Liouvillian Solutions of Certain Differential Equations

by

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A Thesis Submitted to

The Hong Kong University of Science and Technology

in Partial Fulfillment of the Requirements for

the Degree of Master of Philosophy

in Mathematics

August 21, 2018, Hong Kong

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the thesis examination committee have been made.

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Acknowledgment

I would like to express my deep gratitude to my supervisor Prof Edmund Chiang, and my another research advisor Dr Avery Ching, who have given me a lot of advice and kindly support in my research during two years of my MPhil study. I have learnt a lot from the meetings in our research group, as well as in other conferences that I attended.

I would like to thank all my course instructors for their inspirational teaching, and the Department of Mathematics for providing me with postgraduate studentship award so I have the valuable opportunity to study here. It is grateful to learn a lot from discussing with my amiable and humorous classmates during these two years.

Lastly, I would like to thank my parents, my brother and sister, and other family members for their unfailing support.

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Abstract

In this thesis, we apply the Kovacic's algorithm, a tool that is developed from differential Galois theory, to determine whether the Whittaker-Ince equation, ellipsoidal wave equation and the Picard-Fuchs equation of a K3 surface have Liouvillian solutions or not. We have determined the necessary and sufficient conditions of having Liouvillian solutions for the Whittaker-Ince equation when one parameter is equal to zero. Also, we will give a sufficient condition of having a Liouvillian solution for the Whittaker-Ince equation when this parameter is non-zero. On the other hand, we have discovered that the ellipsoidal wave equation has no Liouvillian solution. We generalize a Picard-Fuchs equation for certain K3 surface and show that a particular case of the Picard-Fuchs equation cannot have any Liouvillian solutions.

Chapter 1

Introduction

A *Liouvillian solution* of a homogeneous linear differential equation can be roughly interpreted as a solution that involves solutions of polynomial equations, exponentials or indefinite integrals. Some people also call them *closed-form* solutions. A more precise way to define them will be given in Definition 2.4.7 that based on the *differential Galois theory* (Picard-Vessiot theory). Historically, people have considered whether a differential equation has Liouvllian solutions or not, or equivalently, whether the equation is solvable by *quadratures*. Note that the notion of whether a differential equation is solvable by quadratures is similar to whether a polynomial equation is solvable by *radicals* in the classical Galois theory. We recall that *Riemann's differential equation* is defined as

$$\begin{aligned}&\frac{d^2y}{dx^2} + \left(\frac{1-\rho-\rho'}{x-a} + \frac{1-\sigma-\sigma'}{x-b} + \frac{1-\tau\tau'}{x-c}\right)\frac{dy}{dx}\\&\quad + \left(\frac{\rho\rho'(a-b)(a-c)}{x-a} + \frac{\sigma\sigma'(b-c)(b-a)}{x-b} + \frac{\tau\tau'(c-a)(c-b)}{x-c}\right)\\&\quad \cdot \frac{y}{(x-a)(x-b)(x-c)} = 0,\end{aligned}$$

where $\rho, \rho'; \sigma, \sigma'; \tau, \tau'$ are the exponents belonging to the three regular singularities a, b, c respectively, and satisfy the relation $\rho + \rho' + \sigma + \sigma' + \tau + \tau' = 1$.

In particular, the *hypergeometric differential equation*

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$$

is a special case of Riemann's equation. The work [34] by H.A. Schwarz has determined the class of hypergeometric equations that have algebraic solutions only. It inspired people [28] to determine the class of Riemann's equations that contain algebraic solutions only, which is called the *Schwarz solutions class.* Then in 1950 and 1956, M. Hukuhara and S. Ohasi [16] determined a class of Riemann's equations that are solvable by quadratures. Later in 1969, T. Kimura [20] has shown that if a solution of Riemann's equation is Liouvillian, then this solution belongs to either the class of Schwarz or the class of Hukuhara-Ohasi. When proving this result, Kimura studied the *monodromy groups* of the Riemann's equations. It is because, the Riemann's equation is reducible if and only if a monodromy group of Riemann's equation is reducible to triangular form. And, reducibility of Riemann's equation implies the solvability of the equation by quadratures. Recall that a Riemann's equation is *reducible* if it has a non-trivial solution satisfying a differential equation

$$y' + r(x)y = 0, \quad \text{for some } r(x) \in \mathbb{C}(x).$$

The main task in this thesis is to apply *Kovacic's algorithm* that was constructed in the work of J.J. Kovacic [22], to determine whether the ellipsoidal wave equation, the Whittaker-Ince Equation, and a particular case of the Picard-Fuchs equation of a K3 surface have Liouvillian solutions or not, in other words, whether these differential equations are solvable by quadratures. This algorithm allows us to write down the Liouvillian solutions explicitly if they exist.

Note that the original Kovacic's algorithm can *only* be applied to a second-order homogeneous linear differential equation in this form:

$$y'' + ay' + by = 0, \quad \text{where } a, b \in \mathbb{C}(z). \tag{1.1}$$

For higher-order homogeneous linear differential equations, such as third-order equations, interested readers can refer to the work of M. Singer and F. Ulmer [37], [42] about a new algorithm to compute Liouvillian solutions for third-order differential equations. Detailed explanation will be given in Section 3.4.

For each time, before applying the Kovacic algorithm, we need to transform eqaution (1.1) into the *reduced/normal form*:

$$w'' = (-b + \frac{1}{4}a^2 + \frac{1}{2}a')w$$

by setting $w = e^{\frac{1}{2}\int a}y$ to eliminate the term involving y'. We can see that w is Liouvillian if and only if y is Liouvillian. Also, if equation (1.1) has a Liouvllian solution, then by the method of reduction of order, every solution of (1.1) will be Liouvillian.

Remark. In general, if you have a differential equation in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

then you can set $w = e^{\frac{1}{n}\int a_{n-1}}y$ to eliminate the term involving $y^{(n-1)}$.

On the other hand, it is not surprising that Kovacic's algorithm has been implemented in *Wolfram Mathematica* or MAPLE to calculate Liouvillian solutions for simple differential equation in the form (1.1) whose rational coefficients a, b are explicitly expressed with *fixed* parameters. While, Mathematica is able to express the solutions of the hypergeometric equation in terms of several undetermined parameters, none of these software can be applied in the three types of differential equations that we are encountering because of their complexity. For example, each of the three equations that we are considering is expressed in the general form whose parameters are *undetermined.* Also, none of them can be transformed into a hypergeometric equation. Thus, in order to find the Liouvillian solutions of these three differential equations by Kovacic's algorithm, manual manipulations are required. A thorough discussion of the implementation of Kovacic's algorithm in certain technical computing systems will be provided in Section 3.3.

To begin with, we will consider the *Whittaker-Ince equation*:

$$(1 + a\cos(2z))w'' + \xi\sin(2z)w' + (\eta - p\xi\cos(2z))w = 0,$$

where $a, \xi, \eta, p \in \mathbb{C}$. Especially, when $a = 0$, the Whittaker-Ince equation can be transformed into the *Whittaker–Hill Equation*:

$$W'' + \left[\eta - \frac{\xi^2}{8} - \xi(p+1)\cos(2z) + \frac{\xi^2}{8}\cos(4z)\right]W = 0,$$

after substituting $w = W\exp\left(\frac{\xi\cos(2z)}{4}\right)$. Interested readers can refer to the book [26] written by W. Magnus and S. Winkler, which introduces the periodic solutions of Whittaker-Ince equation. In the book [26], the Whittaker-Ince equation is just referred as Ince equation. In 1923, E.L. Ince [17] appeared to be the main person to study the solutions of the Whittaker-Ince equation with $a = 0$. So, in the literature, e.g. [2], our Whittaker-Ince equation is called *generalized Ince equation*. However, back to 1914, our Whittaker-Ince equation was derived by E.T. Whittaker [43, p. 17], and he has shown that the periodic solutions of a different version of the Whittaker-Ince equation

$$(b - c\sin^2 x)\frac{d^2y}{dx^2} - 2(1-q)c\sin x\cos x\frac{dy}{dx} + [-n(n+2-2q)c\sin^2 x + 4c_0]y = 0$$

are the solutions of the homogeneous integral equation

$$y(x) = \lambda\int_0^{2\pi}(\sqrt{b-c}\sin x\sin s + \sqrt{b}\cos x\cos s)^n(b - c\sin^2 s)^{-q}y(s)\,ds.$$

In order to recognize the contribution of Whittaker, we rename the generalized Ince equation as Whittaker-Ince equation. In particular, when $a = 0$, the accessory parameter η can be chosen so that the Whittaker-Ince equation has solutions of trigonometric polynomials, i.e., the *Ince polynomials*. Later on, we will apply Kovacic's algorithm to the Whittaker-Ince Equation in two separate cases ($a = 0$ and $a \neq 0$). We shall determine the necessary and sufficient conditions of having Liouvillian solutions for Whittaker-Ince equation when assuming $a = 0$, $\xi \neq 0$ and $p = ni + 1$ for some positive integer n. Indeed, when $a = 0$, then $p = \pm ni + 1$ for some positive integer n provided that the Whittaker-Ince equation has a Liouvillian solution. Next, we will give a sufficient condition of having a Liouvillian solution for Whittaker-Ince equation when $a \neq 0$. Besides, there are several physical applications of the Whittaker-Ince equation. For instance, a particular case of the Whittaker-Ince equation has been involved [23] when studying the families of symmetric periodic orbits of the elliptic Sitnikov problem (a restricted version of the three-body problem). Also, the Whittaker–Hill Equation permits [29] to describe the torsional motion of flexible molecules, e.g. hydrogen peroxide molecule.

Next, we shall consider the *Jacobian form of ellipsoidal wave equation*:

$$\frac{d^2w}{dz^2} - (a + bk^2 \operatorname{sn}^2 z + qk^4 \operatorname{sn}^4 z)w = 0,$$

where a, b, q are complex constants and $\operatorname{sn} z = \operatorname{sn}(z, k)$ is the *Jacobi elliptic function with modulus* k, $0 \leq k^2 \leq 1$.. This equation has appeared in a mathematical literature [18] written by E.L. Ince since 1926. It was derived from the general second-order linear differential equation with five regular singularities. On the other hand, this equation emerges when we apply separation of variables to the Helmholtz equation in ellipsoidal coordinates. Therefore the solutions of ellipsoidal wave equation are crucial for many physical applications such as electromagnetic scattering by an ellipsoid, diffraction by elliptic plates etc. Instead of considering its Jacobian form, we will prove the non-existence of Liouvillian solutions for the *algebraic form of ellipsoidal wave equation*:

$$t(t-1)(t-c)\frac{d^2w}{dt^2} + \frac{1}{2}[3t^2 - 2(1+c)t + c]\frac{dw}{dt} + (\lambda + \mu t + \gamma t^2)w = 0,$$

where $c = 1/k^2$, $\lambda = -a/(4k^2)$, $\mu = -b/4$, $\gamma = -qk^2/4$. Especially, the solutions of the algebraic form which are finitely evaluated at the three finite singularities $t = 0, 1, c$ are called the *ellipsoidal wave functions*. They are equivalent to the *doubly-periodic* solutions of the Jacobian form. Some recent investigations of ellipsoidal wave functions can be found in [1], [3].

Furthermore, when $q = 0$, the ellipsoidal wave equation reduces to *Lamé's equation*:

$$\frac{d^2w}{dz^2} + [h - n(n+1)k^2 \operatorname{sn}^2 z]w = 0.$$

The relationship between ellipsoidal wave equation and Lamé's equation, as well as a short review of Lamé's equation, will be discussed in Chapter 5.

Lastly, we will deal with the *Picard-Fuchs equation of a K3 surface* that is defined as a hypersurface in $\mathbb{P}^3$ cut out by

$$\{(x, y, t) \in \mathbb{C}^3 |\, y^2 = 4x^3 - g(t)x - h(t)\}$$

where g and h are polynomials of degrees 8 and 12 respectively with $g^3 - 27h^2 \neq 0$. The main result of this thesis is that we use the brute force method mentioned in C. Schnell's work [33, pp. 2-3] to derive the Picard-Fuchs equation of our K3 surface. The problem of deriving the Picard-Fuchs equation of our K3 surface was suggested by Yum-Tong Siu to my collaborators Edmund Chiang and Avery Ching. In particular, our Picard-Fuchs equation contains the Picard-Fuchs equation [40, p. 207]

$$\frac{d^2 f_1}{dt^2} + \frac{\mathcal{J}'^2 - \mathcal{J}\mathcal{J}''}{\mathcal{J}\mathcal{J}'}\frac{df_1}{dt} + \frac{\mathcal{J}'^2(\frac{31}{144}\mathcal{J} - \frac{1}{36})}{\mathcal{J}^2(\mathcal{J}-1)^2} f_1 = 0, \quad \text{where } \mathcal{J}(t) = \frac{g(t)}{g(t) - 27} \tag{1.2}$$

that was derived by P.F. Stiller when $g = h$ in our K3 surface. It is because, when $g = h$, our generalized Picard-Fuchs equation will reduce to:

$$\frac{d^2 f_1}{dt^2} + \left[\frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'}\right]\frac{df_1}{dt} + \frac{3(g+4)g'^2}{16(g-27)g^2} f_1 = 0. \tag{1.3}$$

This equation is the same as the one derived by Stiller after substituting $g = \frac{27\mathcal{J}}{\mathcal{J}-1}$. Hence, our work provides another way to derive the equation (1.2) and reconfirm its correctness at the same time. We will show that the equation (1.3) has no Liouvillian solutions by Kovacic's algorithm. A short review of K3 surface and the literature [40] will be given in Chapter 6.

Furthermore, there is also a connection between Liouvillian solutions and the *complex oscillation theory*, which is a subject to study the solutions of second-order linear differential equations with transcendental entire coefficients by applying Nevanlinnna's value distribution theory. For instance, Y.-M. Chiang and G. Yu [9] have shown that for a certain Hill equation:

$$f''(z) + (-e^{4z} + K_3 e^{3z} + K_2 e^{2z} + K_1 e^z + K_0) f(z) = 0,$$

where $K_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$, all of its *non-oscillatory solutions* are Liouvillian by applying Kovacic's algorithm. We recall that a non-oscillatory solution f is a solution whose exponent of convergence of zeros is finite, in other words,

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log n(r, f)}{\log r} < \infty,$$

where $n(r, f)$ is the number of zeros of the entire function f in the disk $\{|z| < r\}$. Another similar direction of study can be found in the work [24] by X. Luo, who has investigated the interplay between finite-gap of the Whittaker-Hill equation and Nevanlinna theories. S.B. Bank and I. Laine [5], [6] proposed the complex oscillation theory to investigate the oscillaotroy properties of entire solutions for second-order ordinary differential equations. Besides, Bank [7] has developed an algorithm to determine the existence of non-oscillatory solutions for the class of equations in the generalized form of the above Hill equation:

$$f''(z) + A(z) f(z) = 0,$$

where $A(z)$ is a non-constant periodic entire function, which can be assumed to be a rational function of e^z. If the non-oscillatory solutions exists, this method would allow us to express these non-oscillatory solutions explicitly, which is similar to the purpose of Kovacic's algorithm.

This thesis is organized as follows. In Chapter 2, we discuss preliminary notions about differential Galois theory such as differential fields, Picard-Vessiot Extensions, differential Galois groups and Liouvillian extentions. We devote Chapter

3 to a review of Kovacic's algorithm and its application by computing software etc. Lastly, we demonstrate how to apply Kovacic's algorithm to determine whether the Whittaker-Ince Equation, the ellipsoidal wave equation, and the Picard-Fuchs equation of certain K3 surface have Liouvillian solutions or not in Chapter 4, 5 and 6 respectively.

Chapter 2

Preliminaries

The aim of this chapter is to illustrate some basic notions in differential Galois theory, which can facilitate our understanding of Kovacic's work that we shall apply in Chapter 3. We will see that the Galois theory of linear differential equations is just an analogue of the Galois theory of polynomial equations. Our main reference here is the book [11] written by T. Crespo and Z. Hajto. Interested readers who want to learn more about the applications of differential Galois theory can refer to the book [32], which have investigated certain important topics such as monodromy of differential equations, the Riemann-Hilbert problem, Stokes phenomenon, and moduli for singular differential equations. Other usual old references are [19], [21], [25]. Especially, [15] gives a good comparison between classical Galois theory and differential Galois theory.

2.1 Differential Fields

Definition 2.1.1. A **derivation** of a ring A is a map $\partial : A \to A$ such that

$$\partial(a+b) = \partial(a) + \partial(b), \quad \partial(ab) = (\partial a)b + a(\partial b).$$

We usually write $a' = \partial(a)$ and $a^{(n)} = \partial^n(a)$.

Definition 2.1.2. A **differential ring** is a commutative ring with the identity and a derivation. A **differential field** is a field endowed with a derivation. So, we can define the **ring of constants** to be the subring of a differential ring that contains all the elements with derivative 0. Similarly, we can define the **field of constants** C_K to be the subfield of a differential field K that contains all the elements with derivative 0.

Example 2.1.3. (1) We can endow a **trivial derivation** for every commutative ring A with the identity by setting $\partial(a) = 0$, for all $a \in A$, to make it become a differential ring. For instance, the trivial derivation is the *only* derivation for the ring $\mathbb{Z}$ because $\partial(1) = 0$ and by induction, $\partial(n) = \partial((n-1)+1) = \partial(n-1) + \partial(1) = 0 + 0 = 0$ for all $n \in \mathbb{Z}$.

(2) The field of all rational functions $\mathbb{C}(z)$ is a differential field with our usual differentiation, $f' = \frac{df}{dz}$.

(3) $\mathbb{C}(z, \log z)$, where $(\log z)' = 1/z$.

Definition 2.1.4. A **differential morphism** is a map $\phi : A \to B$ between two differential rings such that

(1) $\phi(a+b) = \phi(a) + \phi(b), \quad \phi(ab) = \phi(a)\phi(b)$, for all $a, b \in A$;

(2) $\phi(1) = 1$; and

(3) $\phi(a)' = \phi(a')$, for all $a \in A$.

2.2 Picard-Vessiot Extensions

Definition 2.2.1. Given two differential fields K and L, then an inclusion $K \subset L$ is a **differential field extension** if the derivation ϕ_L of L restricts to the derivation ϕ_K of K on the subfield K, i.e. $\phi_L|_K = \phi_K$. One can also say L is a differential field extension of K. If S is a subset of L, then we denote the differential subfield of L generated by S over K as, $K\langle S \rangle$.

In particular, given two differential field extensions L and M of a differential field K, if $\phi : L \to M$ is a differential isomorphism that restrict to the identity map on the subfield K, i.e. $\phi|_K = 1_K$, then ϕ is called a **differential K-isomorphism**.

Definition 2.2.2. Let a differential field K with a non-trivial derivation ∂. Then a **linear differential operator $\mathcal{L}$** is a polynomial in ∂ with coefficients in K , i.e.,

$$\mathcal{L} = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0, \quad \text{where } a_i \in K.$$

We say that $\mathcal{L}$ has degree n if $a_n \neq 0$.

Definition 2.2.3. The **ring of linear differential operators** with coeffcients in K is the non-commutative ring $K[\partial]$ of polynomials in ∂ with coefficients in K, such that the product of ∂ and an element $a \in K$ is equal to $\partial a = a' + a\partial$. Be careful of the difference with $\partial(a)$.

Now we consider the homogeneous linear differential equations of order n over a differential field K of *characteristic zero*:

$$\mathcal{L}(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1Y' + a_0Y = 0, \quad \text{where } a_i \in K. \tag{2.1}$$

We know that if $K \subset L$ is a differential field extension, then the set of solutions of $\mathcal{L}(Y) = 0$ in L is a C_L-vector space, where C_L is the field of constants of L.

Proposition 2.2.4. *The dimension of the solution space of $\mathcal{L}(Y) = 0$ is less than or equal to n, i.e., it has at most n solutions in L that are linearly independent over C_L.*

Proof. See [11, p. 127]. $\square$

Definition 2.2.5. If $y_1, \ldots, y_n$ are n solutions of $\mathcal{L}(Y) = 0$ in L such that they are linearly independent over C_L, then $\{y_1, \ldots, y_n\}$ is called a **fundamental set of solutions** of $\mathcal{L}(Y) = 0$ in L.

Definition 2.2.6. Let $\mathcal{L}(Y) = 0$ be a differential equation in the form (2.1). Then a differential field extension $K \subset L$ is a **Picard-Vessiot extension** for $\mathcal{L}$ if

(1) $L = K\langle y_1, \ldots, y_n\rangle$, where $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions of $\mathcal{L}(Y) = 0$ in L; and

(2) $C_L = C_K$.

Remark. The notion of Picard-Vessiot extension is similar to the Galois extension of a polynomial.

Example 2.2.7. (1) If $\mathcal{L} = Y'' + \frac{1}{z}Y$ over the base field $K = \mathbb{C}(z)$, then its Picard-Vessiot extension L is $\mathbb{C}(z, \log z)$.

(2) If $\mathcal{L} = Y' - Y$ over $K = \mathbb{C}(z)$, then $L = \mathbb{C}(z, e^z)$, which is a transcendental extension.

(3) If $\mathcal{L} = Y' + \frac{z}{1-z^2}Y$ over $K = \mathbb{C}(z)$, then $L = \mathbb{C}(z, \sqrt{1-z^2})$, which is not a transcendental extension.

Theorem 2.2.8 (Existence and Uniqueness)**.** *Let $\mathcal{L}(Y) = 0$ be defined as in (2.1), where the field of constants C_K is algebraically closed. Then there exists a Picard-Vessiot extension L of K for $\mathcal{L}$, such that L is unique up to differential K-isomorphisms.*

Proof. See [11, pp. 130-132]. $\square$

2.3 Differential Galois Groups

Definition 2.3.1. Let $K \subset L$ be a differential field extension. Then the **differential Galois group** $G(L|K)$ of the extension $K \subset L$ is the group of all differential K-automorphisms of L. Moreover, if $K \subset L$ is a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$, then we can write $G(L|K)$ as $\mathrm{Gal}_K(\mathcal{L})$, the Galois group of $\mathcal{L}(Y) = 0$ over K.

Example 2.3.2. Let $K \subset L = K\langle\alpha\rangle$ be a differential field extension, where $\alpha' = a \in K$ and a is *not* a derivative in K. Then $K \subset K\langle\alpha\rangle$ is a Picard-Vessiot extension of $Y'' - \frac{a'}{a}Y' = 0$, and the Galois group $G(L|K)$ is isomorphic to the additive group of C_K. Details can be found in [11, pp. 21-22].

Example 2.3.3. Let $K \subset L = K\langle\alpha\rangle$ be a differential field extension, where $\alpha'/\alpha = a \in K \setminus \{0\}$. Then the Galois group $G(L|K)$ is isomorphic to the multiplicative group of C_K when α is transcendental over K; or is isomorphic to a finite cyclic group when α is algebraic over K. Details can be found in [11, p. 22].

Remark. In Example 2.3.2, we see that α is a solution of the non-homogeneous linear equation $Y' = a$, while $K \subset K\langle\alpha\rangle$ is a Picard-Vessiot extension of $Y'' - \frac{a'}{a}Y' = 0$. In general, given a non-homogeneous equation

$$\mathcal{L}(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_1Y' + a_0Y = b,$$

we can associate a homogeneous equation of order increased by one, i.e.,

$$\tilde{\mathcal{L}}(Y) = 0, \quad \text{where } \tilde{\mathcal{L}} = (\partial - \frac{b'}{b})\mathcal{L}.$$

We see that if $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions of $\mathcal{L}(Y) = 0$ and y_0 is a particular solution of $\mathcal{L}(Y) = b$, then $\{y_0, y_1, \ldots, y_n\}$ is a fundamental set of solutions of $\tilde{\mathcal{L}}(Y) = 0$.

Theorem 2.3.4. *Given a differential field K and a Picard-Vessiot extension $L = K\langle y_1, \ldots, y_n\rangle$ of K. Then there exists a set S of polynomials $P(X_{ij})$, $1 \le i, j \le n$, with coefficients in C_K such that*

(1) if ϕ is a differential K-automorphism of L and $\phi(y_j) = \sum_{i=1}^{n} c_{ij} y_i$, then $P(c_{ij}) = 0$, for all $P \in S$; and

(2) if given a matrix $(c_{ij}) \in \mathrm{GL}(n, C_K)$ with $P(c_{ij}) = 0$, for all $P \in S$, then there exists a differential K-automorphism ϕ of L such that $\phi(y_j) = \sum_{i=1}^{n} c_{ij} y_i$.

Proof. See [11, pp. 145-147]. □

Now, let C be an algebraically closed field. Recall that the *affine m-space over C* is the set of all m-tuples of elements of C, which is denoted by $\mathbb{A}^n$. An *algebraic set* is a subset Y of $\mathbb{A}^n$ such that there exists a subset T of the polynomial ring in m variables over C, where

$$Y = Z(T) := \{P \in \mathbb{A}^n | f(P) = 0 \, \text{for all } f \in T\}.$$

Note that, the complements of all algebraic sets satisfy the axioms of open sets in a topology, hence they form a topology for $\mathbb{A}^n$, which is called the *Zariski topology*. For all subsets Y in $\mathbb{A}^n$, we can define the *ideal of Y in the polynomial ring $C[X_1, \ldots, X_n]$* by

$$I(Y) = \{f \in C[X_1, \ldots, X_n] \mid f(P) = 0 \text{ for all } P \in Y\}.$$

If V is an algebraic set, then we define the *coordinate ring $C[V]$ of V* to be the quotient ring $C[X_1, \ldots, X_n]/I(V)$.

Next, if $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are two algebraic sets, then a *morphism* $\phi : V \to W$ is a map such that $\phi(x) = \phi(x_1, \ldots, x_n) = (\phi_1(x), \ldots, \phi_m(x))$ where $\phi_i \in C[V]$ for all i. In our setting, we can consider an *algebraic group over C* as an algebraic set G defined over C such that G is a group, and the multiplication $\mu : G \times G \to G$, $\mu(x, y) = xy$ and the inversion $\iota : G \to G$, $\iota(x) = x^{-1}$ are morphisms. Note that a Zariski closed subgroup of an algebraic group is also an algebraic group. Some simple examples of algebraic groups are finite groups,

$\mathrm{GL}(n, C)$, $\mathrm{SL}(n, C)$, the upper triangular group

$$T(n, C) := \{(a_{ij}) \in \mathrm{GL}(n, C) \,|\, a_{ij} = 0 \,\text{for}\, i > j\}.$$

A *linear algebraic group* is a Zariski closed subgroup of $\mathrm{GL}(n, C)$ for some positive integer n.

Corollary 2.3.5. *The differential Galois group of a Picard-Vessiot extension is a linear algebraic group.*

Proof. It follows from Theorem 2.2.8, we are able to construct a Picard-Vessiot extension L for a homogeneous linear differential equation of order n that defined over a differential field K. By the result of Theorem 2.3.4, we see that $G(L|K)$ is a Zariski closed subgroup of $\mathrm{GL}(n, C_K)$, and hence is a linear algebraic group. □

Next, we can construct a *fundamental theorem* for the differential Galois theory that is similar to the one in the Galois theory for polynomial equations.

Theorem 2.3.6. *Given a Picard-Vessiot extension $K \subset L$ and its differential Galois group $G(L|K)$. Here $L^H := \{x \in L \,|\, \phi(x) = x \,\text{for all}\, \phi \in H\}$ is a differential subfield of L. Then*

(1) The two mutually inverse maps

$$H \mapsto L^H, \quad F \mapsto G(L|F)$$

establishes a one-one correspondence between the set of Zariski closed subgroups H of $G(L|K)$ and the set of intermediate differential fields F with $K \subset F \subset L$, such that the above two maps are also inclusionally inverting. That is, if H_1, H_2 are subgroups $G(L|K)$, then $H_1 \subset H_2$ implies $L^{H_1} \supset L^{H_2}$; while if F_1, F_2 are intermediate differential fields, then $F_1 \subset F_2$ implies $G(L|F_1) \supset G(L|F_2)$.

(2) An intermediate differential field F is a Picard-Vessiot extension of K if and only if the subgroup $H = G(L|F)$ is normal in $G(L|K)$. In particular, the restriction map $G(L|K) \to G(F|K)$ defined by

$$\phi \mapsto \phi|_F$$

produces an isomorphism $G(L|K)/G(L|F) \cong G(F|K)$.

Proof. See [11, pp. 151-158]. □

2.4 Liouvillian Extensions

Definition 2.4.1. Given a differential field extension $K \subset L$, and an element $\alpha \in L$, then we say that an α is a **primitive element** over K if $\alpha' \in K$; an **exponential element** over K if $\alpha'/\alpha \in K$.

Example 2.4.2. Given $K = \mathbb{C}(z)$. Then it follows from the Definition 2.4.1, α_i is an exponential element over F_i is equivalent to $\alpha_i = e^{\int a_i}$ for some $a_i \in F_i$. Similarly, α_i is a primitive element over F_i is equivalent to $\alpha_i = \int a_i$ for some $a_i \in F_i$.

Definition 2.4.3. We define a **Liouvillian extension** to be a differential field extension $K \subset L$ such that there exists a tower of intermediate differential fields $K = F_1 \subset F_2 \subset \cdots \subset F_n = L$ where $F_{i+1} = F_i \langle \alpha_i \rangle$, and each α_i is either an exponential element over F_i or a primitive element over F_i.

Recall that for any elements x, y of a group G, we denote the *commutator* $xyx^{-1}y^{-1}$ as $[x, y]$. While if A, B are subgroups of G, then we denote the subgroup generated by all commutators $[a, b]$ where $a \in A$, $b \in B$, as $[A, B]$. Also the *derived series* D^iG for $i \geq 0$ of a group G is defined as

$$D^0G = G, \quad D^iG = [D^{i-1}G, D^{i-1}G].$$

Then a group G is *solvable* if its derived series terminates as an identity element. Furthermore, if G is an algebraic group, then G is solvable if and only if there is a chain of Zariski closed subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$ such that $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} is abelian, for $i = 0, \ldots, n-1$.

Proposition 2.4.4. *If $K \subset L$ is a Liouvillian extension and $C_L = C_K$, then the differential Galois group $G(L|K)$ of L over K is solvable.*

Proof. See [11, p. 159]. □

Definition 2.4.5. We can also define a **generalized Liouvillian extension** to be a differential field extension $K \subset L$ such that there exists a tower of intermediate differential fields $K = F_1 \subset F_2 \subset \cdots \subset F_n = L$ where $F_{i+1} = F_i \langle \alpha_i \rangle$, and each α_i is either an exponential element over F_i, or a primitive element over F_i, or is algebraic over F_i.

Definition 2.4.6. We define an **elementary extension** to be a differential field extension $K \subset L$ such that there exists a tower of intermediate differential fields $K = F_1 \subset F_2 \subset \cdots \subset F_n = L$ where $F_{i+1} = F_i \langle \alpha_i \rangle$, and either $\alpha_i' = u'/u$ for some $u \neq 0$ in F_i, or $\alpha_i' = u'\alpha_i$ for some u in F_i, or α_i is algebraic over F_i.

Definition 2.4.7. A solution of a homogeneous linear differential equation defined over a differential field K is called a **Liouvillian solution** if it is contained in a generalized Liouvillian extension of K.

Definition 2.4.8. A solution of a homogeneous linear differential equation defined over a differential field K is called an **elementary solution** if it is contained in an elementary extension of K.

Remark. Some authors [12, p. 1] impose an extra condition, namely the fields of constants C_L and C_K are the same, to the Definitions 2.4.5 and 2.4.6. They also refer the generalized Liouvillian extension simply as a *Liouvillian extension.*

Proposition 2.4.9. *Let $K \subset L$ be a Picard-Vessiot extension, where the field of constants C_K is algebraically closed. If L can be embedded in a generalized Liouvillian extension M of K, where $C_M = C_K$, then the identity component G_0 of $G(L|K)$ is solvable.*

Proof. See [11, pp. 160-161]. □

Remark. One can compare the difference between Proposition 2.4.4 and Proposition 2.4.9.

On the other hand, M.F. Singer has given the following two interesting results about the relationship between a *non-homogeneous* equation $\mathcal{L}(Y) = b$ and its associated homogeneous equation $\mathcal{L}(Y) = 0$.

Theorem 2.4.10 ([36], p. 272)**.** *If K is an elementary extension of $\mathbb{C}(x)$ or an algebraic extension of a transcendental Liouvillian extension of $\mathbb{C}(x)$, then we can find a basis for the space of Liouvillian solutions over K, of a linear differential equation $\mathcal{L}(Y) = 0$ with coefficients in differential field K.*

The proof of the above theorem requires our consideration on when does a non-homogeneous equation $\mathcal{L}(Y) = b$ with coefficients in K, have non-zero solutions in K.

Later in 1986, Singer and J.H. Davenport showed that

Theorem 2.4.11 ([12], p. 238)**.** *Let $\mathcal{L}(Y) = b$ be a linear differential equation with coefficients in a differential field K. If $\mathcal{L}(Y) = b$ has a non-trivial Liouvillian solution over K, then either*

(i) $\mathcal{L}(Y) = 0$ *has a non-trivial elementary solution over K; or*

(ii) $\mathcal{L}(Y) = b$ *has a solution in K.*

In summary, one can compare the analogous notions between the classical Galois theory for polynomial equations and the Differential Galois theory as in the table below.

Classical Galois theory	***Differential Galois theory***
Polynomial	Linear differential operator
Root of polynomial	Solution of differential equation
Field extension	Differential field extension
Galois extenion	Picard-Vessiot extension
Galois group (finite)	Differential Galois group (algebraic)
Correspondence Theorem: Intermediate (Galois) field extensions $\leftrightarrow$ (normal) subgroups	Correspondence Theorem: Intermediate (Picard-Vessiot) differential extensions $\leftrightarrow$ (normal) algebraic subgroups
Polynomial equation is solvable by radicals $\leftrightarrow$ Galois extension has a solvable Galois group	Differential equation is solvable by quadratures $\leftrightarrow$ Picard-Vessiot extension has a solvable differential Galois group

Chapter 3

Review of Kovacic's Algorithm

3.1 Kovacic's Algorithm

We quote only partial results that we will use in our computation from Kovacic's paper [22]. First, we consider a second-order homogeneous linear differential equation in the reduced form, i.e.,

$$y'' = ry, \quad r \in \mathbb{C}(x). \tag{3.1}$$

Then, it is natural to ask how does the Galois group of the above differential equation looks like. Indeed, we are able to characterize them.

Proposition 3.1.1. *The Galois group of the differential equation (3.1) is an algebraic subgroup of* SL(2)*.*

Proof. See [22, pp. 5-7]. $\square$

Next, we quote a fundamental fact about algebraic subgroups of SL(2) that we will use later.

Lemma 3.1.2. *Let G be an algebraic subgroup of* $\mathrm{SL}(2)$*. Then exactly one of the following four cases can happen.*

Case 1. *G is triangularizable, i.e., the matrix representing G is triangularizable;*

Case 2. *G is conjugate to a subgroup of*

$$D = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\},$$

and case 1 does not hold;

Case 3. *G is finite and cases 1 and 2 do not hold;*

Case 4. $G = \mathrm{SL}(2)$*.*

Proof. See, for instance, [22, p. 7]. □

Remark. By Proposition 3.1.1 and Lemma 3.1.2, we have classified all kinds of differential Galois groups of equation (3.1).

Theorem 3.1.3. *There are four mutually exclusive cases that can happen for equation (3.1).*

Case 1. *It has a Liouvillian solution of the form $e^{\int w}$ where $w \in \mathbb{C}(x)$.*

Case 2. *It has a Liouvillian solution of the form $e^{\int w}$ where w is algebraic over $\mathbb{C}(x)$ of degree 2, and case 1 doesn't hold.*

Case 3. *All solutions of the (3.1) are algebraic over $\mathbb{C}(x)$ and cases 1 and 2 don't hold.*

Case 4. *The equation (3.1) has no Liouvillian solution.*

Proof. See [22, pp. 7-8]. The results of this theorem are implied by Lemma 3.1.2, where G is the differential Galois group of equation (3.1). Each of the four cases from above corresponds to the respective case in the lemma. □

We recall that if $r(x) = s(x)/t(x)$, where $s, t \in \mathbb{C}(x)$ and are relatively prime, then the *poles* of r are the zeros of $t(x)$, and the *order* of the pole is the multiplicity of the zero of $t(x)$. Similarly, the order of r at ∞ means the order of ∞ as a zero of $r(x)$, hence the order of r at ∞ is equal to $\deg t - \deg s$.

Theorem 3.1.4 ([22], p. 8)**.** *The cases 1, 2 and 3 in Theorem 3.1.3 have the following necessary conditions respectively.*

Case 1. Every pole of r must have order equal to $2v$ for some $v \geq 1$, or just equal to 1. The order of r at ∞ must equal to $2v$ for some $v \leq 1$, or be greater than 2. (Note that the order of r at ∞ can be negative.)

Case 2. r must have at least one pole that either has order equal to $2v + 1$ for some $v \geq 1$, or equal to 2.

Case 3. The order of a pole of r must be smaller than or equal to 2; and the order of r at ∞ must be greater than or equal to 2. If the partial fraction expansion of r is

$$r = \sum_i \frac{\alpha_i}{(x - c_i)^2} + \sum_j \frac{\beta_j}{x - d_j},$$

then $\sqrt{1 + 4\alpha_i} \in \mathbb{Q}$ for all i and $\sum_j \beta_j = 0$; and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1 + 4\gamma} \in \mathbb{Q}$.

Remark. If our equation (3.1) satisfies one of the necessary conditions in Theorem 3.1.4, then we can apply the corresponding algorithm constructed by Kovacic to determine the Liouvillian solutions (if any) for those cases. If none of the three necessary conditions are satisfied, then the equation (3.1) has no Liouvillian solutions. Here we only quote the algorithms for case 1 and case 2 for our later use.

Algorithm for Case 1

Let Γ be the set of all finite poles of r in (3.1).

Step 1. For each $k \in \Gamma \cup \{\infty\}$, we define a rational function $[\sqrt{r}]_k$ and two complex numbers α_k^+, α_k^- as below:

- If k is a pole of order 1, then $[\sqrt{r}]_k = 0$, $\alpha_k^+ = \alpha_k^- = 1$.
- If k is a pole of order 2, then $[\sqrt{r}]_k = 0$. Also, $\alpha_k^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$, where b is the coefficient of $1/(x-k)^2$ in the partial fraction expansion for r.
- If k is a pole of order $2v \geq 4$, then $[\sqrt{r}]_k$ is the sum of terms involving $1/(x-k)^i$ for $2 \leq i \leq v$ in the Laurent series of $\sqrt{r}$ at k, where we can choose either one of them, i.e. $[\sqrt{r}]_k = \frac{c_{-v}}{(x-k)^v} + \cdots + \frac{c_{-2}}{(x-k)^2}$. Also, $\alpha_c^\pm = \frac{1}{2}(\pm\frac{b}{c_{-v}} + v)$, where b be the coefficient of $1/(x-k)^{v+1}$ in r minus the coefficient of $1/(x-k)^{v+1}$ in $([\sqrt{r}]_k)^2$.
- If r has order > 2 at ∞, then $[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1$.
- If r has order 2 at ∞, then $[\sqrt{r}]_\infty = 0$. Also, $\alpha_\infty^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$, where b is the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞ (i.e. if $r = s/t$, where $s, t \in \mathbb{C}[x]$, then b is the leading coefficient of s divided by the leading coefficient of t.)
- If r has order $-2v \leq 0$ at ∞, then $[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series of $\sqrt{r}$ at ∞, where we can choose either one of them, i.e. $[\sqrt{r}]_\infty = c_0 + c_1 x + \cdots + c_v x^v$. Also, $\alpha_\infty^\pm = \frac{1}{2}(\pm\frac{b}{c_v} - v)$, where b is the coefficient of x^{v-1} in r minus the coefficient of x^{v-1} in $([\sqrt{r}]_\infty)^2$.

Step 2. For each family $s = (s(k))_{k \in \Gamma \cup \{\infty\}}$, where $s(k)$ is either $+$ or $-$. So we have totally $2^{\rho+1}$ families s, where ρ is the number of poles of r. Let

$$d = \alpha_\infty^{s(\infty)} - \sum_{k \in \Gamma} \alpha_k^{s(k)}.$$

If $d \in \mathbb{Z}_{\geq 0}$, then let

$$f = \sum_{k \in \Gamma} \left(s(k)[\sqrt{r}]_k + \frac{\alpha_k^{s(k)}}{x-k} \right) + s(\infty)[\sqrt{r}]_\infty.$$

If $d \notin \mathbb{Z}_{\geq 0}$, then the family s is discarded for consideration. If no families s are retained for us to construct f, then case 1 cannot hold.

Step 3. For each family s that are retained from Step 2, we search for a monic polynomial P of degree d (defined in Step 2) that satisfies the differential equation

$$P'' + 2fP' + (f' + f^2 - r)P = 0.$$

If such a polynomial exists, then $y = Pe^{\int f}$ is a Liouvillian solution of equation (3.1). If no such polynomial can be found from the families s that are retained, then case 1 cannot hold.

Algorithm for Case 2

Step 1. For each $k \in \Gamma$ we define E_k as follows.

- If k is a pole of r of order 1, then $E_k = \{4\}$.
- If k is a pole of r of order 2 and if b is the coefficient of $1/(x-k)^2$ in the partial fraction expansion of r, then

$$E_k = \{2 + m\sqrt{1+4b} \,|\, m = 0, \pm 2\} \cap \mathbb{Z}.$$

- If k is a pole of r of order $v > 2$, then $E_k = \{v\}$.
- If r has order > 2 at ∞, then $E_\infty = \{0, 2, 4\}$.
- If r has order 2 at ∞ and b is the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞, then

$$E_\infty = \{2 + m\sqrt{1+4b} \,|\, m = 0, \pm 2\} \cap \mathbb{Z}.$$

- If the order r at ∞ is $v < 2$, then $E_\infty = \{v\}$.

Step 2. We consider all families $(e_k)_{k\in\Gamma\cup\{\infty\}}$ with $e_k \in E_k$. Those families all of whose coordinates are even will be discarded. Let

$$d = \frac{1}{2}\left(e_\infty - \sum_{k\in\Gamma} e_k\right).$$

If $d \in \mathbb{Z}_{\geq 0}$, then the family should be retained, otherwise the family is discarded. If no families remain under consideration, then case 2 cannot hold.

Step 3. For each family retained from Step 2, we construct the rational function

$$\theta = \frac{1}{2}\sum_{k\in\Gamma} \frac{e_k}{x-k}.$$

Then, we search for a monic polynomial P of degree d (as defined in Step 2) such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

If no such polynomial is found for any family retained from Step 2, then case 2 cannot hold.

Suppose such a polynomial P is found, then let $\phi = \theta + P'/P$ and w be a solution of the quadratic equation

$$w^2 + \phi w + (\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r) = 0.$$

Then, $y = e^{\int w}$ is a Liouvillian solution of equation (3.1).

3.2 Examples of Applications

Example 3.2.1. In [22], Kovacic has given several examples of applying his algorithm to different differential equations. One example is that *Bessel's equation*

$$y'' = \left(\frac{4n^2-1}{4x^2} - 1\right)y, \quad n \in \mathbb{C},$$

has a Liouvillian solution if and only if $-1/2 \pm n \in \mathbb{Z}_{\geq 0}$. Also, Bessel's equation belongs to the Case 1 in Theorem 3.1.3.

Example 3.2.2. In 1992, A. Duval and M. Loday-Richaud applied a slightly modified version of Kovacic's algorithm to two second-order differential equations. They showed [13, pp. 226-234] that for the hypergeometric differential equation in the form

$$y'' - \left[\frac{\lambda^2 - 1}{4x^2} + \frac{\mu^2 - 1}{4(1-x)^2} + \frac{\lambda^2 + \mu^2 - \nu^2 - 1}{4x(1-x)}\right] y = 0,$$

if at least two of the three parameters λ, μ, ν belong to $1/2 + \mathbb{Z}$, then the equation has Liouvillian solutions. Besides, the authors have tried to characterize the necessary and sufficient conditions [13, p. 235-241] of having Liouvillian solutions for the *Heun equation*

$$y'' = \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-c} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-c)^2}\right] y$$

by considering its *confluent*, *biconfluent*, *double confluent* and *triconfluent* types separately. They have given a complete answer for the biconfluent, double confluent and triconfluent types without the help of computer, while they only considered one special case of the confluent equations. Each type of the Heun equation corresponds to a particular configuration of the four regular singularities $0, 1, c, \infty$.

Example 3.2.3. We can also apply Kovacic's algorithm to the *Chebyshev differential equation*

$$Y'' = \frac{(\alpha^2 - 1/4)z^2 - \alpha^2 - 1/2}{(z^2-1)^2} Y. \tag{3.2}$$

One can show that when $\alpha = \pm 1/2$, equation (3.2) admits two linearly independent Liouvillian solutions

$$Y_1 = (z-a)^{3/4}(z+1)^{1/4}, \quad Y_2 = (z-1)^{1/4}(z+1)^{3/4},$$

where its differential Galois group over $\mathbb{C}(x)$ will be the cyclic group C_4. When $\alpha \neq \pm 1/2$, we can apply the Algorithm for Case 2 to show that

$$Y = (z^2-1)^{1/4}(z + \sqrt{z^2-1})^{\alpha}$$

is a Liouvillian solution to equation (3.2). Details of computation can be found in [11, p. 187, p. 193].

3.3 Implementation in Computer Software

As mentioned in the introduction, certain simple second-order linear differential equation whose coefficients are in $\mathbb{C}(x)$ and do not contain any undetermined parameters, can be solved by using *Wolfram Mathematica*. For instance, when we want to determine whether the differential equation

$$xy'' + (10x^3 - 1)y' + 5x^2(5x^3 + 1)y = 0 \tag{3.3}$$

has Liouvillian solutions or not, we can input the following command in Mathematica,

```
DSolve[x*y''[x]+(10x^3-1)*y'[x]+5x^2(5x^3+1)*y[x]==0, y[x], x]
```

It will output

$$\left\{\left\{y[x] \to e^{-\frac{5x^3}{3}}\, C[1] + \frac{1}{2}\, e^{-\frac{5x^3}{3}}\, x^2\, C[2]\right\}\right\}$$

i.e., $y_1 = e^{-5x^3/3}$, $y_2 = x^2 e^{-5x^3/3}$ are the two linearly independent Liouvillian solutions of equation (3.3).

Occasionally, we may involve the *exponential integral* $\mathrm{Ei}(x) = -\int_{-x}^{\infty} e^{-t}/t\, dt$ or an unevaluated integral of elementary functions in the solutions of certain second-order differential equations after applying Kovacic's algorithm. For example, inputting the following differential equation in Mathematica

```
DSolve[4 x*y''[x] + (7 x + 12)*y'[x] + 21 y[x] == 0, y[x], x]
```

then it will output the general solution

$$\left\{\left\{y[x] \to e^{-7x/4}\, C[1] - \frac{e^{-7x/4}\, C[2]\left(16\, e^{7x/4} + 28\, e^{7x/4}\, x - 49\, x^2\, \text{ExpIntegralEi}\left[\frac{7x}{4}\right]\right)}{32\, x^2}\right\}\right\}$$

where ExpIntegralEi[.] is the exponential intergral $\mathrm{Ei}(x)$.

The computation process of the both outputs above involves the use of Kovacic's algorithm. Details of the manual of solving second-order linear differential equations with raitonal coefficients by Mathematica can be found in the website: `http://reference.wolfram.com/language/tutorial/DSolveLinearSecondOrderODEsWithRationalCoefficients.html`

On the other hand, the `kovacicsols` routine in MAPLE generates a list of Liouvillian solutions of a second order linear differential equation with rational coefficients. It uses Kovacic's algorithm to determine if such solutions exist, and if so, to determine them.

Below is an example of using `kovacicsols`:

> with(DEtools) :

> $\text{ode} := \frac{d^2}{dx^2} y(x) + \frac{3\left(x^2 - x + 1\right) y(x)}{16 (x-1)^2 x^2}$:

> kovacicsols(ode, $y(x)$)

$$\left[\sqrt{x-1}\left(\frac{-x^{3/2}+x}{-\sqrt{x}-1}\right)^{1/4}, \sqrt{x-1}\left(\frac{x^{3/2}+x}{-1+\sqrt{x}}\right)^{1/4}\right] \qquad (1)$$

Instructions of using `kovacicsols` in MAPLE can be found in the website: `https://www.maplesoft.com/support/help/maple/view.aspx?path=DEtools%2Fkovacicsols`

Remark. Kovacic's algorithm has been implemented in several computer software for solving second-order linear differential equations with rational coefficients, thorough treatises can be found in [8], [38] (for MAPLE), [27], [30] (for MACSYMA), [31] (for SCRATCHPAD).

3.4 Generalization to Higher-Order Differential Equations

In 2001, M.F. Singer and F. Ulmer [37, p. 13-15] extended the Kovacic's algorithm for second-order linear differential equations to *third-order* case. They constructed an algorithm to either determines all Liouvillian solutions of a third-order linear differential equation through a given formula; or asserts that

(i) its differential Galois group G is one of the 12 finite groups (in this case procedures exist to determine the algebraic solutions); or

(ii) G is isomorphic to projective general linear group PGL(2), or $\mathrm{PGL}(2)\times C_3$, where C_3 is the cyclic subgroup of order 3 of SL(3) (in this case we can solve the differential equation using second-order equations); or

(iii) $G = \mathrm{SL}(3)$ (i.e., the equation has no Liouvillian solution and cannot be solved in terms of the solutions of second-order equations).

Two years later, Ulmer [42] refined parts of the algorithm to compute the Liouvillian solutions of third-order equations based on the following fundamental result.

Theorem 3.4.1. *In general, the algorithm of finding Liouvillian solutions of a third-order equation divides into four cases. Given a third-order linear differential equation* $\mathcal{L}(Y) = 0$ *over a differential field* K *with differential Galois group* $G = \mathrm{Gal}_K(\mathcal{L})$*, then* *(i) G is a reducible linear group and $\mathcal{L}$ is decomposable, i.e., $\mathcal{L}(Y) = \mathcal{L}_1(\mathcal{L}_2(Y))$ for some operators $\mathcal{L}_1, \mathcal{L}_2$ of lower order; or*

(ii) G is an imprimitive linear group and there exists a solution $Y = e^{\int u}$ of $\mathcal{L}(Y) = 0$ with index $[K(u) : K] = 3$; or

(iii) G belongs to one of the eight finite primitive linear groups, in particular, all Liouvillian solutions are algebraic; or

(iv) G is an infinite primitive linear group, and $\mathcal{L}(Y) = 0$ has no Liouvillian solution.

Remark. We can observe the similarity between the above theorem and Lemma 3.1.2, Theorem 3.1.3.

As early as in 1981, Singer [35] developed a general decision procedure for determining Liouvillian solutions of a linear differential equation of order n. However, this resulting algorithm was not implementable in computer software as the Kovacic algorithm or the algorithm for third-order case. The construction of Singer's algorithm requires a fact that

Proposition 3.4.2. *If $\mathcal{L}(Y) = 0$ has a Liouvillian solution, then $\mathcal{L}(Y) = 0$ has a solution Y such that $u = Y'/Y$ is algebraic over K.*

Later in 1992, Ulmer [42] obtained sharp bounds $F(n)$ for the algebraic degree of u for linear differential equations of order n by utilizing the action of its differential Galois group on u and the theory of projective representation. For instance, $F(2) = 12$, $F(3) \leq 36$.

Chapter 4

On Whittaker-Ince Equation

In this chapter, we demonstrate how to use Kovacic's algorithm to determine the Liouvillian solutions of the **Whittaker-Ince Equation**:

$$(1 + a\cos(2z))w'' + \xi\sin(2z)w' + (\eta - p\xi\cos(2z))w = 0, \tag{4.1}$$

where $a, \xi, \eta, p \in \mathbb{C}$. First, we shall show that when $a = 0$, then $p = \pm ni + 1$ for some positive integer n provided that the Whittaker-Ince equation has a Liouvillian solution. Then, we will determine the necessary and sufficient conditions of having Liouvillian solutions under the assumptions $a = 0$, $\xi \neq 0$ and $p = ni + 1$ for some positive integer n. Lastly, we will give a sufficient condition of having a Liouvillian solution for the Whittaker-Ince equation when $a \neq 0$.

4.1 When a is zero

We assume $a = 0$. Since $\sin(2z) = \frac{e^{i2z} - e^{-i2z}}{2i}$, $\cos(2z) = \frac{e^{i2z} + e^{-i2z}}{2}$, and by letting

$$x = e^{i2z}, \quad W(x) = w\left(\frac{\ln x}{2i}\right) = w(z), \tag{4.2}$$

then equation (4.1) can be transformed into

$$-4x^2W'' + [-4x + \xi(x^2-1)]W + \left(\eta - p\xi \cdot \frac{x+x^{-1}}{2}\right)W = 0.$$

Dividing $-4x^2$ on both sides, we obtain

$$W'' + \left[\frac{1}{x} - \frac{\xi}{4}\left(1 - \frac{1}{x^2}\right)\right]W' - \frac{1}{4}\left[\frac{\eta}{x^2} - \frac{p\xi}{2}\left(\frac{1}{x} + \frac{1}{x^3}\right)\right]W = 0. \tag{4.3}$$

We recall that for a differential equation of the form:

$$\frac{d^n f}{dz^n} + \sum_{k=1}^{n} a_{n-k}(z)\frac{d^{n-k}f}{dz^{n-k}} = 0, \tag{4.4}$$

where $a_i(z)$ are meromorphic on $\mathbb{C}$ for all i, a *singularity* of equation (4.4) means a singularity of $a_i(z)$ for some i. Let $u = 1/z$. The transformed equation is in u, and $u = 0$ corresponds to $z = \infty$. Equation (4.4) has a singularity at $z = \infty$ if the transformed equation has a singularity at $u = 0$. A singularity $\alpha \in \mathbb{C}$ of equation (4.4) is *regular* if

$$\lim_{z\to\alpha}(z-\alpha)^k a_{n-k}(z)$$

exists for $k = 1, \ldots, n$. Similarly, equation (4.4) has a regular singularity at $z = \infty$ if its transformed equation has a regular singularity at $u = 0$. A singularity which is not regular will be called an *irregular singularity*.

Remark. We can easily see that 0 and ∞ are the irregular singularities of equation (4.3).

Let

$$\alpha = \left[\frac{1}{x} - \frac{\xi}{4}\left(1 - \frac{1}{x^2}\right)\right], \quad \beta = -\frac{1}{4}\left[\frac{\eta}{x^2} - \frac{p\xi}{2}\left(\frac{1}{x} + \frac{1}{x^3}\right)\right]$$

in equation (4.3).

Set $y = e^{\frac{1}{2}\int \alpha} W$. Then we have

$$y'' = \left(-\beta - \frac{\alpha^2}{4} - \frac{\alpha'}{2}\right)y := r(x)y, \tag{4.5}$$

where

$$r(x) = -\frac{\xi^2x^4 + 8\xi(p-1)x^3 - 2x^2\left(8\eta + \xi^2 + 8\right) + 8\xi(p-1)x + \xi^2}{64x^4}.$$

Lemma 4.1.1. *Let $\xi \neq 0$. If the equation (4.5) has a Liouvillian solution, then*

$$p = \pm ni + 1$$

for some positive integer n.

Proof. Since the order of r at ∞ is $4 - 4 = 0$ if $\xi \neq 0$, and the order of r at the only pole 0 is 4, then by Theorem 3.1.4, both case 2 and case 3 cannot hold. Therefore, we try to use the Algorithm for Case 1 to determine the Liouvillian solutions of equation (4.5).

Since 0 is a pole of order 4, we let $[\sqrt{r}]_0 = \frac{c_{-2}}{x^2}$. Then $[\sqrt{r}]_0^2 = \frac{c_{-2}^2}{x^4}$. By comparing the coefficient of $1/x^4$ in r, we have $c_{-2}^2 = -\xi^2/64$. We choose $c_{-2} = \xi i/8$. Also the b here is the coefficient of $1/x^3$ in r minus the coefficient of $1/x^3$ in $[\sqrt{r}]_0^2$, so $b = -\frac{(p-1)\xi}{8} - 0 = (1-p)\xi/8$. Hence,

$$\alpha_0^{\pm} = \frac{1}{2}(\pm\frac{(1-p)\xi/8}{\xi i/8} + 2) = \mp\frac{(1-p)i}{2} + 1.$$

Similarly, let $[\sqrt{r}]_\infty = c_0$, and so $[\sqrt{r}]_\infty^2 = c_0^2$. By comparing the coefficient of x^0 in r, we have $c_0^2 = -\xi^2/64$. We choose $c_0 = \xi i/8$. Also the b here is the coefficient of x^{-1} in r minus the coefficient of x^{-1} in $([\sqrt{r}]_\infty)^2$, i.e., $b = \frac{(1-p)\xi}{8} - 0 = (1-p)\xi/8$. Hence

$$\alpha_\infty^{\pm} = \frac{1}{2}(\pm\frac{(1-p)\xi/8}{\xi i/8} - 0) = \mp\frac{(1-p)i}{2}.$$

Now we move to the Step 2 of the Algorithm for Case 1. There are four possible families.

(1) If $s(0) = +, s(\infty) = +$, then $d = \alpha_\infty^+ - \alpha_0^+ = -1$.

(2) If $s(0) = +, s(\infty) = -$, then $d = \alpha_\infty^- - \alpha_0^+ = (1-p)i - 1$.

(3) If $s(0) = -, s(\infty) = +$, then $d = \alpha_\infty^+ - \alpha_0^- = -(1-p)i - 1$.

(4) If $s(0) = -, s(\infty) = -$, then $d = \alpha_\infty^- - \alpha_0^- = -1$.

In order to construct f, we require our d to be a non-negative integer. That is, if the equation (4.5) has a Liouvillian solution, then it is necessary that $(1-p)i-1$ or $-(1-p)i-1$ are non-negative integers, which is equivalent to saying that $p = \pm ni + 1$ for some positive integer n. $\square$

Lemma 4.1.2. *Let $\xi \neq 0$ and $p = ni + 1$ for some positive integer n. Then the equation (4.5) has a Liouvillian solution if and only if the determinant $|A| = 0$, where A is a $(d+1) \times (d+1)$ tridiagonal matrix*

$$A = \begin{pmatrix} \lambda & \xi i & 0 & 0 & 0 & \dots & 0 \\ d\xi i & \lambda+1[4\cdot 0+8-4(1-p)i] & 2\xi i & 0 & 0 & \dots & 0 \\ 0 & (d-1)\xi i & \lambda+2[4\cdot 1+8-4(1-p)i] & 3\xi i & 0 & \dots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & 2\xi i & \lambda+(d-1)[4(d-2)+8-4(1-p)i] & d\xi i \\ 0 & \dots & \dots & 0 & 0 & \xi i & \lambda+d[4(d-1)+8-4(1-p)i] \end{pmatrix}$$

and $d = (1-p)i - 1$, $\lambda := -p^2 - \eta - 2(1-p)(1+i)$.

This Liouvillian solution can be written as $y = Pe^{\int f}$, where P is a polynomial of degree d such that

$$P = P_d x^d + P_{d-1}x^{d-1} + \cdots + P_1 x + P_0, \quad P_d \neq 0,$$

and $(P_0, P_1, \dots, P_d)$ satisfies the matrix equation

$$A \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

and

$$f = \frac{\xi i}{8x^2} + \frac{2-(1-p)i}{2x} - \frac{\xi i}{8}.$$

Proof. We continue the Step 2 in the Algorithm for Case 1. From the proof in Lemma 4.1.1, we see that only the second family is satisfied because $p = ni + 1$ for some positive integer n. So, for the second family, we let

$$f = [\sqrt{r}]_0 + \frac{\alpha_0^+}{x} - [\sqrt{r}]_\infty = \frac{\xi i}{8x^2} + \frac{2-(1-p)i}{2x} - \frac{\xi i}{8}.$$

Now we move to the Step 3, where we need to find a polynomial of degree $d = (1-p)i - 1$ (not necessary to be monic, since our Liouvillian solution $Pe^{\int f}$ is unique up to scalar)

$$P = P_d x^d + P_{d-1}x^{d-1} + \cdots + P_1 x + P_0, \quad P_d \neq 0,$$

such that

$$P'' + 2fP' + (f' + f^2 - r)P = 0. \tag{4.6}$$

After some computations, we find that

$$f' + f^2 - r = -\frac{\xi(1-p+i)x + p^2 + \eta + 2(1-p)(1+i)}{4x^2}.$$

Multiplying $4x^2$ on both sides in (4.6), we have

$$4x^2P'' + (\xi i + [8 - 4(1-p)i]x - \xi i x^2)P' + [-\xi(1-p+i)x - p^2 - \eta - 2(1-p)(1+i)]P = 0 \tag{4.7}$$

Then, by comparing the coefficients $x^{d+1}, x^d, \ldots, x^1, x^0$ in equation (4.7), we obtain the following $d+1$ three-term recursion relations. Note that the coefficient of x^{d+1} in equation (4.7) is equal to $\xi(-id-1+p-i)P_d = 0$ since $d = (1-p)i-1$.

$$\begin{aligned}
i\xi P_{d-1} + \{\lambda + d[4(d-1) + 8 - 4(1-p)i]\}P_d &= 0\\
2i\xi P_{d-2} + \{\lambda + (d-1)[4(d-2) + 8 - 4(1-p)i]\}P_{d-1} + \xi i d P_d &= 0\\
3i\xi P_{d-3} + \{\lambda + (d-2)[4(d-3) + 8 - 4(1-p)i]\}P_{d-2} + \xi i(d-1)P_{d-1} &= 0\\
&\vdots\\
(d-1)i\xi P_1 + \{\lambda + 2[4\cdot 1 + 8 - 4(1-p)i]\}P_2 + \xi i 3P_3 &= 0\\
di\xi P_1 + \{\lambda + 1[4\cdot 0 + 8 - 4(1-p)i]\}P_1 + \xi i 2P_2 &= 0\\
\lambda P_0 + \xi i P_1 &= 0
\end{aligned}$$

where $\lambda := -p^2 - \eta - 2(1-p)(1+i)$.

Hence, we see that having a non-trivial solution $(P_0, P_1, \ldots, P_d)$ from the above $d+1$ equations is equivalent to the determinant of matrix A equals to zero. Then, for such a non-trivial solution $(P_0, P_1, \ldots, P_d)$, equation (4.5) has a Liouvillian solution $y = Pe^{\int f}$, where P and f are defined as above. $\square$

Remark. Our Lemma 4.1.2 verifies the work in [13, pp. 237-238], since equation (4.5) is indeed a *double confluent Heun equation.* They applied a slightly modified version of Kovacic algorithm to four types of Heun equations directly, while we aim to study the Whittaker-Ince equation by using the *original* Kovacic algorithm.

Theorem 4.1.3. *Let $a = 0$, $\xi \neq 0$ and $p = ni + 1$ for some positive integer n. Then the Whittaker-Ince Equation (4.1) has a Liouvillian solution if and only if the determinant $|A| = 0$, where A is defined as in Lemma 4.1.2.*

This Liouvillian solution can be written as

$$\begin{aligned} w(z) = &(P_d e^{i2zd} + P_{d-1} e^{i2z(d-1)} + \cdots + P_1 e^{i2z} + P_0) \\ &\cdot \exp\left([(1-p)+i]z + \frac{(1-i)\xi}{8}(e^{-i2z} + e^{i2z})\right) \end{aligned}$$

where $(P_0, P_1, \ldots, P_d)$ are defined as in Lemma 4.1.2.

Proof. By the following relationships

$$y(x) = e^{\frac{1}{2}\int \alpha} W(x), \quad W(x) = w\left(\frac{\ln x}{2i}\right) = w(z), \quad x = e^{i2z},$$

$y(x)$ is a Liouvillian solution of equation (4.5) if and only if $w(z)$ is a Liouvillian solution of equation (4.1).

Then by Lemma (4.1.2), we know the expression of $y(x)$, so

$$\begin{aligned} W(x) &= y(x) e^{-\frac{1}{2}\int \alpha} \\ &= P \exp\left(\int (f - \frac{\alpha}{2})\right) \\ &= P \exp\left(\int \left[\frac{\xi(i-1)}{8x^2} + \frac{1-(1-p)i}{2x} + \frac{\xi(1-i)}{8}\right] dx\right) \\ &= CPx^{\frac{1-(1-p)i}{2}} \exp\left(\frac{(1-i)\xi}{8}(x^{-1} + x)\right) \end{aligned}$$

where C is a constant.

Thus,

$$
\begin{aligned}
w(z) &= CP \exp\left([(1-p)+i]z + \frac{(1-i)\xi}{8}(e^{-i2z}+e^{i2z}) \right) \\
&= C(P_d e^{i2zd} + P_{d-1}e^{i2z(d-1)} + \cdots + P_1 e^{i2z} + P_0) \\
&\quad \cdot \exp\left([(1-p)+i]z + \frac{(1-i)\xi}{8}(e^{-i2z}+e^{i2z}) \right)
\end{aligned}
$$

where $(P_0, P_1, \ldots, P_d)$ are defined as in Lemma 4.1.2. □

Corollary 4.1.4. *Let $a = 0$, $\xi \neq 0$ and $p = ni + 1$ for some positive integer n. Then the differential Galois group of the Whittaker-Ince Equation (4.1) over $\mathbb{C}(x)$ is triangularizable if and only if the determinant $|A| = 0$, where A is defined as in Lemma 4.1.2.*

Proof. Direct implication of Lemma 3.1.2 and Theorem 3.1.3. □

4.2 When a is non-zero

Next, we investigate the Whittaker-Ince Equation (4.1) when $a \neq 0$. We apply the same transformation (4.2) to equation (4.1) as in the case when $a = 0$. Then, we obtain

$$
\left(-2ax-2ax^3-4x^2\right)W''+[-4x-2ax^2-2a+\xi(x^2-1)]W'+\left(\eta-p\xi\cdot\frac{x+x^{-1}}{2}\right)W=0.
$$

After that, we divide the coefficient of W'' on both sides and apply the transformation $y = e^{\frac{1}{2}\int \alpha}W$, where

$$
\alpha = [-4x-2ax^2-2a+\xi(x^2-1)] \div \left(-2ax-2ax^3-4x^2\right)
$$

to eliminate the term involving W', then we have

$$
y'' = r(x)y, \tag{4.8}
$$

where

$$r(x) = \frac{s(x)}{16x^2(ax^2+2x+a)^2}$$

and

$$\begin{aligned} s(x) = x^4 &\left(-20a^2+12a\xi-4ap\xi+\xi^2\right)+x^3(-32a+8a\eta+8\xi-8p\xi) \\ &+x^2\left(-24a^2+16+16\eta-4ap\xi+4a\xi+2\xi^2\right) \\ &+x(8a\eta+8\xi-8p\xi)+(-4a^2-4ap\xi-4a\xi-\xi^2) \\ := {} & K_4x^4+K_3x^3+K_2x^2+K_1x+K_0. \end{aligned}$$

Remark. From the above expression of $r(x)$, we see that $0,\ k_\pm := \frac{-1\pm\sqrt{1-a^2}}{a}$ are the three regular singularities of equation (4.8), while ∞ is an irregular singularity.

Theorem 4.2.1. *When $a \neq 0$, the equation (4.8) has a Louvillian solution $y = Pe^{\int f}$, where P is a polynomial of degree*

$$d = -1+\frac{1}{2}\left(\sqrt{1+\frac{4D}{a^2}}+\sqrt{1+\frac{4F}{a^2}}\right)$$

and

$$f = \frac{2a+\sqrt{4a^2+K_0}}{4ax}+\frac{a-\sqrt{a+4D}}{2a(x-k_+)}+\frac{a-\sqrt{a+4F}}{2a(x-k_-)}$$

such that

$$D = \frac{a^2(K_4k_+^4+K_3k_+^3+K_2k_+^2+K_1k_++K_0)}{64k_+^2(1-a^2)},$$
$$F = \frac{a^2(K_4k_-^4+K_3k_-^3+K_2k_-^2+K_1k_-+K_0)}{64k_-^2(1-a^2)}.$$

and

$$k_\pm := \frac{-1\pm\sqrt{1-a^2}}{a}$$

if all of the following conditions are satisfied:

(i) $K_4 \neq 0$,

(ii)
$$\left(\sqrt{1+\frac{4D}{a^2}}+\sqrt{1+\frac{4F}{a^2}}\right) \in \{2,4,6,8,\ldots\},$$

(iii) the polynomial P satisfies the polynomial eqaution

$$16x^2(ax^2+2x+a)^2P''+gP'+hP=0$$

where

$$\begin{aligned} g = &\frac{8x(ax^2+2x+a)^2(2a+\sqrt{4a^2+K_0})}{a} \\ &+16x^2a(x-k_+)^2(x-k_-)(a-\sqrt{a+4F}) \\ &+16x^2a(x-k_+)^2(x-k_-)(a-\sqrt{a+4D}), \end{aligned}$$

and

$$\begin{aligned} h = &\frac{-4(2a+\sqrt{4a^2+K_0})(ax^2+2x+a)^2}{a} \\ &-8x^2a(a-\sqrt{a+4F})(x-k_+)^2-8x^2a(a-\sqrt{a+4D})(x-k_-)^2 \\ &-(K_4x^4+K_3x^3+K_2x^2+K_1x+K_0) \\ &+[(x-k_+)(x-k_-)(2a+\sqrt{4a^2+K_0}) \\ &\quad +2x(x-k_+)(a-\sqrt{a+4F})+2x(x-k_-)(a-\sqrt{a+4D})]^2. \end{aligned}$$

Proof. We suppose all the three conditions stated above are satisfied. We see that $r(x)$ has three poles

$$x=0, \quad x=\frac{-1\pm\sqrt{1-a^2}}{a}:=k_\pm$$

where all of them are of order 2. Since $K_4\neq 0$, so that the order of r at ∞ is equal to $6-4=2$.. Then, all the necessary conditions in Theorem 3.1.4 regarding the orders of poles are satisfied for our $r(x)$. Thus, we can apply all three types of Kovacic's algorithms to check whether the equation (4.8) has Liouvillian solutions or not.

Here we apply the Algorithm for Case 1. We consider that the partial fraction expansion of $r(x)$ is in the form

$$\begin{aligned}
r(x) &= \frac{a^2(K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0)}{16x^2(ax - (-1+\sqrt{1-a^2}))^2(ax - (-1-\sqrt{1-a^2}))^2} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax - (-1+\sqrt{1-a^2})} + \frac{D}{(ax - (-1+\sqrt{1-a^2}))^2} \\
&\quad + \frac{E}{ax - (-1-\sqrt{1-a^2})} + \frac{F}{(ax - (-1-\sqrt{1-a^2}))^2}
\end{aligned}$$

Then, we can compute B, D, F easily by substituting $x = 0, k_+, k_-$ respectively. We find that

$$\begin{aligned}
B &= \frac{K_0}{16a^2}, \quad D = \frac{a^2(K_4k_+^4 + K_3k_+^3 + K_2k_+^2 + K_1k_+ + K_0)}{64k_+^2(1-a^2)}, \\
F &= \frac{a^2(K_4k_-^4 + K_3k_-^3 + K_2k_-^2 + K_1k_- + K_0)}{64k_-^2(1-a^2)}.
\end{aligned}$$

Hence, the coefficients of $1/(x-k_+)^2$ and $1/(x-k_-)^2$ in the partial fraction expansion of $r(x)$ are D/a^2 and K/a^2 respectively. Then, from the Step 1 in the Algorithm for Case 1, we have

$$[\sqrt{r}]_0 = [\sqrt{r}]_{k_+} = [\sqrt{r}]_{k_-} = [\sqrt{r}]_\infty = 0$$

and

$$\begin{aligned}
\alpha_0^\pm = \alpha_\infty^\pm &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4B} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{K_0}{4a^2}}, \\
\alpha_{k_+}^\pm &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4D}{a^2}}, \\
\alpha_{k_-}^\pm &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4F}{a^2}}.
\end{aligned}$$

Then, for the particular family

$$s(\infty) = s(0) = +, \quad s(k_+) = s(k_-) = -$$

we have

$$\begin{aligned}
d &= \alpha_\infty^+ - \alpha_0^+ - \alpha_{k_+}^- - \alpha_{k_-}^- \\
&= -1 + \frac{1}{2}\left(\sqrt{1 + \frac{4D}{a^2}} + \sqrt{1 + \frac{4F}{a^2}}\right) \\
&\geq 0
\end{aligned}$$

since we have assumed that $\left(\sqrt{1+\frac{4D}{a^2}}+\sqrt{1+\frac{4F}{a^2}}\right)$ is a positive even integer.

Thus, we can let

$$\begin{aligned} f &= \frac{\alpha_0^+}{x}+\frac{\alpha_{k_+}^-}{x-k_+}+\frac{\alpha_{k_-}^-}{x-k_-} \\ &= \frac{2a+\sqrt{4a^2+K_0}}{4ax}+\frac{a-\sqrt{a+4D}}{2a(x-k_+)}+\frac{a-\sqrt{a+4F}}{2a(x-k_-)}. \end{aligned}$$

So, from the Step 3 in the Algorithm for Case 1, $y = Pe^{\int f}$ will be a Liouvillian solution of equation (4.8) if a polynomial P of degree d (as defined as above) satisfies the following polynomial equation

$$P'' + 2fP' + (f' + f^2 - r)P = 0. \tag{4.9}$$

After multiplying $16x^2(ax^2+2x+a)^2$ on both sides, equation (4.9) is equivalent to

$$16x^2(ax^2+2x+a)^2P'' + gP' + hP = 0$$

where g and h are defined as in the Lemma. □

Remark. In the above proof, we only use the Algorithm for Case 1 to determine the Liouvillian solutions of the equation (4.8) under certain assumptions. However, since all three cases in Theorem 3.1.3 may happen for equation (4.8), we are able to apply the Algorithms for Case 2 and Case 3 to determine the Liouvillian solutions (if any) when the assumptions in Theorem 4.2.1 fail.

Corollary 4.2.2. *When $a \neq 0$, the Whittaker-Ince equation (4.1) has a Liouvillian solution if all the conditions stated in Theorem 4.2.1 are satisfied.*

Proof. By the relationships

$$y(x) = e^{\frac{1}{2}\int \alpha}W(x), \quad W(x) = w\left(\frac{\ln x}{2i}\right) = w(z), \quad x = e^{i2z},$$

if $y(x)$ is a Liouvillian solution of equation (4.8), then $w(z)$ will also be a Liouvillian solution of equation (4.1). □

Corollary 4.2.3. *When $a \neq 0$, the differential Galois groups of the the equation (4.8) and the Whittaker-Ince equation (4.1) over $\mathbb{C}(x)$ are both triangularizable if all the conditions stated in Theorem 4.2.1 are satisfied.*

Proof. Direct implication of Lemma 3.1.2 and Theorem 3.1.3. $\square$

Chapter 5

On Ellipsoidal Wave Equation

In this chapter, we will apply Kovacic's algorithm to show that the algebraic form of ellipsoidal wave equation has no Liouvillian solution.

The **Jacobian form of ellipsoidal wave equation** is defined as

$$\frac{d^2w}{dz^2} - (a + bk^2 \operatorname{sn}^2 z + qk^4 \operatorname{sn}^4 z)w = 0, \tag{5.1}$$

where a, b, q are complex constants and $\operatorname{sn} z = \operatorname{sn}(z, k)$ is the *Jacobi elliptic function with modulus* k, $0 \leq k^2 \leq 1$.

Remark. When $q = 0$, equation (5.1) will reduce to *Lamé's equation*

$$\frac{d^2w}{dz^2} + [h - n(n+1)k^2 \operatorname{sn}^2 z]w = 0. \tag{5.2}$$

In some special cases, the solutions of Lamé's equation can be written in polynomials, which are the so-called *Lamé polynomials.* Hence, Lamé's equation has Liouvillian solutions, whereas we will show that when $q \neq 0$, the ellipsoidal wave equation (5.1) has *no* Liouvillian solution. Furthermore, in 1915, E.T. Whittaker [44] had shown that the solutions of the homogeneous integral equation

$$w(z) = \lambda \int_0^{4K} P_n(k \operatorname{sn} z \operatorname{sn} s) w(s)\, ds$$

(where n is a positive integer, P_n is Legendre's function, $4K$ is the period of the elliptic function $\operatorname{sn} z$), are the solutions of Lamé's equation (5.2). On the other hand, after suitable change of variable, we can obtain the *algebraic form of Lamé's equation*:

$$\frac{d^2w}{dt^2} + \frac{1}{2}\left(\frac{1}{t-e_1} + \frac{1}{t-e_2} + \frac{1}{t-e_3}\right)\frac{dw}{dt} - \frac{n(n+1)t+B}{4(t-e_1)(t-e_2)(t-e_3)}w, \qquad (5.3)$$

where $e_1, e_2, e_3, B \in \mathbb{C}$, $n \in \mathbb{Q}$, e_i are distinct and $e_1 + e_2 + e_3 = 0$.

In [4], the author has shown that equation (5.3) has only algebraic solutions when $n \notin 1/2\mathbb{Z}$. In addition, [14] has studied the Lamé wave functions and the asymptotics of solutions of Lamé's equation in the complex plane. On the other hand, the author in [10] has classified all the relationship between the monodromy groups or the differential Galois groups of equations (5.2) and (5.3).

Now, we start to investigate the ellipsoidal wave equation. First, we apply the transformation $t = \operatorname{sn}^2 z$ and obtain the **algebraic form of ellipsoidal wave equation** as derived in [3, p. 368]:

$$t(t-1)(t-c)\frac{d^2w}{dt^2} + \frac{1}{2}[3t^2 - 2(1+c)t + c]\frac{dw}{dt} + (\lambda + \mu t + \gamma t^2)w = 0 \qquad (5.4)$$

where $c = 1/k^2$, $\lambda = -a/(4k^2)$, $\mu = -b/4$, $\gamma = -qk^2/4$.

Remark. One can compare the difference between equation (5.3) and equation (5.4). We see that equation (5.4) has three regular singularities at $t = 0, 1, c$, and an irregular singularity at ∞, which can be obtained by the confluence of two other regular singularities. In other words, the equation (5.4) is a confluent case of a general second-order linear differential equation with *five* regular singularities. The ellipsoidal wave equation is known as the first equation in this class to be thoroughly studied. We recall that we are able to roughly classify the second-order linear differential equation in terms of their number and type of singularities. In particular, a *Fuchsian differential equation* is a differential equation which has only regular singularities in $\mathbb{CP}^1$, i.e., the Riemann sphere.

For example, the *hypergeometric type* equations are the equations that have three regular singularities, e.g. Bessel, Legendre. While the *Heun type* equations are the equations that have four regular singularities, e.g. Mathieu, Lamé.

Next, we continue to divide $t(t-1)(t-c)$ on both sides in equation (5.4) to obtain an equation in the form:

$$w'' + \alpha(t)w' + \beta(t)w = 0,$$

where

$$\alpha(t) = \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-c}\right), \quad \beta(t) = \frac{\lambda + \mu t + \gamma t^2}{t(t-1)(t-c)}.$$

Next, we set $y = \exp(\frac{1}{2}\int \alpha)w$ to eliminate the term involving w' and obtain the reduced form:

$$y'' = r(t)y \tag{5.5}$$

such that

$$\begin{aligned} r(t) &= -\beta + \frac{1}{4}\alpha^2 + \frac{1}{2}\alpha' \\ &= \frac{-\lambda - \mu t - \gamma t^2}{t(t-1)(t-c)} + \frac{1}{16}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-c}\right)^2 + \frac{1}{4}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-c}\right)' \\ &= \frac{-\lambda - \mu t - \gamma t^2}{t(t-1)(t-c)} - \frac{3}{16}\left[\frac{1}{t^2} + \frac{1}{(t-1)^2} + \frac{1}{(t-c)^2}\right] \qquad (5.6) \\ &\quad + \frac{1}{8}\left[\frac{1}{t(t-1)} + \frac{1}{t(t-c)} + \frac{1}{(t-1)(t-c)}\right] \\ &= \frac{p(t)}{t^2(t-1)^2(t-c)^2} \qquad (5.7) \end{aligned}$$

where $p(t)$ is a polynomial of degree 5.

Therefore, $r(t)$ is a rational function that has three finite poles (i.e., zeros of the denominator) at $t = 0, 1, c$, with all order 2 (i.e., multiplicity of the zero in the denominator); while the order of r at ∞ is $6 - 5 = 1$.

Proposition 5.0.1. *The reduced form (5.5) of ellipsoidal wave equation has no Liouvillian solution.*

Proof. Since the order of r at ∞ is 1, according to the necessary conditions in the Theorem 3.1.4, case 1 and case 3 cannot hold. Since all finite poles have order 2, which satisfies the necessary condition in case 2, then we may apply the Algorithm for Case 2 to determine whether equation (5.5) has Liouvillian solutions or not.

We consider that when (5.6) is expressed in a partial fraction form, the coefficients of $1/t^2$, $1/(t-1)^2$ and $1/(t-c)^2$ are still all equal to $-3/16$. Then by the Algorithm for Case 2, we have

$$E_0 = E_1 = E_c = \{1, 2, 3\}, \quad E_\infty = 1.$$

Thus, we have 27 families $(e_0, e_1, e_c, e_\infty)$ with $e_k \in E_k$. But

$$d = \frac{1}{2}\left(e_\infty - \sum_{k \in \Gamma} e_k\right)$$

cannot be a non-negative integer for these 27 cases. So no families remain under consideration, then case 2 cannot hold. Hence, the equation (5.5) has no Liouvillian solution. □

Corollary 5.0.2. *The algebraic form (5.4) of ellipsoidal wave equation has no Liouvillian solution.*

Proof. By Proposition 5.0.1, the equation (5.5) has no Liouvillian solution, which means the equation (5.4) also has no Liouvillian solution since $w = \exp(-\frac{1}{2}\int \alpha)y$. □

Corollary 5.0.3. *The Galois groups of the reduced form (5.5) and the algebraic form (5.4) of ellipsoidal wave equation over* $\mathbb{C}(x)$ *are both equal to* $\mathrm{SL}(2)$.

Proof. Direct implication of Lemma 3.1.2 and Theorem 3.1.3. □

Remark. We can see that the Whittaker-Ince equation (4.8) and ellipsoidal wave equation both have three regular singularities and one irregular singularity at

∞. However, Whittaker-Ince equation can have some Liouvllian solutions as shown in Theorem 4.2.1 while the ellipsoidal wave equation cannot have any of them. Indeed, these two differential equations are not equivalent by the mean of Möbius transformation because the Whittaker-Ince equation has two *conjugate* regular singularities $k_{\pm}$ other than zero, while the ellipsoidal wave equation does not have this property.

Chapter 6

On a Picard-Fuchs Equation

In this chapter, we first use the so-called *brute force approach* that mentioned in C. Schnell's work [33, pp. 2-3] to derive the Picard-Fuchs equation of a K3 surface that is defined as a hypersurface in $\mathbb{P}^3$ cut out by

$$\{(x, y, t) \in \mathbb{C}^3 |\, y^2 = 4x^3 - g(t)x - h(t)\} \tag{6.1}$$

where g and h are polynomials of degree 8 and 12 respectively with $g^3 - 27h^2 \neq 0$. Next, we will show that this Picard-Fuchs equation has no Liouvillian solution when $g(t) = h(t)$.

We recall that a *K3 surface* is a simply connected compact complex manifold of complex dimension 2 which has trivial canonical line bundle. For instance, every nonsingular hypersurface of degree 4 in $\mathbb{P}^3$ is a nonrational K3 surface, e.g.,

$$\{x^4 + y^4 + z^4 + w^4 = 0\}.$$

And we know that a nonrational K3 elliptic surface can be birationally given by a Weierstrass equation in the form of (6.1).

Then, the projection to the third coordinate t in (6.1) will give a family of elliptic curves whose periods are

$$f_1(t) = \int_\gamma w = \int_\gamma \frac{dx}{y} = \int_\gamma \frac{dx}{\sqrt{4x^3 - g(t)x - h(t)}}$$

where γ is any loops and $w = w(t) = \frac{dx}{\sqrt{4x^3-g(t)x-h(t)}}$ is a holomorphc 1-form that is defined in all smooth fibers of our K3 surface.

The periods $f_1(t)$ will satisfy a second-order differential equation in the form

$$\frac{d^2 f_1}{dt^2} + B\frac{df_1}{dt} + Cf_1 = 0, \quad B, C \in \mathbb{C}(t), \tag{6.2}$$

which is then the **Picard-Fuchs equation** of our K3 surface.

Remark. Especially, the author in [40, p. 207] has derived a differential equation with solution $f_1(t)$ when our $g(t) = h(t)$ in his Theorem I.3.1. That equation was written in terms of the invariant $\mathcal{J}(t) = \frac{g(t)}{g(t)-27}$ as

$$\frac{d^2 f_1}{dt^2} + \frac{\mathcal{J}'^2 - \mathcal{J}\mathcal{J}''}{\mathcal{J}\mathcal{J}'}\frac{df_1}{dt} + \frac{\mathcal{J}'^2(\frac{31}{144}\mathcal{J} - \frac{1}{36})}{\mathcal{J}^2(\mathcal{J}-1)^2}f_1 = 0. \tag{6.3}$$

The main work in [40] is to study the relationship between the geometric and arithmetic properties of elliptic surface, and the monodromy representation of the differential equation in the form of (6.2). On the other hand, the author in [39] has investigated the Picard-Fuchs equations for families of K3 surfaces. It contains an explanation of the K3 surfaces that they are encountering with many examples. Certain important background knowledge such as Fuchsian differential equations and systems, monodromy representations were also discussed in [39].

Proposition 6.0.1. *The coefficients B and C in equation (6.2) are*

$$B = \frac{-9g^2hg'^2 + 27h^2(5g'h' - 3hg'') + g^3(7g'h' + 3hg'') - 2g^4h'' + 54gh(-2h'^2 + hh'')}{(g^3 - 27h^2)(-3hg' + 2gh')},$$

$$C = \frac{18g^2g'^2h' - 3g(7hg'^3 + 40h'^3) + 8g^3(h'g'' - g'h'') + 108h(-2hh'g'' + g'(h'^2 + 2hh''))}{16(g^3 - 27h^2)(-3hg' + 2gh')}$$

respectively.

Proof. We follow the steps mentioned in [33, p. 3]. Our goal is to determine B and C so that the 1-form

$$\eta = \frac{d^2w}{dt^2} + B\frac{dw}{dt} + Cw$$

is closed, i.e., $\eta = d\phi$ for some function ϕ. Then by Stokes' theorem, we have

$$\frac{d^2 f_1}{dt^2} + B\frac{df_1}{dt} + Cf_1 = \int_\gamma \eta = \int_\gamma d\phi = \int_{\partial\gamma} \phi = 0$$

since

$$\frac{df_1}{dt} = \int_\gamma \frac{dw}{dt}.$$

First, since

$$\frac{dw}{dt} = \frac{g'x + h'}{2y^3}dx, \quad \frac{d^2w}{dt^2} = \frac{2y^2(g''x + h'') + 3(g'x + h')^2}{4y^5}dx,$$

then we have

$$\eta = \frac{P(x)}{y^5}dx$$

where

$$\begin{aligned} P(x) = {} & 16Cx^6 + \frac{1}{4}x^4(-32Cg + 8Bg' + 8g'') + \frac{1}{4}x^3(-32Ch + 8Bh' + 8h'') \\ & + \frac{1}{4}x^2(4Cg^2 - 2Bgg' + 3g'^2 - 2gg'') \\ & + \frac{1}{4}x(8Cgh - 2Bhg' - 2Bgh' + 6g'h' - 2hg'' - 2gh'') \\ & + \frac{1}{4}(4Ch^2 - 2Bhh' + 3h'^2 - 2hh'') \\ & \in \mathbb{C}[t][x]. \end{aligned}$$

Next, we subtract suitable multiples of the exact forms :

$$d(\frac{x^k}{y^3}) = \frac{(4k-18)x^{k+2} + (\frac{3}{2} - k)gx^k - khx^{k-1}}{y^5}dx, \quad k = 0, 1, 2, 4$$

from η in order to reduce the degree of the polynomial $P(x)$ in its numerator into degree *one*, then we should obtain (can be done by, for instance, Mathematica)

$$\eta + 8Cd(\frac{x^4}{y^3}) + \frac{1}{10}(-28Cg + 2Bg' + 2g'')d(\frac{x^2}{y^3}) + \frac{1}{7}(-20Ch + Bh' + h'')d(\frac{x}{y^3})$$
$$+ \frac{1}{18}(\frac{12Cg^2}{5} + \frac{3}{4}g'^2 - \frac{3}{5}g(Bg' + g''))d(\frac{1}{y^3})$$
$$= \frac{Q(x)}{y^5}dx$$

where

$$\begin{aligned} Q(x) = & \frac{1}{560}x[3456Cgh - 504Bhg' + 840g'h' - 504hg'' - 240g(Bh' + h'')] \\ & + \frac{1}{560}[112Cg^3 + 2160Ch^2 + 35gg'^2 - 360Bhh' + 420h'^2 \\ & - 28g^2(Bg' + g'') - 360hh''] \\ := & \; q_1(t)x + q_2(t). \end{aligned}$$

By solving $q_1 = 0$ and $q_2 = 0$ (i.e., $Q(x) = 0$), we then obtain the required B and C that are sufficient to make the 1-form η being closed. □

Remark. In [33, pp. 8-10], it contains another method to derive the Picard-Fuchs equation by using *residues.* The resulting equation will be the same as the one derived by the brute force approach.

Corollary 6.0.2. *When $g(t) = h(t)$, then*

$$B = \frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'}, \quad C = \frac{3(g+4)g'^2}{16(g-27)g^2}.$$

In particular, the Picard-Fuchs equation

$$\frac{d^2 f_1}{dt^2} + \left[\frac{(2g-27)g'}{(g-27)g} - \frac{g''}{g'}\right]\frac{df_1}{dt} + \frac{3(g+4)g'^2}{16(g-27)g^2}f_1 = 0 \qquad (6.4)$$

has no Liouvillian solution for any g.

Proof. We can easily obtain B and C by substituting $h = g$. Then, we let $u = g(t)$ and $F_1(u) = F_1(g(t)) := f_1(t)$, so

$$\frac{df_1}{dt} = g'\frac{dF_1}{du}, \quad \frac{d^2 f_1}{dt^2} = g''\frac{dF_1}{du} + g'^2\frac{d^2F_1}{du^2}.$$

Hence, our equation (6.4) becomes

$$\frac{d^2F_1}{du^2}+\frac{2u-27}{u-27}\frac{dF_1}{du}+\frac{3(u+4)}{16(u-27)u^2}F_1=0. \tag{6.5}$$

Next, if we can show that equation (6.5) has no Liovillian solution $F_1(u)$ by Kovacic's algorithm, then the equation (6.4) also has no Liovillian solution since $f_1(t)=F_1(u)=F_1(g(t))$ and g is just a polynomial.

First, we set $y=\exp(\frac{1}{2}\int\frac{2u-27}{u-27})F_1$, then equation (6.5) becomes

$$\frac{d^2y}{du^2}=r(u)y. \tag{6.6}$$

where

$$\begin{aligned} r(u) &= -\frac{3(u+4)}{16(u-27)u^2}+\frac{1}{4}\left(\frac{2u-27}{u-27}\right)^2+\frac{1}{2}\left(\frac{2u-27}{u-27}\right)' \\ &= \frac{16u^4-432u^3+2697u^2+69u+324}{16(u-27)^2u^2}. \end{aligned}$$

We see that the order of r at ∞ is $4-4=0$, so case 3 cannot hold by Theorem 3.1.4 immediately. While the poles of r are 0 and 27, with all orders 2, so r satisfies the necessary conditions of case 1 and case 2. We then apply the algorithms for both cases to examine whether the equation (6.6) has Liouvillian solution or not.

Case 1. Now $\Gamma=\{0,27\}$. Since 0 and 27 are poles of r of order 2, then from the partial fraction expansion of r:

$$r=1+\frac{675}{4(u-27)^2}+\frac{104945}{3888(u-27)}+\frac{1}{36u^2}+\frac{31}{3888u},$$

we know

$$\alpha_0^{\pm}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4\cdot\frac{1}{36}}=\frac{1}{2}\pm\frac{\sqrt{10}}{6},\quad \alpha_{27}^{\pm}=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+675}=\frac{1}{2}\pm 13.$$

Next, since the order of r at ∞ is 0, then let $[\sqrt{r}]_\infty=c_0$. So $([\sqrt{r}]_\infty)^2=c_0^2=1$ from the expansion of r. We choose $c_0=1$. Then

$$\alpha_\infty^{\pm}=\frac{1}{2}(\pm\frac{31}{3888}-0)=\pm\frac{31}{7776}.$$

Thus,

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{k\in\Gamma} \alpha_k^{s(k)} \notin \mathbb{Z}_{\geq 0}$$

for all families s. Hence case 1 cannot hold.

Case 2. Similarly by the orders of poles of r, the order of r at ∞, and the partial fraction expansion of r, we have

$$\begin{aligned}
E_0 &= \left\{ 2 + m\sqrt{1 + 4 \cdot \frac{1}{36}} \,\middle|\, m = 0, \pm 2 \right\} \cap \mathbb{Z} = \{2\}, \\
E_{27} &= \left\{ 2 + m\sqrt{1 + 4 \cdot \frac{675}{4}} \,\middle|\, m = 0, \pm 2 \right\} \cap \mathbb{Z} = \{2 + 26m \,|m = 0, \pm 2\}, \\
E_{\infty} &= \{0\}.
\end{aligned}$$

Then, all the families $(e_0, e_{27}, e_{\infty})$ with $e_k \in E_k$ have even coordinates, so all the families will be discarded. So, case 2 cannot hold.

Since all first three cases in Theorem 3.1.3 cannot hold, equation (6.6) has no Liouvillian solution. Hence, equation (6.5) also has no Liouvillian solution because $F_1 = \exp(-\frac{1}{2}\int \frac{2u-27}{u-27})y$. $\square$

Remark. Indeed, when $g(t) = h(t)$, the Picard-Fuchs equation (6.4) in Corollary 6.0.2 is the same as the equation (6.3) after substituting $g = \frac{27\mathcal{J}}{\mathcal{J}-1}$.

Corollary 6.0.3. *The Galois group of the Picard-Fuchs equation (6.4) over* $\mathbb{C}(x)$ *is equal to* $\mathrm{SL}(2)$.

Proof. Direct implication of Lemma 3.1.2 and Theorem 3.1.3. $\square$

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