University of Waterloo

MATH 249

Introduction toi Combinatorics (Advanced)

Prof. David Wagner • Fall 2018

Contents

1 Introduction

1.1 Course Outline

MATH 249 has two parts: first part will be **Enumeration** and second part will be **Graph Theory**. We will cover everything in MATH 239, and plenty of extra topics.

Topics in Enumeration: Rational power series, recurrence relations, partial fractions. Binomial Theorem, Vandermonde convolution, multisets, Binomial Series.Generating functions, Sum and Product lemmas, computing averages. Compositions and rational binary languages. Rational languages andblock patterns. The transfer matrix method, domino tilings, etc. Lattice paths, Catalan numbers.

Topics in Graph Theory: Basic terminology, isomorphism, examples. Paths and cycles, connectedness, spanning trees. Planar graphs, Kuratowski's Theorem, graphs on other surfaces. Graph colouring, the Five-Colour Theorem. Bipartite matching, hints at the non-bipartite case. The Matrix-Tree Theorem

1.2 Contact Information

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1.3 Office Hours

To be determined

1.4 Grading Scheme

- Homework - 15% (Due every *second* Friday, starting September 21$^{\text{th}}$, at the start of class)
- Midterm Exam - 30%
- Final Exam - 55%

2 Enumeration

2.1 Generating Functions

Let's start with the definition of the well know Fibonacci Sequence. With $f_0 = f_1 = 1$, we define $f_n \coloneqq f_{n-1} + f_{n-2}$ for $n \geq 2$. We arrive at the following sequence:

Table 1: First 9 values of the Fibonacci Sequence

n	0	1	2	3	4	5	6	7	8	9	...
f_n	1	1	2	3	5	8	13	21	34	55	...

What is f_n as a function of n?

Definition 2.1.1. A generating function $f(x)$ is a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

whose coefficients give the sequence $\{a_0, a_1, \dots\}$.

Example 2.1.2. Consider the Geometric Series, defined by

$$G = 1 + z + z^2 + z^3 + \dots$$

and,

$$zG = z + z^2 + z^3 + \dots$$

so, $G - zG = 1$, thus the generating function for G is

$$G = \frac{1}{1-z}.$$

Now, if we consider the generating function for the Fibonacci Sequence, we have,

$$\begin{aligned} F(x) &= f_0 + f_1 x + \sum_{n=2}^{\infty} f_n x^n \\ &= 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\ &= 1 + x + x \sum_{j=1}^{\infty} f_j x^j + x^2 \sum_{n=0}^{\infty} f_k x^k \\ &= 1 + x + x\big(F(x) - 1\big) + x^2 F(x) \end{aligned}$$

So,

$$F(x) - xF(x) - x^2 F(x) = 1$$

or,

$$F(x) = \frac{1}{1 - x - x^2}$$

We continue by applying partial fractions to $F(x)$. First, we look for the roots of the denominator. Notice, we can factor our denominator as

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

to arrive at the auxiliary function

$$t^2 - t - 1 = (t-\alpha)(t-\beta).$$

The quadratic formula gives the solution for α, β:

$$\alpha, \beta = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot -1}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

Now, there exist $A, B \in \mathbb{C}$ such that

$$F(x) = \frac{1}{1-x-x^2} = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}.$$

We have,

$$1 = A(1-\beta x) + B(1-\alpha x) = (A+B) + (-A\beta - B\alpha)x$$

and the system,

$$\begin{cases} A + B = 1, \\ (-A\beta - B\alpha) = 0 \end{cases} \Rightarrow \begin{cases} A\alpha + B\alpha = \alpha, \\ A(\beta - \alpha) = 0 \end{cases} \Rightarrow \begin{cases} A\beta + B\beta = \beta, \\ B(\alpha - \beta) = \beta \end{cases}$$

so we have

$$A = \frac{\alpha}{\alpha - \beta} = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} = \frac{5+\sqrt{5}}{10}$$

and

$$B = \frac{\beta}{\beta - \alpha} = \frac{1}{-\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2} = \frac{5-\sqrt{5}}{10}.$$

Finally, we arrive at the original generating function,

$$\begin{aligned} F(x) = \sum_{n=0}^{\infty} f_n x^n &= \frac{1}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A\sum_{n=0}^{\infty} (\alpha x)^n + B\sum_{n=0}^{\infty} (\beta x)^n && \text{(by geometric series)} \\ &= \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) x^n. \end{aligned}$$

so, $f_n = A\alpha^n + B\beta^n$ for all $n \in \mathbb{N}$.

2.2 Partial Lists

Theorem 1. *The number of possibilities to arrange n (distinguishable) objects in a row (called permutations) is*

$$n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1.$$

Proof. There are obviously n choices for the first position, then $n-1$ remaining choices for the second position, $n-2$ for the third position, etc. Continuing on, one obtains the stated formula. $\square$

Definition 2.2.1. Let S be a set of size $|S| = n$. A partial set of S of length k is a sequence $s_1, s_2, \dots, s_k$ with each $s_i, s_j \in S$ and $i \neq j$ having $s_i \neq s_j$.

Theorem 2. *The number of partial lists of length k from a set of size n is*

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)\dots(n-k+1).$$

Proof. The proof is essentially the same as for Theorem 1: for the first element, there are n possible choices, then $n-1$ for the second element, etc. For the last element, there are $n-k+1$ choices left. $\square$

Definition 2.2.2. A <u>k-subset of a set S</u> where $|S| = n$ is a collection of elements $\{s_1, \dots, s_k\}$ and no two are equal, for each $s_1 \in S$.

Theorem 3. *The number of k-subsets of a set S where $|S| = n$ is*

$$\binom{n}{k} = \frac{n!}{k!(n-k!)}$$

Proof. Let $C(n,k)$ be the number of possibilities that we are looking for. Once the k elements have been chosen (for which there are $C(n,k)$ possible ways), we has $k!$ possibilities (by Theorem 1) to arrange them in a sequence. Therefore, $C(n,k) \cdot k!$ is exactly the number of possible sequences of k distinct elements, for which we have the formula

$$C(n,k) \cdot k! = \frac{n!}{(n-k)!} \qquad \text{(by Theorem 2)}$$

and the result follows immediately. $\square$

Definition 2.2.3. The <u>multiplicity of an element s in set S</u> is the number of times s is in the set S, denoted by $\mu(s)$.

Definition 2.2.4. A <u>multiset of set S</u> is a pair (S, μ) where $\mu : S \to \mathbb{N}_{>0}$ is a mapping of the multiplicity of each element in S.

Theorem 4. *The number of multisets of size n with t elements is*

$$\binom{n+t-1}{t-1} = \binom{n+t-1}{n}.$$

Proof. This popular proof takes advantage of an illustration with stars and bars. Suppose we represent the multiplicity of each element (up to ordering) with stars, separating each element with stars. Note that since multiplicity is non-negative, we can place bars consecutively. For example, when $n = 7$ and $k = 5$, the tuple $(4, 0, 1, 2, 0)$ representing multiplicity can be illustrated by the following diagram:

$$\star\star\star\star \,||\, \star \,|\, \star\star \,|$$

This establishes a one-to-one correspondence between tuples of the desired form and selections with replacement of $k-1$ gaps from the $n+1$ available gaps, or equivalently, $(k-1)$-element multisets drawn from a set of size $n+1$. Observe that the desired arrangements consist of $n+k-1$ (n stars and $k-1$ bars). Choosing the positions of stars leaves exactly $k-1$ spots for $k-1$ bars. In other words, choosing the position of stars determines the entire arrangement, so by Theorem 3, the arrangement is determined with $\binom{n+k-1}{n}$ selections. But,

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(n-n+k-1)!} = \frac{(n+k-1)!}{(n+k-1-(k-1))!(k-1)!} = \binom{n+k-1}{k-1}$$

reflecting on the fact that choosing the positions for $k-1$ bars also determines the arrangements. $\square$

2.3 Bijections

Definition 2.3.1. Let S, T be sets and $f : S \to T$ be a function. The function f is surjective if for all $t \in T$, there exists some S such that $f(s) = t$

Definition 2.3.2. Let S, T be sets and $f : S \to T$ be a function. The function f is injective if for all $s, s' \in S$ and $f(s) = f(s')$, $s = s'$.

Definition 2.3.3. A function is bijective if f is both injective and surjective. We say f is a bijection.

Proposition 5. *A function f is bijective if and only if for all $t \in T$, there exists exacely one $s \in S$ such that $f(s) = t$.*

Definition 2.3.4. A bijection has an inverse function $f^{-1} : T \to S$ defined by $f^{-1}(t) = s$ if and only if $f(s) = t$.

Remark. The inverse function is unique and $(f^{-1})^{-1}$.

Definition 2.3.5. Two sets, S, T are equicardinal, denoted by $S \rightleftharpoons T$ if there is a bijection $f : S \to T$. This is an equivalence relationship.

Proposition 6. *$S \rightleftharpoons T$ if and only if S and T have the same size.*

2.4 Subsets & Indicator Vectors

Let $\mathcal{P}_n$ be the set of all subsets $S \subseteq \{1, 2, \dots, n\}$. Let $\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}}$. A typical element $\alpha = (a_1, \dots, a_n)$, where each $a_i \in \{0,1\}$. We wish to show that $\mathcal{P}_n \rightleftharpoons \{0,1\}^n$. In other words, we wish to establish mutually inverse bijections. Consider:

$$S \mapsto \alpha \text{ with } \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases} \text{ for } 1 \leq i \leq n$$

Similarly,

$$\alpha \mapsto S = \big\{ i \in \{1, \dots, n\} : a_i = 1 \big\}$$

We call this bijection an indicator function and $\{0,1\}^n$ an indicator vector.

2.5 Binomial Theorem

Consider the set of all subsets $S \subseteq \{1, 2, \dots, n\}$ and summing some variable, x and raising the cardinality of all these subsets to some power. We have

$$\sum_{S \subseteq \{1,2,\dots,n\}} x^{|S|} = \sum_{k=0}^{n} \binom{n}{k} x^k$$

since by Theorem 2 $\{1, 2, \dots, n\}$ has $\binom{n}{k}$ subsets of size k. Also, since $S \rightleftharpoons \{0,1\}^n$,

$$\sum_{S \subseteq \{1,2,\dots,n\}} x^{|S|} = \sum_{\alpha \in \{0,1\}^n} x^{a_1 + \cdots + a_n} \qquad \text{(by Cartisian Product)}$$

$$
\begin{aligned}
&= \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \cdots \sum_{a_n=0}^{1} x^{a_1+\ldots a_n} \\
&= \sum_{a_1=0}^{1} x^{a_1} \cdot \sum_{a_2=0}^{1} x^{a_2} \cdot \cdots \cdot \sum_{a_n=0}^{1} x^{a_n} \\
&= \left(\sum_{c=0}^{1} x^c \right)^n \\
&= (1 + x^n).
\end{aligned}
$$

We have,

Proposition 7 (Binomial Theorem)**.** *For any real number n,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

2.6 Binomial Series

Fix a positive integer t. Let $\mathcal{M}(t)$ be the set of all multisets with elements of t types, of any size. Let $\mu = (m_1, m_2, \ldots, m_t)$ with each $m_i \in \mathbb{N}$. That is,

$$\mathcal{M}(t) = \mathbb{N}^t = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \mathbb{N}}_{t \text{ factors}}$$

and

$$|\mu| = m_1 + \cdots + m_t$$

The number of multisets of size n is $\mathcal{M}(t)$ is $\binom{n+t-1}{t-1}$. Now,

$$\sum_{\mu \in \mathcal{M}(t)} x^{|\mu|} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n \qquad \text{(by Theorem 4)}$$

Also,

$$
\begin{aligned}
\sum_{\mu \in \mathcal{M}(t)} x^{|\mu|} &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_t=0}^{\infty} x^{m_1+m_2\cdots+m_t} \\
&= \sum_{m_1=0}^{\infty} x^{m_1} \cdot \sum_{m_2=0}^{\infty} x^{m_2} \cdots \sum_{m_t=0}^{\infty} x^{m_t} \\
&= \left(\sum_{c=0}^{\infty} x^c \right)^t \\
&= \frac{1}{(1-x)^t}
\end{aligned}
$$

Thus, we arrive at the following identity.

Proposition 8 (Binomial Series Expansion)**.** *For some positive integer t,*

$$\frac{1}{(1+x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$$

This identity is needed for the general case of Partial Fractions.