1. Determine if the following polynomials are irreducible in $\mathbb{Q}[x]$.

   (a) $f(x) = 4x^5 + 28x^4 + 7x^3 - 28x^2 + 14$

   Solution: We can apply Eisenstein's Criterion directly with $p = 7$ to show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (b) $g(x) = x^p + p^2mx + (p-1)$ where $p \in \mathbb{Z}$ is a prime and $m \in \mathbb{Z}$.

   Solution: Consider $g(x+1)$. Notice,

$$\begin{aligned} g(x+1) &= (x+1)^p + p^2m(x+1) + (p-1) \\ &= x^p + \binom{p}{1}x^{p-1} + \cdots + \binom{p}{p-1}x + p^2mx + p^2m + p \\ &= x^p + \binom{p}{1}x^{p-1} + \cdots + \binom{p}{p-1}x + p^2mx + p(pm+1). \end{aligned}$$

   We can see immediately that each non-leading coefficient divides p. It remains to show that $p^2 \mid p(pm+1)$, which is equivalent to $p \mid pm+1$. This follows as

$$pm + 1 \equiv 1 \pmod{p}.$$

   Hence, Eisenstein's Criterion with p gives $g(x+1)$ is irreducible in $\mathbb{Q}[x]$, so $g(x)$ is irreducible in $\mathbb{Q}[x]$.

   (c) $h(x) = x^4 + 4x^3 + 4x^2 + 4x + 5$

   Solution: We can simplify h with the Binomial Theorem. Notice,

$$\begin{aligned} h(x) &= x^4 + 4x^3 + 4x^2 + 4x + 5 \\ &= (x+1)^4 - 2x + 4. \end{aligned}$$

   Now, consider $h(x-1)$, we have

$$\begin{aligned} h(x-1) &= x^4 - 2(x-1)^2 + 4 \\ &= x^4 - 2(x^2 - 2x + 1) + 4 \\ &= x^4 - 2x^2 + 4x + 2 \end{aligned}$$

   and here, $h(x-1)$ satisfies Eisenstein's Criterion with $p = 2$. It follows that $h(x)$ is irreducible in $\mathbb{Q}[x]$.

2. Let $F \subseteq K \subseteq E$ be fields. If ${}^{E}\!/_{K}$ and ${}^{K}\!/_{F}$ are algebraic, prove that ${}^{E}\!/_{F}$ is also algebraic.

Solution:

Proof. This result follows immediately as

$$[E : F] = [E : K][K : F]$$

and ${}^{E}\!/_{K}$ and ${}^{K}\!/_{F}$ are finite extensions, so ${}^{E}\!/_{F}$ is also a finite extension. □

3. Let F be a field. Let α, β be algebraic over F with the minimal polynomial $f(x)$ and $g(x)$ respectively. Prove that $f(x)$ is irreducible over $F(\beta)[x]$ if and only if $g(x)$ is irreducible over $F(\alpha)[x]$.

Solution: Since both directions follow a symmetric proof, it suffices to show one direction, so suppose $f(x)$ is irreducible over $F(\beta)[x]$. We have that $[F(\alpha,\beta) : F(\beta)] = \deg(f)$. Also, since $g(x)$ is the minimal polynomial of β over $F[x]$, we have that $[F(\beta) : F] = \deg(g)$. It follows that $[F(\alpha,\beta) : F] =$

4. (a) Prove that $\alpha = \sqrt[3]{7} + 2i$ is algebraic over $\mathbb{Q}$.

Solution:

(b) Prove that both $\sqrt[3]{7}$ and $2i$ are elements of $\mathbb{Q}(\alpha)$.

Solution:

(c) Compute $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Solution:

(d) Write down the minimal polynomial in $\mathbb{Q}[x]$ for α.

Solution:

5. Let ${}^{E}\!/_{F}$ be a field extension and K, L be intermediate fields. Let KL denote the smallest subfield of E containing both K and L. Suppose ${}^{L}\!/_{F}$ is finite.

(a) Prove that all elements of KL are of the form $\sum_{i=1}^{r} k_i l_i$, where $k_i \in K, l_i \in L$ and $r \in \mathbb{N}$.

Solution:

(b) Prove that $[KL : K] \leq [L : F]$.

Solution:

(c) Give and example of fields $F \subseteq K, L \subseteq E$ which satisfy $[KL : K] < [L : F]$.

Solution:

6. Prove that e is transcendental.

Solution: Consider the function $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with $a_0, a_n \neq 0$ and $a_i \in \mathbb{Z}$ for all $i = 0, \ldots, n$. Now, we assume for a contraction that $f(e) = 0$.