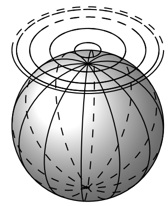
University of Waterloo



PMATH 733

Set and Model Theory

Prof. Rahim Moosa • Fall 2018

Contents

1 Introduction

1.1 Contact Information

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1.2 Office Hours

Every Wednesday, 11:00-12:00 in MC 5018

1.3 Course Description

This course is approximately $1/3$ set theory and $2/3$ model theory. The set theory will be somewhat naive (i.e., not formal) and the model theory with be semantic (i.e., no proof theory). There will be some overlap with PMATH 432/632 (First Order Logic and Computability), but this latter course is neither a pre-requisite nor an anti-requisite. The pre-requisite for this course is a familiarity with algebra: groups, fields, vector spaces, polynomial rings.

Official lecture notes can be purchased for $13.62 from Media.doc in MC 2018. The following reference books are on reserve in the Davis Center Library:

- <u>Set theory: an introduction to independence proofs</u> by K. Kunen, and
- <u>Model Theory: an introduction</u> by D. Marker.

Here are the topics I hope to cover:

- *Set Theory* (four+ϵ weeks): Zermelo-Fraenkel axioms, classes, trans-finite induction/recursion, well-orderings and ordinals, the axiom of choice and equivalents, cardinal arithmetic.
- *Model theory* (eight-ϵ weeks): First-order logic (structures, languages, theories), definable sets, the compactness theorem (via ultraproducts) and its consequences, quantifier elimination, algebraic examples (vector spaces, algebraically closed fields, real closed fields), model companions.

There will be five or six homework assignments worth a total of 30% and a final exam worth 70%. There will be no midterm exam.

2 Zermelo–Fraenkel Set Theory

2.1 Ordinals

We use the natural numbers $0, 1, 2, \ldots$ to count finite sets. Here, count has two related meanings:

- enumerate, order
- measuring size

We wish to develop an infinitary generalization of the natural numbers so that we can $\underbrace{\text{enumerate}}_{\text{ordinals}}$ and $\underbrace{\text{measure}}_{\text{cardinals}}$ arbitrary sets. But first, we want a concrete construction of the natural numbers. We need to start somewhere:

- set
- equality
- membership (denotes $\in$)

We can construct natural numbers as follows:

- $\emptyset = 0 :=$ the empty set, denoted $\emptyset$ (the set with no members)
- $\{\emptyset\} = 1 :=$ the set whose only number is 0, denoted by $\{0\}$
- $\{\emptyset, \{\emptyset\}\} = 2 :=$ the set whose only members are 0 and 1, so $\{0, 1\}$
- and so on...

In general, given a natural number n already constructed we deine the next natural number, the <u>successor of n</u> to be

$$S(n) = n \cup n$$

Before continuing, we should ask do these sets exist? Use axioms - unproved statements.

Axiom 1 (Empty Set Axiom)**.** *There exists a set with no members, denoted $\emptyset$.*

To get 1 from 0, we invoke the following:

Axiom 2 (Pairset Axiom)**.** *Given sets x, y, there exists a set, denoted by $\{x, y\}$, whose only members are x y. That is, $t \in \{x, y\} \iff t = x$ or $t = y$.*

If $x = y$, then $\{x, y\}$ has only x as a member. We recognize that $1 = \{0, 0\}$ (denoted by $\{0\}$). This requires an axiom.

Axiom 3 (Axiom of Extension)**.** *Given sets x, y, $x = y$ <u>if and only if</u> x & y have the same members.*

So far, 0 ($\emptyset$), 1 ($\{\emptyset\}$) and 2 ($\{\emptyset, \{\emptyset\}\}$) exist. What about $3 := \{0, 1, 2\} = \underbrace{\{0, 1\}}_{=2} \cup \{2\}$? In general from n, to get $S(n) = n \cup \{n\}$, we need <u>unions</u>, given by the following axiom.

Axiom 4 (Unionset Axiom)**.** *Given a set x, there exists a set, denoted by Ux, whose members are precisely the members of the members of x. That is,*

$$t \in Ux \iff t \in y$$

for some $y \in x$.

So given n, $S(n) := U\{n, \{n\}\}$. Be definition, $t \in S(n) \iff t \in n$ or $t = n$. With these axioms, we have constructed rigorously each natural number. But, what about the set of <u>all</u> natural numbers?

The set of natural numbers should be the smallest set that contains 0 and is preserved by the successor function. We wish to express this set with definite conditions.

Axiom 5 (Infinity Axiom)**.** *There exists a set I that contains 0 and is preserved by the successor function.*

We can express this axiom in the following definite logical statement.

$$\exists I\big((0 \in I) \wedge \forall x(x \in I(\exists y(\underbrace{\forall t(t \in y \iff (t \in x) \wedge (t = x))}_{y=S(x)} \wedge y \in I)))\big)$$

However, we wish to construct the minimal successor set and we can try to take the intersection of all successor sets.

Definition 2.1.1. If x and y are sets, a <u>subset</u> denoted by $x \subseteq y$ means that every element of x is an element of y.

We can express xy as

$$\forall z(z \in x \implies z \in y)$$

Axiom 6 (Powerset Axiom)**.** *Given a set x, there exists a set $\mathcal{P}(x)$ with the axiomerty that*

$$\forall t(t \in \mathcal{P}(x) \iff t \subseteq x)$$

Axiom 7 ((Bounded) Separation Axiom)**.** *Given a set x and a definite condition P, there exists a set of elements are precisely the members of x that satisfy P.*

$$\exists\{z \in x : P(z)\}$$

Remark. It is necessary for us to have (bounded by) x and P is definite.

$$\forall x \exists y \forall t(t \in y \iff (t \in x \wedge P(t)))$$

Exercise 1. *Given a non-empty set x, there exists a set $\cap x$ satisfying:*

$$\forall t(t \in \cap x \iff \forall y(y \in x \implies t \in y))$$

Definition 2.1.2. Fix a successor set I. The <u>set of natural numbers</u> is

$$\begin{aligned}\omega &\coloneqq \cap\{J \subseteq I : \underbrace{0 \in J \wedge \forall x(x \in J \implies S(c) \in J)}_{J \text{ is a successor set}}\} \\ &= \cap\{J \in \mathcal{P}(I) : J \text{ is a successor set}\}\end{aligned}$$

Exercise 2. *This does not depend on I.*

Another very useful axiom that will be used later:

Axiom 8 (Replacement Axiom)**.** *Suppose P is a binary definite condition such that for every set x there is a unique y satisfying $P(x, y)$. Given a set A there is a set B such that $t \in B$ <u>if and only if</u> there is an $a \in A$ with $P(a, t)$.*

2.2 Classes

We sometimes want to consider collection of sets that do not form a set themselves. For example, there is no set containing all sets.

Proof. Suppose a set U contains all sets. Consider

$$R = \{x \in U : x \notin x\}.$$

If $R \in R$, since $R \in U$, $R \notin R$, so a contradiction. But if $R \notin R$, by definition $R \in R$, so no such U exists. $\square$

2.3 Cartiasian Products & Functions

Definition 2.3.1. Given sets x, y, the <u>ordered pair of x and y</u> is

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Definition 2.3.2. Given two classes X, Y, the <u>cartisian product of X and Y</u>, denoted by $X \times Y$ is

$$X \times Y \coloneqq [[z : z = (x, y), x \in X, y \in Y]].$$

To see that $X \times Y$ exists, we define the cartisian product with a definite statement.

$$\exists x, y\Big((x \in X) \wedge (\in Y) \wedge \exists a, b \underbrace{\forall t(t \in a \iff t = x)}_{a=\{x\}} \wedge \underbrace{\forall t(t \in b \iff t = x \vee t = y)}_{b=\{x,y\}} \wedge \underbrace{\forall(t \in z\ iff t\ = z \vee t = b)}_{z=\{a,b\}}\Big)$$

Remark. If A is a set, B is a class and $B \subseteq A$, then B is a set.

Proof. Notice,

$$B = \{a \in A : a \in B\}$$

So by (bounded) separation B is a set. $\square$

Remark. If X, Y are sets the $\mathcal{P}(\mathcal{P}(X \cup Y))$ is a set and

$$X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$$

so $X \times Y$ is a set.

Definition 2.3.3. Given classes X, Y, a definite operation $f : X \to Y$ is a class $\Gamma(f)$ such that $\Gamma(f) \subseteq X \times Y$ and for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in \Gamma(f)$. We write $f(x) = y$ to mean that $(x, y) \in \Gamma f$

Remark. If X, Y are sets and $f : X \to Y$ is a definite operation, then $\Gamma(f) \subseteq \underbrace{X \times Y}_{\text{set}}$, so $\Gamma(f)$ is a set. In this case, we call f a function.

Definition 2.3.4. A function is a definite operations $f : X \to Y$ where X, Y are both sets.

Proposition 9 (Induction Principle)**.** *Suppose $J \subseteq \omega$, $0 \in J$ and whenever $n \in J$ then $S(n) \in J$. Then $J = \omega$.*

Proof. J is a successor set, so by definition, $\omega \subseteq J$, so $J = \omega$. □

Lemma 10. *Suppose $n \in \omega$.*

(a) Every element of n is a subset of ω.

(b) Every element of n is a subset of n.

(c) $n \notin n$.

(d) Either $n = 0$ or $0 \in n$.

(e) If $y \in n$ then either $S(y) \in n$ or $S(y) = n$.