5 Probability Models

- A **probability measure**, typically denoted P, assigns probabilities to *events* to quantify their relative likelihoods according to the assumptions of the model of the random phenomenon.
- The probability of event A, computed according to probability measure $\mathrm{P}(A)$, is denoted $\mathrm{P}(A)$.
- A valid probability measure P must satisfy the following three logical consistency "axioms".
  - For any event A, $0 \leq \mathrm{P}(A) \leq 1$.
  - If Ω represents the sample space then $\mathrm{P}(\Omega) = 1$.
  - (*Countable additivity.*) If $A_1, A_2, A_3, \ldots$ are disjoint then

$$\mathrm{P}(A_1 \cup A_2 \cup A_2 \cup \cdots) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \mathrm{P}(A_3) + \cdots$$

- Additional properties of a probability measure follow from the axioms
  - *Complement rule.* For any event A, $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$.
  - *Subset rule.* If $A \subseteq B$ then $\mathrm{P}(A) \leq \mathrm{P}(B)$.
  - *Addition rule for two events.* If A and B are any two events

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

  - *Law of total probability.* If $C_1, C_2, C_3 \ldots$ are disjoint events with $C_1 \cup C_2 \cup C_3 \cup \cdots = \Omega$, then

$$\mathrm{P}(A) = \mathrm{P}(A \cap C_1) + \mathrm{P}(A \cap C_2) + \mathrm{P}(A \cap C_3) + \cdots$$

- A **probability model** (or **probability space**) is the collection of all outcomes, events, and random variables associated with a random phenomenon along with the probabilities of all events of interest under the assumptions of the model.
- The axioms of a probability measure are minimal logical consistent requirements that ensure that probabilities of different events fit together in a valid, coherent way.
- A single probability measure corresponds to a particular set of assumptions about the random phenomenon.
- There can be many probability measures defined on a single sample space, each one corresponding to a different probability model for the random phenomenon.
- Probabilities of events can change if the probability measure changes.

Example 5.1. Consider a *single* roll of a four-sided die. (Careful: don't confuse these examples with other examples that involve two rolls.) The sample space is $\{1, 2, 3, 4\}$. Table 5.1 lists all possible *events*.

1. Add a description in words for each of the events

2. Suppose the die is fair, and let P denote the correponding probability measure. Compute $\mathrm{P}(A)$ for each event in Table 5.1.

Table 5.1: All possible events associated with a single roll of a four-sided die.

A	Description	$\mathrm{P}(A)$	$\mathrm{Q}(A)$	$\tilde{\mathrm{Q}}(A)$?
$\emptyset$				
$\{1\}$				
$\{2\}$				
$\{3\}$				
$\{4\}$				
$\{1,2\}$				
$\{1,3\}$				
$\{1,4\}$				
$\{2,3\}$			0.5	
$\{2,4\}$				
$\{3,4\}$			0.7	
$\{1,2,3\}$			0.6	
$\{1,2,4\}$				
$\{1,3,4\}$				
$\{2,3,4\}$				
$\{1,2,3,4\}$				

Example 5.2. Consider a *single* roll of a four-sided die, but suppose the die is weighted so that the outcomes are no longer equally likely. Let Q denote the probability measure corresponding to a particular weighting. A few probabilities are provided in Table 5.1;

compute $\mathrm{Q}(A)$ for all other events in Table 5.1. In what particular way is the die weighted? That is, what is the probability of each the four possible outcomes?

Example 5.3. Consider a single roll of a different weighted four-sided die. Suppose that

- Rolling a 1 is twice as likely as rolling a 4
- Rolling a 2 is three times as likely as rolling a 4
- Rolling a 3 is 1.5 times as likely as rolling a 4

Let $\tilde{\mathrm{Q}}$ denote the probability measure corresponding to this particular weighting. Compute $\tilde{\mathrm{Q}}(A)$ for all events in Table 5.1.

In what particular way is the die weighted? That is, what is the probability of each the four possible outcomes?

Example 5.4. The general meeting problem involves multiple people, but we'll first consider the arrival time of just a single person, who we'll call Han. Suppose that Han arrives "uniformly at random" at a time in $[0, 60]$.

1. Compute the probability that Han arrives before 12:15.

2. Compute the probability that Han arrives after 12:45.

3. Let P denote the corresponding probability measure. Suggest a general formula for $\mathrm{P}([a, b])$, the probability that Han arrives between a and b minutes after noon for $0 \leq a < b \leq 60$.

4. Compute the probability that Han's arrival time, truncated to the nearest minute, is 0 minutes after noon; that is, find the probability that Han arrives between 12:00 and 12:01.

5. Continue to compute the probability that Han's arrival time truncated to the nearest minute is $1, 2, 3, \ldots, 59$, and sketch a plot with arrival time (truncated minutes after noon) on the horizontal axis and probability on the vertical axis. Is the plot what you would expect for arriving "uniformly at random"?

Example 5.5. Assume that the probability that Han arrives between a and b minutes after noon is $(b/60)^2 - (a/60)^2$, for $0 \leq a < b \leq 60$. Let Q denote the corresponding probability measure; notice that $\mathrm{Q}([0, 60]) = (60/60)^2 - (0/60)^2 = 1$. Compute the following probabilities and compare your answers to the corresponding parts from Example 5.4.

1. Compute the probability that Han arrives before 12:15.

2. Compute the probability that Han arrives after 12:45.

3. Compute the probability Han's arrival time, truncated to the nearest minute, is 0 minutes after noon; that is, find the probability that Han arrives between 12:00 and 12:01.

4. Compute the probability Han's arrival time, truncated to the nearest minute, is 59 minutes after noon; that is, find the probability that Han arrives between 12:59 and 1:00.

5. Continue to compute the probability that Han's arrival time truncated to the nearest minute is $1, 2, 3, \ldots, 59$, and sketch a plot with arrival time (truncated minutes after noon) on the horizontal axis and probability on the vertical axis. What assumptions about Han's arrival time does this probability measure reflect?

Example 5.6. Continuing with the uniform probability measure of Example 5.4.

1. Compute the probability that Han arrives between 12:00 and 12:01, within 1 minute after noon.

2. Compute the probability that Han arrives between 12:00:00 and 12:00:01, within 1 second after noon.

3. Compute the probability that Han arrives between 12:00:00.000 and 12:01:00.001, within 1 millisecond after noon.

4. Compute the probability that Han arrives at the exact time 12:00:00.00000... (with infinite precision).

5. What is the probability that Han arrives "at noon"? Discuss.

Example 5.7. Continuing with the non-uniform probability measure of Example 5.5.

1. Compute the probability that Han arrives between 12:00 and 12:01, within 1 minute after noon.

2. Compute the probability that Han arrives between 12:00:00 and 12:00:01, within 1 second after noon.

3. Compute the probability that Han arrives between 12:00:00.000 and 12:01:00.001, within 1 millisecond after noon.

4. Compute the probability that Han arrives at the exact time 12:00:00.00000... (with infinite precision).

5. Compute the probability that Han arrives between 12:59 and 1:00, within 1 minute before 1:00.

6. Compute the probability that Han arrives between 12:59:59 and 1:00:00, within 1 second before 1:00.

7. Compute the probability that Han arrives between 12:59:59.999 and 1:00:00.000, within 1 millisecond before 1:00.

8. Compute the probability that Han arrives at the exact time 1:00:00.00000… (with infinite precision).

9. Which is more likely: that Han arrives "at noon" or "at 1:00"? Discuss.

- For a continuous sample space, the probability of any particular outcome is 0.
- Particular outcomes represent "infinite precision" which is not practical in real applications
- For continuous sample spaces it makes more sense to consider "close to" probabilities

rather than "equals to" probabilities.

- "Close to" events correspond to *intervals* of reasonable practical precision and these intervals can have non-zero probability.
- Certain outcomes can be more likely than others in the "close to" sense.

Example 5.8. Back to the Regina, Cady meeting problem. Assume that Regina and Cady each arrive at a time uniformly at random between noon and 1:00, independently of each other, so that they arrive "uniformly at random" in the sample space of Example 4.3. Let P denote the corresponding probability measure.

Let R be the random variable representing Regina's arrival time (minutes after noon), and Y for Cady, and let $T = \min(R, Y)$ and $W = |R - Y|$.

Compute and interpret the following.

1. $\mathrm{P}(R > Y)$

2. $\mathrm{P}(T < 30)$

3. $\mathrm{P}(W < 15)$

4. $\mathrm{P}(R < 24)$

5. $\mathrm{P}(W < 1)$

6. $\mathrm{P}(W = 0)$

7. What is the probability that Regina and Cady arrive "at the same time"? Discuss.