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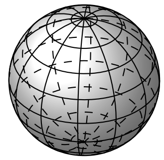

PMATH 348

Complex Analysis

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Contents

1 Complex Numbers

Definition 1.0.1. A complex number is a vector in $\mathbb{R}^2$. The complex plane denoted by $\mathbb{C}$ is the set of complex numbers.

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write,

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$iy = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

with $x, y \in \mathbb{R}$. If $z = x + iy, x, y \in \mathbb{R}$, then x is called the real part of z and y the imaginary part of z and write

$$\Re(z) = x \qquad \Im(z) = y$$

Definition 1.0.2. We define the sum of two complex numbers to be the vector sum.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \end{aligned}$$

We define the product of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative, associative and distributive over the sum. So,

$$\begin{aligned} (a + bi)(c + di) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + (ad + bc) \end{aligned}$$

Proposition 1 (Mulitplicative Inverses)**.** *Every complex number has a unique multiplicative inverse denoted by z^{-1}.*

Proof. Let $z = a+i, a, b \in \mathbb{R}$ with $a^2+b^2 = 0$. We want to solve for x and y such that $(a+ib)(x+iy) = 1$. In other words,

$$\begin{aligned} &(ax - by) + i(ay + bx) = 1 \\ \Rightarrow &\begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = (1, 0) \\ \Rightarrow &\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (1, 0) \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a}{a^2+b^2} \\ \frac{b}{a^2+b^2} \end{pmatrix}$$

This is unique as the inverse matrix is unique. □

Remark. The set of complex numbers is a field under the operations of addition and multiplication as operations are associative, commutative and distributive and every element has a unique inverse as before.

Definition 1.0.3. If $z = x + iy, x, y, \in \mathbb{R}$, then the conjugate of z is $\bar{z} = x - iy$.

Definition 1.0.4. We define the modulus (or length or magnitude) of $z = x + iy, x, y \in \mathbb{R}$ to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

Remark. For any $z, w \in \mathbb{C}$,

$$\begin{aligned}
\bar{\bar{z}} &= z \\
z + \bar{z} &= 2\Re(z) \\
z - \bar{z} &= 2\Im(z) \\
z \cdot \bar{z} &= |z|^2 \\
|z| &= |\bar{z}| \\
\overline{z+w} &= \bar{z} + \overline{w} \\
\overline{zw} &= \bar{z} \cdot \overline{w} \\
|zw| &= |z||w|
\end{aligned}$$

Proposition 2. *The following inequalities hold for any $z \in \mathbb{C}$.*

1. $|\Re(z)| \leq |z|$
2. $|\Im(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$
4. $|z + w| \geq \Big| |z| - |w| \Big|$

Proof. (1) and (2) follows as

$$|z|^2 = \Re(z)^2 + \Im(z)^2.$$

(3). Notice,

$$\begin{aligned}
|x + iy|^2 &= (x + iy)\overline{(x + iy)} \\
&= (x + iy)(\overline{x} + \overline{iy}) \\
&= x\overline{x} + y\overline{y} + x\overline{y} + y\overline{x}
\end{aligned}$$

$$
\begin{aligned}
&= |x|^2 + |y|^2 + x\bar{y} + y\bar{x} \\
&= |x|^2 + |y|^2 + 2\Re(x\bar{y}) \\
&\leq |x|^2 + |y|^2 + 2|x\bar{y}| \\
&= |x|^2 + |y|^2 + 2|x| \cdot |\bar{y}| \\
&= |x + y|^2
\end{aligned}
$$

Taking the square root of both sides gives the result.

(4). From (3), we have that

$$
\begin{aligned}
|z| &= |z - w + w| \leq |z - w| + |w| \\
|w| &= |w - z + z| \leq |w - z| + |z|
\end{aligned}
$$

Then, isolating $|z - w|$ implies the result. More specifically since we have the simulateous inequality,

$$
\begin{cases} |z| - |w| \leq |z - w| \\ |w| - |z| \leq |z - w| \end{cases} \Rightarrow |z - w| \geq \Big| |z| - |w| \Big|
$$

as desired. $\square$

Proposition 3. *Every non-zero complex number has exactly 2 square roots.*

Proof. Let $z = x + iy \in \mathbb{C}$ with $x^2 + y^2 \neq 0, x, y, \in \mathbb{R}$. We want to solve $w^2 = z$ for $w \in \mathbb{C}$. Say w takes the form $w = u + iv, u, v \in \mathbb{R}$. Then

$$
\begin{aligned}
&w^2 = z \\
\Rightarrow\ &(u + iv)^2 = x + iy \\
\Rightarrow\ &(u^2 - v^2) + i2uv = x + iy
\end{aligned}
$$

So we have that $x = u^2 - v^2$ and $y = 2uv^2$. We can solve for u and v. Take the square of both sides of the second equation to get $4u^2v^2 = y^2$. Now, we multiply the first equation by $4u^2$ to get

$$
\begin{aligned}
&4u^4 - 4u^2v^2 = 4xu^2 \\
\Rightarrow\ &4u^4 - 4xu^2 - y^2 = 0
\end{aligned}
$$

This is a quadratic equation over u^2 so,

$$
u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}
$$

Suppose that $y \neq 0$. Then we must take the positive solution above to get

$$
u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}
$$

Under the assumption that $x^2 + y^2 > 0$, this solution exists. Notice we cannot take the negative solution as it yields a negative u^2 which is impossible. We can use a similar procedure to find that

$$
v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}
$$

Rooting u and v gives 2 solutions for each. However, if y is positive, since $2uv = y$, u and v must take the same sign. Similarly, if y is negative, they must take different signs. In each of these cases, there are 2 solutions for u and v. So,

$$w = \begin{cases} \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) + i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y > 0 \\ \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) - i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y < 0 \\ \pm\sqrt{x} & , x > 0,\ y = 0 \\ \pm i\sqrt{-x} & , x < 0,\ y = 0 \end{cases}$$

□

Remark. Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots.

Remark. The square root doesn't distribute. Consider $z = w = -1 \in \mathbb{C}$. $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.

Remark. The Quadratic Formula holds true for complex polynomials. In other words, if $a, b, c \in \mathbb{C}, a \neq 0$,

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 1.0.5. If $z \in \mathbb{C} \setminus \{0\}$, we define the angle (or argument) of z to be the angle $\theta(z)$ from the positive x-axis counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big).$$

Remark. For $\theta \in \mathbb{R}$ (or for $\theta \in \mathbb{R}/2\pi$), we have that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Remark. If $z \neq 0$, we have $x = \Re(z)$, $y = \Im(z)$, $r = |z|$ and

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \tan\theta &= \frac{y}{x}, \text{ if } x \neq 0 \\ z &= rei\theta \\ \bar{z} &= re^{-i\theta} \\ z^{-1} &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

Remark. We now have 2 representations of a complex number $z \in \mathbb{C}$. We say that $z = x + iy$ is the cartesian coordinates of z and $z = re^{i\theta}$, where $r = |z|$, is the polar form of z.

Consider $z = re^{i\alpha}$ and $w = se^{i\beta}$. We have,

$$\begin{aligned} zw &= rs(\cos\alpha + i\sin\alpha)(\sin\beta + i\cos\beta) \\ &= rs\big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) \\ &= rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big) \end{aligned}$$

$$= e^{i(\alpha+\beta)}$$

which defines a formula for multiplication in polar coordinates. Notice that the following identity known as De Moivre's Law follows.

$$(re^{i\theta})^n = r^n e^{in\theta}$$

for all $r, \theta \in \mathbb{R}$, $n \in \mathbb{Z}$. We can use this identity to find the n^{th} roots of z. In other words, we solve $w^n = z$. We have,

$$\begin{aligned} & w^n = z \\ \Rightarrow & (se^{i\alpha})^n = re^{i\theta} \\ \Rightarrow & s^n e^{in\alpha} = re^{i\theta} \end{aligned}$$

so $s^n = r$ and $n\alpha = \theta + 2\pi k$ for $k \in \mathbb{Z}$. In other words, we have

$$(re^{i\theta}) = \sqrt[n]{r} e^{i(\theta+2\pi k)/n}, \quad k = 0, \ldots, n-1$$

Remark. When working with complex numbers, for $0 \neq z \in \mathbb{C}$, and for $0 < n \in \mathbb{Z}$, $\sqrt[n]{z}$ or $z^{1/n}$ denotes either one of the n roots, or the set of all n^{th} roots.

Example 1.0.6. Consider the $n-1$ diagonals of a regular n-gon inscribed in a circle of radius 1 obtained by connecting one vector with all the others. Show that the product of these diagonals is n.

Notice that $z_2, \ldots, z_n$ are the n^{th} roots of unity other than 1. Let z be the variable and consider the polynomial

$$P(z) \coloneqq 1 + z + \cdots + z^{n-1}.$$

Since the roots of $P(z)$ are n^{th} roots of unity other than 1, we can factorize

$$\begin{aligned} P(z) &= 1 + z + \cdots + z^{n-1} \\ &= (z - z_2) \ldots (z - z_n) \end{aligned}$$

and setting $z = 1$, the result follows. In particular, we have

$$|1 - z_2| \ldots |1 - z_n| = n.$$

2 Complex Functions

2.1 Limits

Definition 2.1.1. A sequence of complex numbers $z_1, z_2 \ldots$ converges to $z \in C$ if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

Equivalently, given any $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ sufficiently large such that $|z_n - z| < \epsilon$ whenever $n > N$.

Remark. If $\{z_n\}_n$ converges to z, we write

$$\lim_{n\to\infty} z_n = z$$

or $z_n \to z$ as $n \to \infty$.

Example 2.1.2. For $|z| > 1$, show that $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converges.

Notice,

$$\lim_{n\to\infty}\left|\frac{1}{z^n} - 0\right| = \lim_{n\to\infty}\left|\frac{1}{z^n}\right| = 0$$

as $|z| > 1$.

Example 2.1.3. Show that $\{i^n\}_{n-1}^{\infty}$ does not converge.

Definition 2.1.4. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say

$$\lim_{z\to z_0} f(z) = L$$

if for every sequence $\{z_n\}_n \subseteq \Omega$ we have that $z_n \to z \Rightarrow f(z_n) \to L$.

Remark. Here, z_0 need not to be in Ω.

Example 2.1.5. Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_z \to 0 f(z)$.

If $z = x \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{x}{x} = 1$. So $\lim_{x\to 0} f(x) = 1$. If $z = iy, y \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{-iy}{iy} = -1$. So $\lim_{y\to 0} f(iy) = -1$. Hence, the limit does not exist.

Example 2.1.6. Show that $z_n \to z$ if and only if $\Re z_n \to \Re z$ and $\Im z_n \to z$.

2.2 Function Continuity

Definition 2.2.1. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say f is <u>continuous at $z_0 \in \Omega$</u> if for every sequence $\{z_n\} \subseteq \Omega$, we have $z_0 \to z \Rightarrow f(z_0) \to f(z)$. Equivalently, given any $\epsilon > 0, \exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Remark. f is continuous on Ω if it is continuous at ever point of Ω.

Remark. We may split f into its real and imaginary parts

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

2.3 Holomorphic Functions

Definition 2.3.1. An open disk of radius r at z_0 with $r > 0$ is the <u>neighborhood</u> around z_0 denoted by $D(z_0, r)$ with

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 2.3.2. Let $f(z)$ be defined in a neighborhood of z_0. We say f is <u>complex differentiable</u> (or holomorphic) at z_0 if

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote the limit by $f'(z_0)$.

Remark. Here, $h \in \mathbb{C}$ can approach zero from any direction in $\mathbb{C}$.

Example 2.3.3. Where is $f(z) = \frac{1}{z}, z \neq 0$ holomorphic?

Notice,

$$\lim_{h\to 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h\to 0} \frac{-h}{(z_0+h)(z_0)} = -\frac{1}{z_0^2}$$

So, f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and $f'(z) = -\frac{1}{z_0^2}$.

Example 2.3.4. $f(z) = \bar{z}$ is not holomorphic at any $z \in \mathbb{C}$.

Notice,

$$\lim_{h\to 0} \frac{\overline{z_0 + h} - \overline{z_0}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}$$

which does not exist from Example 2.1.5. However, the function can be though of a map from $\mathbb{R}^2 \to \mathbb{R}^2$ defined as $(x, y) \mapsto (x, -y)$ which is differentiable as all partial derivatives exist. This distinguishes real analysis from complex analysis.

Remark. If f and g are holomorphic, so are $f + g$, fg and $\frac{f}{g}$ (when $g \neq 0$). The proof is identical to the statements of real functions.

We can now generalize when a function is complex differentiable. If the complex function $f(z) = u + iv$. If the complex derivative $f'(z)$ is to exist, then it must be that the limit exists approaching from both the real and imaginary axis. Thus, we have,

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it}$$

where t is a real number. In terms of u and v, taking the derivative along the real line gives,

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(z+t) - f(z)}{t} \\
&= \lim_{t\to 0} \frac{u(x+t, y) + iv(x+t, y) - u(x, y) - iv(x, y)}{t} \\
&= \lim_{t\to 0} \frac{u(x+t, y) - u(x, y)}{t} + i \lim_{t\to 0} \frac{v(x+t, y) - v(x, y)}{t} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.
\end{aligned}$$

Taking the derivative along the vertical line gives,

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(z+it) - f(z)}{it} \\
&= -i \lim_{t\to 0} \frac{u(x, y+t) + iv(x, y+t) - u(x, y) - iv(x, y)}{t} \\
&= -i \lim_{t\to 0} \frac{u(x, y+t) - u(x, y)}{t} + \lim_{t\to 0} \frac{v(x, y+t) - v(x, y)}{t} \\
&= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.
\end{aligned}$$

Equating real and imaginary parts, we arrive at the following theorem.

Theorem 4 (Cauchy-Riemann Equations)**.** *If a function $f(z) = u + iv$ is holomorphic in a neighborhood around $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \textit{at } (x_0, y_0)$$

with

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Example 2.3.5. Show that

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{, if } z \neq 0 \\ 0 & \text{, if } z = 0 \end{cases}$$

is not holomorphic at $z = 0$ and that the Cauchy-Reimann Equations hold at $z = 0$.

Notice,

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\bar{h}^2}{h^2} = \lim_{h\to 0} \left(\frac{\bar{x} - iy}{x + iy}\right)^2$$

Let $h = x + imx$, $m \neq 0, x \to 0$. We get

$$\lim_{x\to 0} \left(\frac{x - imx}{x + imx}\right)^2 \left(\frac{1 - im}{1 + im}\right)^2$$

which is dependent of m and thus the limit does not exist. Now, notice we have

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{-3x^2y + y^3}{x^2 + y^2}$$

So we have

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & \text{, if } (x, y) \neq (0, 0) \\ 0 & \text{, if } (x, y) = (0, 0) \end{cases}$$

and

$$v(x, y) = \begin{cases} \frac{-3x^2y + y^3}{x^2 + y^2} & \text{, if } (x, y) \neq (0, 0) \\ 0 & \text{, if } (x, y) = (0, 0) \end{cases}.$$

Now, we can verify that the Cauchy-Riemann Equations hold. Indeed,

$$\begin{aligned}
\frac{\partial u}{\partial x}\left(\frac{x^3 - 3xy^2}{x^2 + y^2}\right) &= \frac{\left(\frac{\partial}{\partial x}(x^3 - 3xy^2)\right)(x^2 + y^2) - (x^3 - 3xy^2)\left(\frac{\partial}{\partial x}(x^2 + y^2)\right)}{(x^2 + y^2)^2} \\
&= \frac{(3x^2 - 3y^2)(x^2 + y^2) - (x^3 - 3xy^2)(2x)}{(x^2 + y^2)^2} \\
&= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} \\
\frac{\partial v}{\partial y}\left(\frac{y^3 - 3x^2y}{x^2 + y^2}\right) &= \frac{\left(\frac{\partial}{\partial y}(y^3 - 3x^2y)\right)(x^2 + y^2) - (y^3 - 3x^2y)\left(\frac{\partial}{\partial y}(x^2 + y^2)\right)}{(x^2 + y^2)^2} \\
&= \frac{(3y^2 - 3x^2)(x^2 + y^2) - (y^3 - 3x^2y)(y^2)}{(x^2 + y^2)^2} \\
&= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2}
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial y}\left(\frac{x^3-3xy^2}{x^2+y^2}\right) &= \frac{\left(\frac{\partial}{\partial y}(x^3-3xy^2)\right)(x^2+y^2)-(x^3-3xy^2)\left(\frac{\partial}{\partial y}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2)-(x^3-3xy^2)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2} \\
\frac{\partial v}{\partial x}\left(\frac{y^3-3x^2y}{x^2+y^2}\right) &= \frac{\left(\frac{\partial}{\partial x}(y^3-3x^2y)\right)(x^2+y^2)-(y^3-3x^2y)\left(\frac{\partial}{\partial x}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2)-(y^3-3x^2y)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2}
\end{aligned}
$$

Thus, we can consider the converse statement of Theorem 4.

Theorem 5. *Let $f = u + iv : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in \Omega$. If*

1. *the partials of u, v exist in a neighborhood of (x_0, y_0)*
2. *the partials of u, v are continuous at (x_0, y_0)*
3. *$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0)*

then, f is holomorphic at z_0.

TODO: find proof online.

Example 2.3.6. Consider the power series, an expression of the form

$$\sum_{n=0}^{\infty} c_n z^n$$

where $c_n \in \mathbb{C}$. This expression converges if the sequence of partials sums, $\{s_N\}$ defined by

$$s_N \coloneqq \sum_{n=0}^{N} c_n z^n$$

converges as $N \to \infty$. This is quite a strong condition, so we consider the following definition.

Definition 2.3.7. A power series expression converges absolutely if

$$\sum_{n=0}^{\infty} |c_n||z|^n$$

converges.

Remark. Absolute convergence implies converges. Notice,

$$\left|\sum_{n=0}^{N} c_n z^n\right| = \sum_{n=0}^{N} |c_n||z|^n$$

for each $N \in \mathbb{N}$.

Theorem 6. *For any power series* $\sum_{n=0}^{\infty} c_n z^n, \exists 0 \leq R \leq \infty$*, such that*

1. *If* $|z| < R$*, the series converges absolutely*
2. *If* $|z| > R$*, the series diverges.*

Moreover, R *is given by Hadamard's formula:* $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$

Remark. R is called the radius of convergence of the series and $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.

Remark. Recall,

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{m\leq n} a_m\right)$$

and is the "highest peak reached by a_n's as $n \to \infty$".

Proposition 7 (Property of lim sup)**.** *If* $L = \limsup_{n\to\infty} a_n$*, then for any* $\epsilon > 0, \exists N > 0$ *such that* $\forall n \leq N, a_n < L + \epsilon$

Proof of Theorem 6. Let $L := \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$ Clearly, $L \leq 0$.

1. Suppose $|z| < R$. So, there exists some $\epsilon > 0$ such that $r := |z|(L+\epsilon) < 1$ and $0 < r < 1$. By Proposition 7, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|c_n|^{\frac{1}{n}} < L + \epsilon$. Now,

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty}\left(|c_n|^{\frac{1}{n}}|z|\right)^n < \sum_{n=N}^{\infty} r^n$$

   converges as $0 < r < 1$. By the comparison test, $\sum_{n=N}^{\infty} |c_n||z|^n$ is monotonic and bounded and thus converges by Bolzano-Weierstrass.

2. This follows from the proof above. Specifically, this time, notice that there exists some $\epsilon > 0$ such that $r := |z|(L-\epsilon) > 1$. Again, by Proposition 7, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|c_n|^{\frac{1}{n}} > L - \epsilon$ so that

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty}\left(|c_n|^{\frac{1}{n}}|z|\right)^n > \sum_{n=N}^{\infty} r^n$$

   follows and so the series diverges.

□

Theorem 8. *Suppose* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *has radius of convergence* R*. Then* $f'(z)$ *exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$*. Moreover,* f' *has the same radius of convergence as* f*.*

Proof. f' has some radius of convergence because

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{1}{n}}|c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Let $|z_0| \le r < R$, $g(z_0) \coloneqq \sum_{n=1}^{\infty} nc_n z_0^{n-1}$. We want to show

$$\lim_{k\to 0} \frac{f(z_0+h)-f(z_0)}{h} = g$$

or equivalently, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| < \epsilon.$$

For any fixed $\epsilon > 0$, we write

$$f(z) = \underbrace{\sum_{n=0}^{N} c_n z^n}_{\coloneqq S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{\coloneqq E_N(z)}$$

We have $S'_N = \sum_{n=1}^{N} nc_n z^{n-1}$ and

$$\begin{aligned}
&\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| \\
&= \left|\frac{S_N(z_0+h)-S_N(z_0)}{h} + \frac{E_N(z_0+h)-E_N(z_0)}{h} - g(z_0) + S'_N(z_0) - S'_N(z_0)\right| \\
&= \left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S'_N(z_0)\right| + \left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| + \left|S'_N(z_0) - g(z_0)\right|.
\end{aligned}$$

We have

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| = \left|\frac{1}{h}\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^h - z_0^n\big)\right|$$

As $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ in any ring, we have

$$\left|\frac{1}{h}\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^h - z_0^n\big)\right| = \left|\sum_{n=N+1}^{\infty} c_n\big((z_0+h)^{n-1} + (z_0+h)^{n+2}z_0 + \cdots + (z_0+h)z_0^{n-2} + z_0^{n-1}\big)\right|$$

Now, by choosing δ relatively small so that $|z_0| \le r$, we have $|z_0|, |z_0+h| \le r$ and so

$$(z_0+h)^{n-1} + (z_0+h)^{n+2}z_0 + \cdots + (z_0+h)z_0^{n-2} + z_0^{n-1} \le nr^{n-1}$$

So,

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| \le \sum_{n=N+1}^{\infty} nc_n r^{n-1} < \frac{\epsilon}{3}.$$

for a large enough N. $\square$