7 Conditioning

- Conditioning concerns how probabilities of events or distributions of random variables are influenced by information about the occurrence of events or the values of random variables.
- A probability is a measure of the likelihood or degree of uncertainty of an event. A *conditional probability* revises this measure to reflect any "new" information about the outcome of the underlying random phenomenon.

Example 7.1. The probability that a randomly selected American adult believes in human-driven climate change[1] is 0.54.

The probability[2] that a randomly selected American adult is a Democrat is 0.28.

1. Donny Don't says that the probability that a randomly selected American adult both (1) is a Democrat, *and* (2) believes in human-driven climate change is equal to 0.28×0.54. Do you agree?

2. Suppose that the probability that a randomly selected American adult both is a Democrat and believes in human-driven climate change is 0.19. Construct an appropriate two-way table of probabilities.

3. Compute the probability that a randomly selected American adult *who is a Democrat* believes in human-driven climate change.

[1]Probabilities are estimated based on this 2024 survey.
[2]Estimate based on Gallup poll

4. Compute the probability that a randomly selected American adult *who believes in human-driven climate change* is a Democrat.

5. How can the probability in the two previous parts be written in terms of the probabilities provided (0.54, 0.28, 0.19)?

- The **conditional probability of event A given event B**, denoted $\mathrm{P}(A|B)$, is defined as (provided $\mathrm{P}(B) > 0$)
$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$
- The conditional probability $\mathrm{P}(A|B)$ represents how the likelihood or degree of uncertainty of event A should be updated to reflect information that event B has occurred.
- In general, knowing whether or not event B occurs influences the probability of event A. That is,
$$\text{In general, } \mathrm{P}(A|B) \neq \mathrm{P}(A)$$
- Be careful: order is essential in conditioning. That is,
$$\text{In general, } \mathrm{P}(A|B) \neq \mathrm{P}(B|A)$$
- Within the context of two events, we have joint, conditional, and marginal probabilities.
  - Joint: unconditional probability involving both events, $\mathrm{P}(A \cap B)$.
  - Conditional: conditional probability of one event given the other, $\mathrm{P}(A|B)$, $\mathrm{P}(B|A)$.
  - Marginal: unconditional probability of a single event $\mathrm{P}(A)$, $\mathrm{P}(B)$.
- The relationship $\mathrm{P}(A|B) = \mathrm{P}(A \cap B)/\mathrm{P}(B)$ can be stated generically as
$$\text{conditional} = \frac{\text{joint}}{\text{marginal}}$$
- In many problems conditional probabilities are provided or can be determined directly.

Example 7.2. Suppose that

- 67% of Democrats believe in human-driven climate
- 46% of Independents believe in human-driven climate
- 34% of Republicans believe in human-driven climate

Also suppose that

- 28% of American adults are Democrats
- 42% of American adults are Independents
- 30% of American adults are Republicans

1. Define the event A to represent "believes in human-driven climate change" and D, I, R to correspond to affiliation in each of the parties. If the probability measure P corresponds to selecting an American adult uniformly at random, write all the percentages above as probabilities using proper notation.

2. Construct an appropriate two-way table of probabilities.

3. Now suppose that the randomly selected American believes in human-driven climate change. How does this information change the probability that the selected American belongs to each political party? Answer by computing appropriate probabilities (and using proper notation).

4. How many times more likely is it for an American *adult* to believe in human-driven climate change and be Independent than to:

   a. believe in human-driven climate change and be Democrat

b. believe in human-driven climate change and be Republican

5. How many times more likely is it for an American adult *who believes in human-driven climate change* to be Independent than to be:

   a. Democrat

   b. Republican

6. What do you notice about the answers to the two previous parts?

- The process of conditioning can be thought of as **"slicing and renormalizing".**
  - Extract the "slice" corresponding to the event being conditioned on (and discard the rest). For example, a slice might correspond to a particular row or column of a two-way table.
  - "Renormalize" the values in the slice so that corresponding probabilities add up to 1.
- Slicing determines the *shape*; renormalizing determines the *scale*.
- Slicing determines relative probabilities; renormalizing just makes sure they add up to 1.

Example 7.3. Roll a fair four-sided die twice. Let X be the sum of the two dice, and let Y be the larger of the two rolls (or the common value if both rolls are the same). The table below represents the joint distribution of X and Y.

$x \setminus y$	1	2	3	4
2	1/16	0	0	0
3	0	2/16	0	0
4	0	1/16	2/16	0
5	0	0	2/16	2/16
6	0	0	1/16	2/16
7	0	0	0	2/16
8	0	0	0	1/16

1. Compute and interpret in context $\mathrm{P}(X = 5|Y = 4)$.

2. Construct a table to represent the conditional distribution of X given $Y = 4$ by "slicing and renormalizing".

3. Interpret the the conditional distribution of X given $Y = 4$ as a long run relative frequency distribution.

4. Compute $\mathrm{E}(X|Y = 4)$.

5. Interpret the value from the previous part as a long run average value in context.

6. Construct a table to represent the conditional distribution of X given $Y = 3$, and

compute $\mathrm{E}(X|Y=3)$

7. Construct a table to represent the conditional distribution of X given $Y = 2$, and compute $\mathrm{E}(X|Y=2)$

8. Construct a table to represent the conditional distribution of X given $Y = 1$, and compute $\mathrm{E}(X|Y=1)$