University of Waterloo

CO 255

Introduction to Optimization (Advanced)

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1 Introduction

1.1 Types of Optimization Problems

Given a set S and an objective function $f : S \to \mathbb{R}$, an optimization problem looks like,

$$\max_{\text{subject to (s.t.)}} f(x)_{x\in S}$$

which means to find some $x* \in S$ such that $f(x) \leq f(x*)$ for all $x \in S$. Here, S is called the "feasible region", and a point $\bar{x} \in S$ is called a "feasible solution" and $f(\bar{x})$ is called the "objective value".

Definition 1.1.1. For some feasible region S, $x*$ is an optimal solution if for all $x \in S$ that $f(x) \leq f(x*)$.

We can use different notation like

$$\max\{f(x) : x \in S\}$$

or

$$\max_{x\in S} f(x).$$

Also, there is a correspondence with minimization problems as,

$$\max_{x\in S} f(x) = -\left(\min_{x\in S} -f(x)\right)$$

A number of problems can arise,

- $S = \emptyset$ (This problem is always infeasible)
- If for all $a \in \mathbb{R}$, there always exists some $\bar{x} \in S : f(\bar{x}) > a$ (This problem is unbounded)
- $\max_{x<1} x$ (Here, an optimal solution does not exist)

Definition 1.1.2. A supremum is defined as

$$\sup\{f(x) : x \in S\} = \begin{cases} -\infty, & \text{, if infeasible} \\ +\infty, & \text{, if unbouded} \\ \min\limits_{f(x)\leq\lambda,\ \forall x\in S} \lambda & \text{, otherwise} \end{cases}$$

Definition 1.1.3. The infimum can be defined as

$$\inf_{x\in S} f(x) = -\sup_{x\in S} -f(x)$$

So now by replacing maximization problems with the supremum (and minimization problems with infimum), we would never fall into the case where "an optimal solution doesn't exist" as long as we consider the supremum (or infimum). In other words, if the problem is feasible and unbounded, we can always find an optimal solution.

Definition 1.1.4. A set $S \subseteq \mathbb{R}^n$ is convex if $t \in [0, 1]$ and for any $x, y \in S$, $tx + (1 - t)y \in S$.

Definition 1.1.5. For some set S, $f : S \to \mathbb{R}$ is <u>convex</u> if $t \in [0,1]$ and for any $x, y \in S$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

We can begin to define a few types of optimization problems:

1. **Linear Programming**

$$f(x) = c^T x \text{ and } S = \{x \in \mathbb{R}^n : Ax \leq B\} \text{ where } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \text{ and } c \in \mathbb{R}^n$$

2. **Integer Linear Programming**

$$f(x) = c^T \text{ and } S = \{x \in \mathbb{Z}^n : Ax \leq B\} \text{ where } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \text{ and } c \in \mathbb{R}^n$$

3. **Convex Optimization**

$$S \subseteq \mathbb{R}^n \text{ is convex and } f : S \to \mathbb{R} \text{ is convex}$$

Definition 1.1.6. The <u>convex hull</u> of $S \subseteq \mathbb{R}^n$, denoted by $\text{conv}(S)$ is the minimal convex set that contains S.

Remark. The convex hull of S is unique.

Consider the optimization problem $\min(f(x) : x \in S)$ where $S \subseteq R^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. We can "reduce" this problem to a convex optimization problem with a linear objective function.

<u>**Step 1:**</u> Linearize the objective function. Let

$$\hat{S} = \{(x, y) : x \in S, y = f(x)\} \subseteq \mathbb{R}^{n+1}$$

Then,

$$\min(f(x) : x \in S) = \min(y : (x, y) \in \hat{S})$$

<u>**Step 2:**</u> Convexify S. If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, $\min(f(x) : x \in S) = \min(f(x) : x \in \text{conv}(S))$.

Remark. This is true from a theoretical point of view, but it is not very practical in application. Taking the convex hull doesn't produce any better solutions.