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Liouvillian solutions of Whittaker-Ince equation

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ABSTRACT

We apply Kovacic's algorithm, a tool that is developed from differential Galois theory, to discuss the existence of Liouvillian solutions of Whittaker-Ince equation, ellipsoidal wave equation and the Picard-Fuchs equation of a K3 surface. We determine the necessary and sufficient conditions of having Liouvillian solutions for Whittaker-Ince equation when a parameter vanishes; as well as a sufficient condition of having Liouvillian solution when this parameter is non-zero. On the other hand, we prove that ellipsoidal wave equation has no Liouvillian solution. We generalize a Picard-Fuchs equation for certain K3 surface and show that a particular case of this Picard-Fuchs equation cannot have any Liouvillian solutions.

1. Introduction

When a *Liouvillian function* is a solution of a homogeneous linear differential equation, then the solution is called a *Liouvillian solution*. Some people also call them *closed-form* solutions. While Liouvillian functions can be roughly interpreted as functions that can be obtained from rational functions by a finite process of exponentiations, integrations and algebraic functions. A more precise way to define them can be found in the book by Crespo and Hajto (2011). Interested readers who want to learn more about the applications of differential Galois theory can also refer to the book by van der Put and Singer (2003). While other usual old references are Kaplansky (1957), Kolchin (1973), Magid (1994). Especially, Hubbard and Lundell (2011) gives a good comparison between classical Galois theory and differential Galois theory.

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Historically, people have considered whether a differential equation has Liouvllian solutions or not, or equivalently, whether the equation is solvable by *quadratures*. Note that the notion of whether a differential equation is solvable by quadratures is similar to whether a polynomial equation is solvable by *radicals* in the classical Galois theory.

The main task in this paper is to apply *Kovacic's algorithm* that was constructed in the work of Kovacic (1986), to determine whether the ellipsoidal wave equation, the Whittaker-Ince equation, and a particular case of the Picard-Fuchs equation of a K3 surface have Liouvillian solutions or not. This algorithm allows us to write down the Liouvillian solutions explicitly if they exist.

Note that the original Kovacic's algorithm is designed explicitly for a second-order homogeneous linear differential equation in this form:

$$y'' + ay' + by = 0, \quad \text{where } a, b \in \mathbb{C}(z). \tag{1}$$

For higher-order homogeneous linear differential equations, such as third-order equations, interested readers can refer to the work of Singer and Ulmer (2001), Ulmer (2003) about a new algorithm to compute Liouvillian solutions for third-order differential equations.

For each time, before applying Kovacic's algorithm, we need to transform equation (1) into the *reduced/normal form*:

$$w'' = (-b + \frac{1}{4}a^2 + \frac{1}{2}a')w$$

by setting $w = e^{\frac{1}{2}\int a} y$ to eliminate the term involving y'. We can see that w is Liouvillian if and only if y is Liouvillian. Also, if equation (1) has a Liouvllian solution, then by the method of reduction of order, every solution of (1) will be Liouvillian.

Remark 1.1. In general, if you have a differential equation in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

then you can set $w = e^{\frac{1}{n}\int a_{n-1}} y$ to eliminate the term involving $y^{(n-1)}$.

On the other hand, it is not surprising that Kovacic's algorithm has been implemented in *Wolfram Mathematica* or MAPLE to calculate Liouvillian solutions for simple differential equation in the form (1) whose rational coefficients a, b are explicitly expressed with *fixed* parameters. While, Mathematica is able to express the solutions of the hypergeometric equation in terms of several undetermined parameters, none of these software can be applied in the three types of differential equations that we are encountering because of their complexity. For example, each of the three equations that we are considering is expressed in the general form whose parameters are *undetermined* but none of them can be transformed into a hypergeometric equation. Thus, in order to find the Liouvillian solutions of these three differential equations by Kovacic's algorithm, manual manipulations are required.

To begin with, we will consider the *Whittaker-Ince equation*:

$$(1 + a\cos(2z))w'' + \xi\sin(2z)w' + (\eta - p\xi\cos(2z))w = 0, \tag{2}$$

where $a, \xi, \eta, p \in \mathbb{C}$. Especially, when $a = 0$, the Whittaker-Ince equation can be transformed into the *Whittaker–Hill equation*:

$$W'' + \left[\eta - \frac{\xi^2}{8} - \xi(p+1)\cos(2z) + \frac{\xi^2}{8}\cos(4z)\right]W = 0,$$

after substituting $w = W\exp\left(\frac{\xi\cos(2z)}{4}\right)$. Interested readers can refer to the book by Magnus and Winkler (2013), which introduces the periodic solutions of Whittaker-Hill equation. In the book by Magnus and Winkler (2013), our equation (2) is just referred as Ince equation. In 1923, Ince (1923) appeared to be the main person to study the solutions of equation (2) with $a = 0$. So, in the literature, e.g. Arscott (1964), equation (2) is called *generalized Ince equation*. However, back to 1914, Whittaker derived equation (2) in (Whittaker, 1914, p. 17), and he has shown that the periodic solutions of a different version of equation (2):

$$(b - c\sin^2 x)\frac{d^2y}{dx^2} - 2(1-q)c\sin x\cos x\frac{dy}{dx} + [-n(n+2-2q)c\sin^2 x + 4c_0]y = 0$$

are the solutions of certain homogeneous integral equation. So in order to recognize the contribution of Whittaker, we rename the generalized Ince equation as Whittaker-Ince equation.

In particular, when $a = 0$, the accessory parameter η can be chosen so that the Whittaker-Ince equation has solutions of trigonometric polynomials, i.e., the *Ince polynomials*. Later on, we will apply Kovacic's algorithm to the Whittaker-Ince equation in two separate cases ($a = 0$ and $a \neq 0$). We shall determine the necessary and sufficient conditions of having Liouvillian solutions for Whittaker-Ince equation when assuming $a = 0$, $\xi \neq 0$ and p is a non-negative integer. Indeed, when $a = 0$, then $p \in \mathbb{Z} \setminus \{-1\}$ if the Whittaker-Ince equation has a Liouvillian solution. Next, we will give a sufficient condition of having a Liouvillian solution for Whittaker-Ince equation when $a \neq 0$. Besides, there are several physical applications of the Whittaker-Ince equation. For instance, a particular case of the Whittaker-Ince equation has been involved in Llibre and Ortega (2008) when studying the families of symmetric periodic orbits of the elliptic Sitnikov problem (a restricted version of the three-body problem). Also, the Whittaker–Hill equation permits Roncaratti and Aquilanti (2009) to describe the torsional motion of flexible molecules, e.g. hydrogen peroxide molecule.

Next, we shall consider the *Jacobian form of ellipsoidal wave equation*:

$$\frac{d^2w}{dz^2} - (a + bk^2\,\mathrm{sn}^2 z + qk^4\,\mathrm{sn}^4 z)w = 0,$$

where a, b, q are complex constants and $\mathrm{sn}\, z = \mathrm{sn}(z, k)$ is the *Jacobi elliptic function with modulus* k, $0 \le k^2 \le 1$. This equation has appeared in a mathematical literature written by Ince (1926) since 1926. On the other hand, this equation emerges when we apply separation of variables to the Helmholtz equation in ellipsoidal coordinates. Therefore the solutions of ellipsoidal wave equation are crucial for many physical applications such as electromagnetic scattering by an ellipsoid, diffraction by elliptic plates etc. Instead of considering its Jacobian form, we will prove the non-existence of Liouvillian solutions for the *algebraic form of ellipsoidal wave equation*:

$$t(t-1)(t-c)\frac{d^2w}{dt^2} + \frac{1}{2}[3t^2 - 2(1+c)t + c]\frac{dw}{dt} + (\lambda + \mu t + \gamma t^2)w = 0,$$

where $c = 1/k^2$, $\lambda = -a/(4k^2)$, $\mu = -b/4$, $\gamma = -qk^2/4$. Especially, the solutions of the algebraic form which are finitely evaluated at the three finite singularities $t = 0, 1, c$ are called the *ellipsoidal wave functions*. They are equivalent to the *doubly-periodic* solutions of the Jacobian form. Some recent investigations of ellipsoidal wave functions can be found in Arscott (1959), Arscott et al. (1983).

Lastly, we will deal with the *Picard-Fuchs equation of a K3 surface* that is defined as a hypersurface in $\mathbb{P}^3$ cut out by

$$\{(x, y, t) \in \mathbb{C}^3 \,|\, y^2 = 4x^3 - g(t)x - h(t)\}$$

where g and h are polynomials of degrees 8 and 12 respectively with $g^3 - 27h^2 \neq 0$. We use the brute force method mentioned in (Schnell, 2012, pp. 2-3) to derive the Picard-Fuchs equation of our K3 surface. The problem of deriving the Picard-Fuchs equation of our K3 surface was suggested by Yum-Tong Siu. In particular, we can achieve the same Picard-Fuchs equation

$$\frac{d^2f_1}{dt^2} + \frac{\mathcal{J}'^2 - \mathcal{J}\mathcal{J}''}{\mathcal{J}\mathcal{J}'}\frac{df_1}{dt} + \frac{\mathcal{J}'^2(\frac{31}{144}\mathcal{J} - \frac{1}{36})}{\mathcal{J}^2(\mathcal{J}-1)^2}f_1 = 0, \quad \text{where } \mathcal{J}(t) = \frac{g(t)}{g(t) - 27} \tag{3}$$

that was derived by (Stiller, 1981, p. 207) when $g = h$ in our K3 surface. It is because, when $g = h$, our generalized Picard-Fuchs equation will reduce to:

$$\frac{d^2 f_1}{dt^2} + \left[\frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'}\right]\frac{df_1}{dt} + \frac{3(g+4)g'^2}{16(g-27)g^2} f_1 = 0. \tag{4}$$

This equation is the same as the one derived by Stiller after substituting $g = \frac{27\mathcal{J}}{\mathcal{J}-1}$. Hence, our work provides another way to derive the equation (3) and reconfirm its correctness at the same time. We will show that the equation (4) has no Liouvillian solutions by Kovacic's algorithm.

This paper is organized as follows. We devote Section 2 to a review of Kovacic's algorithm. Then, we demonstrate how to apply Kovacic's algorithm to discuss the existence of Liouvillian solutions of Whittaker-Ince equation, ellipsoidal wave equation, and the Picard-Fuchs equation of certain K3 surface in Section 3, 4 and 5 respectively. In particular, the results in this paper are part of Cheung (2018)'s MPhil thesis.

2. Review of Kovacic's algorithm

We quote only partial results that we will use in our computation from Kovacic (1986)'s paper. First, we consider a second-order homogeneous linear differential equation in the reduced form, i.e.,

$$y'' = ry, \quad r \in \mathbb{C}(x). \tag{5}$$

Then, it is natural to ask how does the Galois group of the above differential equation look like. Indeed, we are able to characterize them.

Proposition 2.1 *(Kovacic (1986), p. 5). The Galois group of the differential equation (5) is an algebraic subgroup of* SL(2).

Lemma 2.2 *(Kovacic (1986), p. 7). Let G be an algebraic subgroup of* SL(2). *Then exactly one of the following four cases can happen.*

Case 1. *G is triangularizable, i.e., the matrix representing G is triangularizable;*
Case 2. *G is conjugate to a subgroup of*

$$D = \left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix} : c \in \mathbb{C}, c \neq 0\right\} \cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0\end{pmatrix} : c \in \mathbb{C}, c \neq 0\right\},$$

and case 1 does not hold;
Case 3. *G is finite and cases 1 and 2 do not hold;*
Case 4. $G = \mathrm{SL}(2)$.

Remark 2.3. By Proposition 2.1 and Lemma 2.2, we have classified all kinds of differential Galois groups of equation (5).

Theorem 2.4 *(Kovacic (1986), p. 7). There are four mutually exclusive cases that can happen for equation (5).*

Case 1. *It has a Liouvillian solution of the form $e^{\int w}$ where $w \in \mathbb{C}(x)$.*
Case 2. *It has a Liouvillian solution of the form $e^{\int w}$ where w is algebraic over $\mathbb{C}(x)$ of degree 2, and case 1 doesn't hold.*
Case 3. *All solutions of the (5) are algebraic over $\mathbb{C}(x)$ and cases 1 and 2 don't hold.*
Case 4. *The equation (5) has no Liouvillian solution.*

Remark 2.5. The results of Theorem 2.4 are implied by Lemma 2.2, where G is then the differential Galois group of equation (5). Each of the four cases from above corresponds to the respective case in the lemma.

We recall that if $r(x) = s(x)/t(x)$, where $s, t \in \mathbb{C}(x)$ and are relatively prime, then the *poles* of r are the zeros of $t(x)$, and the *order* of the pole is the multiplicity of the zero of $t(x)$. Similarly, the order of r at ∞ means the order of ∞ as a zero of $r(x)$, hence the order of r at ∞ is equal to $\deg t - \deg s$.

Theorem 2.6 *(Kovacic (1986), p. 8). The cases 1, 2 and 3 in Theorem 2.4 have the following necessary conditions respectively.*

Case 1. *Every pole of r must have order equal to 2ν for some $\nu \geq 1$, or just equal to 1. The order of r at ∞ must equal to 2ν for some $\nu \leq 1$, or be greater than 2. (Note that the order of r at ∞ can be negative.)*

Case 2. *r must have at least one pole that either has order equal to $2\nu+1$ for some $\nu \geq 1$, or equal to 2.*

Case 3. *The order of a pole of r must be smaller than or equal to 2; and the order of r at ∞ must be greater than or equal to 2. If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j},$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$ for all i and $\sum_j \beta_j = 0$; and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1+4\gamma} \in \mathbb{Q}$.

Remark 2.7. If our equation (5) satisfies one of the necessary conditions in Theorem 2.6, then we can apply Kovacic's algorithm to determine the Liouvillian solutions (if any) for those cases. If none of the three necessary conditions are satisfied, then the equation (5) has no Liouvillian solutions. Here we only quote the algorithms for case 1 and case 2 for our later use.

Algorithm for Case 1.

Let Γ be the set of all finite poles of r in (5).

Step 1. For each $k \in \Gamma \cup \{\infty\}$, we define a rational function $[\sqrt{r}]_k$ and two complex numbers α_k^+, α_k^- as below:

- If k is a pole of order 1, then $[\sqrt{r}]_k = 0$, $\alpha_k^+ = \alpha_k^- = 1$.
- If k is a pole of order 2, then $[\sqrt{r}]_k = 0$. Also, $\alpha_k^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$, where b is the coefficient of $1/(x-k)^2$ in the partial fraction expansion for r.
- If k is a pole of order $2\nu \geq 4$, then $[\sqrt{r}]_k$ is the sum of terms involving $1/(x-k)^i$ for $2 \leq i \leq \nu$ in the Laurent series of $\sqrt{r}$ at k, where we can choose either one of them, i.e. $[\sqrt{r}]_k = \frac{c_{-\nu}}{(x-k)^\nu} + \cdots + \frac{c_{-2}}{(x-k)^2}$. Also, $\alpha_c^\pm = \frac{1}{2}(\pm\frac{b}{c_{-\nu}} + \nu)$, where b be the coefficient of $1/(x-k)^{\nu+1}$ in r minus the coefficient of $1/(x-k)^{\nu+1}$ in $([\sqrt{r}]_k)^2$.
- If r has order > 2 at ∞, then $[\sqrt{r}]_\infty = 0$, $\alpha_\infty^+ = 0$, $\alpha_\infty^- = 1$.
- If r has order 2 at ∞, then $[\sqrt{r}]_\infty = 0$. Also, $\alpha_\infty^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$, where b is the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞ (i.e. if $r = s/t$, where $s, t \in \mathbb{C}[x]$, then b is the leading coefficient of s divided by the leading coefficient of t).
- If r has order $-2\nu \leq 0$ at ∞, then $[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq \nu$ in the Laurent series of $\sqrt{r}$ at ∞, where we can choose either one of them, i.e. $[\sqrt{r}]_\infty = c_0 + c_1 x + \cdots + c_\nu x^\nu$. Also, $\alpha_\infty^\pm = \frac{1}{2}(\pm\frac{b}{c_\nu} - \nu)$, where b is the coefficient of $x^{\nu-1}$ in r minus the coefficient of $x^{\nu-1}$ in $([\sqrt{r}]_\infty)^2$.

Step 2. For each family $s = (s(k))_{k\in\Gamma\cup\{\infty\}}$, where $s(k)$ is either $+$ or $-$. So we have totally $2^{\rho+1}$ families s, where ρ is the number of poles of r. Let

$$d = \alpha_\infty^{s(\infty)} - \sum_{k\in\Gamma} \alpha_k^{s(k)}.$$

If $d \in \mathbb{Z}_{\geq 0}$, then let

$$f=\sum_{k\in\Gamma}\left(s(k)[\sqrt{r}]_k+\frac{\alpha_k^{s(k)}}{x-k}\right)+s(\infty)[\sqrt{r}]_\infty.$$

If $d\notin\mathbb{Z}_{\geq 0}$, then the family s is discarded for consideration. If no families s are retained for us to construct f, then case 1 cannot hold.

Step 3. For each family s that are retained from Step 2, we search for a monic polynomial P of degree d (defined in Step 2) that satisfies the differential equation

$$P''+2fP'+(f'+f^2-r)P=0.$$

If such a polynomial exists, then $y=Pe^{\int f}$ is a Liouvillian solution of equation (5). If no such polynomial can be found from the families s that are retained, then case 1 cannot hold.

Algorithm for Case 2.

Step 1. For each $k\in\Gamma$ we define E_k as follows.

- If k is a pole of r of order 1, then $E_k=\{4\}$.
- If k is a pole of r of order 2 and if b is the coefficient of $1/(x-k)^2$ in the partial fraction expansion of r, then

$$E_k=\{2+m\sqrt{1+4b}\,|\,m=0,\pm 2\}\cap\mathbb{Z}.$$

- If k is a pole of r of order $v>2$, then $E_k=\{v\}$.
- If r has order >2 at ∞, then $E_\infty=\{0,2,4\}$.
- If r has order 2 at ∞ and b is the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞, then

$$E_\infty=\{2+m\sqrt{1+4b}\,|\,m=0,\pm 2\}\cap\mathbb{Z}.$$

- If the order r at ∞ is $v<2$, then $E_\infty=\{v\}$.

Step 2. We consider all families $(e_k)_{k\in\Gamma\cup\{\infty\}}$ with $e_k\in E_k$. Those families all of whose coordinates are even will be discarded. Let

$$d=\frac{1}{2}\left(e_\infty-\sum_{k\in\Gamma}e_k\right).$$

If $d\in\mathbb{Z}_{\geq 0}$, then the family should be retained, otherwise the family is discarded. If no families remain under consideration, then case 2 cannot hold.

Step 3. For each family retained from Step 2, we construct the rational function

$$\theta=\frac{1}{2}\sum_{k\in\Gamma}\frac{e_k}{x-k}.$$

Then, we search for a monic polynomial P of degree d (as defined in Step 2) such that

$$P'''+3\theta P''+(3\theta^2+3\theta'-4r)P'+(\theta''+3\theta\theta'+\theta^3-4r\theta-2r')P=0.$$

If no such polynomial is found for any family retained from Step 2, then case 2 cannot hold.

Suppose such a polynomial P is found, then let $\phi=\theta+P'/P$ and w be a solution of the quadratic equation

$$w^2-\phi w+(\frac{1}{2}\phi'+\frac{1}{2}\phi^2-r)=0. \tag{6}$$

Then, $y=e^{\int w}$ is a Liouvillian solution of equation (5).

Remark 2.8. Note that there is a typo in (Kovacic, 1986, p. 18)'s paper when he was describing the Algorithm for Case 2, where he stated the second term of equation (6) as $+\phi w$ instead of $-\phi w$. But in the proof of Algorithm for Case 2 in (Kovacic, 1986, p. 21), the second term is $-\phi w$ which is the correct one.

Example 2.9. We can apply Kovacic's algorithm to the *Chebyshev differential equation*

$$Y'' = \frac{(\alpha^2 - 1/4)z^2 - \alpha^2 - 1/2}{(z^2-1)^2} Y. \tag{7}$$

One can show that when $\alpha = \pm 1/2$, equation (7) admits two linearly independent Liouvillian solutions

$$Y_1 = (z-1)^{3/4}(z+1)^{1/4}, \quad Y_2 = (z-1)^{1/4}(z+1)^{3/4},$$

where its differential Galois group over $\mathbb{C}(x)$ will be the cyclic group C_4. When $\alpha \neq \pm 1/2$, we can apply the Algorithm for Case 2 to show that

$$Y = (z^2-1)^{1/4}(z + \sqrt{z^2-1})^{\alpha}$$

is a Liouvillian solution to equation (7). Details of computation can be found in (Crespo and Hajto, 2011, p. 187, p. 193).

3. On Whittaker-Ince equation

In this Section, we demonstrate how to use Kovacic's algorithm to determine the Liouvillian solutions of the **Whittaker-Ince equation**:

$$(1 + a\cos(2z))w'' + \xi \sin(2z)w' + (\eta - p\xi \cos(2z))w = 0, \tag{8}$$

where $a, \xi, \eta, p \in \mathbb{C}$. First, we shall show that when $a = 0$, then $p \in \mathbb{Z} \setminus \{-1\}$ if the Whittaker-Ince equation has a Liouvillian solution. Then, we will determine the necessary and sufficient conditions of having Liouvillian solutions under the assumptions $a = 0$, $\xi \neq 0$ and p is a non-negative integer. Lastly, we will give a sufficient condition of having a Liouvillian solution for the Whittaker-Ince equation when $a \neq 0$.

3.1. When a vanishes

We assume $a = 0$. Since $\sin(2z) = \frac{e^{i2z} - e^{-i2z}}{2i}$, $\cos(2z) = \frac{e^{i2z} + e^{-i2z}}{2}$, and by letting

$$x = e^{i2z}, \quad W(x) = w\left(\frac{\ln x}{2i}\right) = w(z), \tag{9}$$

then equation (8) can be transformed into

$$-4x^2 W'' + [-4x + \xi(x^2-1)]W' + \left(\eta - p\xi \cdot \frac{x + x^{-1}}{2}\right) W = 0.$$

Dividing $-4x^2$ on both sides, we obtain

$$W'' + \left[\frac{1}{x} - \frac{\xi}{4}\left(1 - \frac{1}{x^2}\right)\right] W' - \frac{1}{4}\left[\frac{\eta}{x^2} - \frac{p\xi}{2}\left(\frac{1}{x} + \frac{1}{x^3}\right)\right] W = 0. \tag{10}$$

We recall that for a differential equation of the form:

$$\frac{d^n f}{dz^n} + \sum_{k=1}^{n} a_{n-k}(z) \frac{d^{n-k} f}{dz^{n-k}} = 0, \tag{11}$$

where $a_i(z)$ are meromorphic on $\mathbb{C}$ for all i, a *singularity* of equation (11) means a singularity of $a_i(z)$ for some i. Let $u = 1/z$. The transformed equation is in u, and $u = 0$ corresponds to $z = \infty$. Equation (11) has a singularity at $z = \infty$ if the transformed equation has a singularity at $u = 0$. A singularity $\alpha \in \mathbb{C}$ of equation (11) is *regular* if

$$\lim_{z\to\alpha} (z-\alpha)^k a_{n-k}(z)$$

exists for $k = 1, \dots, n$. Similarly, equation (11) has a regular singularity at $z = \infty$ if its transformed equation has a regular singularity at $u = 0$. A singularity which is not regular will be called an *irregular singularity*.

Remark 3.1. We can easily see that 0 and ∞ are the irregular singularities of equation (10).

Let

$$\alpha = \left[\frac{1}{x} - \frac{\xi}{4}\left(1 - \frac{1}{x^2}\right)\right], \quad \beta = -\frac{1}{4}\left[\frac{\eta}{x^2} - \frac{p\xi}{2}\left(\frac{1}{x} + \frac{1}{x^3}\right)\right]$$

in equation (10).

Set $y = e^{\frac{1}{2}\int \alpha} W$. Then we have

$$y'' = \left(-\beta + \frac{\alpha^2}{4} + \frac{\alpha'}{2}\right) y := r(x) y, \tag{12}$$

where

$$r(x) = \frac{\xi^2 x^4 - 8\xi(p+1)x^3 + 2x^2\left(8\eta - \xi^2 - 8\right) - 8\xi(p+1)x + \xi^2}{64x^4}.$$

Lemma 3.2. *Let $\xi \neq 0$. If the equation (12) has a Liouvillian solution, then*

$$p \in \mathbb{Z} \setminus \{-1\}$$

where $\mathbb{Z}$ is the set of all integers.

Proof. Since the order of r at ∞ is $4 - 4 = 0$ if $\xi \neq 0$, and the order of r at the only pole 0 is 4, then by Theorem 2.6, both case 2 and case 3 cannot hold. Therefore, we try to use the Algorithm for Case 1 to determine the Liouvillian solutions of equation (12).

Since 0 is a pole of order 4, we let $[\sqrt{r}]_0 = \frac{c_{-2}}{x^2}$. Then $[\sqrt{r}]_0^2 = \frac{c_{-2}^2}{x^4}$. By comparing the coefficient of $1/x^4$ in r, we have $c_{-2}^2 = \xi^2/64$. We choose $c_{-2} = \xi/8$. Also the b here is the coefficient of $1/x^3$ in r minus the coefficient of $1/x^3$ in $[\sqrt{r}]_0^2$, so $b = -\frac{(p+1)\xi}{8} - 0 = -(p+1)\xi/8$. Hence,

$$\alpha_0^{\pm} = \frac{1}{2}(\pm\frac{-(p+1)\xi/8}{\xi/8} + 2) = \mp\frac{p+1}{2} + 1.$$

Similarly, let $[\sqrt{r}]_\infty = c_0$, and so $[\sqrt{r}]_\infty^2 = c_0^2$. By comparing the coefficient of x^0 in r, we have $c_0^2 = \xi^2/64$. We choose $c_0 = \xi/8$. Also the b here is the coefficient of x^{-1} in r minus the coefficient of x^{-1} in $([\sqrt{r}]_\infty)^2$, i.e., $b = -\frac{(p+1)\xi}{8} - 0 = -(p+1)\xi/8$. Hence

$$\alpha_0^{\pm} = \frac{1}{2}(\pm\frac{-(p+1)\xi/8}{\xi/8} - 0) = \mp\frac{p+1}{2}.$$

Now we move to the Step 2 of the Algorithm for Case 1. There are four possible families.

(1) If $s(0) = +, s(\infty) = +$, then $d = \alpha_\infty^+ - \alpha_0^+ = -1$.

(2) If $s(0)=+, s(\infty)=-$, then $d=\alpha_\infty^- - \alpha_0^+ = p$.
(3) If $s(0)=-, s(\infty)=+$, then $d=\alpha_\infty^+ - \alpha_0^- = -p-2$.
(4) If $s(0)=-, s(\infty)=-$, then $d=\alpha_\infty^- - \alpha_0^- = -1$.

In order to construct f, we require our d to be a non-negative integer. That is, if the equation (12) has a Liouvillian solution, then it is necessary that p or $-p-2$ are non-negative integers, which is equivalent to saying that $p \in \mathbb{Z} \setminus \{-1\}$. $\square$

Lemma 3.3. *Let $\xi \neq 0$ and p be a non-negative integer. Then the equation (12) has a Liouvillian solution if and only if the determinant $|A|=0$, where A is a $(d+1)\times(d+1)$ tridiagonal matrix*

$$A=\begin{pmatrix} p^2-\eta & \xi & 0 & 0 & 0 & \dots & 0 \\ d\xi & p^2-\eta-4(1)(d-1) & 2\xi & 0 & 0 & \dots & 0 \\ 0 & (d-1)\xi & p^2-\eta-4(2)(d-2) & 3\xi & 0 & \dots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & 2\xi & p^2-\eta-4(d-1)(1) & d\xi \\ 0 & \dots & \dots & 0 & 0 & \xi & p^2-\eta \end{pmatrix}$$

and $d=p$.

This Liouvillian solution can be written as $y=Pe^{\int f}$, where P is a polynomial of degree d such that

$$P = P_d x^d + P_{d-1}x^{d-1} + \dots + P_1 x + P_0, \quad P_d \neq 0,$$

and $(P_0, P_1, \dots, P_d)$ satisfies the matrix equation

$$A\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

and

$$f = \frac{\xi}{8x^2} + \frac{1-p}{2x} - \frac{\xi}{8}.$$

Proof. We continue the Step 2 in the Algorithm for Case 1. From the proof in Lemma 3.2, we see that only the second family is satisfied because p is assumed to be a non-negative integer. So, for the second family, we let

$$f = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} - [\sqrt{r}]_\infty = \frac{\xi}{8x^2} + \frac{1-p}{2x} - \frac{\xi}{8}.$$

Now we move to the Step 3, where we need to find a polynomial of degree $d=p$ (not necessary to be monic, since our Liouvillian solution $Pe^{\int f}$ is unique up to scalar)

$$P = P_d x^d + P_{d-1}x^{d-1} + \dots + P_1 x + P_0, \quad P_d \neq 0,$$

such that

$$P'' + 2fP' + (f' + f^2 - r)P = 0. \tag{13}$$

After some computations, we find that

$$f' + f^2 - r = \frac{p\xi x + p^2 - \eta}{4x^2}.$$

Multiplying $4x^2$ on both sides in (13), we have

$$4x^2P'' + (\xi + 4x(1-p) - \xi x^2)P' + (p\xi x + p^2 - \eta)P = 0 \tag{14}$$

Then, by comparing the coefficients $x^{d+1}, x^d, \ldots, x^1, x^0$ in equation (14), we obtain the following $d+1$ three-term recursion relations. Note that the coefficient of x^{d+1} in equation (14) is equal to $(-\xi d + p\xi)P_d = 0 \cdot P_d = 0$ since $d = p$.

$$\begin{aligned}
\xi P_{d-1} + (p^2-\eta)P_d + 0 &= 0\\
2\xi P_{d-2} + [p^2-\eta-4(d-1)(1)]P_{d-1} + \xi d P_d &= 0\\
3\xi P_{d-3} + [p^2-\eta-4(d-2)(2)]P_{d-2} + \xi(d-1)P_{d-1} &= 0\\
&\vdots\\
(d-1)\xi P_1 + [p^2-\eta-4(2)(d-2)]P_2 + \xi 3P_3 &= 0\\
d\xi P_1 + [p^2-\eta-4(1)(d-1)]P_1 + \xi 2P_2 &= 0\\
(p^2-\eta)P_0 + \xi P_1 &= 0
\end{aligned}$$

Hence, we see that having a non-trivial solution $(P_0, P_1, \ldots, P_d)$ from the above $d+1$ equations is equivalent to the determinant of matrix A equals to zero. Then, for such a non-trivial solution $(P_0, P_1, \ldots, P_d)$, equation (12) has a Liouvillian solution $y = Pe^{\int f}$, where P and f are defined as above. $\square$

Corollary 3.4. *In particular when $p = 0$ and $\eta = 0$, then equation (12) has a Liouvillian solution in the form*

$$y(x) = P_0 x^{1/2}\exp\left(-\frac{\xi}{8}(\frac{1}{x}+x)\right)$$

where P_0 is a constant.

Remark 3.5. Our Lemma 3.3 verifies the work in (Duval and Loday-Richaud, 1992, pp. 237-238), since equation (12) is indeed a *double confluent Heun equation*. They applied a slightly modified version of Kovacic algorithm to four types of Heun equations directly, while we aim to study the Whittaker-Ince equation by using the *original* Kovacic algorithm.

Theorem 3.6. *Let $a = 0$, $\xi \neq 0$ and p be a non-negative integer. Then the Whittaker-Ince equation (8) has a Liouvillian solution if and only if the determinant $|A| = 0$, where A is defined as in Lemma 3.3.*

This Liouvillian solution can be written as

$$w(z) = (P_d e^{i2dz} + P_{d-1}e^{i2(d-1)z} + \cdots + P_1 e^{i2z} + P_0)\cdot \exp(-ipz)$$

where $d = p$ and $(P_0, P_1, \ldots, P_d)$ are defined as in Lemma 3.3.

Proof. By the following relationships

$$y(x) = e^{\frac{1}{2}\int\alpha}W(x), \quad W(x) = w\left(\frac{\ln x}{2i}\right) = w(z), \quad x = e^{i2z},$$

$y(x)$ is a Liouvillian solution of equation (12) if and only if $w(z)$ is a Liouvillian solution of equation (8).

Then by Lemma (3.3), we know the expression of $y(x)$, so

$$\begin{aligned} W(x) &= y(x)e^{-\frac{1}{2}\int\alpha} \\ &= P\exp\left(\int(f-\frac{\alpha}{2})\right) \\ &= P\exp\left(\int\frac{-p}{2x}\,dx\right) \\ &= CPx^{-p/2} \end{aligned}$$

where C is a constant.

Thus,

$$\begin{aligned} w(z) &= CP\exp(-ipz) \\ &= C(P_d e^{i2dz} + P_{d-1}e^{i2(d-1)z} + \cdots + P_1 e^{i2z} + P_0)\cdot\exp(-ipz) \end{aligned}$$

where $(P_0, P_1, \ldots, P_d)$ are defined as in Lemma 3.3. □

Corollary 3.7. *Let $a = 0$, $\xi \neq 0$ and p be a non-negative integer. Then the differential Galois group of the Whittaker-Ince equation (8) over $\mathbb{C}(x)$ is triangularizable if and only if the determinant $|A| = 0$, where A is defined as in Lemma 3.3.*

Proof. Direct implication of Lemma 2.2 and Theorem 2.4. □

3.2. When a is non-zero

Next, we investigate the Whittaker-Ince equation (8) when $a \neq 0$. We apply the same transformation (9) to equation (8) as in the case when $a = 0$. Then, we obtain

$$\left(-2ax-2ax^3-4x^2\right)W'' + [-4x-2ax^2-2a+\xi(x^2-1)]W' + \left(\eta - p\xi\cdot\frac{x+x^{-1}}{2}\right)W = 0.$$

After that, we divide the coefficient of W'' on both sides and apply the transformation $y = e^{\frac{1}{2}\int\alpha}W$, where

$$\alpha = [-4x-2ax^2-2a+\xi(x^2-1)] \div \left(-2ax-2ax^3-4x^2\right)$$

to eliminate the term involving W', then we have

$$y'' = r(x)y, \tag{15}$$

where

$$r(x) = \frac{s(x)}{16x^2(ax^2+2x+a)^2}$$

and

$$\begin{aligned} s(x) = {} & x^4\left(4a^2-12a\xi-4ap\xi+\xi^2\right) + x^3(8a\eta-8\xi-8p\xi) \\ & + x^2\left(8a^2-16+16\eta-8ap\xi-8a\xi-2\xi^2\right) \\ & + x(8a\eta+8\xi-8p\xi-16\xi) + (4a^2-4ap\xi+4a\xi+\xi^2) \\ := {} & K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0. \end{aligned}$$

Remark 3.8. From the above expression of $r(x)$, we see that 0, $k_\pm := \frac{-1\pm\sqrt{1-a^2}}{a}$ are the three regular singularities of equation (15), while ∞ is an irregular singularity.

Theorem 3.9. *When $a \neq 0$, the equation (15) has a Louvillian solution $y = Pe^{\int f}$, where P is a polynomial of degree*

$$d = -1 + \frac{1}{2}\left(\sqrt{1+\frac{4D}{a^2}} + \sqrt{1+\frac{4F}{a^2}}\right)$$

and

$$f = \frac{2a+\sqrt{4a^2+K_0}}{4ax} + \frac{a-\sqrt{a+4D}}{2a(x-k_+)} + \frac{a-\sqrt{a+4F}}{2a(x-k_-)}$$

such that

$$D = \frac{a^2(K_4k_+^4 + K_3k_+^3 + K_2k_+^2 + K_1k_+ + K_0)}{64k_+^2(1-a^2)},$$
$$F = \frac{a^2(K_4k_-^4 + K_3k_-^3 + K_2k_-^2 + K_1k_- + K_0)}{64k_-^2(1-a^2)},$$

and

$$k_\pm := \frac{-1\pm\sqrt{1-a^2}}{a}$$

if all of the following conditions are satisfied:

(i) $K_4 \neq 0$,

(ii)

$$\left(\sqrt{1+\frac{4D}{a^2}} + \sqrt{1+\frac{4F}{a^2}}\right) \in \{2, 4, 6, 8, \dots\},$$

(iii) *the polynomial P satisfies the polynomial equation*

$$16x^2(ax^2+2x+a)^2P'' + gP' + hP = 0$$

where

$$\begin{aligned} g = {} & \frac{8x(ax^2+2x+a)^2(2a+\sqrt{4a^2+K_0})}{a} \\ & + 16x^2a(x-k_+)^2(x-k_-)(a-\sqrt{a+4F}) \\ & + 16x^2a(x-k_+)^2(x-k_-)(a-\sqrt{a+4D}), \end{aligned}$$

and

$$\begin{aligned} h = {} & \frac{-4(2a+\sqrt{4a^2+K_0})(ax^2+2x+a)^2}{a} \\ & - 8x^2a(a-\sqrt{a+4F})(x-k_+)^2 - 8x^2a(a-\sqrt{a+4D})(x-k_-)^2 \\ & - (K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0) \\ & + [(x-k_+)(x-k_-)(2a+\sqrt{4a^2+K_0}) \\ & \quad + 2x(x-k_+)(a-\sqrt{a+4F}) + 2x(x-k_-)(a-\sqrt{a+4D})]^2. \end{aligned}$$

Proof. We suppose all the three conditions stated above are satisfied. We see that $r(x)$ has three poles

$$x=0, \quad x=\frac{-1\pm\sqrt{1-a^2}}{a}:=k_{\pm}$$

where all of them are of order 2. Since $K_4\neq 0$, so that the order of r at ∞ is equal to $6-4=2$. Then, all the necessary conditions in Theorem 2.6 regarding the orders of poles are satisfied for our $r(x)$. Thus, we can apply all three types of Kovacic's algorithms to check whether the equation (15) has Liouvillian solutions or not.

Here we apply the Algorithm for Case 1. We consider that the partial fraction expansion of $r(x)$ is in the form

$$\begin{aligned}
r(x)&=\frac{a^2(K_4x^4+K_3x^3+K_2x^2+K_1x+K_0)}{16x^2(ax-(-1+\sqrt{1-a^2}))^2(ax-(-1-\sqrt{1-a^2}))^2}\\
&=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{ax-(-1+\sqrt{1-a^2})}+\frac{D}{(ax-(-1+\sqrt{1-a^2}))^2}\\
&\quad+\frac{E}{ax-(-1-\sqrt{1-a^2})}+\frac{F}{(ax-(-1-\sqrt{1-a^2}))^2}
\end{aligned}$$

Then, we can compute B, D, F easily by substituting $x=0, k_+, k_-$ respectively. We find that

$$\begin{aligned}
B&=\frac{K_0}{16a^2}, \quad D=\frac{a^2(K_4k_+^4+K_3k_+^3+K_2k_+^2+K_1k_++K_0)}{64k_+^2(1-a^2)},\\
F&=\frac{a^2(K_4k_-^4+K_3k_-^3+K_2k_-^2+K_1k_-+K_0)}{64k_-^2(1-a^2)}.
\end{aligned}$$

Hence, the coefficients of $1/(x-k_+)^2$ and $1/(x-k_-)^2$ in the partial fraction expansion of $r(x)$ are D/a^2 and K/a^2 respectively. Then, from the Step 1 in the Algorithm for Case 1, we have

$$[\sqrt{r}]_0=[\sqrt{r}]_{k_+}=[\sqrt{r}]_{k_-}=[\sqrt{r}]_\infty=0$$

and

$$\begin{aligned}
\alpha_0^{\pm}=\alpha_\infty^{\pm}&=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4B}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{K_0}{4a^2}},\\
\alpha_{k_+}^{\pm}&=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{4D}{a^2}},\\
\alpha_{k_-}^{\pm}&=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{4F}{a^2}}.
\end{aligned}$$

Then, for the particular family

$$s(\infty)=s(0)=+, \quad s(k_+)=s(k_-)=-$$

we have

$$\begin{aligned}
d&=\alpha_\infty^+-\alpha_0^+-\alpha_{k_+}^--\alpha_{k_-}^-\\
&=-1+\frac{1}{2}\left(\sqrt{1+\frac{4D}{a^2}}+\sqrt{1+\frac{4F}{a^2}}\right)\\
&\geq 0
\end{aligned}$$

since we have assumed that $\left(\sqrt{1+\frac{4D}{a^2}}+\sqrt{1+\frac{4F}{a^2}}\right)$ is a positive even integer.

Thus, we can let

$$\begin{aligned} f &= \frac{\alpha_0^+}{x}+\frac{\alpha_{k_+}^-}{x-k_+}+\frac{\alpha_{k_-}^-}{x-k_-} \\ &= \frac{2a+\sqrt{4a^2+K_0}}{4ax}+\frac{a-\sqrt{a+4D}}{2a(x-k_+)}+\frac{a-\sqrt{a+4F}}{2a(x-k_-)}. \end{aligned}$$

So, from the Step 3 in the Algorithm for Case 1, $y=Pe^{\int f}$ will be a Liouvillian solution of equation (15) if a polynomial P of degree d (as defined as above) satisfies the following polynomial equation

$$P''+2fP'+(f'+f^2-r)P=0. \tag{16}$$

After multiplying $16x^2(ax^2+2x+a)^2$ on both sides, equation (16) is equivalent to

$$16x^2(ax^2+2x+a)^2P''+gP'+hP=0$$

where g and h are defined as in the Lemma. □

Remark 3.10. In the above proof, we only use the Algorithm for Case 1 to determine the Liouvillian solutions of the equation (15) under certain assumptions. However, since all three cases in Theorem 2.4 may happen for equation (15), we are able to apply the Algorithms for Case 2 and Case 3 to determine the Liouvillian solutions (if any) when the assumptions in Theorem 3.9 fail.

Corollary 3.11. *When $a\neq 0$, the Whittaker-Ince equation (8) has a Liouvillian solution if all the conditions stated in Theorem 3.9 are satisfied.*

Proof. By the relationships

$$y(x)=e^{\frac{1}{2}\int\alpha}W(x),\quad W(x)=w\left(\frac{\ln x}{2i}\right)=w(z),\quad x=e^{i2z},$$

if $y(x)$ is a Liouvillian solution of equation (15), then $w(z)$ will also be a Liouvillian solution of equation (8). □

Corollary 3.12. *When $a\neq 0$, the differential Galois groups of the equation (15) and the Whittaker-Ince equation (8) over $\mathbb{C}(x)$ are both triangularizable if all the conditions stated in Theorem 3.9 are satisfied.*

Proof. Direct implication of Lemma 2.2 and Theorem 2.4. □

4. On ellipsoidal wave equation

In this Section, we will apply Kovacic's algorithm to show that the algebraic form of ellipsoidal wave equation has no Liouvillian solution.

The **Jacobian form of ellipsoidal wave equation** is defined as

$$\frac{d^2w}{dz^2}-(a+bk^2\,\mathrm{sn}^2\,z+qk^4\,\mathrm{sn}^4\,z)w=0, \tag{17}$$

where a,b,q are complex constants and $\mathrm{sn}\,z=\mathrm{sn}(z,k)$ is the *Jacobi elliptic function with modulus k*, $0\leq k^2\leq 1$.

Remark 4.1. When $q=0$, equation (17) will reduce to *Lamé's equation*

$$\frac{d^2w}{dz^2}+[h-n(n+1)k^2\operatorname{sn}^2 z]w=0. \tag{18}$$

In some special cases, the solutions of Lamé's equation can be written in polynomials, which are the so-called *Lamé polynomials*. Hence, Lamé's equation has Liouvillian solutions, whereas we will show that when $q\neq 0$, the ellipsoidal wave equation (17) has *no* Liouvillian solution. Furthermore, in 1915, Whittaker (1915) had shown that the solutions of the homogeneous integral equation

$$w(z)=\lambda\int_0^{4K} P_n(k\operatorname{sn}z\operatorname{sn}s)w(s)\,ds$$

(where n is a positive integer, P_n is Legendre's function, $4K$ is the period of the elliptic function $\operatorname{sn}z$), are the solutions of Lamé's equation (18). On the other hand, after suitable change of variable, we can obtain the *algebraic form of Lamé's equation*:

$$\frac{d^2w}{dt^2}+\frac{1}{2}\left(\frac{1}{t-e_1}+\frac{1}{t-e_2}+\frac{1}{t-e_3}\right)\frac{dw}{dt}-\frac{n(n+1)t+B}{4(t-e_1)(t-e_2)(t-e_3)}w, \tag{19}$$

where $e_1,e_2,e_3,B\in\mathbb{C}$, $n\in\mathbb{Q}$, e_i are distinct and $e_1+e_2+e_3=0$.

In particular, Baldassarri (1981) has shown that equation (19) has only algebraic solutions when $n\notin 1/2\mathbb{Z}$. In addition, Fedoryuk (1989) has studied the Lamé wave functions and the asymptotics of solutions of Lamé's equation in the complex plane. On the other hand, Churchill (1999) has classified all the relationship between the monodromy groups or the differential Galois groups of equations (18) and (19).

Now, we start to investigate the ellipsoidal wave equation. First, we apply the transformation $t=\operatorname{sn}^2 z$ and obtain the **algebraic form of ellipsoidal wave equation** as derived in (Arscott et al., 1983, p. 368):

$$t(t-1)(t-c)\frac{d^2w}{dt^2}+\frac{1}{2}[3t^2-2(1+c)t+c]\frac{dw}{dt}+(\lambda+\mu t+\gamma t^2)w=0 \tag{20}$$

where $c=1/k^2$, $\lambda=-a/(4k^2)$, $\mu=-b/4$, $\gamma=-qk^2/4$.

Remark 4.2. One can compare the difference between equation (19) and equation (20). We see that equation (20) has three regular singularities at $t=0,1,c$, and an irregular singularity at ∞, which can be obtained by the confluence of two other regular singularities. In other words, the equation (20) is a confluent case of a general second-order linear differential equation with *five* regular singularities. The ellipsoidal wave equation is known as the first equation in this class to be thoroughly studied. We recall that we are able to roughly classify the second-order linear differential equation in terms of their number and type of singularities. In particular, a *Fuchsian differential equation* is a differential equation which has only regular singularities in $\mathbb{CP}^1$, i.e., the Riemann sphere. For example, the *hypergeometric type* equations are the equations that have three regular singularities, e.g. Bessel, Legendre. While the *Heun type* equations are the equations that have four regular singularities, e.g. Mathieu, Lamé.

Next, we continue to divide $t(t-1)(t-c)$ on both sides in equation (20) to obtain an equation in the form:

$$w''+\alpha(t)w'+\beta(t)w=0,$$

where

$$\alpha(t)=\frac{1}{2}\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-c}\right), \quad \beta(t)=\frac{\lambda+\mu t+\gamma t^2}{t(t-1)(t-c)}.$$

Next, we set $y=\exp(\frac{1}{2}\int\alpha)w$ to eliminate the term involving w' and obtain the reduced form:

$$y''=r(t)y \tag{21}$$

such that

$$\begin{aligned} r(t)&=-\beta+\frac{1}{4}\alpha^2+\frac{1}{2}\alpha' \\ &=\frac{-\lambda-\mu t-\gamma t^2}{t(t-1)(t-c)}+\frac{1}{16}\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-c}\right)^2+\frac{1}{4}\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-c}\right)' \\ &=\frac{-\lambda-\mu t-\gamma t^2}{t(t-1)(t-c)}-\frac{3}{16}\left[\frac{1}{t^2}+\frac{1}{(t-1)^2}+\frac{1}{(t-c)^2}\right] \qquad (22)\\ &\quad+\frac{1}{8}\left[\frac{1}{t(t-1)}+\frac{1}{t(t-c)}+\frac{1}{(t-1)(t-c)}\right] \\ &=\frac{p(t)}{t^2(t-1)^2(t-c)^2} \qquad (23)\end{aligned}$$

where $p(t)$ is a polynomial of degree 5.

Therefore, $r(t)$ is a rational function that has three finite poles (i.e., zeros of the denominator) at $t=0,1,c$, with all order 2 (i.e., multiplicity of the zero in the denominator); while the order of r at ∞ is $6-5=1$.

Proposition 4.3. *The reduced form (21) of ellipsoidal wave equation has no Liouvillian solution.*

Proof. Since the order of r at ∞ is 1, according to the necessary conditions in the Theorem 2.6, case 1 and case 3 cannot hold. Since all finite poles have order 2, which satisfies the necessary condition in case 2, then we may apply the Algorithm for Case 2 to determine whether equation (21) has Liouvillian solutions or not.

We consider that when (22) is expressed in a partial fraction form, the coefficients of $1/t^2$, $1/(t-1)^2$ and $1/(t-c)^2$ are still all equal to $-3/16$. Then by the Algorithm for Case 2, we have

$$E_0=E_1=E_c=\{1,2,3\}, \quad E_\infty=1.$$

Thus, we have 27 families (e_0,e_1,e_c,e_∞) with $e_k\in E_k$. But

$$d=\frac{1}{2}\left(e_\infty-\sum_{k\in\Gamma}e_k\right)$$

cannot be a non-negative integer for these 27 cases. So no families remain under consideration, then case 2 cannot hold. Hence, the equation (21) has no Liouvillian solution. □

Corollary 4.4. *The algebraic form (20) of ellipsoidal wave equation has no Liouvillian solution.*

Proof. By Proposition 4.3, the equation (21) has no Liouvillian solution, which means the equation (20) also has no Liouvillian solution since $w=\exp(-\frac{1}{2}\int\alpha)y$. □

Corollary 4.5. *The Galois groups of the reduced form (21) and the algebraic form (20) of ellipsoidal wave equation over $\mathbb{C}(x)$ are both equal to* SL(2).

Proof. Direct implication of Lemma 2.2 and Theorem 2.4. □

Remark 4.6. We can see that the Whittaker-Ince equation (15) and ellipsoidal wave equation both have three regular singularities and one irregular singularity at ∞. However, Whittaker-Ince equation can have some Liouvllian solutions as shown in Theorem 3.9 while the ellipsoidal wave equation cannot have any of them. Indeed, these two differential equations are not equivalent by the mean of Möbius transformation because the Whittaker-Ince equation has two *conjugate* regular singularities $k_\pm$ other than zero, while the ellipsoidal wave equation does not have this property.

5. On the Picard-Fuchs equation of a K3 surface

In this Section, we first use the so-called *brute force approach* that mentioned in (Schnell, 2012, pp. 2-3) to derive the Picard-Fuchs equation of a K3 surface that is defined as a hypersurface in $\mathbb{P}^3$ cut out by

$$\{(x,y,t)\in\mathbb{C}^3 \mid y^2=4x^3-g(t)x-h(t)\} \tag{24}$$

where g and h are polynomials of degree 8 and 12 respectively with $g^3-27h^2\neq 0$. Next, we will show that this Picard-Fuchs equation has no Liouvillian solution when $g(t)=h(t)$.

We recall that a *K3 surface* is a simply connected compact complex manifold of complex dimension 2 which has trivial canonical line bundle. For instance, every nonsingular hypersurface of degree 4 in $\mathbb{P}^3$ is a nonrational K3 surface, e.g.,

$$\{x^4+y^4+z^4+w^4=0\}.$$

And we know that a nonrational K3 elliptic surface can be birationally given by a Weierstrass equation in the form of (24).

Then, the projection to the third coordinate t in (24) will give a family of elliptic curves whose periods are

$$f_1(t)=\int_\gamma w=\int_\gamma \frac{dx}{y}=\int_\gamma \frac{dx}{\sqrt{4x^3-g(t)x-h(t)}}$$

where γ is any loops and $w=w(t)=\frac{dx}{\sqrt{4x^3-g(t)x-h(t)}}$ is a holomorphic 1-form that is defined in all smooth fibers of our K3 surface.

The periods $f_1(t)$ will satisfy a second-order differential equation in the form

$$\frac{d^2f_1}{dt^2}+B\frac{df_1}{dt}+Cf_1=0,\quad B,C\in\mathbb{C}(t), \tag{25}$$

which is then the **Picard-Fuchs equation** of our K3 surface.

Remark 5.1. Especially, (Stiller, 1981, p. 207) has derived a differential equation with solution $f_1(t)$ when our $g(t)=h(t)$ in his Theorem I.3.1. That equation was written in terms of the invariant $\mathcal{J}(t)=\frac{g(t)}{g(t)-27}$ as

$$\frac{d^2f_1}{dt^2}+\frac{\mathcal{J}'^2-\mathcal{J}\mathcal{J}''}{\mathcal{J}\mathcal{J}'}\frac{df_1}{dt}+\frac{\mathcal{J}'^2(\frac{31}{144}\mathcal{J}-\frac{1}{36})}{\mathcal{J}^2(\mathcal{J}-1)^2}f_1=0. \tag{26}$$

Proposition 5.2. *The coefficients B and C in equation (25) are*

$$B=\frac{-9g^2hg'^2+27h^2(5g'h'-3hg'')+g^3(7g'h'+3hg'')-2g^4h''+54gh(-2h'^2+hh'')}{(g^3-27h^2)(-3hg'+2gh')},$$

$$C=\frac{18g^2g'^2h'-3g(7hg'^3+40h'^3)+8g^3(h'g''-g'h'')+108h(-2hh'g''+g'(h'^2+2hh''))}{16(g^3-27h^2)(-3hg'+2gh')}$$

respectively.

Proof. We follow the steps mentioned in (Schnell, 2012, p. 3). Our goal is to determine B and C so that the 1-form

$$\eta = \frac{d^2 w}{dt^2} + B\frac{dw}{dt} + Cw$$

is closed, i.e., $\eta = d\phi$ for some function ϕ. Then by Stokes' theorem, we have

$$\frac{d^2 f_1}{dt^2} + B\frac{df_1}{dt} + Cf_1 = \int_\gamma \eta = \int_\gamma d\phi = \int_{\partial\gamma} \phi = 0$$

since

$$\frac{df_1}{dt} = \int_\gamma \frac{dw}{dt}.$$

First, since

$$\frac{dw}{dt} = \frac{g'x+h'}{2y^3}dx, \quad \frac{d^2w}{dt^2} = \frac{2y^2(g''x+h'')+3(g'x+h')^2}{4y^5}dx,$$

then we have

$$\eta = \frac{P(x)}{y^5}dx$$

where

$$\begin{aligned}
P(x) = {} & 16Cx^6 + \frac{1}{4}x^4(-32Cg + 8Bg' + 8g'') + \frac{1}{4}x^3(-32Ch + 8Bh' + 8h'') \\
& + \frac{1}{4}x^2(4Cg^2 - 2Bgg' + 3g'^2 - 2gg'') \\
& + \frac{1}{4}x(8Cgh - 2Bhg' - 2Bgh' + 6g'h' - 2hg'' - 2gh'') \\
& + \frac{1}{4}(4Ch^2 - 2Bhh' + 3h'^2 - 2hh'') \\
& \in \mathbb{C}[t][x].
\end{aligned}$$

Next, we subtract suitable multiples of the exact forms:

$$d(\frac{x^k}{y^3}) = \frac{(4k-18)x^{k+2} + (\frac{3}{2}-k)gx^k - khx^{k-1}}{y^5}dx, \quad k = 0,1,2,4$$

from η in order to reduce the degree of the polynomial $P(x)$ in its numerator into degree *one*, then we should obtain (can be done by, for instance, Mathematica)

$$\begin{aligned}
& \eta + 8Cd(\frac{x^4}{y^3}) + \frac{1}{10}(-28Cg + 2Bg' + 2g'')d(\frac{x^2}{y^3}) + \frac{1}{7}(-20Ch + Bh' + h'')d(\frac{x}{y^3}) \\
& + \frac{1}{18}(\frac{12Cg^2}{5} + \frac{3}{4}g'^2 - \frac{3}{5}g(Bg' + g''))d(\frac{1}{y^3}) \\
& = \frac{Q(x)}{y^5}dx
\end{aligned}$$

where

$$
\begin{aligned}
Q(x) = & \frac{1}{560} x[3456Cgh - 504Bhg' + 840g'h' - 504hg'' - 240g(Bh' + h'')] \\
& + \frac{1}{560}[112Cg^3 + 2160Ch^2 + 35gg'^2 - 360Bhh' + 420h'^2 \\
& - 28g^2(Bg' + g'') - 360hh''] \\
:= & q_1(t)x + q_2(t).
\end{aligned}
$$

By solving $q_1 = 0$ and $q_2 = 0$ (i.e., $Q(x) = 0$), we then obtain the required B and C that are sufficient to make the 1-form η being closed. $\square$

Corollary 5.3. *When $g(t) = h(t)$, then*

$$
B = \frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'}, \quad C = \frac{3(g+4)g'^2}{16(g-27)g^2}.
$$

In particular, the Picard-Fuchs equation

$$
\frac{d^2 f_1}{dt^2} + \left[\frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'} \right] \frac{df_1}{dt} + \frac{3(g+4)g'^2}{16(g-27)g^2} f_1 = 0 \tag{27}
$$

has no Liouvillian solution for any g.

Proof. We can easily obtain B and C by substituting $h = g$. Then, we let $u = g(t)$ and $F_1(u) = F_1(g(t)) := f_1(t)$, so

$$
\frac{df_1}{dt} = g' \frac{dF_1}{du}, \quad \frac{d^2 f_1}{dt^2} = g'' \frac{dF_1}{du} + g'^2 \frac{d^2 F_1}{du^2}.
$$

Hence, our equation (27) becomes

$$
\frac{d^2 F_1}{du^2} + \frac{2u-27}{u-27} \frac{dF_1}{du} + \frac{3(u+4)}{16(u-27)u^2} F_1 = 0. \tag{28}
$$

Next, if we can show that equation (28) has no Liovillian solution $F_1(u)$ by Kovacic's algorithm, then the equation (27) also has no Liovillian solution since $f_1(t) = F_1(u) = F_1(g(t))$ and g is just a polynomial.

First, we set $y = \exp(\frac{1}{2} \int \frac{2u-27}{u-27}) F_1$, then equation (28) becomes

$$
\frac{d^2 y}{du^2} = r(u) y, \tag{29}
$$

where

$$
\begin{aligned}
r(u) &= -\frac{3(u+4)}{16(u-27)u^2} + \frac{1}{4}\left(\frac{2u-27}{u-27}\right)^2 + \frac{1}{2}\left(\frac{2u-27}{u-27}\right)' \\
&= \frac{16u^4 - 432u^3 + 2697u^2 + 69u + 324}{16(u-27)^2 u^2}.
\end{aligned}
$$

We see that the order of r at ∞ is $4 - 4 = 0$, so case 3 cannot hold by Theorem 2.6 immediately. While the poles of r are 0 and 27, with all orders 2, so r satisfies the necessary conditions of case 1 and case 2. We then apply the algorithms for both cases to examine whether the equation (29) has Liouvillian solution or not.

Case 1. Now $\Gamma = \{0, 27\}$. Since 0 and 27 are poles of r of order 2, then from the partial fraction expansion of r:

$$
r = 1 + \frac{675}{4(u-27)^2} + \frac{104945}{3888(u-27)} + \frac{1}{36u^2} + \frac{31}{3888u},
$$

we know

$$\alpha_0^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\cdot\frac{1}{36}} = \frac{1}{2} \pm \frac{\sqrt{10}}{6}, \quad \alpha_{27}^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+675} = \frac{1}{2} \pm 13.$$

Next, since the order of r at ∞ is 0, then let $[\sqrt{r}]_\infty = c_0$. So $([\sqrt{r}]_\infty)^2 = c_0^2 = 1$ from the expansion of r. We choose $c_0 = 1$. Then

$$\alpha_\infty^{\pm} = \frac{1}{2}(\pm\frac{31}{3888} - 0) = \pm\frac{31}{7776}.$$

Thus,

$$d = \alpha_\infty^{s(\infty)} - \sum_{k\in\Gamma} \alpha_k^{s(k)} \notin \mathbb{Z}_{\geq 0}$$

for all families s. Hence case 1 cannot hold.

Case 2. Similarly by the orders of poles of r, the order of r at ∞, and the partial fraction expansion of r, we have

$$E_0 = \left\{2 + m\sqrt{1+4\cdot\frac{1}{36}}\,\middle|\, m = 0, \pm 2\right\} \cap \mathbb{Z} = \{2\},$$
$$E_{27} = \left\{2 + m\sqrt{1+4\cdot\frac{675}{4}}\,\middle|\, m = 0, \pm 2\right\} \cap \mathbb{Z} = \{2 + 26m \,|\, m = 0, \pm 2\},$$
$$E_\infty = \{0\}.$$

Then, all the families (e_0, e_{27}, e_∞) with $e_k \in E_k$ have even coordinates, so all the families will be discarded. So, case 2 cannot hold.

Since all first three cases in Theorem 2.4 cannot hold, equation (29) has no Liouvillian solution. Hence, equation (28) also has no Liouvillian solution because $F_1 = \exp(-\frac{1}{2}\int\frac{2u-27}{u-27})y$. □

Remark 5.4. Indeed, when $g(t) = h(t)$, the Picard-Fuchs equation (27) in Corollary 5.3 is the same as the equation (26) after substituting $g = \frac{27\mathcal{J}}{\mathcal{J}-1}$. Hence we reconfirm the correctness of Stiller's work.

Corollary 5.5. *The Galois group of the Picard-Fuchs equation (27) over $\mathbb{C}(x)$ is equal to* SL(2).

Proof. Direct implication of Lemma 2.2 and Theorem 2.4. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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