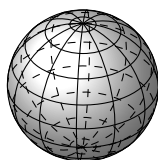


UNIVERSITY OF WATERLOO



# PMATH 348

## COMPLEX ANALYSIS

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# 1 Complex Numbers

**Definition 1.0.1.** A complex number is a vector in  $\mathbb{R}^2$ . The complex plane denoted by  $\mathbb{C}$  is the set of complex numbers.

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In  $\mathbb{C}$ , we usually write,

$$\begin{aligned} 0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x &= \begin{pmatrix} x \\ 0 \end{pmatrix} \\ iy &= \begin{pmatrix} 0 \\ y \end{pmatrix} \end{aligned}$$

with  $x, y \in \mathbb{R}$ . If  $z = x + iy, x, y \in \mathbb{R}$ , then  $x$  is called the real part of  $z$  and  $y$  the imaginary part of  $z$  and write

$$\Re(z) = x \quad \Im(z) = y$$

**Definition 1.0.2.** We define the sum of two complex numbers to be the vector sum.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \end{aligned}$$

We define the product of two complex numbers by setting  $i^2 = -1$  and by requiring the product to be commutative, associative and distributive over the sum. So,

$$\begin{aligned} (a + bi)(c + di) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

**Proposition 1** (Multiplicative Inverses). *Every complex number has a unique multiplicative inverse denoted by  $z^{-1}$ .*

*Proof.* Let  $z = a + bi, a, b \in \mathbb{R}$  with  $a^2 + b^2 \neq 0$ . We want to solve for  $x$  and  $y$  such that  $(a + bi)(x + iy) = 1$ . In other words,

$$\begin{aligned} (ax - by) + i(ay + bx) &= 1 \\ \Rightarrow \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} &= (1, 0) \\ \Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= (1, 0) \end{aligned}$$

$$\begin{aligned}
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{a}{a^2 + b^2} \\ \frac{b}{a^2 + b^2} \end{pmatrix}
\end{aligned}$$

This is unique as the inverse matrix is unique.  $\square$

*Remark.* The set of complex numbers is a field under the operations of addition and multiplication as operations are associative, commutative and distributive and every element has a unique inverse as before.

**Definition 1.0.3.** If  $z = x + iy, x, y, \in \mathbb{R}$ , then the conjugate of  $z$  is  $\bar{z} = x - iy$ .

**Definition 1.0.4.** We define the modulus (or length or magnitude) of  $z = x + iy, x, y \in \mathbb{R}$  to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

*Remark.* For any  $z, w \in \mathbb{C}$ ,

$$\begin{aligned}
\bar{\bar{z}} &= z \\
z + \bar{z} &= 2\Re(z) \\
z - \bar{z} &= 2\Im(z) \\
z \cdot \bar{z} &= |z|^2 \\
|z| &= |\bar{z}| \\
\overline{z + w} &= \bar{z} + \bar{w} \\
\overline{zw} &= \bar{z} \cdot \bar{w} \\
|zw| &= |z||w|
\end{aligned}$$

**Proposition 2.** *The following inequalities hold for any  $z \in \mathbb{C}$ .*

1.  $|\Re(z)| \leq |z|$
2.  $|\Im(z)| \leq |z|$
3.  $|z + w| \leq |z| + |w|$
4.  $|z + w| \geq \left| |z| - |w| \right|$

*Proof.* (1) and (2) follows as

$$|z|^2 = \Re(z)^2 + \Im(z)^2.$$

(3). Notice,

$$\begin{aligned}
|x + iy|^2 &= (x + iy)\overline{(x + iy)} \\
&= (x + iy)(\bar{x} + \bar{iy}) \\
&= x\bar{x} + y\bar{y} + x\bar{y} + y\bar{x}
\end{aligned}$$

$$\begin{aligned}
&= |x|^2 + |y|^2 + x\bar{y} + y\bar{x} \\
&= |x|^2 + |y|^2 + 2\Re(x\bar{y}) \\
&\leq |x|^2 + |y|^2 + 2|x\bar{y}| \\
&= |x|^2 + |y|^2 + 2|x| \cdot |\bar{y}| \\
&= |x + y|^2
\end{aligned}$$

Taking the square root of both sides gives the result.

(4). From (3), we have that

$$\begin{aligned}
|z| &= |z - w + w| \leq |z - w| + |w| \\
|w| &= |w - z + z| \leq |w - z| + |z|
\end{aligned}$$

Then, isolating  $|z - w|$  implies the result. More specifically since we have the simultaneous inequality,

$$\begin{cases} |z| - |w| \leq |z - w| \\ |w| - |z| \leq |z - w| \end{cases} \Rightarrow |z - w| \geq \left| |z| - |w| \right|$$

as desired. □

**Proposition 3.** *Every non-zero complex number has exactly 2 square roots.*

*Proof.* Let  $z = x + iy \in \mathbb{C}$  with  $x^2 + y^2 \neq 0, x, y \in \mathbb{R}$ . We want to solve  $w^2 = z$  for  $w \in \mathbb{C}$ . Say  $w$  takes the form  $w = u + iv, u, v \in \mathbb{R}$ . Then

$$\begin{aligned}
w^2 &= z \\
\Rightarrow (u + iv)^2 &= x + iy \\
\Rightarrow (u^2 - v^2) + i2uv &= x + iy
\end{aligned}$$

So we have that  $x = u^2 - v^2$  and  $y = 2uv^2$ . We can solve for  $u$  and  $v$ . Take the square of both sides of the second equation to get  $4u^2v^2 = y^2$ . Now, we multiply the first equation by  $4u^2$  to get

$$\begin{aligned}
4u^4 - 4u^2v^2 &= 4xu^2 \\
\Rightarrow 4u^4 - 4xu^2 - y^2 &= 0
\end{aligned}$$

This is a quadratic equation over  $u^2$  so,

$$u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}$$

Suppose that  $y \neq 0$ . Then we must take the positive solution above to get

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}$$

Under the assumption that  $x^2 + y^2 > 0$ , this solution exists. Notice we cannot take the negative solution as it yields a negative  $u^2$  which is impossible. We can use a similar procedure to find that

$$v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}$$

Rooting  $u$  and  $v$  gives 2 solutions for each. However, if  $y$  is positive, since  $2uv = y$ ,  $u$  and  $v$  must take the same sign. Similarly, if  $y$  is negative, they must take different signs. In each of these cases, there are 2 solutions for  $u$  and  $v$ . So,

$$w = \begin{cases} \pm \left[ \left( \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \right) + i \left( \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}} \right) \right] & , y > 0 \\ \pm \left[ \left( \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \right) - i \left( \sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}} \right) \right] & , y < 0 \\ \pm \sqrt{x} & , x > 0, y = 0 \\ \pm i \sqrt{-x} & , x < 0, y = 0 \end{cases}$$

□

*Remark.* Let  $z \in \mathbb{C}$ . The notation  $\sqrt{z}$  may represent either one of the square roots of  $z$  or both of the square roots.

*Remark.* The square root doesn't distribute. Consider  $z = w = -1 \in \mathbb{C}$ .  $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$ .

*Remark.* The Quadratic Formula holds true for complex polynomials. In other words, if  $a, b, c \in \mathbb{C}, a \neq 0$ ,

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Definition 1.0.5.** If  $z \in \mathbb{C} \setminus \{0\}$ , we define the angle (or argument) of  $z$  to be the angle  $\theta(z)$  from the positive  $x$ -axis counterclockwise to  $z$ . In other words,  $\theta(z)$  is the angle such that

$$z = |z|(\cos \theta(z) + i \sin \theta(z)).$$

*Remark.* For  $\theta \in \mathbb{R}$  (or for  $\theta \in \mathbb{R}/2\pi$ ), we have that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

*Remark.* If  $z \neq 0$ , we have  $x = \Re(z)$ ,  $y = \Im(z)$ ,  $r = |z|$  and

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ \tan \theta &= \frac{y}{x}, \text{ if } x \neq 0 \\ z &= re^{i\theta} \\ \bar{z} &= re^{-i\theta} \\ z^{-1} &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

*Remark.* We now have 2 representations of a complex number  $z \in \mathbb{C}$ . We say that  $z = x + iy$  is the cartesian coordinates of  $z$  and  $z = re^{i\theta}$ , where  $r = |z|$ , is the polar form of  $z$ .

Consider  $z = re^{i\alpha}$  and  $w = se^{i\beta}$ . We have,

$$\begin{aligned} zw &= rs(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= rs((\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)) \\ &= rs(\cos(\alpha + \beta) + i \sin(\alpha + \beta)) \end{aligned}$$

$$= e^{i(\alpha+\beta)}$$

which defines a formula for multiplication in polar coordinates. Notice that the following identity known as De Moivre's Law follows.

$$(re^{i\theta})^n = r^n e^{in\theta}$$

for all  $r, \theta \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ . We can use this identity to find the  $n^{\text{th}}$  roots of  $z$ . In other words, we solve  $w^n = z$ . We have,

$$\begin{aligned} w^n &= z \\ \Rightarrow (se^{i\alpha})^n &= re^{i\theta} \\ \Rightarrow s^n e^{in\alpha} &= re^{i\theta} \end{aligned}$$

so  $s^n = r$  and  $n\alpha = \theta + 2\pi k$  for  $k \in \mathbb{Z}$ . In other words, we have

$$(re^{i\theta}) = \sqrt[n]{r} e^{i(\theta+2\pi k)/n}, \quad k = 0, \dots, n-1$$

*Remark.* When working with complex numbers, for  $0 \neq z \in \mathbb{C}$ , and for  $0 < n \in \mathbb{Z}$ ,  $\sqrt[n]{z}$  or  $z^{1/n}$  denotes either one of the  $n$  roots, or the set of all  $n^{\text{th}}$  roots.