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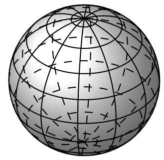

PMATH 352

Complex Analysis

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Contents

1 Complex Numbers

Definition 1.0.1. A complex number is a vector in $\mathbb{R}^2$. The complex plane denoted by $\mathbb{C}$ is the set of complex numbers.

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write,

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$x = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$iy = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

with $x, y \in \mathbb{R}$. If $z = x + iy, x, y \in \mathbb{R}$, then x is called the real part of z and y the imaginary part of z and write

$$\Re(z) = x \qquad \Im(z) = y$$

Definition 1.0.2. We define the sum of two complex numbers to be the vector sum.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \end{aligned}$$

We define the product of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative, associative and distributive over the sum. So,

$$\begin{aligned} (a + bi)(c + di) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + (ad + bc) \end{aligned}$$

Proposition 1 (Mulitplicative Inverses)**.** *Every complex number has a unique multiplicative inverse denoted by* z^{-1}.

Proof. Let $z = a+i, a, b \in \mathbb{R}$ with $a^2+b^2 = 0$. We want to solve for x and y such that $(a+ib)(x+iy) = 1$. In other words,

$$\begin{aligned} &(ax - by) + i(ay + bx) = 1 \\ \Rightarrow &\begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = (1, 0) \\ \Rightarrow &\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (1, 0) \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \dfrac{a}{a^2+b^2} \\ \dfrac{b}{a^2+b^2} \end{pmatrix}
\end{aligned}
$$

This is unique as the inverse matrix is unique. □

Remark. The set of complex numbers is a field under the operations of addition and multiplication as operations are associative, commutative and distributive and every element has a unique inverse as before.

Definition 1.0.3. If $z = x + iy, x, y, \in \mathbb{R}$, then the conjugate of z is $\bar{z} = x - iy$.

Definition 1.0.4. We define the modulus (or length or magnitude) of $z = x + iy, x, y \in \mathbb{R}$ to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

Remark. For any $z, w \in \mathbb{C}$,

$$
\begin{aligned}
\bar{\bar{z}} &= z \\
z + \bar{z} &= 2\Re(z) \\
z - \bar{z} &= 2\Im(z) \\
z \cdot \bar{z} &= |z|^2 \\
|z| &= |\bar{z}| \\
\overline{z + w} &= \bar{z} + \overline{w} \\
\overline{zw} &= \bar{z} \cdot \overline{w} \\
|zw| &= |z||w|
\end{aligned}
$$

Proposition 2. *The following inequalities hold for any $z \in \mathbb{C}$.*

(1) $|\Re(z)| \leq |z|$

(2) $|\Im(z)| \leq |z|$

(3) $|z + w| \leq |z| + |w|$

(4) $|z + w| \geq \Big| |z| - |w| \Big|$

Proof. (1) and (2) follows as

$$|z|^2 = \Re(z)^2 + \Im(z)^2.$$

(3). Notice,

$$
\begin{aligned}
|x + iy|^2 &= (x + iy)\overline{(x + iy)} \\
&= (x + iy)(\overline{x} + \overline{iy}) \\
&= x\overline{x} + y\overline{y} + x\overline{y} + y\overline{x}
\end{aligned}
$$

$$
\begin{aligned}
&= |x|^2 + |y|^2 + x\bar{y} + y\bar{x} \\
&= |x|^2 + |y|^2 + 2\Re(x\bar{y}) \\
&\leq |x|^2 + |y|^2 + 2|x\bar{y}| \\
&= |x|^2 + |y|^2 + 2|x| \cdot |\bar{y}| \\
&= |x+y|^2
\end{aligned}
$$

Taking the square root of both sides gives the result.

(4). From (3), we have that

$$
\begin{aligned}
|z| = |z - w + w| \leq |z - w| + |w| \\
|w| = |w - z + z| \leq |w - z| + |z|
\end{aligned}
$$

Then, isolating $|z-w|$ implies the result. More specifically since we have the simultaneous inequality,

$$
\begin{cases} |z| - |w| \leq |z - w| \\ |w| - |z| \leq |z - w| \end{cases} \Rightarrow |z - w| \geq \Big| |z| - |w| \Big|
$$

as desired. ☐

Proposition 3. *Every non-zero complex number has exactly 2 square roots.*

Proof. Let $z = x + iy \in \mathbb{C}$ with $x^2 + y^2 \neq 0, x, y, \in \mathbb{R}$. We want to solve $w^2 = z$ for $w \in \mathbb{C}$. Say w takes the form $w = u + iv, u, v \in \mathbb{R}$. Then

$$
\begin{aligned}
&w^2 = z \\
\Rightarrow\ &(u + iv)^2 = x + iy \\
\Rightarrow\ &(u^2 - v^2) + i2uv = x + iy
\end{aligned}
$$

So we have that $x = u^2 - v^2$ and $y = 2uv^2$. We can solve for u and v. Take the square of both sides of the second equation to get $4u^2v^2 = y^2$. Now, we multiply the first equation by $4u^2$ to get

$$
\begin{aligned}
&4u^4 - 4u^2v^2 = 4xu^2 \\
\Rightarrow\ &4u^4 - 4xu^2 - y^2 = 0
\end{aligned}
$$

This is a quadratic equation over u^2 so,

$$
u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}
$$

Suppose that $y \neq 0$. Then we must take the positive solution above to get

$$
u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}
$$

Under the assumption that $x^2 + y^2 > 0$, this solution exists. Notice we cannot take the negative solution as it yields a negative u^2 which is impossible. We can use a similar procedure to find that

$$
v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}
$$

Rooting u and v gives 2 solutions for each. However, if y is positive, since $2uv = y$, u and v must take the same sign. Similarly, if y is negative, they must take different signs. In each of these cases, there are 2 solutions for u and v. So,

$$w = \begin{cases} \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) + i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y > 0 \\ \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) - i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y < 0 \\ \pm\sqrt{x} & , x > 0,\ y = 0 \\ \pm i\sqrt{-x} & , x < 0,\ y = 0 \end{cases}$$

□

Remark. Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots.

Remark. The square root doesn't distribute. Consider $z = w = -1 \in \mathbb{C}$. $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.

Remark. The Quadratic Formula holds true for complex polynomials. In other words, if $a, b, c \in \mathbb{C}, a \neq 0$,

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 1.0.5. If $z \in \mathbb{C} \setminus \{0\}$, we define the <u>angle</u> (or <u>argument</u>) of z to be the angle $\theta(z)$ from the positive x-axis counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big).$$

Remark. For $\theta \in \mathbb{R}$ (or for $\theta \in \mathbb{R}/2\pi$), we have that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Remark. If $z \neq 0$, we have $x = \Re(z)$, $y = \Im(z)$, $r = |z|$ and

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \tan\theta &= \frac{y}{x}, \text{ if } x \neq 0 \\ z &= rei\theta \\ \bar{z} &= re^{-i\theta} \\ z^{-1} &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

Remark. We now have 2 representations of a complex number $z \in \mathbb{C}$. We say that $z = x + iy$ is the <u>cartesian coordinates</u> of z and $z = re^{i\theta}$, where $r = |z|$, is the <u>polar form</u> of z.

Consider $z = re^{i\alpha}$ and $w = se^{i\beta}$. We have,

$$\begin{aligned} zw &= rs(\cos\alpha + i\sin\alpha)(\sin\beta + i\cos\beta) \\ &= rs\big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) \end{aligned}$$

$$
\begin{aligned}
&= rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big) \\
&= e^{i(\alpha+\beta)}
\end{aligned}
$$

which defines a formula for multiplication in polar coordinates. Notice that the following identity known as De Moivre's Law follows.

$$(re^{i\theta})^n = r^n e^{in\theta}$$

for all $r, \theta \in \mathbb{R}$, $n \in \mathbb{Z}$. We can use this identity to find the n^{th} roots of z. In other words, we solve $w^n = z$. We have,

$$
\begin{aligned}
& w^n = z \\
\Rightarrow\ & (se^{i\alpha})^n = re^{i\theta} \\
\Rightarrow\ & s^n e^{in\alpha} = re^{i\theta}
\end{aligned}
$$

so $s^n = r$ and $n\alpha = \theta + 2\pi k$ for $k \in \mathbb{Z}$. In other words, we have

$$(re^{i\theta}) = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}, \quad k = 0, \dots, n-1$$

Remark. When working with complex numbers, for $0 \neq z \in \mathbb{C}$, and for $0 < n \in \mathbb{Z}$, $\sqrt[n]{z}$ or $z^{1/n}$ denotes either one of the n roots, or the set of all n^{th} roots.

Example 1.0.6. Consider the $n-1$ diagonals of a regular n-gon inscribed in a circle of radius 1 obtained by connecting one vector with all the others. Show that the product of these diagonals is n.

Notice that $z_2, \dots, z_n$ are the n^{th} roots of unity other than 1. Let z be the variable and consider the polynomial

$$P(z) \coloneqq 1 + z + \dots + z^{n-1}.$$

Since the roots of $P(z)$ are n^{th} roots of unity other than 1, we can factorize

$$
\begin{aligned}
P(z) &= 1 + z + \dots + z^{n-1} \\
&= (z - z_2) \dots (z - z_n)
\end{aligned}
$$

and setting $z = 1$, the result follows. In particular, we have

$$|1 - z_2| \dots |1 - z_n| = n.$$

2 Complex Functions

2.1 Limits

Definition 2.1.1. A sequence of complex numbers $z_1, z_2 \dots$ converges to $z \in C$ if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

Equivalently, given any $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ sufficiently large such that $|z_n - z| < \epsilon$ whenever $n > N$.

Remark. If $\{z_n\}_n$ converges to z, we write

$$\lim_{n\to\infty} z_n = z$$

or $z_n \to z$ as $n \to \infty$.

Example 2.1.2. For $|z| > 1$, show that $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converges.

Notice,

$$\lim_{n\to\infty} \left|\frac{1}{z^n} - 0\right| = \lim_{n\to\infty} \left|\frac{1}{z^n}\right| = 0$$

as $|z| > 1$.

Example 2.1.3. Show that $\{i^n\}_{n-1}^{\infty}$ does not converge.

Definition 2.1.4. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say

$$\lim_{z\to z_0} f(z) = L$$

if for every sequence $\{z_n\}_n \subseteq \Omega$ we have that $z_n \to z \Rightarrow f(z_n) \to L$.

Remark. Here, z_0 need not to be in Ω.

Example 2.1.5. Let $f(z) = \dfrac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim\limits_{z} \to 0 f(z)$.

If $z = x \in \mathbb{R} \setminus \{0\}$, then $f(z) = \dfrac{x}{x} = 1$. So $\lim\limits_{x\to 0} f(x) = 1$. If $z = iy, y \in \mathbb{R} \setminus \{0\}$, then $f(z) = \dfrac{-iy}{iy} = -1$. So $\lim\limits_{y\to 0} f(iy) = -1$. Hence, the limit does not exist.

Example 2.1.6. Show that $z_n \to z$ if and only if $\Re z_n \to \Re z$ and $\Im z_n \to z$.

2.2 Function Continuity

Definition 2.2.1. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say f is <u>continuous at $z_0 \in \Omega$</u> if for every sequence $\{z_n\} \subseteq \Omega$, we have $z_0 \to z \Rightarrow f(z_0) \to f(z)$. Equivalently, given any $\epsilon > 0, \exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Remark. f is continuous on Ω if it is continuous at ever point of Ω.

Remark. We may split f into its real and imaginary parts

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

2.3 Holomorphic Functions

Definition 2.3.1. An open disk of radius r at z_0 with $r > 0$ is the <u>neighborhood</u> around z_0 denoted by $D(z_0, r)$ with

$$D(z_0, r) \coloneqq \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 2.3.2. Let $f(z)$ be defined in a neighborhood of z_0. We say f is complex differentiable (or holomorphic) at z_0 if

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

exists. If it does, we denote the limit by $f'(z_0)$.

Remark. Here, $h\in\mathbb{C}$ can approach zero from any direction in $\mathbb{C}$.

Example 2.3.3. Where is $f(z)=\dfrac{1}{z}, z\neq 0$ holomorphic?

Notice,

$$\lim_{h\to 0}\frac{\frac{1}{z_0+h}-\frac{1}{z_0}}{h}=\lim_{h\to 0}\frac{-h}{(z_0+h)(z_0)}=-\frac{1}{z_0^2}$$

So, f is holomorphic at any $z\in\mathbb{C}\setminus\{0\}$, and $f'(z)=-\dfrac{1}{z_0^2}$.

Example 2.3.4. $f(z)=\bar{z}$ is not holomorphic at any $z\in\mathbb{C}$.

Notice,

$$\lim_{h\to 0}\frac{\overline{z_0+h}-\overline{z_0}}{h}=\lim_{h\to 0}\frac{\bar{h}}{h}$$

which does not exist from Example 2.1.5. However, the function can be though of a map from $\mathbb{R}^2\to\mathbb{R}^2$ defined as $(x,y)\mapsto(x,-y)$ which is differentiable as all partial derivatives exist. This distinguishes real analysis from complex analysis.

Remark. If f and g are holomorphic, so are $f+g$, fg and $\dfrac{f}{g}$ (when $g\neq 0$). The proof is identical to the statements of real functions.

We can now generalize when a function is complex differentiable. If the complex function $f(z)=u+iv$. If the complex derivative $f'(z)$ is to exist, then it must be that the limit exists approaching from both the real and imaginary axis. Thus, we have,

$$f'(z)=\lim_{t\to 0}\frac{f(z+t)-f(z)}{t}=\lim_{t\to 0}\frac{f(z+it)-f(z)}{it}$$

where t is a real number. In terms of u and v, taking the derivative along the real line gives,

$$\begin{aligned}&\lim_{t\to 0}\frac{f(z+t)-f(z)}{t}\\&=\lim_{t\to 0}\frac{u(x+t,y)+iv(x+t,y)-u(x,y)-iv(x,y)}{t}\\&=\lim_{t\to 0}\frac{u(x+t,y)-u(x,y)}{t}+i\lim_{t\to 0}\frac{v(x+t,y)-v(x,y)}{t}\\&=\frac{u}{x}+i\frac{v}{x}.\end{aligned}$$

Taking the derivative along the vertical line gives,

$$\lim_{t\to 0}\frac{f(z+it)-f(z)}{it}$$

$$
\begin{aligned}
&= -\, i \lim_{t\to 0} \frac{u(x,y+t)+iv(x,y+t)-u(x,y)-iv(x,y)}{t} \\
&= -\, i \lim_{t\to 0} \frac{u(x,y+t)-u(x,y)}{t} + \lim_{t\to 0} \frac{v(x,y+t)-v(x,y)}{t} \\
&= -\, i\frac{u}{y} + \frac{v}{y}.
\end{aligned}
$$

Equating real and imaginary parts, we arrive at the following theorem.

Theorem 4 (Cauchy-Riemann Equations)**.** *If a function $f(z) = u + iv$ is holomorphic in a neighborhood around $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy*

$$\frac{u}{x} = \frac{v}{y} \quad \textit{and} \quad \frac{v}{x} = -\frac{u}{y} \quad \textit{at } (x_0, y_0)$$

with

$$f'(z_0) = \frac{u}{x} + i\frac{v}{x} = \frac{v}{y} - i\frac{u}{y}.$$

Example 2.3.5. Show that

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{, if } z \neq 0 \\ 0 & \text{, if } z = 0 \end{cases}$$

is not holomorphic at $z = 0$ and that the Cauchy-Reimann Equations hold at $z = 0$.

Notice,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\bar{h}^2}{h^2} = \lim_{h\to 0} \left(\frac{\bar{x}-iy}{x+iy}\right)^2$$

Let $h = x + imx$, $m \neq 0, x \to 0$. We get

$$\lim_{x\to 0} \left(\frac{x-imx}{x+imx}\right)^2 \left(\frac{1-im}{1+im}\right)^2$$

which is dependent of m and thus the limit does not exist. Now, notice we have

$$\frac{\bar{z}^2}{z} = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\frac{-3x^2y+y^3}{x^2+y^2}$$

So we have

$$u(x,y) = \begin{cases} \dfrac{x^3-3xy^2}{x^2+y^2} & \text{, if } (x,y) \neq (0,0) \\ 0 & \text{, if } (x,y) = (0,0) \end{cases}$$

and

$$v(x,y) = \begin{cases} \dfrac{-3x^2y+y^3}{x^2+y^2} & \text{, if } (x,y) \neq (0,0) \\ 0 & \text{, if } (x,y) = (0,0) \end{cases}.$$

Now, we can verify that the Cauchy-Riemann Equations hold. Indeed,

$$
\begin{aligned}
\frac{u}{x}\left(\frac{x^3-3xy^2}{x^2+y^2}\right) &= \frac{\left(\frac{}{x}(x^3-3xy^2)\right)(x^2+y^2) - (x^3-3xy^2)\left(\frac{}{x}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(3x^2-3y^2)(x^2+y^2) - (x^3-3xy^2)(2x)}{(x^2+y^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2} \\
\frac{v}{y}\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right) &= \frac{\left(\frac{}{y}(y^3 - 3x^2y)\right)(x^2+y^2) - (y^3 - 3x^2y)\left(\frac{}{y}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(3y^2 - 3x^2)(x^2+y^2) - (y^3 - 3x^2y)(y^2)}{(x^2+y^2)^2} \\
&= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{u}{y}\left(\frac{x^3 - 3xy^2}{x^2+y^2}\right) &= \frac{\left(\frac{}{y}(x^3 - 3xy^2)\right)(x^2+y^2) - (x^3 - 3xy^2)\left(\frac{}{y}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2) - (x^3 - 3xy^2)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2} \\
\frac{v}{x}\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right) &= \frac{\left(\frac{}{x}(y^3 - 3x^2y)\right)(x^2+y^2) - (y^3 - 3x^2y)\left(\frac{}{x}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2) - (y^3 - 3x^2y)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2}
\end{aligned}
$$

Thus, we can consider the converse statement of Theorem 4.

Theorem 5. *Let $f = u + iv : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in \Omega$. If*

(1) the partials of u, v exist in a neighborhood of (x_0, y_0)

(2) the partials of u, v are continuous at (x_0, y_0)

(3) $\frac{u}{x} = \frac{v}{y}$ and $\frac{v}{x} = -\frac{u}{y}$ at (x_0, y_0)

then, f is holomorphic at z_0.

TODO: find proof online.

3 Power Series

3.1 Convergence and Divergence

Example 3.1.1. Consider the power series, an expression of the form

$$\sum_{n=0}^{\infty} c_n z^n$$

where $c_n \in \mathbb{C}$. This expression converges if the sequence of partials sums, $\{s_N\}$ defined by

$$s_N := \sum_{n=0}^{N} c_n z^n$$

converges as $N \to \infty$. This is quite a strong condition, so we consider the following definition.

Definition 3.1.2. A power series expression converges absolutely if

$$\sum_{n=0}^{\infty} |c_n||z|^n$$

converges.

Remark. Absolute convergence implies converges. Notice,

$$\left|\sum_{n=0}^{N} c_n z^n\right| = \sum_{n=0}^{N} |c_n||z|^n$$

for each $N \in \mathbb{N}$.

3.2 Radius of Convergence

Theorem 6. *For any power series* $\sum_{n=0}^{\infty} c_n z^n, \exists 0 \leq R \leq \infty$*, such that*

(1) If $|z| < R$, the series converges absolutely

(2) If $|z| > R$, the series diverges.

Moreover, R is given by Hadamard's formula: $\dfrac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$

Remark. R is called the radius of convergence of the series and $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.

Remark. Recall,

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{m\leq n} a_m\right)$$

and is the "highest peak reached by a_n's as $n \to \infty$".

Proposition 7 (Property of lim sup)**.** *If* $L = \limsup_{n\to\infty} a_n$*, then for any* $\epsilon > 0, \exists N > 0$ *such that* $\forall n \leq N, a_n < L + \epsilon$

Proof of Theorem 6. Let $L \coloneqq \dfrac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$ Clearly, $L \leq 0$.

(1) Suppose $|z| < R$. So, there exists some $\epsilon > 0$ such that $r \coloneqq |z|(L+\epsilon) < 1$ and $0 < r < 1$. By Proposition 7, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|c_n|^{\frac{1}{n}} < L + \epsilon$. Now,

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty} \left(|c_n|^{\frac{1}{n}}|z|\right)^n < \sum_{n=N}^{\infty} r^n$$

converges as $0 < r < 1$. By the comparison test, $\displaystyle\sum_{n=N}^{\infty} |c_n||z|^n$ is monotonic and bounded and thus converges by Bolzano-Weierstrass.

(2) This follows from the proof above. Specifically, this time, notice that there exists some $\epsilon > 0$ such that $r \coloneqq |z|(L-\epsilon) > 1$. Again, by Proposition 7, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|c_n|^{\frac{1}{n}} > L - \epsilon$ so that

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty} \left(|c_n|^{\frac{1}{n}}|z|\right)^n > \sum_{n=N}^{\infty} r^n$$

follows and so the series diverges.

□

Theorem 8. *Suppose $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence R. Then $f'(z)$ exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$. Moreover, f' has the same radius of convergence as f.

Proof. f' has some radius of convergence because

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{1}{n}}|c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Let $|z_0| \leq r < R$, $g(z_0) \coloneqq \displaystyle\sum_{n=1}^{\infty} n c_n z_0^{n-1}$. We want to show

$$\lim_{k\to 0} \frac{f(z_0+h) - f(z_0)}{h} = g$$

or equivalently, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < \epsilon.$$

For any fixed $\epsilon > 0$, we write

$$f(z) = \underbrace{\sum_{n=0}^{N} c_n z^n}_{\coloneqq S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{\coloneqq E_N(z)}$$

We have $S_N' = \sum_{n=1}^{N} nc_n z^{n-1}$ and

$$\begin{aligned}
&\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| \\
&= \left|\frac{S_N(z_0+h)-S_N(z_0)}{h} + \frac{E_N(z_0+h)-E_N(z_0)}{h} - g(z_0) + S_N'(z_0) - S_N'(z_0)\right| \\
&= \left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right| + \left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| + \left|S_N'(z_0) - g(z_0)\right|.
\end{aligned}$$

We have

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| = \left|\frac{1}{h}\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^h - z_0^n\big)\right|$$

As $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ in any ring, we have

$$\begin{aligned}
&\left|\frac{1}{h}\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^h - z_0^n\big)\right| \\
&= \left|\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^{n-1} + (z_0+h)^{n+2}z_0 + \cdots + (z_0+h)z_0^{n-2} + z_0^{n-1}\big)\right|
\end{aligned}$$

Now, by choosing δ relatively small so that $|z_0| \le r$, we have $|z_0|, |z_0+h| \le r$ and so

$$(z_0+h)^{n-1} + (z_0+h)^{n+2}z_0 + \cdots + (z_0+h)z_0^{n-2} + z_0^{n-1} \le nr^{n-1}$$

So,

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| \le \sum_{n=N+1}^{\infty} nc_n r^{n-1} < \frac{\epsilon}{3}.$$

for a large enough N_1.

Now, observe that by definition,

$$S_N'(z_0) = \sum_{n=1}^{N} nc_n z^{n-1}.$$

Since,

$$\lim_{N\to\infty} S_N'(z_0) = \lim_{N\to\infty}\sum_{n=1}^{N} nc_n z^{n-1} = \sum_{n=1}^{\infty} nc_n z^{n-1} = g(z_0)$$

we can pick some $\frac{\epsilon}{3} > 0$ such that there exists some $N_2 \in \mathbb{N}$, for all $n > N_2$, we have

$$|S_N'(z_0) - g(z_0)| < \frac{\epsilon}{3}.$$

Finally, let $\frac{\epsilon}{3} > 0$. Observe that there exists some $\delta > 0$ such that there exists some $N > \max N_1, N_2$, for all $n > N$,

$$\left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right| < \frac{\epsilon}{3}.$$

as $|h| < \delta$. It follows that
$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \epsilon$$
as desired. □

Example 3.2.1. Consider $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula,

$$\begin{aligned} \frac{1}{R} &= \limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} n^{\frac{1}{n}} \\ &= 1 \end{aligned}$$

Thus, $R = 1$. By Theorem 6, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, in other words, when $|z| = 1$, consider the follwoing two cases:

1. If $z = 1$, then $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series, and hence f diverges.

2. If $z = i$, then

$$\begin{aligned} f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots\right) i \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right) \end{aligned}$$

   which both real and imaginary parts converge by the alternating series test. Therefore, we observe that both convergence and divergence may occur on the boundary, depending on the value of z.

Remark. The positions of lim and $\sum_{n=0}^{\infty}$ cannot be exchanged when we consider infinite sums. Consider the following example.

Example 3.2.2. Consider for $|x| > 1$, $\sum_{n=1}^{\infty} \lim_{x\to 1} (x^n - x^{n+1}) = \sum_{n=1}^{\infty} (1-1) = 0$. Now,

$$\begin{aligned} &\lim_{x\to 1} \lim_{N\to\infty} \sum_{n-1}^{N} (x^n - x^{n+1}) \\ &= \lim_{x\to 1} \lim_{N\to\infty} (x - x^2 + x^2 - x^3 + \dots + x^{N-1} - x^N + x^N - x^{N+1}) \\ &= \lim_{x\to 1} \lim_{N\to\infty} (x - x^{N+1}) \\ &= \lim_{x\to 1} x \\ &= 1 \end{aligned}$$

so that

$$\sum_{n=1}^{\infty} \lim_{x\to 1}(x^n - x^{n+1}) \neq \lim_{x\to 1} \sum_{n=1}^{\infty}(x^n - x^{n+1}).$$

Definition 3.2.3. A function f is said to be <u>entire</u> if f is holomorphic in the entire complex plane.

Example 3.2.4. Define $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that the radius of convergence of this series is ∞ (which implies e^z is entire) and $(e^z)' = e^z$.

Consider Stirling's formula, which says as $n \to \infty, n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Then, we have

$$e^z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{n}\right)^n.$$

Now,

$$\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty} \left| \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right) \right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty} \left| \frac{1}{2\pi n} \right|^{\frac{1}{n}} \limsup_{n\to\infty} \left| \frac{e}{n} \right|^{\frac{1}{n}} \\
&= 0
\end{aligned}$$

Thus, $R \to \infty$ as $n \to \infty$. By Theorem 6, e^z is an entire function. Now, we can show the derivative by the limit defintition. Notice,

$$\begin{aligned}
\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} &= \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= \lim_{h\to 0} \frac{e^h}{h} \lim_{h\to 0} \frac{1}{h}
\end{aligned}$$

Corollary 9 (Corollary of Theorem 8)**.** *A power series is infinitely complex-differentiable in its radius of convergence. All its derivatives are also power series, obtained by termwise differentiation. If* $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$, *then*

$$f^{(N)}(z) = \sum_{n=N}^{\infty} n(n-1)\cdots(n-N)c_n z^{n-N}$$

for some $N \in \mathbb{N}$.

In general, we have have $\displaystyle\sum_{n=0}^{\infty} c_n(z - z_0)^n$, which is the power series centered at $z \in \mathbb{C}$. Then as before, the readius of convergence R is given by

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}.$$

But the disc of convergence is now centered around z_0. We have shown that $f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ in some neighborhood of z_0) with radius of convergence $R > 0$. This implies that $f(z)$ is holomorphic at z_0. In fact, the converse is true; any function holomorphic at z_0 is infinitely holomorphic at z_0.
However, for this, we need the concept of integration over paths of curves.

4 Integration

4.1 Curves and Paths

Definition 4.1.1. A curve in $\mathbb{C}$ is a continuous function $\gamma(t) : [a, b] \to \mathbb{C}$ with $a, b \in \mathbb{R}$. The image of γ in $\mathbb{C}$ is called γ^*.

Example 4.1.2. Let $z_0 \in \mathbb{C}$, $r > 0$. Take $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $t \mapsto z_0 + re^{it}$. This is a circle of radius r centered at z_0, oriented counterclockwise.

Example 4.1.3. Consider $\hat{\gamma} : [0, 1] \to \mathbb{C}$ defined by $t \mapsto z_0 + re^{2\pi it}$. This is identical to the curve γ defined above with the same oriented path and shows that curves have different parameterizations.

Definition 4.1.4. We say γ is smooth on the interval $[a, b]$ if γ' exists, is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for any $t \in [a, b]$.

Definition 4.1.5. $\gamma : [a, b] \to \mathbb{C}$ is piecewise-smooth if it is smooth on $[a, b]$ except at finitely many points in $[a, b]$.

Remark. Piecewise smooth curves are called paths.

4.2 The Integral

Definition 4.2.1. Given a path $\gamma : [a, b] \to \mathbb{C}$ and $f(z)$ is a continuous function on γ, the integral of f along γ, (called the contour) is defined by

$$\int_\gamma f(z)\, dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)\, dt.$$

where $z = \gamma(t)$, so $dz = \gamma'(t)\, dt$.

Remark. If g is complex valued, then

$$\int_a^b g(t)\, dt = \int_a^b \Re\big(g(t)\big)\, dt + i \int_a^b \Im\big(g(t)\big)\, dt.$$

Remark. The integral $\displaystyle\int_\gamma f(z)\, dz$ can be shown to be independent of the parameterization chosen for $\gamma*$.

Example 4.2.2. For all $n \in \mathbb{Z}$, evaluate $\displaystyle\int_\gamma z^n\, dz$. That is, continue on the path γ that describes any circle centered at origin oriented anticlockwise.

Let $R \in \mathbb{R}$ and define

$$\gamma : [0, 1] \to \mathbb{C} \quad t \mapsto Re^{2\pi it}$$
$$\gamma'(t) = 2R\pi ie^{2\pi it} = 2\pi i\gamma(t).$$

Then,

$$\int_\gamma z^n\, dz = \int_0^1 R^n e^{2\pi int} \cdot 2\pi i \cdot Re^{2\pi it}\, dt$$

$$
\begin{aligned}
&= 2\pi i R^{n+1} \int_0^1 e^{2\pi i(n+1)t}\,dt \\
&= \begin{cases} \dfrac{R^{n+1}}{n+1} e^{2\pi i(n+1)t}\Big|_0^1 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i t\Big|_0^1 & \text{if } n = -1 \end{cases} \\
&= \begin{cases} \dfrac{R^{n+1}}{n+1}\left(e^{2\pi i(n+1)} - 1\right) & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases} \qquad \text{(since } e^{2\pi k i} \equiv 1 \pmod{2\pi}\text{)} \\
&= \begin{cases} 0 & \text{if } n \in \mathbb{Z}\setminus\{-1\} \\ 2\pi i & \text{if } n = -1 \end{cases}
\end{aligned}
$$

Note that our final answer does not depend on R, the radius of the circle.

Theorem 10. *For any complex valued functions $f(z), g(z)$, and $\alpha, \beta \in \mathbb{C}$, the following hold:*

(1) Integration is linear; For any curve $\gamma : [\alpha, \beta] \to \mathbb{C}$,

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz$$

(2) If $\beta \leq \alpha$,

$$\left|\int_a^b g(z)\,dz\right| \leq \int_a^b |g(z)|\,dz$$

(3) If $f(x)$ is continuous on the path $\gamma : [\alpha, \beta] \to \mathbb{C}$,

$$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in\gamma} |f(x)| \underbrace{\int_a^b |\gamma(t)|\,dt}_{\textit{length of the path}}$$

(4) If γ^- is the reverse direction of the path $\gamma : [a, b] \to \mathbb{C}$, then

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz$$

Proof.

(1) This follows as we can write $f(x) = \Re(f) + i\Im(f)$ and apply linearity of real-valued integrals.

(2) Since we have $-|g(z)| \leq g(z) \leq |g(z)|$, for all $z \in [a, b]$, we have

$$-\int_a^b |g(z)|\,dz \leq \int_a^b g(z)\,dz \leq \int_a^b |g(z)|\,dz$$

and the result follows.

(3) Notice,

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right|$$

$$
\begin{aligned}
&\leq \int_a^b |f(\gamma(t))\gamma'(t)|\,dt && \text{(from (2))} \\
&\leq \int_a^b \sup_{z\in\gamma} |f(z)||\gamma'(t)|\,dt && (|f(z)| \leq \sup_{z\in\gamma} |f(z)|) \\
&= \sup_{z\in\gamma} |f(z)| \int_a^b |\gamma'(t)|\,dt
\end{aligned}
$$

as desired.

(4) This follows trivially from the Fundamental Theorem of Calculus. We can define $\gamma^- : [b, a] \to \mathbb{C}$ and

$$\int_{\gamma^-} f(z)\,dz = \int_b^a f(z)\,dz = F(a) - F(a) = -(F(a) - F(b)) = \int_a^b f(z)\,dz \int_\gamma f(z)\,dz$$

where $F(z) \coloneqq \int f(z)\,dz$ is called the indefinite integral.

□

At this point, we generalize the Fundamental Theorem of Calculus for $\mathbb{C}$.

4.3 Fundamental Theorem of Calculus

Remark. We denote the set of all holomorphic functions $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ by $H(\Omega)$ where Ω is an open set. In other words, f is holomorphic in Ω if and only if $f \in H(\Omega)$.

Theorem 11 (Fundamental Theorem of Calculus)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a path inside an open set $\Omega \subseteq \mathbb{C}$. Suppose $f(z)$ is continuous on γ, and has an antiderivative $F \in \Omega$. Then,*

$$\int_\gamma f(z)dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big) \tag{1}$$

Proof. Let $G = F \circ \gamma$ and suppose γ is a smooth function. Since γ is smooth, γ' exists and is continuous on $[a, b]$ and $\gamma'(t) \neq 0$ for all $t \in [a, b]$, and since f is continuous on $[a, b]$, $G(t) = F'(\gamma(t))\gamma'(t)$ is continuous as well. Now

$$
\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt \\
&= \int_a^b F'\big(\gamma(t)\big)\gamma'(t)\,dt \\
&= \int_a^b G'(t)\,dt
\end{aligned}
$$

Now, we can write $G'(t) = \Re(G) + i\Im(G)$ and apply the Fundamental Theorem of Calculus in $\mathbb{R}$ to arrive at

$$= \int_a^b G'(t)\,dt$$

$$
\begin{aligned}
&= \int_a^b \Re(G)\,dt + i\int_a^b \Im(G)\,dt \\
&= \Re(G(b)) + i\Im(G(b)) - \Re(G(a)) - i\Im(G(b)) \\
&= G(b) - G(a) \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}
$$

If γ is piecewise smooth, then we can simply apply the above to each of the smooth paths separately and sum up all of the integrals. $\square$

Definition 4.3.1. A path $\gamma : [a, b] \to \mathbb{C}$ is said to be closed if $\gamma(a) = \gamma(b)$.

Corollary 12. *If $f \in H(\Omega)$, $\Omega \in \mathbb{C}$ open, then*

$$\int_\gamma F'(z)\,dz = 0$$

on any closed path $\gamma : [a, b] \to \mathbb{C}$.

Proof. By the Fundamental Theorem of Calculus, we have

$$\int_\gamma F'(z)\,dz = F(\gamma(a)) - F(\gamma(b)) = 0$$

as desired. $\square$

Example 4.3.2. Take $f(z) = z^n$ where $n \in \mathbb{Z} \setminus \{-1\}$. Then f is continuous on $\mathbb{C} \setminus \{0\}$. Then $f = F'$ for $F = \dfrac{z^{n+1}}{n+1}$ and $F = H(\mathbb{C} \setminus \{0\})$. Therefore, $\displaystyle\int_\gamma z^n\,dz = 0$ for any closed path γ not passing through 0 by Corollary 12.

If $n = -1$, we know from Example 4.2.2 that F' is not continuous and thus we cannot invoke Corollary 12. In this particular case, we have $\displaystyle\int_\gamma \frac{1}{z}\,dz = 2\pi i$.

Definition 4.3.3. The interior of a set Ω is defined as

$$\Omega^\circ := \{z \in \Omega : \exists \epsilon \in \mathbb{R}, B_\epsilon(z) \subseteq \Omega\}.$$

Theorem 13 (Cauchy-Goursat Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a open set and $f : \Omega \to \mathbb{C}$ such that $f \in H(\Omega)$. Then*

$$\int_\Delta f(z)\,dz = 0$$

for any triangular path $\Delta \in \Omega$.

Remark. Given any two points in $\mathbb{C}$, if we can connect these two points by two paths, then the integrals of any given holomorphic function over these paths are the same.

Proof. We begin with the assumption that

$$\left|\int_\Delta f(z)\,dz\right| = c \geq 0.$$

We construct $\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_1^{(3)}, \Delta_1^{(4)}$ be the smaller triangles by bisecting each side of Δ. Then, it is true that

$$\int_\Delta f(z)\,dz = \sum_{i=1}^{4} \int_{\Delta_i^{(1)}} f(z)\,dz$$

which gives the inequality

$$c = \left| \int_\Delta f(z)\,dz \right| \leq \sum_{i=1}^{4} \left| \int_{\Delta_i^{(1)}} f(z)\,dz \right|.$$

Now, we can choose some $i \in \{1,2,3,4\}$ such that

$$\left| \int_{\Delta_i^{(1)}} f(z)\,dz \right| \geq \frac{1}{4}c$$

and fix $\Delta^{(1)} := \Delta_i^{(1)}$. Here, we have that $L(\Delta^{(1)}) = \dfrac{1}{2}L(\Delta)$ where $L(\gamma)$ is the length of the curve. We can repeat this process of subdividing the triangular paths so that we get a sequence of triangles

$$\Delta \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \cdots \supseteq \Delta^{(n)} \supseteq \ldots$$

satisfying both

$$\left| \int_{\Delta^{(n)}} f(z)\,dz \right| \geq \left(\frac{1}{4}\right)^n c \quad \text{and} \quad L(\Delta^{(n)}) = \left(\frac{1}{2}\right)^n L(\Delta)$$

for all $n \in \mathbb{N} \setminus \{0\}$.

Claim (Nested Triangles Theorem)**.** *The nested sequence* $\Delta \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \cdots \supseteq \Delta^{(n)} \supseteq \ldots$ *has a limit point. In other words, there exists some* $z_0 \in \displaystyle\bigcap_{n=1}^{\infty} \Delta^{(n)}$.

Suppose that there was no fixed point. Then $(\Delta^{(1)})^\mathsf{c}, (\Delta^{(2)})^\mathsf{c}, \ldots$ form an open cover for Δ. By Heine-Borel, Δ is compact, so this open cover admits some finite subcover, say $(\Delta^{(n_1)})^\mathsf{c}$, $(\Delta^{(n_2)})^\mathsf{c}$, $\ldots$, $(\Delta^{(n_k)})^\mathsf{c}$, where $n_1 < n_2 < \cdots < n_k$. But, $\displaystyle\bigcup_{r=1}^{k} (\Delta^{(n_r)})^\mathsf{c} = (\Delta^{(n_k)})^\mathsf{c}$, which means $\Delta \subseteq (\Delta^{(n_k)})^\mathsf{c}$, but since $(\Delta^{(n_k)})^\mathsf{c} \neq \emptyset$, this implies that $(\Delta^{(n_k)})^\mathsf{c} \supset \Delta^{(n_k)}$, which is a contradiction.

Now, since f is holomorphic, at z_0, for a given $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

or,

$$0 < |z - z_0| < \delta \implies \big| f(z) - f(z_0) - f'(z_0)(z - z_0) \big| < \epsilon |z - z_0|$$

for all $z \in \Delta$. Now, there exists some $m \in \mathbb{N} \setminus \{0\}$ such that $\Delta^{(m)} \subseteq D(z_0, \delta)$. Also, by Corollary 12, we have that

$$\int_{\Delta^{(m)}} f(z_0)\,dz = \int_{\Delta^{(m)}} f'(z_0)(z - z_0)\,dz = 0.$$

Then,

$$\int_{\Delta^{(m)}} f(z)\,dz = \int_{\Delta^{(m)}} \Big(f(z) - f(z_0) - f'(z_0)(z - z_0) \Big)\,dz$$

It follows by Theorem 10, we have

$$\begin{aligned}
&\left|\int_{\Delta^{(m)}} \Big(f(z) - f(z_0) - f'(z_0)(z - z_0)\Big)\, dz\right| \\
&\leq \int_{\Delta^{(m)}} \Big|\Big(f(z) - f(z_0) - f'(z_0)(z - z_0)\Big)\Big|\, dz \\
&\leq \int_{\Delta^{(m)}} \epsilon |z - z_0|\, dz \\
&\leq \epsilon L(\Delta^{(m)}) \int_{\Delta^{(m)}} dz && (z \in \Delta \implies |z - z_0| \leq L(\Delta^{(m)})) \\
&\leq \epsilon L^2(\Delta^{(m)})
\end{aligned}$$

Notice that

$$\left(\frac{1}{4}\right)^m c \leq \left|\int_{\Delta^{(m)}} f(z)\, dz\right| \leq \epsilon L^2(\Delta^{(m)}) = \left(\frac{1}{4}\right)^m \epsilon L^2(\Delta^{(m)})$$

which yields

$$c \leq \epsilon L^2(\Delta^{(m)}).$$

Since $\epsilon > 0$ can be chosen arbitrarily small, c = 0. □

5 Practice Problems

Remark. Consider the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\frac{1}{R} := \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in [0,\infty)$.

- If $|z-z_0| < R$, the series converges absolutely
- If $|z-z_0| > R$, the series diverges
- If $0 < r < R$, the series converges the series converges uniformly on $\{z : |z-z_0| < r\}$

Exercise 1. *Parameterize the semi-circle $|z-4-5i| = 3$ clockwise, starting from $z = 4+8i$ to $z = 4+2i$.*

Let $\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{C}$ such that $\gamma(t) = 3e^{-it} + 4 + 5i$. Notice,

$$\begin{aligned}
\gamma\left(-\frac{\pi}{2}\right) &= 4+8i \\
\gamma(0) &= 7+5i \\
\gamma\left(\frac{\pi}{2}\right) &= 4+2i
\end{aligned}$$

which parameterizes the given semicircle.

Exercise 2. *If the power series $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ centered at z_0 has a non-zero radius of convergence, then show that*

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

for any $m \in \mathbb{Z}$, $m \geq 0$, where $f^{(m)}(z_0)$ denotes the m^{th} derivative of f at z_0.

Since $f(z)$ is a power series and the radius of convergence $R \neq 0$ by Theorem 8, $f(z)$ is $\mathbb{C}$-differentiable and each derivative has the same radius of convergence. By induction, it can be shown that

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z-z_0)^{n-m}.$$

Evaluating $f^{(m)}$ at z_0, we have

$$\begin{aligned}
f^{(m)}(z_0) &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z_0-z_0)^{n-m} \\
&= m!\, c_m
\end{aligned}$$

as all terms $n > m$ are 0. Then, we obtain

$$c_m = \frac{f^{(m)}(z_0)}{m!}$$

as desired.

Exercise 3. *Let γ be the arc of the unit circle centered at the origin in the first quadrant oriented clockwise (from i to 1). Evaluate the integral*

$$\int_\gamma \bar{z}^2 \, dz$$

by parameterizing the curve.

Conisder the parameterization $\gamma : [-\frac{\pi}{0}] \to \mathbb{C}$ given by $\gamma)^t = e^{-it}$. Not that $\overline{e^{-it}} = e^{it}$. Then,

$$\begin{aligned}\int_\gamma \bar{z}^2 \, dz &= \int_{-\frac{\pi}{2}}^0 e^{2it} \cdot (-ie^{-it}) \, dz \\ &= -i \int_{-\frac{\pi}{2}}^0 e^{it} \, dz \\ &= -e^{it}\Big|_{-\frac{\pi}{2}}^0 \\ &= -1 - i.\end{aligned}$$

Exercise 4. *Evaluate the above integral by finding an anti-derivative.*

Note that $z\bar{z} = |z|^2$, so on the circle, we have $\bar{z} = \frac{1}{z}$. Thus, the integral is equivalent to $\int_\gamma \frac{1}{z^2} \, dz$. Now, the anti-derivative of $\frac{1}{z^2}$ is $-\frac{1}{z}$. Thus, by the Fundamental Theorem of Calculus, we have,

$$\int_\gamma \frac{1}{z^2} \, dz = F(\gamma(0)) - F\left(\gamma\left(\frac{\pi}{2}\right)\right) = -\frac{1}{e^{-i(0)}} + \frac{1}{e^{-i(-pi/2)}} = -1 - i.$$

Exercise 5. *Let $\{c_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that*

$$L = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

exists. Show that $\lim_{n\to} c_n^{\frac{1}{n}} = L$.

We have that for every $\epsilon > 0$, there esists some $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \frac{c_n}{c_{n-1}} - L \right| < \epsilon.$$

$$\begin{aligned}c_n^{\frac{1}{n}} &= \left(\frac{c_n}{c_{n-1}} \cdot \frac{c_{n-1}}{c_{n-2}} \cdots \frac{c_N}{c_{n-1}} \cdot c_{n-1} \right)^{\frac{1}{n}} \\ &= \left(\frac{c_n}{c_{n-1}} \right)^{\frac{1}{n}} \left(\frac{c_{n-1}}{c_{n-2}} \right)^{\frac{1}{n}} \cdots \left(\frac{c_N}{c_{n-1}} \right)^{\frac{1}{n}} c_{n-1}^{\frac{1}{n}}\end{aligned}$$

Now,

$$\underbrace{(L-\epsilon)^{\frac{1}{n}} \ldots (L-\epsilon)^{\frac{1}{n}}}_{n \text{ times}} c_{n+1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq \underbrace{(L+\epsilon)^{\frac{1}{n}} \ldots (L+\epsilon)^{\frac{1}{n}}}_{n \text{ times}}$$

$$\implies \qquad (L-\epsilon)^{\frac{n-N+1}{n}} c_{n-1}^{\frac{1}{n}} \leq c_n^{\frac{1}{n}} \leq (L+\epsilon)^{\frac{n-N+1}{n}} c_{n-1}^{\frac{1}{n}}$$

so,

$$\lim_{n\to\infty} (L-\epsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L-\epsilon$$
$$\lim_{n\to\infty} (L+\epsilon)^{\frac{n-N+1}{n}} c_{N-1}^{\frac{1}{n}} = L+\epsilon$$

and it follows that

$$L-\epsilon \leq c_n^{\frac{1}{n}} \leq L+\epsilon \implies \left| c_n^{\frac{1}{n}} - L \right| \leq \epsilon$$

as desired.

6 Cauchy's Integral Formula

Definition 6.0.1. A set $S \subseteq \mathbb{C}$ is called a convex set if the line segment joining any pair of points in S lies entirely in S.

Theorem 14 (Cauchy's Theorem for Convex Sets)**.** *Let $\Omega \subseteq \mathbb{C}$ be a convex open set, and $f \in H(\Omega)$. Then,*

(1) $f = F'$ for some $F \in H(\Omega)$.

(2) $\displaystyle\int_\gamma f(z)\,dz = 0$ for any closed path $\gamma \in \Omega$.

Proof. Let $a \in \Omega$ and $[a,z]$ denote the straight line from a to z. Since Ω is a convex set, $[a,z]$ is in Ω. Define $F(z) = \displaystyle\int_{[a,z]} f(z)\,dz$. We wish to show that $F \in H(\Omega)$ and $F'(z_0) = f(z_0)$ for any $z_0 \in \Omega$. By Theorem 13,

$$\begin{aligned}
0 &= \int_\gamma f(z)\,dz \\
&= \int_{[a,z]} f(z)\,dz + \int_{[z,z_0]} f(z)\,dz + \int_{[z_0,a]} f(z)\,dz \\
&= F(z) + \int_{[z,z_0]} f(z)\,dz - F(z_0) \\
\implies \qquad F(z) - F(z_0) &= \int_{[z_0,z]} f(z)\,dz \\
\implies \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z)\,dz - f(z_0) \\
\implies \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} f(z) - f(z_0).
\end{aligned}$$

The last line follows as $\displaystyle\int_{[z_0,z]} dz = z - z_0$. Since $f \in H(\Omega)$ and is hence continuous, for ever $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Notice,

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| = \left|\frac{1}{z-z_0}\int_{[z_0,z]} [f(z)-f(z_0)]\,dz\right| \leq \frac{1}{z-z_0}\left|\int_{[z_0,z]} \epsilon\ dz\right| = \epsilon$$

so $F'(z_0) = f(z_0)$. By Corollary 12, $\displaystyle\int_\gamma f(z)\,dz = \int_\gamma F'(z)\,dz$ for any closed path $\gamma \in \Omega$. ☐

Definition 6.0.2. Let $\gamma : [\alpha,\beta] \to \mathbb{C}$ be a closed path and let Ω be the set complement of $\mathrm{Im}(\gamma)$, that is, $\Omega :- \mathbb{C}\backslash\gamma([\alpha,\beta])$. Then, the index of z with respects to γ (or the winding number) $\mathrm{Ind}_\gamma : \Omega \to \mathbb{C}$ is defined by

$$\mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

and denotes the number of times the contour C winds around the point w.

Theorem 15 (Cauchy's Integral Formula)**.** *Let $\Omega \subseteq \mathbb{C}$ be a convex open set, C be a closed circle path in Ω. If $w \in \Omega \setminus \mathbb{C}$ and $f \in H(\Omega)$. Then*

$$f(w)\operatorname{Ind}_C(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz.$$

Proof. For $z \in \Omega \setminus \{z\}$, define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{, if } z \neq w \\ f'(z) & \text{, if } z = w. \end{cases}$$

Then, g is continuous on Ω and holomorphic on $\Omega \setminus \{z\}$. Thus, by the Cauchy's Theorem for Convex Sets, we have $\displaystyle\int_C g(z)\,dz = 0$. Rearranging this, we get

$$\int_C \frac{f(z)}{z-w}\,dz = \int_C \frac{f(w)}{z-w}\,dz = f(w)\int_C \frac{1}{z-w}\,dz = 2\pi i\operatorname{Ind}_C(w)f(w).$$

□

Theorem 16. *Let $\Omega \subseteq \mathbb{C}$ be an open set, $f \in H(\Omega)$. Then f can be expressed as a power series.*

Proof. □