MSBA Workshop: Calculus — A Few Things about Derivatives

Maximixing a Function

Here's a pretty standard calculus problem written by ChatGPT:

A company sells a specialized product, and market research shows that the revenue (in thousands of dollars) from producing and selling the product for x months is modeled by[1]

$$f(x) = x^8 e^{-5x}$$

At what time x (in months) should the company end its sales campaign to maximize revenue? What is the maximum revenue (in thousands of dollars)?

Note: We will see a very similar problem in a probability/statistics context in GSB 518.

1. How you should you *first* try to solve this problem?

2. What is the general idea of the calculus method for solving such a problem? (Don't do it yet.)

3. Plot $\log f(x)$. What do you notice about where the maximum occurs?

4. Write an expression for $\log f(x)$. Which would you rather differentiate: $f(x)$ or $\log f(x)$?

5. Use calculus to find x^* that maximizes $f(x)$.

6. What is the maximum value of $f(x)$?

- A derivative is usually interpreted as the slope of the tangent line to a curve
- Maximizing a function f usually involves taking its derivative, setting the derivative equal to 0, and solving
- By try graphical methods first!
- Finding the value x^* which maximizes $f(x)$ is equivalent to finding the value x^* which maximizes $\log f(x)$ (provided that $f(x) > 0$ for all x under consideration)
- $\log f(x)$ is often easier to differentiate than $f(x)$, especially if f is a product of several terms
- For example, we will see in GSB518 that when finding a *maximum likelihood estimator* we usually work with the log-likelihood function rather than the likelihood function itself.

[1]The factor x^8 represents the rapid growth in sales during the early stages due to advertising and word-of-mouth. The factor e^{-5x} represents the loss of market interest and increasing competition over time.

Derivative of Exponential and Logarithmic Functions

The derivative of e^x is itself

$$\frac{d}{dx}e^x = e^x$$

The derivative of $\log(x)$ is $1/x$

$$\frac{d}{dx}\log(x) = \frac{1}{x}$$

Derivatives in Python

```
from sympy import symbols, diff, exp

# Define the variable
x = symbols('x')

# Define the function
f = x ** 8 * exp(5 * x)

# Partial derivative with respect to x
diff(f, x)
```

$5x^8e^{5x} + 8x^7e^{5x}$

Partial Derivatives

- For a function of multiple variables, we can compute *partial derivatives*.
- To compute the partial derivative with respect to one variable, treat all other variables like constants

Example

$$f(x, y) = x^5y^2$$

1. If $y = 1$, what is the derivative of f with respect to x?

2. If $y = 2$, what is the derivative of f with respect to x?

3. If $y = 3$, what is the derivative of f with respect to x?

4. For general y, what is the partial derivative of f with respect to x?

5. What is the partial derivative of f with respect to y?

Exercise

$$f(x, y, z) = e^x y^3 + 2z^4$$

1. Find the partial derivative of f with respect to x.

2. Find the partial derivative of f with respect to y.

3. Find the partial derivative of f with respect to z.

Partial Derivatives in Python

```
from sympy import symbols, diff, exp

# Define the variable
x = symbols('x')
y = symbols('y')
z = symbols('z')

# Define the function
f = exp(x) * y ** 3 + 2 * z ** 4

# Partial derivative with respect to x
diff(f, x)
```

$y^3 e^x$

```
# Partial derivative with respect to y
diff(f, y)
```

$3y^2 e^x$

```
# Partial derivative with respect to z
diff(f, z)
```

$8z^3$

Online Derivative Tools

- WolframAlpha
- Symbolab
- Desmos