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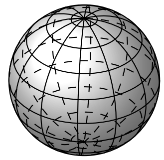

PMATH 348

Complex Analysis

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1 Complex Numbers

Definition 1.0.1. A complex number is a vector in $\mathbb{R}^2$. The complex plane denoted by $\mathbb{C}$ is the set of complex numbers.

$$\mathbb{C} = \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

In $\mathbb{C}$, we usually write,

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$x = \begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$iy = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

with $x, y \in \mathbb{R}$. If $z = x + iy, x, y \in \mathbb{R}$, then x is called the real part of z and y the imaginary part of z and write

$$\Re(z) = x \qquad \Im(z) = y$$

Definition 1.0.2. We define the sum of two complex numbers to be the vector sum.

$$\begin{aligned} (a + ib) + (c + id) &= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a + c \\ b + d \end{pmatrix} \end{aligned}$$

We define the product of two complex numbers by setting $i^2 = -1$ and by requiring the product to be commutative, associative and distributive over the sum. So,

$$\begin{aligned} (a + bi)(c + di) &= ac + iad + ibc + i^2bd \\ &= (ac - bd) + (ad + bc) \end{aligned}$$

Proposition 1 (Mulitplicative Inverses)**.** *Every complex number has a unique multiplicative inverse denoted by z^{-1}.*

Proof. Let $z = a+i, a, b \in \mathbb{R}$ with $a^2+b^2 = 0$. We want to solve for x and y such that $(a+ib)(x+iy) = 1$. In other words,

$$\begin{aligned} &(ax - by) + i(ay + bx) = 1 \\ \Rightarrow &\begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = (1, 0) \\ \Rightarrow &\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (1, 0) \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{a}{a^2+b^2} \\ \frac{b}{a^2+b^2} \end{pmatrix}
\end{aligned}
$$

This is unique as the inverse matrix is unique. □

Remark. The set of complex numbers is a field under the operations of addition and multiplication as operations are associative, commutative and distributive and every element has a unique inverse as before.

Definition 1.0.3. If $z = x + iy, x, y, \in \mathbb{R}$, then the conjugate of z is $\bar{z} = x - iy$.

Definition 1.0.4. We define the modulus (or length or magnitude) of $z = x + iy, x, y \in \mathbb{R}$ to be

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

Remark. For any $z, w \in \mathbb{C}$,

$$
\begin{aligned}
\bar{\bar{z}} &= z \\
z + \bar{z} &= 2\Re(z) \\
z - \bar{z} &= 2\Im(z) \\
z \cdot \bar{z} &= |z|^2 \\
|z| &= |\bar{z}| \\
\overline{z+w} &= \bar{z} + \overline{w} \\
\overline{zw} &= \bar{z} \cdot \overline{w} \\
|zw| &= |z||w|
\end{aligned}
$$

Proposition 2. *The following inequalities hold for any $z \in \mathbb{C}$.*

1. $|\Re(z)| \leq |z|$
2. $|\Im(z)| \leq |z|$
3. $|z + w| \leq |z| + |w|$
4. $|z + w| \geq \Big| |z| - |w| \Big|$

Proof. (1) and (2) follows as

$$|z|^2 = \Re(z)^2 + \Im(z)^2.$$

(3). Notice,

$$
\begin{aligned}
|x + iy|^2 &= (x + iy)\overline{(x + iy)} \\
&= (x + iy)(\overline{x} + \overline{iy}) \\
&= x\overline{x} + y\overline{y} + x\overline{y} + y\overline{x}
\end{aligned}
$$

$$
\begin{aligned}
&= |x|^2 + |y|^2 + x\bar{y} + y\bar{x} \\
&= |x|^2 + |y|^2 + 2\Re(x\bar{y}) \\
&\le |x|^2 + |y|^2 + 2|x\bar{y}| \\
&= |x|^2 + |y|^2 + 2|x| \cdot |\bar{y}| \\
&= |x + y|^2
\end{aligned}
$$

Taking the square root of both sides gives the result.

(4). From (3), we have that

$$
\begin{aligned}
|z| = |z - w + w| \le |z - w| + |w| \\
|w| = |w - z + z| \le |w - z| + |z|
\end{aligned}
$$

Then, isolating $|z - w|$ implies the result. More specifically since we have the simulateous inequality,

$$
\begin{cases} |z| - |w| \le |z - w| \\ |w| - |z| \le |z - w| \end{cases} \Rightarrow |z - w| \ge \Big| |z| - |w| \Big|
$$

as desired. $\square$

Proposition 3. *Every non-zero complex number has exactly 2 square roots.*

Proof. Let $z = x + iy \in \mathbb{C}$ with $x^2 + y^2 \neq 0, x, y, \in \mathbb{R}$. We want to solve $w^2 = z$ for $w \in \mathbb{C}$. Say w takes the form $w = u + iv, u, v \in \mathbb{R}$. Then

$$
\begin{aligned}
&w^2 = z \\
\Rightarrow\ &(u + iv)^2 = x + iy \\
\Rightarrow\ &(u^2 - v^2) + i2uv = x + iy
\end{aligned}
$$

So we have that $x = u^2 - v^2$ and $y = 2uv^2$. We can solve for u and v. Take the square of both sides of the second equation to get $4u^2v^2 = y^2$. Now, we multiply the first equation by $4u^2$ to get

$$
\begin{aligned}
&4u^4 - 4u^2v^2 = 4xu^2 \\
\Rightarrow\ &4u^4 - 4xu^2 - y^2 = 0
\end{aligned}
$$

This is a quadratic equation over u^2 so,

$$
u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}
$$

Suppose that $y \neq 0$. Then we must take the positive solution above to get

$$
u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}
$$

Under the assumption that $x^2 + y^2 > 0$, this solution exists. Notice we cannot take the negative solution as it yields a negative u^2 which is impossible. We can use a similar procedure to find that

$$
v^2 = \frac{-x + \sqrt{x^2 + y^2}}{2}
$$

Rooting u and v gives 2 solutions for each. However, if y is positive, since $2uv = y$, u and v must take the same sign. Similarly, if y is negative, they must take different signs. In each of these cases, there are 2 solutions for u and v. So,

$$w = \begin{cases} \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) + i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y > 0 \\ \pm\left[\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right) - i\left(\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)\right] & , y < 0 \\ \pm\sqrt{x} & , x > 0,\ y = 0 \\ \pm i\sqrt{-x} & , x < 0,\ y = 0 \end{cases}$$

□

Remark. Let $z \in \mathbb{C}$. The notation $\sqrt{z}$ may represent either one of the square roots of z or both of the square roots.

Remark. The square root doesn't distribute. Consider $z = w = -1 \in \mathbb{C}$. $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$.

Remark. The Quadratic Formula holds true for complex polynomials. In other words, if $a, b, c \in \mathbb{C}, a \neq 0$,

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 1.0.5. If $z \in \mathbb{C} \setminus \{0\}$, we define the <u>angle</u> (or <u>argument</u>) of z to be the angle $\theta(z)$ from the positive x-axis counterclockwise to z. In other words, $\theta(z)$ is the angle such that

$$z = |z|\big(\cos\theta(z) + i\sin\theta(z)\big).$$

Remark. For $\theta \in \mathbb{R}$ (or for $\theta \in \mathbb{R}/2\pi$), we have that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Remark. If $z \neq 0$, we have $x = \Re(z)$, $y = \Im(z)$, $r = |z|$ and

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \tan\theta &= \frac{y}{x}, \text{ if } x \neq 0 \\ z &= rei\theta \\ \bar{z} &= re^{-i\theta} \\ z^{-1} &= \frac{1}{r}e^{-i\theta} \end{aligned}$$

Remark. We now have 2 representations of a complex number $z \in \mathbb{C}$. We say that $z = x + iy$ is the <u>cartesian coordinates</u> of z and $z = re^{i\theta}$, where $r = |z|$, is the <u>polar form</u> of z.

Consider $z = re^{i\alpha}$ and $w = se^{i\beta}$. We have,

$$\begin{aligned} zw &= rs(\cos\alpha + i\sin\alpha)(\sin\beta + i\cos\beta) \\ &= rs\big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\big) \\ &= rs\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big) \end{aligned}$$

$$= e^{i(\alpha+\beta)}$$

which defines a formula for multiplication in polar coordinates. Notice that the following identity known as De Moivre's Law follows.

$$(re^{i\theta})^n = r^n e^{in\theta}$$

for all $r, \theta \in \mathbb{R}$, $n \in \mathbb{Z}$. We can use this identity to find the n^{th} roots of z. In other words, we solve $w^n = z$. We have,

$$\begin{aligned} w^n &= z \\ \Rightarrow (se^{i\alpha})^n &= re^{i\theta} \\ \Rightarrow s^n e^{in\alpha} &= re^{i\theta} \end{aligned}$$

so $s^n = r$ and $n\alpha = \theta + 2\pi k$ for $k \in \mathbb{Z}$. In other words, we have

$$(re^{i\theta}) = \sqrt[n]{r} e^{i(\theta+2\pi k)/n}, \quad k = 0, \ldots, n-1$$

Remark. When working with complex numbers, for $0 \neq z \in \mathbb{C}$, and for $0 < n \in \mathbb{Z}$, $\sqrt[n]{z}$ or $z^{1/n}$ denotes either one of the n roots, or the set of all n^{th} roots.

Example 1.0.6. Consider the $n-1$ diagonals of a regular n-gon inscribed in a circle of radius 1 obtained by connecting one vector with all the others. Show that the product of these diagonals is n.

Notice that $z_2, \ldots, z_n$ are the n^{th} roots of unity other than 1. Let z be the variable and consider the polynomial

$$P(z) \coloneqq 1 + z + \cdots + z^{n-1}.$$

Since the roots of $P(z)$ are n^{th} roots of unity other than 1, we can factorize

$$\begin{aligned} P(z) &= 1 + z + \cdots + z^{n-1} \\ &= (z - z_2) \ldots (z - z_n) \end{aligned}$$

and setting $z = 1$, the result follows. In particular, we have

$$|1 - z_2| \ldots |1 - z_n| = n.$$

2 Complex Functions

2.1 Limits

Definition 2.1.1. A sequence of complex numbers $z_1, z_2 \ldots$ converges to $z \in C$ if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

Equivalently, given any $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ sufficiently large such that $|z_n - z| < \epsilon$ whenever $n > N$.

Remark. If $\{z_n\}_n$ converges to z, we write

$$\lim_{n\to\infty} z_n = z$$

or $z_n \to z$ as $n \to \infty$.

Example 2.1.2. For $|z| > 1$, show that $\{\frac{1}{z^n}\}_{n=1}^{\infty}$ converges.

Notice,

$$\lim_{n\to\infty} \left| \frac{1}{z^n} - 0 \right| = \lim_{n\to\infty} \left| \frac{1}{z^n} \right| = 0$$

as $|z| > 1$.

Example 2.1.3. Show that $\{i^n\}_{n-1}^{\infty}$ does not converge.

Definition 2.1.4. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say

$$\lim_{z\to z_0} f(z) = L$$

if for every sequence $\{z_n\}_n \subseteq \Omega$ we have that $z_n \to z \Rightarrow f(z_n) \to L$.

Remark. Here, z_0 need not to be in Ω.

Example 2.1.5. Let $f(z) = \frac{\bar{z}}{z}, z \in \mathbb{C} \setminus \{0\}$. Find $\lim_z \to 0 f(z)$.

If $z = x \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{x}{x} = 1$. So $\lim_{x\to 0} f(x) = 1$. If $z = iy, y \in \mathbb{R} \setminus \{0\}$, then $f(z) = \frac{-iy}{iy} = -1$. So $\lim_{y\to 0} f(iy) = -1$. Hence, the limit does not exist.

Example 2.1.6. Show that $z_n \to z$ if and only if $\Re z_n \to \Re z$ and $\Im z_n \to z$.

2.2 Function Continuity

Definition 2.2.1. Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. We say f is <u>continuous at $z_0 \in \Omega$</u> if for every sequence $\{z_n\} \subseteq \Omega$, we have $z_0 \to z \Rightarrow f(z_0) \to f(z)$. Equivalently, given any $\epsilon > 0, \exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Remark. f is continuous on Ω if it is continuous at ever point of Ω.

Remark. We may split f into its real and imaginary parts

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

2.3 Holomorphic Functions

Definition 2.3.1. An open disk of radius r at z_0 with $r > 0$ is the <u>neighborhood</u> around z_0 denoted by $D(z_0, r)$ with

$$D(z_0, r) \coloneqq \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 2.3.2. Let $f(z)$ be defined in a neighborhood of z_0. We say f is <u>complex differentiable</u> (or holomorphic) at z_0 if

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote the limit by $f'(z_0)$.

Remark. Here, $h \in \mathbb{C}$ can approach zero from any direction in $\mathbb{C}$.

Example 2.3.3. Where is $f(z) = \frac{1}{z}, z \neq 0$ holomorphic?

Notice,

$$\lim_{h\to 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h\to 0} \frac{-h}{(z_0+h)(z_0)} = -\frac{1}{z_0^2}$$

So, f is holomorphic at any $z \in \mathbb{C} \setminus \{0\}$, and $f'(z) = -\frac{1}{z_0^2}$.

Example 2.3.4. $f(z) = \bar{z}$ is not holomorphic at any $z \in \mathbb{C}$.

Notice,

$$\lim_{h\to 0} \frac{\overline{z_0+h} - \overline{z_0}}{h} = \lim_{h\to 0} \frac{\bar{h}}{h}$$

which does not exist from Example 2.1.5. However, the function can be though of a map from $\mathbb{R}^2 \to \mathbb{R}^2$ defined as $(x, y) \mapsto (x, -y)$ which is differentiable as all partial derivatives exist. This distinguishes real analysis from complex analysis.

Remark. If f and g are holomorphic, so are $f+g$, fg and $\frac{f}{g}$ (when $g \neq 0$). The proof is identical to the statements of real functions.

We can now generalize when a function is complex differentiable. If the complex function $f(z) = u + iv$. If the complex derivative $f'(z)$ is to exist, then it must be that the limit exists approaching from both the real and imaginary axis. Thus, we have,

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it}$$

where t is a real number. In terms of u and v, taking the derivative along the real line gives,

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(z+t) - f(z)}{t} \\
&= \lim_{t\to 0} \frac{u(x+t, y) + iv(x+t, y) - u(x, y) - iv(x, y)}{t} \\
&= \lim_{t\to 0} \frac{u(x+t, y) - u(x, y)}{t} + i \lim_{t\to 0} \frac{v(x+t, y) - v(x, y)}{t} \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\end{aligned}$$

Taking the derivative along the vertical line gives,

$$\begin{aligned}
&\lim_{t\to 0} \frac{f(z+it) - f(z)}{it} \\
&= -i \lim_{t\to 0} \frac{u(x, y+t) + iv(x, y+t) - u(x, y) - iv(x, y)}{t} \\
&= -i \lim_{t\to 0} \frac{u(x, y+t) - u(x, y)}{t} + \lim_{t\to 0} \frac{v(x, y+t) - v(x, y)}{t} \\
&= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.
\end{aligned}$$

Equating real and imaginary parts, we arrive at the following theorem.

Theorem 4 (Cauchy-Riemann Equations)**.** *If a function $f(z) = u + iv$ is holomorphic in a neighborhood around $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \textit{at } (x_0, y_0)$$

with

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Example 2.3.5. Show that

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & , \text{if } z \neq 0 \\ 0 & , \text{if } z = 0 \end{cases}$$

is not holomorphic at $z = 0$ and that the Cauchy-Reimann Equations hold at $z = 0$.

Notice,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\bar{h}^2}{h^2} = \lim_{h \to 0} \left(\frac{\bar{x} - iy}{x + iy} \right)^2$$

Let $h = x + imx$, $m \neq 0, x \to 0$. We get

$$\lim_{x \to 0} \left(\frac{x - imx}{x + imx} \right)^2 \left(\frac{1 - im}{1 + im} \right)^2$$

which is dependent of m and thus the limit does not exist. Now, notice we have

$$\frac{\bar{z}^2}{z} = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{-3x^2y + y^3}{x^2 + y^2}$$

So we have

$$u(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2} & , \text{if } (x, y) \neq (0, 0) \\ 0 & , \text{if } (x, y) = (0, 0) \end{cases}$$

and

$$v(x, y) = \begin{cases} \frac{-3x^2y + y^3}{x^2 + y^2} & , \text{if } (x, y) \neq (0, 0) \\ 0 & , \text{if } (x, y) = (0, 0) \end{cases}.$$

Now, we can verify that the Cauchy-Riemann Equations hold. Indeed,

$$\begin{aligned} \frac{\partial u}{\partial x}\left(\frac{x^3 - 3xy^2}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial x}(x^3 - 3xy^2)\right)(x^2 + y^2) - (x^3 - 3xy^2)\left(\frac{\partial}{\partial x}(x^2 + y^2)\right)}{(x^2 + y^2)^2} \\ &= \frac{(3x^2 - 3y^2)(x^2 + y^2) - (x^3 - 3xy^2)(2x)}{(x^2 + y^2)^2} \\ &= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} \\ \frac{\partial v}{\partial y}\left(\frac{y^3 - 3x^2y}{x^2 + y^2} \right) &= \frac{\left(\frac{\partial}{\partial y}(y^3 - 3x^2y)\right)(x^2 + y^2) - (y^3 - 3x^2y)\left(\frac{\partial}{\partial y}(x^2 + y^2)\right)}{(x^2 + y^2)^2} \\ &= \frac{(3y^2 - 3x^2)(x^2 + y^2) - (y^3 - 3x^2y)(y^2)}{(x^2 + y^2)^2} \\ &= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2 + y^2)^2} \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial u}{\partial y}\left(\frac{x^3-3xy^2}{x^2+y^2}\right) &= \frac{\left(\frac{\partial}{\partial y}(x^3-3xy^2)\right)(x^2+y^2)-(x^3-3xy^2)\left(\frac{\partial}{\partial y}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2)-(x^3-3xy^2)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2} \\
\frac{\partial v}{\partial x}\left(\frac{y^3-3x^2y}{x^2+y^2}\right) &= \frac{\left(\frac{\partial}{\partial x}(y^3-3x^2y)\right)(x^2+y^2)-(y^3-3x^2y)\left(\frac{\partial}{\partial x}(x^2+y^2)\right)}{(x^2+y^2)^2} \\
&= \frac{(-6xy)(x^2+y^2)-(y^3-3x^2y)(2y)}{(x^2+y^2)^2} \\
&= -\frac{8x^3y}{x^2+y^2}
\end{aligned}
$$

Thus, we can consider the converse statement of Theorem 4.

Theorem 5. *Let $f = u + iv : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in \Omega$. If*

1. *the partials of u, v exist in a neighborhood of (x_0, y_0)*
2. *the partials of u, v are continuous at (x_0, y_0)*
3. *$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0)*

then, f is holomorphic at z_0.

TODO: find proof online.

Example 2.3.6. Consider the <u>power series</u>, an expression of the form

$$\sum_{n=0}^{\infty} c_n z^n$$

where $c_n \in \mathbb{C}$. This expression <u>converges</u> if the sequence of partials sums, $\{s_N\}$ defined by

$$s_N \coloneqq \sum_{n=0}^{N} c_n z^n$$

converges as $N \to \infty$. This is quite a strong condition, so we consider the following definition.

Definition 2.3.7. A power series expression <u>converges absolutely</u> if

$$\sum_{n=0}^{\infty} |c_n||z|^n$$

converges.

Remark. Absolute convergence implies converges. Notice,

$$\left|\sum_{n=0}^{N} c_n z^n\right| = \sum_{n=0}^{N} |c_n||z|^n$$

for each $N \in \mathbb{N}$.

Theorem 6. *For any power series* $\sum_{n=0}^{\infty} c_n z^n, \exists 0 \leq R \leq \infty$, *such that*

1. *If* $|z| < R$, *the series converges absolutely*
2. *If* $|z| > R$, *the series diverges.*

Moreover, R *is given by Hadamard's formula:* $\dfrac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$

Remark. R is called the radius of convergence of the series and $\{z \in \mathbb{C} : |z| < R\}$ is called the disk of convergence of the series.

Remark. Recall,

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{m\leq n} a_m\right)$$

and is the "highest peak reached by a_n's as $n \to \infty$".

Proposition 7 (Property of lim sup)**.** *If* $L = \limsup_{n\to\infty} a_n$, *then for any* $\epsilon > 0, \exists N > 0$ *such that* $\forall n \leq N, a_n < L + \epsilon$

Proof of Theorem 6. Let $L := \dfrac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$ Clearly, $L \leq 0$.

1. Suppose $|z| < R$. So, there exists some $\epsilon > 0$ such that $r := |z|(L + \epsilon) < 1$ and $0 < r < 1$. By Proposition 7, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|c_n|^{\frac{1}{n}} < L + \epsilon$. Now,

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty}\left(|c_n|^{\frac{1}{n}}|z|\right)^n < \sum_{n=N}^{\infty} r^n$$

   converges as $0 < r < 1$. By the comparison test, $\sum_{n=N}^{\infty} |c_n||z|^n$ is monotonic and bounded and thus converges by Bolzano-Weierstrass.

2. This follows from the proof above. Specifically, this time, notice that there exists some $\epsilon > 0$ such that $r := |z|(L - \epsilon) > 1$. Again, by Proposition 7, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|c_n|^{\frac{1}{n}} > L - \epsilon$ so that

$$\sum_{n=N}^{\infty} |c_n||z|^n = \sum_{n=N}^{\infty}\left(|c_n|^{\frac{1}{n}}|z|\right)^n > \sum_{n=N}^{\infty} r^n$$

   follows and so the series diverges.

□

Theorem 8. *Suppose* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *has radius of convergence* R. *Then* $f'(z)$ *exists and equals*

$$\sum_{n=1}^{\infty} n c_n z^{n-1}$$

throughout $|z| < R$. *Moreover,* f' *has the same radius of convergence as* f.

Proof. f' has some radius of convergence because

$$\limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \limsup_{n\to\infty} n^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Let $|z_0| \le r < R$, $g(z_0) := \sum_{n=1}^{\infty} nc_n z_0^{n-1}$. We want to show

$$\lim_{k\to 0} \frac{f(z_0+h) - f(z_0)}{h} = g$$

or equivalently, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|h| < \delta \Rightarrow \left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < \epsilon.$$

For any fixed $\epsilon > 0$, we write

$$f(z) = \underbrace{\sum_{n=0}^{N} c_n z^n}_{:=S_N(z)} + \underbrace{\sum_{n=N+1}^{\infty} c_n z^n}_{:=E_N(z)}$$

We have $S_N' = \sum_{n=1}^{N} nc_n z^{n-1}$ and

$$\begin{aligned}
&\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| \\
=& \left| \frac{S_N(z_0+h) - S_N(z_0)}{h} + \frac{E_N(z_0+h) - E_N(z_0)}{h} - g(z_0) + S_N'(z_0) - S_N'(z_0) \right| \\
=& \left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| + \left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| + \left| S_N'(z_0) - g(z_0) \right|.
\end{aligned}$$

We have

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| = \left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n \big((z_0+h)^h - z_0^n \big) \right|$$

As $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ in any ring, we have

$$\begin{aligned}
&\left| \frac{1}{h} \sum_{n=N+1}^{\infty} c_n \big((z_0+h)^h - z_0^n \big) \right| \\
=& \left| \sum_{n=N+1}^{\infty} c_n \big((z_0+h)^{n-1} + (z_0+h)^{n+2} z_0 + \cdots + (z_0+h) z_0^{n-2} + z_0^{n-1} \big) \right|
\end{aligned}$$

Now, by choosing δ relatively small so that $|z_0| \le r$, we have $|z_0|, |z_0 + h| \le r$ and so

$$(z_0+h)^{n-1} + (z_0+h)^{n+2} z_0 + \cdots + (z_0+h) z_0^{n-2} + z_0^{n-1} \le nr^{n-1}$$

So,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| \le \sum_{n=N+1}^{\infty} nc_n r^{n-1} < \frac{\epsilon}{3}.$$

for a large enough N_1.

Now, observe that by definition,

$$S'_N(z_0) = \sum_{n=1}^{N} nc_n z^{n-1}.$$

Since,

$$\lim_{N\to\infty} S'_N(z_0) = \lim_{N\to\infty} \sum_{n=1}^{N} nc_n z^{n-1} = \sum_{n=1}^{\infty} nc_n z^{n-1} = g(z_0)$$

we can pick some $\frac{\epsilon}{3} > 0$ such that there exists some $N_2 \in \mathbb{N}$, for all $n > N_2$, we have

$$|S'_N(z_0) - g(z_0)| < \frac{\epsilon}{3}.$$

Finally, let $\frac{\epsilon}{3} > 0$. Observe that there exists some $\delta > 0$ such that there exists some $N > \max N_1, N_2$, for all $n > N$,

$$\left| \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \frac{\epsilon}{3}.$$

as $|h| < \delta$. It follows that

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \epsilon$$

as desired. □

Example 2.3.8. Consider $f(z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$. To find the radius of convergence, we use Hadamard's Formula,

$$\begin{aligned}\frac{1}{R} &= \limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} n^{\frac{1}{n}} \\ &= 1\end{aligned}$$

Thus, $R = 1$. By Theorem 6, f converges absolutely when $|z| < 1$ and diverges when $|z| > 1$. As for the boundary, in other words, when $|z| = 1$, consider the follwoing two cases:

1. If $z = 1$, then $f(1) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series, and hence f diverges.

2. If $z = i$, then

$$\begin{aligned}f(i) &= \sum_{n=1}^{\infty} \frac{i^n}{n} \\ &= i - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} \\ &= \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots\right) i \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)\end{aligned}$$

which both real and imaginary parts converge by the alternating series test. Therefore, we observe that both convergence and divergence may occur on the boundary, depending on the value of z.

Remark. The positions of lim and $\displaystyle\sum_{n=0}^{\infty}$ cannot be exchanged when we consider infinite sums. Consider the following example.

Example 2.3.9. Consider for $|x| > 1$, $\displaystyle\sum_{n=1}^{\infty} \lim_{x\to 1}(x^n - x^{n+1}) = \sum_{n=1}^{\infty}(1-1) = 0$. Now,

$$\begin{aligned}
&\lim_{x\to 1}\lim_{N\to\infty}\sum_{n-1}^{N}(x^n - x^{n+1}) \\
&= \lim_{x\to 1}\lim_{N\to\infty}(x - x^2 + x^2 - x^3 + \cdots + x^{N-1} - x^N + x^N - x^{N+1}) \\
&= \lim_{x\to 1}\lim_{N\to\infty}(x - x^{N+1}) \\
&= \lim_{x\to 1} x \\
&= 1
\end{aligned}$$

so that

$$\sum_{n=1}^{\infty}\lim_{x\to 1}(x^n - x^{n+1}) \neq \lim_{x\to 1}\sum_{n=1}^{\infty}(x^n - x^{n+1}).$$

Definition 2.3.10. A function f is said to be <u>entire</u> if f is holomorphic in the entire complex plane.

Example 2.3.11. Define $e^z = \displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}$. Show that the radius of convergence of this series is ∞ (which implies e^z is entire) and $(e^z)' = e^z$.

Consider Stirling's formula, which says as $n \to \infty, n\,! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Then, we have

$$e^z = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi n}}\left(\frac{ez}{n}\right)^n.$$

Now,

$$\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty}\left|\frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)\right|^{\frac{1}{n}} \\
&= \limsup_{n\to\infty}\left|\frac{1}{2\pi n}\right|^{\frac{1}{n}}\limsup_{n\to\infty}\left|\frac{e}{n}\right|^{\frac{1}{n}} \\
&= 0
\end{aligned}$$

Thus, $R \to \infty$ as $n \to \infty$. By Theorem 6

Corollary 9 (Corollary of Theorem 8)**.** *A power series is infinitely complex-differentiable in its radius of convergence. All its derivitives are also power series, obtained by termwise differentiation. If* $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$*, then*

$$f^{(N)}(z) = \sum_{n=N}^{\infty} n(n-1)\cdots(n-N)c_n z^{n-N}$$

for some $N \in \mathbb{N}$.

In general, we have have $\sum_{n=0}^{\infty} c_n(z-z_0)^n$, which is the power series centered at $z \in \mathbb{C}$. Then as before, the readius of convergence R is given by

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}.$$

But the disc of convergence is now centered around z_0. We have shown that $f(z)$ has a power series expansion at z_0 (i.e. $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ in some neighborhood of z_0) with radius of convergence $R > 0$. This implies that $f(z)$ is holomorphic at z_0. In fact, the converse is true; any function holomorphic at z_0 is infinitely holomorphic at z_0.
However, for this, we need the concept of integration over paths of curves.

Definition 2.3.12. A curve in $\mathbb{C}$ is a continuous function $\gamma(t) : [a,b] \to \mathbb{C}$ with $a, b \in \mathbb{R}$. The image of γ in $\mathbb{C}$ is called γ^*.

Example 2.3.13. Let $z_0 \in \mathbb{C}$, $r > 0$. Take $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $t \mapsto z_0 + re^{it}$. This is a circle of radius r centered at z_0, oriented counterclockwise.

Example 2.3.14. Consider $\hat{\gamma} : [0,1] \to \mathbb{C}$ defined by $t \mapsto z_0 + re^{2\pi it}$. This is identical to the curve γ defined above with the same oriented path and shows that curves have different parameterizations.

Definition 2.3.15. We say γ is smooth on the interval $[a,b]$ if γ' exists, is continuous on $[a,b]$ and $\gamma'(t) \neq 0$ for any $t \in [a,b]$.

Definition 2.3.16. $\gamma : [a,b] \to \mathbb{C}$ is piecewise-smooth if it is smooth on $[a,b]$ except at finitely many points in $[a,b]$.

Remark. Piecewise smooth curves are called paths.

Definition 2.3.17. Given a path $\gamma : [a,b] \to \mathbb{C}$ and $f(z)$ is a continuous function on γ, the integral of f along γ is defined by

$$\int_\gamma f(z)\,dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt.$$

Remark. If g is complex valued, then

$$\int_a^b g(t)\,dt = \int_a^b \Re\big(g(t)\big)\,dt + i\int_a^b \Im\big(g(t)\big)\,dt.$$

Remark. The integral $\int_\gamma f(z)\,dz$ can be shown to be independent of the parameterization chosen for $\gamma*$.

Theorem 10. *Integration is linear.*

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