then x is called the real part of z and y the imaginary part of z and write

$$\Re(z) = x \quad \Im(z) = y$$

Definition 1.0.2. We define the sum of two complex numbers to be the vector sum.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \end{aligned}$$

We define the product of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative, associative and distributive over the sum. So,

$$\begin{aligned} (a + bi)(c + di) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Proposition 1 (Multiplicative Inverses). *Every complex number has a unique multiplicative inverse denoted by z^{-1} .*

Proof. Let $z = a + bi, a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. We want to solve for x and y such that $(a + bi)(x + iy) = 1$. In other words,

$$\begin{aligned} (ax - by) + i(ay + bx) &= 1 \\ \Rightarrow \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} &= (1, 0) \\ \Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= (1, 0) \end{aligned}$$

$$\begin{aligned}
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{a}{a^2 + b^2} \\ \frac{b}{a^2 + b^2} \end{pmatrix}
\end{aligned}$$

This is unique as the inverse matrix is unique. \square

Remark. The set of complex numbers is a field under the operations of addition and multiplication as operations are associative, commutative and distributive and every element has a unique inverse as before.

Definition 1.0.3. If $z = x + iy, x, y, \in \mathbb{R}$, then the conjugate of z is $\bar{z} = x - iy$.

Definition 1.0.4. We define the modulus (or length or magnitude) of $z = x + iy, x, y \in \mathbb{R}$ to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

Remark. For any $z, w \in \mathbb{C}$,

$$\begin{aligned}
\bar{\bar{z}} &= z \\
z + \bar{z} &= 2\Re(z) \\
z - \bar{z} &= 2\Im(z) \\
z \cdot \bar{z} &= |z|^2 \\
|z| &= |\bar{z}| \\
\overline{z + w} &= \bar{z} + \bar{w} \\
\overline{zw} &= \bar{z} \cdot \bar{w} \\
|zw| &= |z||w|
\end{aligned}$$

Proposition 2. *The following inequalities hold for any $z \in \mathbb{C}$.*

- (1) $|\Re(z)| \leq |z|$
- (2) $|\Im(z)| \leq |z|$
- (3) $|z + w| \leq |z| + |w|$
- (4) $|z + w| \geq \left| |z| - |w| \right|$

Proof. (1) and (2) follows as

$$|z|^2 = \Re(z)^2 + \Im(z)^2.$$

(3). Notice,

$$\begin{aligned}
|x + iy|^2 &= (x + iy)\overline{(x + iy)} \\
&= (x + iy)(\bar{x} + \bar{iy}) \\
&= x\bar{x} + y\bar{y} + x\bar{y} + y\bar{x}
\end{aligned}$$

$$\begin{aligned}
&= |x|^2 + |y|^2 + x\bar{y} + y\bar{x} \\
&= |x|^2 + |y|^2 + 2\Re(x\bar{y}) \\
&\leq |x|^2 + |y|^2 + 2|x\bar{y}| \\
&= |x|^2 + |y|^2 + 2|x| \cdot |\bar{y}| \\
&= |x + y|^2
\end{aligned}$$

Taking the square root of both sides gives the result.

(4). From (3), we have that

$$\begin{aligned}
|z| &= |z - w + w| \leq |z - w| + |w| \\
|w| &= |w - z + z| \leq |w - z| + |z|
\end{aligned}$$

Then, isolating $|z - w|$ implies the result. More specifically since we have the simultaneous inequality,

$$\begin{cases} |z| - |w| \leq |z - w| \\ |w| - |z| \leq |z - w| \end{cases} \Rightarrow |z - w| \geq \left| |z| - |w| \right|$$

as desired. □

Proposition 3. *Every non-zero complex number has exactly 2 square roots.*

Proof. Let $z = x + iy \in \mathbb{C}$ with $x^2 + y^2 \neq 0, x, y \in \mathbb{R}$. We want to solve $w^2 = z$ for $w \in \mathbb{C}$. Say w takes the form $w = u + iv, u, v \in \mathbb{R}$. Then

$$\begin{aligned}
w^2 &= z \\
\Rightarrow (u + iv)^2 &= x + iy \\
\Rightarrow (u^2 - v^2) + i2uv &= x + iy
\end{aligned}$$

So we have that $x = u^2 - v^2$ and $y = 2uv^2$. We can solve for u and v . Take the square of both sides of the second equation to get $4u^2v^2 = y^2$. Now, we multiply the first equation by $4u^2$ to get

$$\begin{aligned}
4u^4 - 4u^2v^2 &= 4xu^2 \\
\Rightarrow 4u^4 - 4xu^2 - y^2 &= 0
\end{aligned}$$

This is a quadratic equation over u^2 so,

$$u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Suppose that $y \neq 0$. Then we must take the positive solution above to get

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}$$

Under the assumption that $x^2 + y^2 > 0$, this solution exists. Notice we cannot take the negative solution as it yields a negative u^2 which is impossible. We can use a similar procedure to find that

$$v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}$$

Rooting u and v gives 2 solutions for each. However, if y is positive, since $2uv = y$, u and v must take the same sign. Similarly, if y is negative, they must take different signs. In each of these cases, there are 2 solutions for u and v . So,

$$w = \begin{cases} \pm \left[\left(\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \right) + i \left(\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}} \right) \right] & , y > 0 \\ \pm \left[\left(\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \right) - i \left(\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}} \right) \right] & , y < 0 \\ \pm \sqrt{x} & , x > 0, y = 0 \\ \pm i \sqrt{-x} & , x < 0, y = 0 \end{cases}$$

□

Remark. Let $z \in \mathbb{C}$. The notation \sqrt{z} may represent either one of the square roots of z or both of the square roots.

Remark. The square root doesn't distribute. Consider $z = w = -1 \in \mathbb{C}$. $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.

Remark. The Quadratic Formula holds true for complex polynomials. In other words, if $a, b, c \in \mathbb{C}, a \neq 0$,

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 1.0.5. If $z \in \mathbb{C} \setminus \{0\}$, we define the angle (or argument) of z to be the angle $\theta(z)$ from the positive x -axis counterclockwise to z . In other words, $\theta(z)$ is the angle such that

$$z = |z|(\cos \theta(z) + i \sin \theta(z)).$$

Remark. For $\theta \in \mathbb{R}$ (or for $\theta \in \mathbb{R}/2\pi$), we have that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Remark. If $z \neq 0$, we have $x = \Re(z)$, $y = \Im(z)$, $r = |z|$ and

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ \tan \theta &= \frac{y}{x}, \text{ if } x \neq 0 \\ z &= re^{i\theta} \\ \bar{z} &= re^{-i\theta} \\ z^{-1} &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

Remark. We now have 2 representations of a complex number $z \in \mathbb{C}$. We say that $z = x + iy$ is the cartesian coordinates of z and $z = re^{i\theta}$, where $r = |z|$, is the polar form of z .

Consider $z = re^{i\alpha}$ and $w = se^{i\beta}$. We have,

$$\begin{aligned} zw &= rs(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= rs((\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)) \\ &= rs(\cos(\alpha + \beta) + i \sin(\alpha + \beta)) \end{aligned}$$

$$= e^{i(\alpha+\beta)}$$

which defines a formula for multiplication in polar coordinates. Notice that the following identity known as De Moivre's Law follows.

$$(re^{i\theta})^n = r^n e^{in\theta}$$

for all $r, \theta \in \mathbb{R}$, $n \in \mathbb{Z}$. We can use this identity to find the n^{th} roots of z . In other words, we solve $w^n = z$. We have,

$$\begin{aligned} w^n &= z \\ \Rightarrow (se^{i\alpha})^n &= re^{i\theta} \\ \Rightarrow s^n e^{in\alpha} &= re^{i\theta} \end{aligned}$$

so $s^n = r$ and $n\alpha = \theta + 2\pi k$ for $k \in \mathbb{Z}$. In other words, we have

$$(re^{i\theta}) = \sqrt[n]{r} e^{i(\theta+2\pi k)/n}, \quad k = 0, \dots, n-1$$

Remark. When working with complex numbers, for $0 \neq z \in \mathbb{C}$, and for $0 < n \in \mathbb{Z}$, $\sqrt[n]{z}$ or $z^{1/n}$ denotes either one of the n roots, or the set of all n^{th} roots.

Example 1.0.6. Consider the $n-1$ diagonals of a regular n -gon inscribed in a circle of radius 1 obtained by connecting one vertex with all the others. Show that the product of these diagonals is n .

Notice that z_2, \dots, z_n are the n^{th} roots of unity other than 1. Let z be the variable and consider the polynomial

$$P(z) := 1 + z + \dots + z^{n-1}.$$

Since the roots of $P(z)$ are n^{th} roots of unity other than 1, we can factorize

$$\begin{aligned} P(z) &= 1 + z + \dots + z^{n-1} \\ &= (z - z_2) \dots (z - z_n) \end{aligned}$$

and setting $z = 1$, the result follows. In particular, we have

$$|1 - z_2| \dots |1 - z_n| = n.$$

2 Complex Functions

2.1 Limits

Definition 2.1.1. A sequence of complex numbers z_1, z_2, \dots converges to $z \in \mathbb{C}$ if

$$\lim_{n \rightarrow \infty} |z_n - z| = 0.$$

Equivalently, given any $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ sufficiently large such that $|z_n - z| < \epsilon$ whenever $n > N$.

Remark. If $\{z_n\}_n$ converges to z , we write

$$\lim_{n \rightarrow \infty} z_n = z$$

or $z_n \rightarrow z$ as $n \rightarrow \infty$.

Example 2.1.2. For $|z| > 1$, show that $\{\frac{1}{z^n}\}_{n=1}^\infty$ converges.

Notice,

$$\lim_{n \rightarrow \infty} \left| \frac{1}{z^n} - 0 \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{z^n} \right| = 0$$

as $|z| > 1$.

Example 2.1.3. Show that $\{i^n\}_{n=1}^\infty$ does not converge.

Definition 2.1.4. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$. We say

$$\lim_{z \rightarrow z_0} f(z) = L$$

if for every sequence $\{z_n\}_n \subseteq \Omega$ we have that $z_n \rightarrow z \Rightarrow f(z_n) \rightarrow L$.

Remark. Here, z_0 need not to be in Ω .

Example 2.1.5. Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_{z \rightarrow 0} f(z)$.

If $z = x \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{x}{x} = 1$. So $\lim_{x \rightarrow 0} f(x) = 1$. If $z = iy, y \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{-iy}{iy} = -1$. So $\lim_{y \rightarrow 0} f(iy) = -1$. Hence, the limit does not exist.

Example 2.1.6. Show that $z_n \rightarrow z$ if and only if $\Re z_n \rightarrow \Re z$ and $\Im z_n \rightarrow \Im z$.

2.2 Function Continuity

Definition 2.2.1. Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$. We say f is continuous at $z_0 \in \Omega$ if for every sequence $\{z_n\} \subseteq \Omega$, we have $z_n \rightarrow z_0 \Rightarrow f(z_n) \rightarrow f(z_0)$. Equivalently, given any $\epsilon > 0, \exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Remark. f is continuous on Ω if it is continuous at every point of Ω .

Remark. We may split f into its real and imaginary parts

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

2.3 Holomorphic Functions

Definition 2.3.1. An open disk of radius r at z_0 with $r > 0$ is the neighborhood around z_0 denoted by $D(z_0, r)$ with

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 2.3.2. Let $f(z)$ be defined in a neighborhood of z_0 . We say f is complex differentiable (or holomorphic) at z_0 if

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote the limit by $f'(z_0)$.

Remark. Here, $h \in \mathbb{C}$ can approach zero from any direction in \mathbb{C} .

Example 2.3.3. Where is $f(z) = \frac{1}{z}, z \neq 0$ holomorphic?

Notice,

$$\lim_{h \rightarrow 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h \rightarrow 0} \frac{-h}{(z_0+h)(z_0)} = -\frac{1}{z_0^2}$$

So, f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and $f'(z) = -\frac{1}{z^2}$.

Example 2.3.4. $f(z) = \bar{z}$ is not holomorphic at any $z \in \mathbb{C}$.

Notice,

$$\lim_{h \rightarrow 0} \frac{\overline{z_0+h} - \bar{z}_0}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}$$

which does not exist from Example 2.1.5. However, the function can be thought of a map from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as $(x, y) \mapsto (x, -y)$ which is differentiable as all partial derivatives exist. This distinguishes real analysis from complex analysis.

Remark. If f and g are holomorphic, so are $f + g$, fg and $\frac{f}{g}$ (when $g \neq 0$). The proof is identical to the statements of real functions.



We can now generalize when a function is complex differentiable. If the complex function $f(z) = u + iv$. If the complex derivative $f'(z)$ is to exist, then it must be that the limit exists approaching from both the real and imaginary axis. Thus, we have,

$$f'(z) = \lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{t} = \lim_{t \rightarrow 0} \frac{f(z+it) - f(z)}{it}$$

where t is a real number. In terms of u and v , taking the derivative along the real line gives,

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{t} \\ &= \lim_{t \rightarrow 0} \frac{u(x+t, y) + iv(x+t, y) - u(x, y) - iv(x, y)}{t} \\ &= \lim_{t \rightarrow 0} \frac{u(x+t, y) - u(x, y)}{t} + i \lim_{t \rightarrow 0} \frac{v(x+t, y) - v(x, y)}{t} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \end{aligned}$$

Taking the derivative along the vertical line gives,

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{f(z+it) - f(z)}{it} \\ &= -i \lim_{t \rightarrow 0} \frac{u(x, y+t) + iv(x, y+t) - u(x, y) - iv(x, y)}{t} \\ &= -i \lim_{t \rightarrow 0} \frac{u(x, y+t) - u(x, y)}{t} + \lim_{t \rightarrow 0} \frac{v(x, y+t) - v(x, y)}{t} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

Equating real and imaginary parts, we arrive at the following theorem.

Theorem 4 (Cauchy-Riemann Equations). *If a function $f(z) = u + iv$ is holomorphic in a neighborhood around $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{at } (x_0, y_0)$$

with

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Example 2.3.5. Show that

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & , \text{ if } z \neq 0 \\ 0 & , \text{ if } z = 0 \end{cases}$$

is not holomorphic at $z = 0$ and that the Cauchy-Reimann Equations hold at $z = 0$.

Notice,

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}^2}{h^2} = \lim_{h \rightarrow 0} \left(\frac{\bar{x} - iy}{x + iy} \right)^2$$

Let $h = x + imx$, $m \neq 0, x \rightarrow 0$. We get

$$\lim_{x \rightarrow 0} \left(\frac{x - imx}{x + imx} \right)^2 \left(\frac{1 - im}{1 + im} \right)^2$$

which is dependent of m and thus the limit does not exist. Now, notice we have

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \frac{-3x^2y + y^3}{x^2 + y^2}$$

So we have

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & , \text{ if } (x, y) \neq (0, 0) \\ 0 & , \text{ if } (x, y) = (0, 0) \end{cases}$$

and

$$v(x, y) = \begin{cases} \frac{-3x^2y + y^3}{x^2 + y^2} & , \text{ if } (x, y) \neq (0, 0) \\ 0 & , \text{ if } (x, y) = (0, 0) \end{cases}.$$

Now, we can verify that the Cauchy-Riemann Equations hold. Indeed,

$$\begin{aligned} \frac{\partial u}{\partial x} \left(\frac{x^3 - 3xy^2}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial x} (x^3 - 3xy^2) \right) (x^2 + y^2) - (x^3 - 3xy^2) \left(\frac{\partial}{\partial x} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= \frac{(3x^2 - 3y^2)(x^2 + y^2) - (x^3 - 3xy^2)(2x)}{(x^2 + y^2)^2} \\ &= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} \\ \frac{\partial v}{\partial y} \left(\frac{y^3 - 3x^2y}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial y} (y^3 - 3x^2y) \right) (x^2 + y^2) - (y^3 - 3x^2y) \left(\frac{\partial}{\partial y} (x^2 + y^2) \right)}{(x^2 + y^2)^2} \\ &= \frac{(3y^2 - 3x^2)(x^2 + y^2) - (y^3 - 3x^2y)(y^2)}{(x^2 + y^2)^2} \\ &= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial y} \left(\frac{x^3 - 3xy^2}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial y}(x^3 - 3xy^2) \right)(x^2 + y^2) - (x^3 - 3xy^2) \left(\frac{\partial}{\partial y}(x^2 + y^2) \right)}{(x^2 + y^2)^2} \\
&= \frac{(-6xy)(x^2 + y^2) - (x^3 - 3xy^2)(2y)}{(x^2 + y^2)^2} \\
&= -\frac{8x^3y}{x^2 + y^2} \\
\frac{\partial v}{\partial x} \left(\frac{y^3 - 3x^2y}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial x}(y^3 - 3x^2y) \right)(x^2 + y^2) - (y^3 - 3x^2y) \left(\frac{\partial}{\partial x}(x^2 + y^2) \right)}{(x^2 + y^2)^2} \\
&= \frac{(-6xy)(x^2 + y^2) - (y^3 - 3x^2y)(2y)}{(x^2 + y^2)^2} \\
&= -\frac{8x^3y}{x^2 + y^2}
\end{aligned}$$

Thus, we can consider the converse statement of Theorem 4.

Theorem 5. Let $f = u + iv : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$, $z_0 = x_0 + iy_0 \in \Omega$. If

- (1) the partials of u, v exist in a neighborhood of (x_0, y_0)
- (2) the partials of u, v are continuous at (x_0, y_0)
- (3) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0)

then, f is holomorphic at z_0 .

TODO: find proof online.

Example 2.3.6. Consider the power series, an expression of the form

$$\sum_{n=0}^{\infty} c_n z^n$$

where $c_n \in \mathbb{C}$. This expression converges if the sequence of partials sums, $\{s_N\}$ defined by

$$s_N := \sum_{n=0}^N c_n z^n$$

converges as $N \rightarrow \infty$. This is quite a strong condition, so we consider the following definition.

Definition 2.3.7. A power series expression converges absolutely if

$$\sum_{n=0}^{\infty} |c_n| |z|^n$$

converges.

Remark. Absolute convergence implies converges. Notice,

$$\left| \sum_{n=0}^N c_n z^n \right| \leq \sum_{n=0}^N |c_n| |z|^n$$

for each $N \in \mathbb{N}$.

Theorem 6. For any power series $\sum_{n=0}^{\infty} c_n z^n$, $\exists 0 \leq R \leq \infty$, such that

(1) If $|z| < R$, the series converges absolutely

(2) If $|z| > R$, the series diverges.

Moreover, R is given by Hadamard's formula: $\frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$

Remark. R is called the radius of convergence of the series and $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.

Remark. Recall,

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \left(\sup_{m \leq n} a_m \right)$$

and is the “highest peak reached by a_n 's as $n \rightarrow \infty$ ”.

Proposition 7 (Property of \limsup). If $L = \limsup_{n \rightarrow \infty} a_n$, then for any $\epsilon > 0$, $\exists N > 0$ such that $\forall n \geq N, a_n < L + \epsilon$

Proof of Theorem 6. Let $L := \frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$. Clearly, $L \geq 0$.

(1) Suppose $|z| < R$. So, there exists some $\epsilon > 0$ such that $r := |z|(L + \epsilon) < 1$ and $0 < r < 1$. By Proposition 7, $\exists N \in \mathbb{N}$ such that $\forall n > N, |c_n|^{\frac{1}{n}} < L + \epsilon$. Now,

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} \left(|c_n|^{\frac{1}{n}} |z| \right)^n < \sum_{n=N}^{\infty} r^n$$

converges as $0 < r < 1$. By the comparison test, $\sum_{n=N}^{\infty} |c_n| |z|^n$ is monotonic and bounded and thus converges by Bolzano-Weierstrass.

(2) This follows from the proof above. Specifically, this time, notice that there exists some $\epsilon > 0$ such that $r := |z|(L - \epsilon) > 1$. Again, by Proposition 7, there exists some $N \in \mathbb{N}$ such that for all $n > N, |c_n|^{\frac{1}{n}} > L - \epsilon$ so that

$$\sum_{n=N}^{\infty} |c_n| |z|^n = \sum_{n=N}^{\infty} \left(|c_n|^{\frac{1}{n}} |z| \right)^n > \sum_{n=N}^{\infty} r^n$$

follows and so the series diverges.

□

Theorem 8. Suppose $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has radius of convergence R . Then $f'(z)$ exists and equals

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$. Moreover, f' has the same radius of convergence as f .

Proof. f' has some radius of convergence because

$$\limsup_{n \rightarrow \infty} |nc_n|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} n^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$$

since $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$. Let $|z_0| \leq r < R$, $g(z_0) := \sum_{n=1}^{\infty} nc_n z_0^{n-1}$. We want to show

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = g$$

or equivalently, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \epsilon.$$

For any fixed $\epsilon > 0$, we write

$$f(z) = \underbrace{\sum_{n=0}^N c_n z^n}_{:=S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{:=E_N(z)}$$

We have $S'_N = \sum_{n=1}^N nc_n z^{n-1}$ and

$$\begin{aligned} & \left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| \\ &= \left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} + \frac{E_N(z_0 + h) - E_N(z_0)}{h} - g(z_0) + S'_N(z_0) - S'_N(z_0) \right| \\ &= \left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| + \left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| + |S'_N(z_0) - g(z_0)|. \end{aligned}$$

We have

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| = \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n ((z_0 + h)^n - z_0^n) \right|$$

As $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$ in any ring, we have

$$\begin{aligned} & \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n ((z_0 + h)^n - z_0^n) \right| \\ &= \left| \sum_{n=N+1}^{\infty} c_n ((z_0 + h)^{n-1} + (z_0 + h)^{n+2}z_0 + \dots + (z_0 + h)z_0^{n-2} + z_0^{n-1}) \right| \end{aligned}$$

Now, by choosing δ relatively small so that $|z_0| \leq r$, we have $|z_0|, |z_0 + h| \leq r$ and so

$$(z_0 + h)^{n-1} + (z_0 + h)^{n+2}z_0 + \dots + (z_0 + h)z_0^{n-2} + z_0^{n-1} \leq nr^{n-1}$$

So,

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| \leq \sum_{n=N+1}^{\infty} nc_n r^{n-1} < \frac{\epsilon}{3}.$$

for a large enough N_1 .

Now, observe that by definition,

$$S'_N(z_0) = \sum_{n=1}^N n c_n z^{n-1}.$$

Since,

$$\lim_{N \rightarrow \infty} S'_N(z_0) = \lim_{N \rightarrow \infty} \sum_{n=1}^N n c_n z^{n-1} = \sum_{n=1}^{\infty} n c_n z^{n-1} = g(z_0)$$

we can pick some $\frac{\epsilon}{3} > 0$ such that there exists some $N_2 \in \mathbb{N}$, for all $n > N_2$, we have

$$|S'_N(z_0) - g(z_0)| < \frac{\epsilon}{3}.$$

Finally, let $\frac{\epsilon}{3} > 0$. Observe that there exists some $\delta > 0$ such that there exists some $N > \max N_1, N_2$, for all $n > N$,

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \frac{\epsilon}{3}.$$

as $|h| < \delta$. It follows that

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \epsilon$$

as desired. \square

Example 2.3.8. Consider $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula,

$$\begin{aligned} \frac{1}{R} &= \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \right)^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} n^{\frac{1}{n}} \\ &= 1 \end{aligned}$$

Thus, $R = 1$. By Theorem 6, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, in other words, when $|z| = 1$, consider the following two cases:

1. If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series, and hence f diverges.

2. If $z = i$, then

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots \right) i \left(1 - \frac{1}{3} + \frac{1}{5} - \dots \right) \end{aligned}$$

which both real and imaginary parts converge by the alternating series test. Therefore, we observe that both convergence and divergence may occur on the boundary, depending on the value of z .

Remark. The positions of \lim and $\sum_{n=0}^{\infty}$ cannot be exchanged when we consider infinite sums. Consider the following example.

Example 2.3.9. Consider for $|x| > 1$, $\sum_{n=1}^{\infty} \lim_{x \rightarrow 1} (x^n - x^{n+1}) = \sum_{n=1}^{\infty} (1 - 1) = 0$. Now,

$$\begin{aligned} & \lim_{x \rightarrow 1} \lim_{N \rightarrow \infty} \sum_{n=1}^N (x^n - x^{n+1}) \\ &= \lim_{x \rightarrow 1} \lim_{N \rightarrow \infty} (x - x^2 + x^2 - x^3 + \cdots + x^{N-1} - x^N + x^N - x^{N+1}) \\ &= \lim_{x \rightarrow 1} \lim_{N \rightarrow \infty} (x - x^{N+1}) \\ &= \lim_{x \rightarrow 1} x \\ &= 1 \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} \lim_{x \rightarrow 1} (x^n - x^{n+1}) \neq \lim_{x \rightarrow 1} \sum_{n=1}^{\infty} (x^n - x^{n+1}).$$

Definition 2.3.10. A function f is said to be entire if f is holomorphic in the entire complex plane.

Example 2.3.11. Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that the radius of convergence of this series is ∞ (which implies e^z is entire) and $(e^z)' = e^z$.

Consider Stirling's formula, which says as $n \rightarrow \infty$, $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Then, we have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{n}\right)^n.$$

Now,

$$\begin{aligned} \frac{1}{R} &= \limsup_{n \rightarrow \infty} \left| \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right) \right|^{\frac{1}{n}} \\ &= \limsup_{n \rightarrow \infty} \left| \frac{1}{2\pi n} \right|^{\frac{1}{n}} \limsup_{n \rightarrow \infty} \left| \frac{e}{n} \right|^{\frac{1}{n}} \\ &= 0 \end{aligned}$$

Thus, $R \rightarrow \infty$ as $n \rightarrow \infty$. By Theorem 6, e^z is an entire function. Now, we can show the derivative by the limit definition. Notice,

$$\lim_{h \rightarrow 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\
&= \lim_{h \rightarrow 0} \frac{e^h}{h} \lim_{h \rightarrow 0} \frac{1}{h}
\end{aligned}$$

Corollary 9 (Corollary of Theorem 8). *A power series is infinitely complex-differentiable in its radius of convergence. All its derivatives are also power series, obtained by termwise differentiation.*

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$f^{(N)}(z) = \sum_{n=N}^{\infty} n(n-1) \cdots (n-N) c_n z^{n-N}$$

for some $N \in \mathbb{N}$.

In general, we have $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which is the power series centered at $z \in \mathbb{C}$. Then as before, the radius of convergence R is given by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}.$$

But the disc of convergence is now centered around z_0 . We have shown that $f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in some neighborhood of z_0) with radius of convergence $R > 0$. This implies that $f(z)$ is holomorphic at z_0 . In fact, the converse is true; any function holomorphic at z_0 is infinitely holomorphic at z_0 . However, for this, we need the concept of integration over paths of curves.

Definition 2.3.12. A curve in \mathbb{C} is a continuous function $\gamma(t) : [a, b] \rightarrow \mathbb{C}$ with $a, b \in \mathbb{R}$. The image of γ in \mathbb{C} is called γ^* .

Example 2.3.13. Let $z_0 \in \mathbb{C}$, $r > 0$. Take $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ defined by $t \mapsto z_0 + re^{it}$. This is a circle of radius r centered at z_0 , oriented counterclockwise.

Example 2.3.14. Consider $\hat{\gamma} : [0, 1] \rightarrow \mathbb{C}$ defined by $t \mapsto z_0 + re^{2\pi it}$. This is identical to the curve γ defined above with the same oriented path and shows that curves have different parameterizations.

Definition 2.3.15. We say γ is smooth on the interval $[a, b]$ if γ' exists, is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for any $t \in [a, b]$.

Definition 2.3.16. $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise-smooth if it is smooth on $[a, b]$ except at finitely many points in $[a, b]$.

Remark. Piecewise smooth curves are called paths.

Definition 2.3.17. Given a path $\gamma : [a, b] \rightarrow \mathbb{C}$ and $f(z)$ is a continuous function on γ , the integral of f along γ is defined by

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Remark. If g is complex valued, then

$$\int_a^b g(t) dt = \int_a^b \Re(g(t)) dt + i \int_a^b \Im(g(t)) dt.$$

Remark. The integral $\int_\gamma f(z) dz$ can be shown to be independent of the parameterization chosen for γ .

Theorem 10.

(1) *Integration is linear.*