Project Euler Solutions

Kevin Chen

December 30, 2021

Solutions to Project Euler problems that are particularly enlightening are explained here. Questions labeled by a $\star$ can be solved by hand with a calculator.

1$\star$ – Multiples of 3 and 5

The sum of the first n numbers is $n(n+1)/2$. So the sum of the multiples of k under 1000 is $S_k = kn_k(n_k+1)/2$ where $n_k = \lfloor 999/k \rfloor$. The answer is $S = S_3 + S_5 - S_{15}$, or written explicitly,

$$\boxed{3\frac{333 \cdot 334}{2} + 5\frac{199 \cdot 200}{2} - 15\frac{66 \cdot 67}{2}}$$

2$\star$ – Even Fibonacci numbers

The closed-form expression for Fibonacci numbers is $F_n = (\varphi_+^n - \varphi_-^n)/\sqrt{5}$, where $\varphi_\pm = (1 \pm \sqrt{5})/2$. This can be proved by induction, using the fact that $\varphi_\pm$ are the solutions to $x^2 = x + 1$. As $|\varphi_-| < 1$, for large n we have a good approximation $F_n \approx \varphi_+^n/\sqrt{5}$. Using this, we can find indices n where F_n is less than four million. $4 \cdot 10^6 > F_n \approx \varphi_+^n/\sqrt{5}$ implies that $n < (\ln 4 + 6 \ln 10 + \ln \sqrt{5})/(\ln \varphi_+) \approx 33.263$. Therefore, we are considering $1 \leq n \leq 33$. Every third Fibonacci number is even, so we are finding the sum $S = F_3 + F_6 + \cdots + F_{33}$. However, by definition of Fibonacci numbers, $S = (F_1 + F_2) + (F_4 + F_5) + \cdots + (F_{31} + F_{32})$ as well. Therefore $S = (F_1 + F_2 + F_3 + \cdots + F_{33})/2$. The sum of the first n Fibonacci numbers is $F_{n+2} - 1$, so given that $F_{35} = 9227465$ the sum is $\boxed{(9227465 - 1)/2}$.

5$\star$ – Smallest multiple

This is the product of the all p^k, where p is a prime and k is the largest integer such that $p^k \leq 20$. The answer is $\boxed{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$.

6$\star$ – Sum square difference

The sum of the first n numbers is $\frac{n(n+1)}{2}$. The sum of the squares of the first n numbers is $\frac{n(n+1)(2n+1)}{6}$. Therefore the answer is

$$\boxed{\left(\frac{100 \cdot 101}{2}\right)^2 - \frac{100 \cdot 101 \cdot 201}{6}}$$

9$\star$ – Special Pythagorean triplet

A primitive Pythagorean triplet $a^2 + b^2 = c^2$ is one where all a, b, and c do not share a common factor. These are in a one-to-one correspondence with coprime integers m, n with opposite parity and $0 < n < m$, where we take

$$a = m^2 - n^2 , \qquad b = 2mn , \qquad c = m^2 + n^2$$

All Pythagorean triplets are integer multiples of primitive Pythagorean triplets, i.e. $(ka)^2 + (kb)^2 = (kc)^2$. Therefore, for a generic triplet, if $a + b + c = 2km(m + n) = 1000$ then $km(m + n) = 500 = 5^3 2^2$. $m + n$ must be odd so we just check each case:

(i) $m + n = 1$. We cannot satisfy $0 < n < m$.

(ii) $m + n = 5$. We can have $m = 1, 2, 4$ but $n < m$ only allows $m = 4, n = 1, k = 25$. We have a solution $a = 200, b = 375, c = 425$.

(iii) $m + n = 25$. We can have $m = 1, 2, 4, 5, 10, 20$ but $n < m$ only allows $m = 20, n = 5$. However, this contradicts m, n coprime.

(iv) $m + n = 125$. We can have $m = 1, 2, 4$ but we cannot satisfy $n < m$.

Therefore the answer is $\boxed{200 \cdot 375 \cdot 425}$.

10 – Summation of primes

A very fast way of generating prime numbers up to some N is to use a Sieve of Eratosthenes. Create a list of the first N numbers, and cross-out 0 and 1. The first uncrossed integer is 2, which is the first prime. Next, cross-out all multiples of 2. The next uncrossed integer is 3, which is the second prime. Next, cross-out all multiples of 3. Continuing in this way for all integers up to $\sqrt{N}$, we generate all primes less than or equal to N.

12 – Highly divisible triangular number

There is a fast way to count the divisors of an integer. Given an integer n and its prime factorization, $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, the number of ways we can multiply the primes together gives the number of divisors of n. Thus the number of divisors of n is $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$.

14 – Longest Collatz sequence

A useful programming technique in dynamical programming is memoization. In a process where we anticipate that a function would be called multiple times with the same input, it is often more efficient to remember a table of input-output values, which we should consult first. In Python, this can be easily done using the `@cache` decorator.

15⋆ – Lattice paths

This is a combinatorics problem. The number of paths through the grid is equivalent to the number of ways of choosing 20 "right" moves from 40 total moves, setting the other moves to "down". Written explicitly, the answer is

$$\boxed{\binom{40}{20} = \frac{40!}{20! \cdot 20!}}$$

17⋆ – Number letter counts

This problem is actually not too tedious by hand. The numbers $1 \to 9$ use 36 letters and the numbers $10 \to 19$ use 70 letters. The numbers $20 \to 29$ then use $6 \cdot 10 + 36$ letters as "twenty" uses 6 letters. Continuing in this fashion, the numbers $1 \to 99$ use $(6 + 6 + 5 + 5 + 5 + 7 + 6 + 6) \cdot 10 + 36 \cdot 9 + 70 = 854$ letters. Next, the numbers $100 \to 199$ use $(3 + 10) \cdot 100 + 854 - 3$ letters. The $(3 + 10)$ factor comes from "one hundred and" using $3 + 10$ letters, and we subtract 3 letters at the end because we drop the "and" for 100. Continuing in this fashion, the numbers $1 \to 999$ use $(36 + 10 \cdot 9) \cdot 100 + 854 \cdot 10 - 3 \cdot 9 = 21113$ letters. Then add 11 for "one thousand" to get a total of $\boxed{21124}$.

18⋆ – Maximum path sum I

It is possible to do this by hand by working from the bottom and moving upwards. For each row, we add to each integer the bigger of the two integers just below. For instance, the second row from the bottom becomes

$$125, 164, 102, 95, 112, 123, 165, 128, 166, 109, 122, 147, 100, 54$$

19⋆ – Counting Sundays

Very interesting to use John Conway's Doomsday algorithm, but unsurprisingly the answer is 1200/7 rounded to the nearest integer, $\boxed{171}$.

21 – Amicable numbers

There is a fast way to sum the divisors of an integer. Let n be a number and suppose its prime factorization is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then $(1 + p_1 + p_1^2 + \cdots + p_1^{a_1})(1 + p_2 + p_2^2 + \cdots + p_2^{a_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{a_k})$ expanded would be the sum of all numbers that divide n. But $(1 + p_i + p_i^2 + \cdots + p_i^{a_i}) = (p_i^{a_i+1} - 1)/(p_i - 1)$, so

$$\sum_{d|n} d = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

24⋆ – Lexicographic permutations

Our goal is to find the unique coefficients for $10^6 = a_9 \cdot 9! + a_8 \cdot 8! + \cdots + a_1 \cdot 1!$, which we can find using a long-division-like algorithm. 9! goes into 10^6 two times, so a_9 is the 3rd smallest unused digit, i.e. $a_9 = 2$. $10^6 - 2 \cdot 9! = 274240$, which 8! goes into 6 times. So a_8 is the 7th smallest unused digit, i.e. $a_8 = 7$. We repeat to create our permutation. However, because of indexing convention, we are looking for the coefficients of $10^6 - 1$ instead.

$$\begin{array}{r|l|l}
(10^6 - 1) - 2 \cdot 9! = 274239 & 01\underline{2}3456789 & a_9 = 2 \\
274239 - 6 \cdot 8! = 32319 & 01\cancel{2}3456\underline{7}89 & a_8 = 7 \\
32319 - 6 \cdot 7! = 2079 & 01\cancel{2}3456\cancel{7}\underline{8}9 & a_7 = 8 \\
2079 - 2 \cdot 6! = 639 & 01\cancel{2}\underline{3}456\cancel{7}\cancel{8}9 & a_6 = 3 \\
639 - 5 \cdot 5! = 39 & 01\cancel{2}\cancel{3}456\cancel{7}\cancel{8}\underline{9} & a_5 = 9 \\
39 - 1 \cdot 4! = 15 & 0\underline{1}\cancel{2}\cancel{3}456\cancel{7}\cancel{8}\cancel{9} & a_4 = 1 \\
15 - 2 \cdot 3! = 3 & 0\cancel{1}\cancel{2}\cancel{3}4\underline{5}6\cancel{7}\cancel{8}\cancel{9} & a_3 = 5 \\
3 - 1 \cdot 2! = 1 & 0\cancel{1}\cancel{2}\cancel{3}\underline{4}\cancel{5}6\cancel{7}\cancel{8}\cancel{9} & a_2 = 4 \\
1 - 1 \cdot 1! = 0 & 0\cancel{1}\cancel{2}\cancel{3}\cancel{4}\cancel{5}\underline{6}\cancel{7}\cancel{8}\cancel{9} & a_1 = 6 \\
 & \underline{0}\cancel{1}\cancel{2}\cancel{3}\cancel{4}\cancel{5}\cancel{6}\cancel{7}\cancel{8}\cancel{9} & a_0 = 0
\end{array} \quad \Longrightarrow \quad \boxed{2783915460}$$

25⋆ – 1000-digit Fibonacci number

As explained in Problem 2, for large n we can approximate $F_n \approx \varphi_+^n/\sqrt{5}$. So if some F_n has a thousand digits, $10^{999} \leq F_n \approx \varphi_+^n/\sqrt{5}$ implies that $n \geq (999 \ln 10 + \ln \sqrt{5})/\ln \varphi_+ \approx 4781.859$. This gives that $n = \boxed{4782}$ as the first Fibonacci number with more than a thousand digits.

28⋆ – Number spiral diagonals

For a shell around the center with dimensions $n \times n$, the values at the corners are n^2, $n^2 - (n-1)$, $n^2 - 2(n-1)$, and $n^2 - 3(n-1)$. So we are summing $4n^2 - 6n + 6$ for odd n from 3 to 1001. We can rewrite this slightly by letting $n = 2m + 1$ so that now we are summing $16m^2 + 4m + 4$ for integer m from 1 to 500. Now, recall that the sum of the first m digits is $\frac{m(m+1)}{2}$ and the sum of the squares of the first m digits is $\frac{m(m+1)(2m+1)}{6}$. Therefore the answer is

$$\boxed{16\frac{500 \cdot 501 \cdot 1001}{6} + 4\frac{500 \cdot 501}{2} + 4 \cdot 500}$$

43⋆ – Sub-string divisibility

Rule 3 implies that $d_6 = 0$ or 5. But if $d_6 = 0$, then by Rule 5, $d_7 = d_8$. Therefore $d_6 = 5$. Given that $500 \equiv 5 \pmod{11}$ we have the following possibilities for $d_6d_7d_8$: 506, 517, 528, 539, 550 (ignored), 561, 572, 583, and 594. For each of these values, we check what value of d_9 allows $d_7d_8d_9$ to be divisible by 13. For example, take 506 and $060 \equiv 8 \pmod{13}$, so 065 is a possibility, but repeats 5 so it is ignored. Continuing this way, we get only four candidates for $d_6d_7d_8d_9$: 5286, 5390, 5728, and 5832. Repeating for d_{10} for divisibility by 17, we get three candidates for $d_6d_7 \dots d_{10}$: 52867, 53901, and 57289.

Next, we find d_5 such that $d_5d_6d_7$ is divisible by 7. We do this by recalling that $100 \equiv 2 \pmod 7$. Then for $d_6d_7 = 52$, $52 \equiv 3 \pmod 7$ so 252 and 952 are possible, but the first is ignored. For $d_6d_7 = 53$, $53 \equiv 4 \pmod 7$ so 553 is possible but is ignored. For $d_6d_7 = 57$, $57 \equiv 1 \pmod 7$ so 357 is possible. Thus we have two candidates for $d_5d_6 \dots d_{10}$: 952867 and 357289.

Considering the first candidate, Rule 1 allows 0 and 4 as possible values for d_4. If $d_4 = 4$, then it is impossible to make $d_3d_4d_5$ divisible by 3 from our remaining digits 0, 1, and 3 (a number if divisible by 3 if its digits add up to 3). Therefore $d_4 = 0$, which forces $d_3 = 0$. d_1 and d_2 are 1 and 4 with no restrictions.

Considering the second candidate, possible values for d_4 are 0, 4, and 6. For the same reasons above, we can deduce that d_3 and d_4 are 0 and 6 with no restrictions, and likewise d_1 and d_2 are 1 and 4.

We have discovered six numbers that satisfy every condition, and their sum is

$$\boxed{(14 + 41)10^8 + 30952867 \cdot 2 + (1406 + 1460 + 4106 + 4160)10^6 + 357289 \cdot 4}$$

45⋆ – Triangular, pentagonal, and hexagonal

Every hexagonal number is a triangle number, $T_{2n-1} = H_n$, so it suffices to solve $H_n = P_m$ or $n(2n-1) = m(3m-1)/2$ for positive integer solutions. After some rearrangement, we get $(6m-1)^2 - 3(4n-1)^2 = -2$. Letting $x = 6m-1$ and $y = 4n-1$, we are looking for positive integer solutions to the generalized Pell's equation $x^2 - 3y^2 = -2$. Solutions are $x_k + \sqrt{3}y_k = (1+\sqrt{3})(2+\sqrt{3})^k$ for $k = 0, 1, 2, \dots$ and can be generated recursively. What remains is to check that $x_k \equiv 5 \pmod 6$ and $y_k \equiv 3 \pmod 4$, which we can prove only happens when $k \equiv 1 \pmod 4$. Since $k = 1$ is 1 and $k = 5$ is the given 40755, we want $k = 9$.

k	x_k	$x_k \mod 6$	y_k	$y_k \mod 4$	
0	1	1	1	1	
1	5	5	3	3	$T_1 = P_1 = H_1 = 1$
2	19	1	11	3	
3	71	5	41	1	
4	265	1	153	1	
5	989	5	571	3	$T_{285} = P_{165} = H_{143} = 40755$
6	3691	1	2131	3	
7	13775	5	7953	1	
8	51409	1	29681	1	
9	191861	5	110771	3	$T_{55385} = P_{31977} = H_{27693} = \boxed{1533776805}$

57 – Square root convergents

Given a continued fraction expansion $[a_0; a_1, a_2, \dots]$, the convergents can be calculated by the recurrence relation,

$$
\begin{aligned}
p_{-1} &= 1 & q_{-1} &= 0 \\
p_0 &= a_0 & q_0 &= 1 \\
p_n &= a_n p_{n-1} + p_{n-2} & q_n &= a_n q_{n-1} + q_{n-2}
\end{aligned}
$$

Then the fraction p_n/q_n gives the nth convergent.

63⋆ – Powerful digit counts

10^n has $n+1$ digits, so we only need to consider bases 1 through 9. If we take 9, 9^n will never have more than n digits, so the point where 9^n has less than n digits is when $9^n < 10^{n-1}$. This simplifies to $n > \frac{1}{1-\log_{10} 9} \approx 21.854$. Thus for $1 \leq n \leq 21$, 9^n has n digits. We can calculate this for each digit and add them up to get the answer. Written explicitly, the answer is

$$\boxed{\sum_{n=1}^{9} \left\lfloor \frac{1}{1-\log_{10} n} \right\rfloor}$$

64 – Odd period square roots

The continued fraction expansion for an irrational square root $\sqrt{n}$ is a repeating non-terminating sequence $[a_0; \overline{a_1, a_2, \dots, a_p}]$. The coeffcients a_n can be calculated using the recurrence relation,

$$
\begin{aligned}
m_0 &= 0 & d_0 &= 1 & a_0 &= \lfloor \sqrt{n} \rfloor \\
m_{n+1} &= d_n a_n - m_n & d_{n+1} &= \frac{n - m_{n+1}^2}{d_n} & a_{n+1} &= \left\lfloor \frac{\sqrt{n} + m_{n+1}}{d_{n+1}} \right\rfloor
\end{aligned}
$$

Then once $a_n = 2a_0$, one period has passed.

69⋆ – Totient maximum

The totient of n is given by $\phi(n) = n \prod (1 - 1/p_i)$, where the product is over all distinct primes p_i that divide n. However, notice that $n/\phi(n) = \prod p_i/(p_i - 1)$. Each term is greater than 1 and smaller primes give larger terms, so $n/\phi(n)$ is maximized by multiplying together small primes in increasing order. The largest product under 1 million is $\boxed{510510} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$.

72 – Counting fractions

The number of reduced fractions with denominator d is precisely the totatives of d. Therefore the answer is

$$\sum_{n=2}^{10^6} \phi(n)$$

76 – Counting summations

This problem is basically asking for the integer partition of 100. This can be calculated using the recurrence relation,

$$p(n) = \sum_{k \neq 0} (-1)^{k-1} p(n - g_k)$$

where $g_k = k(3k-1)/2$ is the kth pentagonal number, $p(0) = 1$, and $p(x) = 0$ for $x < 0$.

79$\star$ – Passcode derivation

Easier to figure out by hand using trial-and-error.

83 – Path sum: four ways

This can be solved by an implementation of Dijkstra's algorithm. We traverse through the graph by keeping track of nodes we have visited and the minimum distance to that node, always advancing a step from the node with smallest distance.

84 – Monopoly odds

We can model the monopoly board as a Markov chain. Then multiply the matrix over and over on any vector to get the steady state.

86 – Cuboid route

Here we present a new method for generating Pythagorean triples. Given a primitive triple (a, b, c), we can generate three different primitive triples using the following linear transformations,

$$\begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Using an initial seed $(3, 4, 5)$ every primitive triple can be generated using this method.

94$\star$ – Almost equilateral triangles

Given a triangle with side lengths $(a, a, a+\eta)$ where $\eta = \pm 1$, we can derive the area using Heron's formula to be $A = \frac{a+\eta}{4}\sqrt{(3a+\eta)(a-\eta)}$. For this to be an integer, a must be odd. If we let $a = 2n+1$ and $1+\eta = 2\delta$, we can write $A = (n+\delta)\sqrt{(3n+1+\delta)(n+1-\delta)}$. For this to be an integer, we need $(3n+1+\delta)(n+1-\delta) = y^2$ for some integer y. We can rearrange this as $(3n+2-\delta)^2 - 3y^2 = 1$. If we let $x = 3n+2-\delta$, then we are solving the Pell's equation $x^2 - 3y^2 = 1$. Solutions are $x_k + \sqrt{3}y_k = (2+\sqrt{3})^{1+k}$ for $k = 0, 1, 2, \ldots$ and can be generated recursively.[1] What remains is to check that $x_k \equiv 1 \pmod 3$, in which case $\delta = 1$, or $x_k \equiv 2 \pmod 3$, in which case $\delta = 0$. We can prove that x_k is not divisible by 3 for all k. The perimeter is then $P = 2(x_k + 2\delta - 1)$ (although note that $k = 0$ is not a triangle, for other reasons).

k	x_k	P
0	2	–
1	7	16
2	26	50
3	97	196
4	362	722
5	1351	2704
6	5042	10082
7	18817	37636
8	70226	140450
9	262087	524176
10	978122	1956242
11	3650401	7300804
12	13623482	27246962
13	50843527	101687056
14	189750626	379501250

We stop at $k = 14$ as the perimeter for $k = 15$ exceeds 10^9. The sum over the last column is $\boxed{518408346}$.

[1] For an explanation on how to solve these equations, see https://acollectionofelectrons.wordpress.com/2016/11/24/almost-equilateral-triangles-part-i/

100$\star$ – Arranged probability

Let b be the number of blue disks and t be the total number of disks. We are looking for solutions where $\frac{b}{t} \cdot \frac{b-1}{t-1} = \frac{1}{2}$. Expanding and rearranging we can get $(2t-1)^2 - 2(2b-1)^2 = -1$. Letting $x = 2t-1$ and $y = 2b-1$ we get the negative Pell's equation $x^2 - 2y^2 = -1$. Solutions are $x_k + \sqrt{2}y_k = (1+\sqrt{2})^{1+2k}$ for $k = 0, 1, 2, \ldots$ and can be generated recursively. What remains is to check that x_k and $y_k \equiv 1 \pmod 2$, which we can prove is true for all k. The answer can be reached in $k = 16$ iterations, at which point $t = (x_{16}+1)/2$ is the first to exceed 10^{12}. The answer is then $b = (y_{16}+1)/2 = \boxed{(1513744654945+1)/2}$.

101 – Optimum polynomial

The optimum polynomial $OP(k, n)$ is the polynomial P interpolated on the points $n = 1, 2, \ldots, k$ such that $P(n) = u_n$. We can find P by Lagrange interpolation. Given a set of points $\{x_1, x_2, \ldots, x_k\}$ and values $\{u_1, u_2, \ldots, u_k\}$, we find polynomials P_i such that $P_i(x_j) = \delta_{ij}$. An explicit formula is:

$$P_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

Then our interpolation is simply $P(x) = \sum\limits_{i=1}^{k} u_i P_i(x)$.

106$\star$ – Special subset sums: meta-testing

This problem can then be solved by hand, which we demonstrate by an example. Note that as a consequence of property (ii), we only need to test subset pairs of equal numbers of elements. Take $n = 7$. First let us consider all disjoint subset pairs A, B with 3 elements each. There are $\binom{7}{6} = 7$ ways to pick these 6 elements. Next, order our 6 elements in increasing magnitude, and for each element let "+" represent set-inclusion by A and "−" represent set-inclusion by B. The corresponding string of "+" and "−" is defined to be our *pattern*. For instance if our 6 elements are $\{1, 2, 3, 4, 5, 6\}$ and our pattern is $+-++--$, then $A = \{1, 3, 4\}$ and $B = \{2, 5, 6\}$. Without loss of generality we can assume that the smallest number belongs to A and all patterns start with a "+". We then take $+ \mapsto 1$ and $- \mapsto -1$ and compute the partial sums of the sequence from left to right. For the same example above, the sequence of partial sums is $(1, 0, 1, 2, 1, 0)$. Subset pairs whose partial sums never become negative do not need to be tested for equality, as each element of A is strictly less than a unique element from B and $S(A) < S(B)$. Therefore, with 6 elements we only need to test subset pairs that have one of the 5 patterns: $++---+, +-+--+, +--++-, +--+-+, +---++$. Next, we consider subset pairs with 2 elements each, where there are $\binom{7}{4} = 35$ different combinations and only 1 pattern: $+--+$. Therefore the total number of tests is $7 \cdot 5 + 35 \cdot 1 = 70$.

We can count the number of patterns to test for general n and subset pairs with k elements each (total $2k$ elements). The total number of patterns that start with a "+" is $\frac{1}{2}\binom{2k}{k}$. The number of patterns that have non-negative partial sums is given by the kth Catalan number, $C_k = \binom{2k}{k}\frac{1}{k+1}$. Thus the number of paths to test is the difference, $\frac{1}{2}\binom{2k}{k}\frac{k-1}{k+1}$. This is multiplied by $\binom{n}{2k}$ and added for all $2 \leq k \leq n/2$. Written explicitly, the number of subset pairs to test is,

$$\boxed{\frac{1}{2}\sum_{k=2}^{\lfloor n/2 \rfloor}\binom{n}{2k}\binom{2k}{k}\frac{k-1}{k+1}}$$

In particular, for this question $n = 12$.

107 – Minimal network

This can be solved using Kruskal's algorithm. We start with a graph with no edges and add edges one-by-one in increasing order of weight, skipping those that do not combine two trees, until the graph is connected.

108 – Diophantine reciprocals I

$n < x, y$ so let $x = n + a$ and $y = n + b$. Then with some algebra we can show that $\frac{1}{n} = \frac{1}{n+a} + \frac{1}{n+b} \implies n^2 = ab$. Therefore the number of solutions is the number of ways to represent n^2 as the product of two numbers, or half its number of divisors. This is fast to calculate, as mentioned in Problem 12.

113⋆ – Non-bouncy numbers

Given d digits, there are $\binom{8+d}{8}$ increasing numbers. To derive this fact, we can work out that for d nested sums and $k \geq 1$ that

$$\sum_{n_1=1}^{k}\left(\sum_{n_2=1}^{n_1}\left(\cdots\left(\sum_{n_d=1}^{n_{d-1}} 1\right)\cdots\right)\right) = \binom{k+d-1}{d} = \binom{k+d-1}{k-1}$$

and then letting $k = 9$ for the digits 1 through 9. Alternatively, we can explain this fact inductively: for $d = 1$, we have the set of digits $\{1, 2, 3, \ldots, 9\}$, from which we choose one. For $d = 2$, we have the set $\{1, 2, 3, \ldots, 9, \#\}$ where $\#$ means that we double the first digit. From this set, we pick two elements. For $d = 3$, we have the set $\{1, 2, 3, \ldots, 9, \#_1, \#_2\}$ where $\#_1$ means we double the first digit selected and $\#_2$ means we double the second digit selected, and we pick three elements. This continues for higher d, so the number of increasing numbers is $\binom{8+d}{d}$.

The number of decreasing numbers is very similar as we are allowed digits 0 through 9, so for d digits we have $\binom{9+d}{9} - 1$ possibilities, minus 1 as we cannot start a number with 0. But we overcount numbers that are just repeating digits, which there are 9 of. Thus for d digits there are $\binom{8+d}{8} + \binom{9+d}{9} - 10$ non-bouncy numbers. If we sum this up for $1 \leq d \leq 100$, we get

$$\begin{aligned}
&\sum_{d=1}^{100}\binom{8+d}{8} + \sum_{d=1}^{100}\binom{9+d}{9} - \sum_{d=1}^{100} 10 \\
&= \left(\binom{8+100+1}{8+1} - 1\right) + \left(\binom{9+100+1}{9+1} - 1\right) - 1000 \\
&= \boxed{\binom{109}{9} + \binom{110}{10} - 1002}
\end{aligned}$$

Here we used the formula $\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$.

114⋆ – Counting block combinations I

Let $f(n)$ be the number of ways to fill a row n units in length. We can work out that $f(n) = f(-1) + f(0) + f(1) + f(2) + \cdots + f(n-4) + f(n-1)$ where $f(-1) = f(0) = 1$. $f(n-1)$ represents starting with a grey square, $f(n-4)$ represents starting with 3 red blocks, and so forth. $f(-1)$ represents all red blocks. This recurrence relation is equivalent to $f(n) = 2f(n-1) - f(n-2) + f(n-4)$ where $f(0) = f(1) = f(2) = 1$ and $f(3) = 2$. The answer is $f(50) = \boxed{16475640049}$.

115 – Counting block combinations II

Nearly identical to Problem 114, but generalized. Now $F(m, n) = 2F(m, n-1) - F(m, n-2) + F(m, n-m-1)$ where $F(m, n) = 1$ when $n < m$ and $F(m, m) = 2$.

116⋆ – Red, green or blue tiles

The number of ways to tile with red tiles is $f_2(n) - 1$, where $f_2(n) = f_2(n-1) + f_2(n-2)$ with initial values $f_2(1) = 1$ and $f_2(2) = 2$. These are precisely the Fibonacci numbers. Similarly, the number of ways to tile with green tiles is $f_3(n)-1$, where $f_3(n)=f_3(n-1)+f_3(n-3)$ with initial values $f_3(1)=f_3(2)=1$ and $f_3(3)=2$. Finally, the number of ways to tile with blue tiles is $f_4(n)-1$, where $f_4(n)=f_4(n-1)+f_4(n-4)$ with initial values $f_4(1)=f_4(2)=f_4(3)=1$ and $f_4(4)=2$. It is possible to compute the answer by hand, which is

$$f_2(50)+f_3(50)+f_4(50)-3=\boxed{20365011074+122106097+5453761-3}$$

117⋆ – Red, green, and blue tiles

The number of ways to tile is $f(n)$, where $f(n)=f(n-1)+f(n-2)+f(n-3)+f(n-4)$ with initial values $f(0)=1$ and $f(n<0)=0$. These are the tetranacci numbers. The answer is $f(50)=\boxed{100808458960497}$.

120⋆ – Square remainders

By expanding, we can determine that for odd n, $(a\pm1)^n\equiv an \pmod{a^2}$, and for even n, $(a\pm1)^n\equiv 1 \pmod{a^2}$. Thus we are trying to maximize $2an \mod a^2$ for odd n, or equivalently finding largest even multiple of a less than a^2. For odd a this is a^2-a, and for even a this is a^2-2a. We are summing over all $3\le a\le 1000$, so the answer is the sum of $(2k-1)^2-(2k-1)+(2k)^2-2(2k)=8k^2-10k+2$ for $2\le k\le 500$. Recall that the sum of the first n numbers is $n(n+1)/2$ and the sum of the squares of the first n numbers is $n(n+1)(2n+1)/6$. Therefore the answer is

$$\boxed{\left(8\cdot\frac{500\cdot501\cdot1001}{6}-10\cdot\frac{500\cdot501}{2}+2\cdot500\right)-(8-10+2)}$$

121⋆ – Disc game prize fund

With $n=4$ turns, the number of ways to draw exactly $k=2$ red disks is

$$\sum_{1\le i<j\le4} ij=1\cdot2+1\cdot3+1\cdot4+2\cdot3+2\cdot4+3\cdot4\equiv\begin{bmatrix}5\\3\end{bmatrix}$$

where $\left[{a \atop b}\right]$ is the unsigned Stirling number of the first kind. In general, for n turns, the number of ways to draw k red disks is equal to $\left[{n+1 \atop k+1}\right]$. Thus for $n=15$ turns, the number of ways to win is $W=\left[{16 \atop 16}\right]+\left[{16 \atop 15}\right]+\cdots+\left[{16 \atop 9}\right]$. There are $(n+1)!$ total outcomes, so if we award x pounds for a win, our expected profit is $-(x-1)\cdot\frac{W}{16!}+1\cdot\frac{16!-W}{16!}$. The break-even point is when our profit equals 0, which we can then solve for x to get $x=\frac{16!-W}{W}+1=\frac{16!}{W}$ and take the floor. Written explicitly, the answer is

$$\boxed{\left\lfloor 16! \middle/ \sum_{k=9}^{16}\begin{bmatrix}16\\k\end{bmatrix}\right\rfloor}$$

The unsigned Stirling numbers of the first kind are easily generated using the recurrence relation $\left[{n+1 \atop k}\right]=n\left[{n \atop k}\right]+\left[{n \atop k-1}\right]$ with the initial conditions $\left[{0 \atop 0}\right]=1$ and $\left[{0 \atop n}\right]=\left[{n \atop 0}\right]=0$ for $n>0$. It may help to use the identity $n!=\sum_{k=0}^{n}\left[{n \atop k}\right]$.

129 – Repunit divisibility

We can generate $R(k) \mod n$ by the recurrence relation $R(k+1)\equiv 10R(k)+1 \pmod n$ with initial value $R(1)=1$. Because n and 10 are coprime, the recurrence relation creates a cycle. Since $10\cdot0+1=1=R(1)$, so there exists a $k>0$ such that $R(k)\equiv 0 \pmod n$. $A(n)$ denotes the the smallest such k. Additionally, since the cycle must have length $\le n$, so we have $A(n)\le n$. Therefore, for this problem we can start finding $A(n)$ from $n=10^6$.

131 – Prime cube partnership

We claim that n is a perfect cube. Assume for a contradiction that $n = x \cdot y^3$ where $x \neq 1$ and is cubefree. As $n^3 + n^2 p = n^2(n+p)$ is a perfect cube, so $x | n+p$. But $x|n$, so $x|p$ and $x = p$ as p is prime. But then $n^2(n+p) = 2p^3y^6$ which is not a perfect cube. Therefore n is a perfect cube, which implies that $n+p$ is a perfect cube as well. Therefore p is the difference of two perfect cubes. But if $p = k^3 - (k-a)^3 = a(3k^2 - 3ka + a^2)$, we require $a = 1$ and p is the difference of two consecutive cubes. What remains is to generate consecutive cubes and test for primality.

132 – Large repunit factors

$R(k) = (10^k - 1)/9$, so if we are checking divisibility of $R(k)$ by some prime p, $(10^k - 1)/9 \equiv 0 \pmod{p}$ if and only if $10^k \equiv 1 \pmod{p}$, for $p \neq 3$. Therefore all we need to test is that $10^{10^9} \equiv 1 \pmod{p}$. We can simplify a little more using Fermat's little theorem, which says $10^{p-1} \equiv 1 \pmod{p}$. Therefore it suffices that we check $10^{\gcd(10^9,\, p-1)} \equiv 1 \pmod{p}$.

137⋆ – Fibonacci golden nuggets

Here we work out the generating function for the Fibonacci numbers.

$$\begin{aligned}
A_F(x) &= \sum_{k=1}^{\infty} F_k x^k \\
&= F_1 x + F_2 x^2 + \sum_{k=3}^{\infty} F_k x^k \\
&= x + x^2 + \sum_{k=3}^{\infty} F_{k-1} x^k + \sum_{k=3}^{\infty} F_{k-2} x^k \\
&= x + x^2 - F_1 x^2 + x \sum_{k=1}^{\infty} F_k x^k + x^2 \sum_{k=1}^{\infty} F_k x^k \\
&= x + x A_F(x) + x^2 A_F(x)
\end{aligned}$$

So the generating function for the Fibonacci numbers if $A_F(x) = \frac{x}{1-x-x^2}$. If we set $A_F(x) = n$, we can rearrange to get the equation $nx^2 + (n+1)x - n = 0$. By the quadratic formula, x is rational when $(n+1)^2 + 4n^2$ equals a square number y^2. We can rearrange again to get $(5n+1)^2 + 4 = 5y^2$. Letting $x = 5n+1$, we are solving the generalized Pell's equation $x^2 - 5y^2 = -4$. Solution are $x_k + \sqrt{5} y_k = 2(1/2 + \sqrt{5}/2)^{1+2k}$ for $k = 0, 1, 2, \ldots$ and can be generated recursively. What remains is to check that $x_k \equiv 1 \pmod{5}$, which we can prove only happens when $k \equiv 0 \pmod{2}$. Since $k = 0$ is not a golden nugget, we want $k = 30$ and the answer is $(x_{30} - 1)/5 = \boxed{(5600748293801 - 1)/5}$.

138⋆ – Special isosceles triangles

Suppose we have a triangle with sides lengths $L, L, 2b$ and height $h = b + \eta$ perpendicular to side $2b$ where $\eta = \pm 1$. From the Pythagorean Theorem we have $b^2 + (2b+\eta)^2 = L^2$, which can be rearranged to $(5b+2\eta)^2 + 1 = 5L^2$. Letting $x = 5b + 2\eta$ and $y = L$ we are solving the negative Pell's equation $x^2 - 5y^2 = -1$. Solutions are $x_k + \sqrt{5} y_k = (1/2 + \sqrt{5}/2)^{3(1+2k)}$ for $k = 0, 1, 2, \ldots$ and can be generated recursively. We can also prove that $x_k \equiv 2$ or $3 \pmod{5}$ for all k.

k	y_k
0	1
1	17
2	305
3	5473
4	98209
5	1762289
6	31622993
7	567451585
8	10182505537
9	182717648081
10	3278735159921
11	58834515230497
12	1055742538989025

Ignoring $k = 0$, which is not a triangle, the answer is $y_1 + y_2 + \cdots + y_{12} = \boxed{1118049290473932}$.

139⋆ – Pythagorean tiles

Suppose $a^2 + (a+n)^2 = c^2$ is a primitive Pythagorean triplet with $n > 0$. We are looking for cases where n divides c. Expanding, have $n(2a+n) = c^2 - 2a^2$ but as $n|c$, so $n = 1$, $n = 2$, or $n|a$. The second case is impossible as a and $a+n$ would have the same parity, since for primitive triplet a and b must have opposite parity. The third case is also impossible as n would divide a and $a+n$, contradicting the triplet being primitive. Therefore $n = 1$ and we are searching for primitive triplets where a and b differ by 1.

Suppose $a^2 + (a+1)^2 = c^2$. We can rearrange this as $(2a+1)^2 + 1 = 2c^2$. Letting $x = 2a+1$ and $y = c$, we are solving the negative Pell equation $x^2 - 2y^2 = -1$. Solutions are $x_k + \sqrt{2}y_k = (1+\sqrt{2})^{1+2k}$ for $k = 0, 1, 2, \ldots$ and can be generated recursively. We can also prove that $x_k \equiv 1 \pmod 2$ for all k. But since we only generate primitive triplets, for each perimeter $P = x_k + y_k$ of a primitive triplet there are $\lfloor 10^8/P \rfloor$ triangles.

k	x_k	y_k	P	$\lfloor 10^8/P \rfloor$
0	1	1	–	–
1	7	5	12	8333333
2	41	29	70	1428571
3	239	169	408	245098
4	1393	985	2378	42052
5	8119	5741	13860	7215
6	47321	33461	80782	1237
7	275807	195025	470832	212
8	1607521	1136689	2744210	36
9	9369319	6625109	15994428	6
10	54608393	38613965	93222358	1

The sum over the last column is $\boxed{10057761}$.

140⋆ – Modified Fibonacci golden nuggets

Using the same steps as Problem 137, the generating function is $A_G(x) = x(1+3x)/(1-x-x^2)$. We are eventually solving the generalized Pell's equation $x^2 - 5y^2 = 44$ for $x = 5n + 7$. There are two families of solutions,

$$x_k^{(1)} + \sqrt{5}y_k^{(1)} = (7+\sqrt{5})(1/2+\sqrt{5}/2)^{2k}$$
$$x_k^{(2)} + \sqrt{5}y_k^{(2)} = (8+2\sqrt{5})(1/2+\sqrt{5}/2)^{2k}$$

We can prove that $x_k^{(1)} \equiv 2 \pmod 5$ only when $k \equiv 0 \pmod 2$, and $x_k^{(2)} \equiv 2 \pmod 5$ only when $k \equiv 1 \pmod 2$. So we need to generate both families up to $k = 30$.

145⋆ – How many reversible numbers are there below one-billion?

Let us define two types of pairs of digits with opposite parity: α-pairs, where the two digits sum less than 10, and β-pairs, where the two digits sum greater than 10. There are 30 α-pairs and 20 β-pairs, but only 20 α-pairs if we exclude those that contain the digit 0. Next let x be any number from 0 to 4, which there are 5 of. Then the table below summarizes the allowed patterns for reversible numbers.

# of digits	Pattern	Count
1	–	0
2	$\alpha\alpha$	20
3	$\beta x \beta$	$20 \cdot 5$
4	$\alpha\alpha\alpha\alpha$	$20 \cdot 30$
5	–	0
6	$\alpha\alpha\alpha\alpha\alpha\alpha$	$20 \cdot 30^2$
7	$\beta x \beta x \beta x \beta$	$20^2 \cdot 5^3$
8	$\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha$	$20 \cdot 30^3$
9	–	0

Therefore the answer is $\boxed{20 + 20 \cdot 5 + 20 \cdot 30 + 20 \cdot 30^2 + 20^2 \cdot 5^3 + 20 \cdot 30^3}$.

148⋆ – Exploring Pascal's triangle

It is best to start by drawing the non-multiples of 7 on Pascal's triangle. Rows 1–7 form a triangle with 28 non-multiples of 7. Call this X_1. Rows 1–14 form a "tri-force" symbol, containing three copies of X_1. This pattern continues for rows 1–49, where we have 28 copies of X_1 forming a "generalized tri-force" with seven triangles on its bottom layer. Call this X_2, which contains a total of $28^2 = 784$ non-multiples of 7. This pattern continues again, where rows 1–98 form a tri-force symbol, containing three copies of X_2, and so forth.

Using this picture, we can come up with an algorithm of computing the number of non-multiples of 7 in the first 10^9 rows. Let us list the first 7 triangle numbers:

$$T_0 = 0\ , \qquad T_1 = 1\ , \qquad T_2 = 3\ , \qquad T_3 = 6\ , \qquad T_4 = 10\ , \qquad T_5 = 15\ , \qquad T_6 = 21$$

First, we factorize 10^9 in base 7: $10^9 = 33531600616_7$. The highest digit is 3×7^{10}, so we expect there to be $T_3 = 6$ completed X_{10}s, and $3 + 1 = 4$ incomplete X_{10}s. the next highest digit is 3×7^9, so each of these incomplete X_{10}s contains $T_3 = 6$ completed X_9s, and $3 + 1$ incomplete X_8s. This continues, and the total number of non-multiples of 7 is

$$\boxed{28^{10}T_3 + 4\big(28^9T_3 + 4\big(28^8T_5 + 6\big(28^7T_3 + 4\big(28^6T_1 + 2\big(28^5T_6 + 7\big(28^2T_6 + 7\big(28T_1 + 2\big(T_6\big)\big)\big)\big)\big)\big)\big)\big)}$$

162⋆ – Hexadecimal numbers

For a given number of digits n, we can count the number of possible hexadecimal numbers using the inclusion-exclusion principle. If boldface **1** denotes "the set of numbers that contains at least one 1", and likewise for **0** and **A**, then $|\mathbf{0} \cap \mathbf{1} \cap \mathbf{A}| = |\mathbf{0}| + |\mathbf{1}| + |\mathbf{A}| - |\mathbf{0} \cup \mathbf{1}| - |\mathbf{0} \cup \mathbf{A}| - |\mathbf{1} \cup \mathbf{A}| + |\mathbf{0} \cup \mathbf{1} \cup \mathbf{A}|$. Add this up for $3 \le n \le 16$ to get the solution,

$$\boxed{\sum_{n=3}^{16} \left(15 \cdot 16^{n-1} - 43 \cdot 15^{n-1} + 41 \cdot 14^{n-1} - 13^n\right)}$$

169 – Exploring the number of different ways a number can be expressed as a sum of powers of 2

The function $f(n)$ is equivalent to the number of ways of expressing n as a binary string, but also permitting the usage of the digit 2. For instance, $f(10) = f(1010_2) = \{1010, 210, 1002, 202, 122\}$ and $f(11) = f(1011_2) = \{1011, 211\}$. We can establish two properties of the function $f(n)$:

$$f(2n) = f(n) + f(n-1)\ , \qquad f(2n+1) = f(n)$$

- if n is even (i.e. $n = 2k$) its binary string ends with a 0. Then its binary string representations are those of k with a 0 digit appended at the end and those of $k-1$ with a 2 digit appended. For instance, $f(10) = f(1010_2) = \{1010, 210, 1002, 202, 122\} = \{101, 21\}0 + \{100, 20, 12\}2 = f(101_2) + f(100_2) = f(5) + f(4)$.
- if n is odd (i.e. $n = 2k+1$) its binary string ends with a 1. Then its binary string representations are those of k with a 1 digit appended. For instance, $f(11) = f(1011_2) = \{1011, 211\} = \{101, 21\}_1 = f(5)$.

With this, we can recursively find $f(n)$ for all n.

183 – Maximum product of parts

For a given N, we want to maximize $(N/k)^k$. We can do this by differentiating with respect to k to get $(N/k)^k[\ln(N/k) - 1]$, so the maximum is when $k = N/e$. But this is not an integer, so we test the integers above and below for the larger value of $(N/k)^k$. Once we know our k, if $k/\gcd(k, N)$ contains prime factors other than 2 or 5, then the decimal does not terminate.

188 – The hyperexponentiation of a number

We are finding $1777 \uparrow\uparrow 1855 \equiv 1777^{1777\uparrow\uparrow 1854} \pmod{10^8}$. The generalized Fermat's little theorem says that $a^{\phi(n)} \equiv 1 \pmod n$ for any a and n coprime, where $\phi(n)$ is the totient function. Since 1777 is prime, so $1777^{1777\uparrow\uparrow 1854} \equiv 1777^{a_1} \pmod{10^8}$ for some integer $a_1 \equiv 1777 \uparrow\uparrow 1854 \pmod{\phi(10^8)}$. We can repeat this process by taking successive totients, defining a sequence $a_1, a_2, \ldots$. Eventually we will reach a totient $\phi^{(k)}(10^8) = 2$. As 1777 is odd, any power of 1777 is odd so $a_k = 1$. We can then work backwards using the recursive formula $a_i \equiv 1777^{a_{i+1}} \pmod{\phi^{(i)}(10^8)}$ until we find a_0, which is the answer.

197 – Investigating the behaviour of a recursively defined sequence

The function given is very close to the analytic function $f(x) = 1.42 \cdot 2^{-x^2}$. Let us look for stable fixed-points for the function $f(x)$, i.e. points x_0 where $f(x_0) = x_0$ and $|f'(x_0)| < 1$. There is one fixed-point for $f(x)$ at $x_0 \approx 0.855$, but $f'(x_0) \approx -1.014$ so it is not stable. Next we have to look for fixed-points of the double map, $f\big(f(x)\big)$, which are $x_0 \approx 1.029$ and $x_1 \approx 0.681$. These fixed-points are stable and satisfy $f(x_0) = x_1$ and $f(x_1) = x_0$. If an initial point u_0 is picked near these values, iteratively calling $u_{n+1} = f(u_n)$ will converge upon these two fixed-points very quickly.

216 – Investigating the primality of numbers of the form $2n^2 - 1$

We can use a sieve method to generate all primes. This relies on two claims:

Claim 1: If d is a non-trivial (i.e. $d \neq 1$) divisor of $t(n) = 2n^2 - 1$, then

(a) d does not divide n, $n+1$, or $n-1$. Equivalently, for no $k \in \mathbb{Z}$ will $n + kd$ be 0 or ± 1.

(b) d divides $t(\pm n + kd)$ for all $k \in \mathbb{Z}$.

Proof. For (a), note that d cannot divide 1 and $t(n) = 2(n \pm 1)^2 \mp 4(n \pm 1) + 1$. For (b), note that $t(\pm n + kd) = t(n) \pm 4nkd + 2k^2d^2$. ■

Claim 2: Let d be the smallest non-trivial divisor of $t(n) = 2n^2 - 1$. If $d \neq t(n)$ then d divides $t(n')$ for some $2 \leq n' < n$.

Proof. First, we prove that $d < 2n$. Since d is the smallest non-trivial divisor of $t(n)$ but $d \neq t(n)$, we necessarily have $d^2 \leq t(n)$.

$$d^2 \leq t(n) = 2n^2 - 1 < 2n^2 \qquad \Longrightarrow \qquad d < \sqrt{2}n < 2n$$

This implies $|n - d| < n$. From Claim 1a, we also have $2 \leq |n - d|$. From Claim 1b, d divides $t(|n - d|)$. ■

The sieve works as follows: starting from $n = 2$, by Claim 2 we know that $t(2) = 7$ is prime. Then all $n = 7 - 2, 7 + 2, 14 - 2, 14 + 2, \ldots$ correspond to composite $t(n)$. So $t(3)$ and $t(4)$ are prime and $t(5)$ is composite. For each composite we encounter, if it contains a factor that we have not encountered before, we apply the same procedure to find more composites. For instance, $t(9) = 7 \cdot 23$ is composite and 23 has not been encountered before. Then all $n = 23 - 9, 23 + 9, 46 - 9, 46 + 9, \ldots$ correspond to composite $t(n)$.

231 – The prime factorisation of binomial coefficients

Kummer's theorem on binomial coefficients states that given integers $n \geq m \geq 0$ and a prime p that the maximum integer k such that p^k divides $\binom{n}{m}$ is equal to the number of carries when m is added to $n - m$ in base p.

235 – An Arithmetic Geometric sequence

The sum of the first n terms of a arithmetico-geometric sequence is given by

$$\sum_{k=1}^{n} [a + (k-1)d]r^{k-1} = \frac{a + (d-a)r - (a+nd)r^n + (a+nd-d)r^{n+1}}{(1-r)^2}$$

301⋆ – Nim

Given k piles with a_k stones, if $a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0$ then we have a losing game, where $\oplus$ denotes the bit-wise XOR operation on the binary representation. For 3 piles, $a_1 \oplus a_2 \oplus a_3 = 0 \iff a_1 \oplus a_2 = a_3$. Then for this question, our requirement becomes $n \oplus 2n = 3n = n + 2n$. This is only true if the binary representation for n does not have two adjacent "1" digits. For instance if $n = 1011_2$, $2n = 10110_2$ and $n \oplus 2n = 11101_2 \neq 100001_2 = n + 2n$. So we are finding the number of binary strings of length m with no adjacent "1" digits, which is given by the mth Fibonacci number, with $F_1 = 2, F_2 = 3$. Thus using these seeds, $F_{30} = \boxed{2178309}$ gives our answer.

317⋆ – Firecracker

This problem has a closed-form answer from physics. Let us place the exploding firecracker at the coordinate $(0, H)$, with the y-axis extending upwards and the x-axis extending in a horizontal direction. A projectile launched from this position at an angle θ to the vertical with speed v follows the usual parabolic trajectory,

$$y(t) = H + vt\cos\theta - \frac{1}{2}gt^2 \ , \qquad x(t) = vt\sin\theta$$

where t is the time elapsed ($t = 0$ at moment of explosion) and $g = 9.81\,\mathrm{m/s^2}$ is the acceleration due to gravity. The height y as a function of x is

$$y(x) = H + x\cot\theta - \frac{x^2 g}{2v^2}(1 + \cot^2\theta)$$

Maximizing $y(x)$ with respect to θ will give the maximum height $y_{\max}(x)$ of any projectile at that x. So differentiating with respect to θ and setting $\mathrm{d}y/\mathrm{d}\theta = 0$, we obtain the condition,

$$\cot\theta = \frac{v^2}{xg}$$

Thus,

$$y_{\max}(x) = H + \frac{v^2}{2g} - \frac{x^2 g}{2v^2} = H + h_0 - \frac{x^2}{4h_0}$$

where we have defined $h_0 \equiv v^2/2g$. The volume enclosed by the function $y_{\max}(x)$, rotated around the y-axis, and the $y = 0$ plane gives the region that the firework fragments travel through. To calculate this volume, we are integrating cylindrical shells with volume $2\pi x y_{\max}(x)\,\mathrm{d}x$ from $x : 0 \to sqrt 4h_0(H + h_0)$.

$$\begin{aligned}
V &= \int_0^{\sqrt{4h_0(H+h_0)}} 2\pi x y_{\max}(x)\,\mathrm{d}x \\
&= 2\pi \int_0^{\sqrt{4h_0(H+h_0)}} \left[(H+h_0)x - \frac{1}{4h_0}x^3\right] \mathrm{d}x \\
&= 2\pi\left[2h_0(H+h_0)^2 - h_0(H+h_0)^2\right] \\
&= 2\pi h_0 (H+h_0)^3
\end{aligned}$$

If we plug-in the values $h_0 = 20^2/(2 \cdot 9.81)$ and $H = 100$, we have the answer,

$$\boxed{2\pi\left(\frac{20^2}{2\cdot 9.81}\right)\left(100 + \frac{20^2}{2\cdot 9.81}\right)^2}$$

381 – (prime-k) factorial

Wilson's Theorem states that for prime p, we have $(p-1)! \equiv -1 \pmod p$. Thus we can deduce that $(p-2)! \equiv 1$, $(p-3)! \equiv -1/2$, $(p-4)! \equiv 1/6$, and $(p-5)! \equiv -1/24 \pmod p$. Fractions are to be interpreted as modular inverses, since $\mathbb{Z}_p$ is a field. Thus $S(p) \equiv -3/8 \pmod p$.

389⋆ – Platonic Dice

We are interested in finding generating functions,

$$G(x) = \sum_{n=1} P_n x^n$$

where P_n is the probability of obtaining the outcome n. For the normalization of probabilities, we require $G(x) = 1$. For instance, the generating function of a single k-sided die is

$$g_k(x) = \frac{1}{k}(x + x^2 + \cdots + x^k)$$

The generating function for T is simply $G = g_4$, since a single 4-sided die is rolled. For the generating function of C, we note that because the generating function for rolling an k-sided die N times is $[g_k(x)]^N$, the replacement $x \mapsto g_6(x)$ in the function $g_4(x)$ represents rolling a 6-sided die based on the outcome of a 4-sided die. On other words, the generating function of C is $G(x) = g_4(g_6(x)) = g_4 \circ g_6(x)$. Continuing in this fashion, the generating function for I is

$$G = g_4 \circ g_6 \circ g_8 \circ g_{12} \circ g_{20}$$

We want to calculate the variance of I, given by $\sigma^2 = \langle I^2 \rangle - \langle I \rangle^2$. This can be calculated from our generating functions,

$$\langle I \rangle = \sum_{n=1} nP_n = \left(\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1} P_n x^n \right) \Bigg|_{x=1} = G'(1)$$
$$\langle I^2 \rangle = \sum_{n=1} n^2 P_n = \left(\frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} G(x) \right) \right) \Bigg|_{x=1} = G'(1) + G''(1)$$

Using the chain rule in calculus,

$$G'(x) = (g'_{20}(x))(g'_{12} \circ g_{20}(x))(g'_8 \circ g_{12} \circ g_{20}(x))(g'_6 \circ g_8 \circ g_{12} \circ g_{20}(x))(g'_4 \circ g_6 \circ g_8 \circ g_{12} \circ g_{20}(x))$$

Noting that $g_k(1) = 1$, we have

$$G'(1) = g'_4(1) \cdot g'_6(1) \cdot g'_8(1) \cdot g'_{12}(1) \cdot g'_{20}(1)$$

Similarly, we can show that,

$$G''(1) = G'(1) \left(\frac{g''_{20}}{g'_{20}} + g'_{20} \left(\frac{g''_{12}}{g'_{12}} + g'_{12} \left(\frac{g''_8}{g'_8} + g'_8 \left(\frac{g''_6}{g'_6} + g'_6 \cdot \frac{g''_4}{g'_4} \right) \right) \right) \right) \Bigg|_{x=1}$$

Now to simplify out expressions for $G'(1)$ and $G''(1)$, we can show that

$$g'_k(1) = \frac{1}{2}(k+1) \ , \qquad \frac{g''_k(1)}{g'_k(1)} = \frac{2}{3}(k-1)$$

Plugging everything in, this gives us

$$G'(1) = \frac{85995}{32} \quad , \quad G''(1) = G'(1) \cdot \frac{85963}{24}$$

and the final answer for σ^2 is

$$\frac{2464129395}{1024} \approx \boxed{2406376.3623}$$

407 – Idempotents

For integer $n > 1$, let its prime factorization be $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for distinct primes $p_1, \ldots, p_k$. The Chinese Remainder Theorem defines a ring isomorphism,

$$\mathbf{Z}_n \cong \mathbf{Z}_{p_1^{a_1}} \oplus \mathbf{Z}_{p_2^{a_2}} \oplus \cdots \oplus \mathbf{Z}_{p_k^{a_k}}$$

The only idempotents in $\mathbf{Z}_{p^a}$ for any prime p are 0 and 1. Therefore, $\mathbf{Z}_n$ has 2^k idempotents corresponding to every possible selection of 0 or 1 in each $\mathbf{Z}_{p_i^{a_i}}$, i.e.

$$\underbrace{(0,0,\ldots,0)}_{k\text{-tuple}} \ , \qquad (1,0,\ldots,0) \ , \qquad (0,1,\ldots,0) \ , \quad \ldots \quad , \qquad (1,1,\ldots,1)$$

We use the isomorphism to convert elements of $\bigoplus_i \mathbf{Z}_{p_i^{a_i}}$ into elements of $\mathbf{Z}_n$.

500 – Problem 500!!

Recall that given the prime factorization of a number $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, the number of divisors of n is $(a_1+1)(a_2+1)\cdots(a_k+1)$. So a number with 2^{500500} divisors is $2 \times 3 \times \cdots \times p_{500500}$. However, this is not the smallest number, as we can replace p_{500500} with 2^2 to get $2^3 \times 3 \times \cdots \times p_{500499}$. We repeat this, replacing the largest primes in the product with powers of smaller primes.

577⋆ – Counting hexagons

With length $n = 3$, we are able to construct a hexagon from the center point with "radius 1" (i.e. the vertices are a unit distance away from the center). If the side length is increased to $n = 4$, then we can construct 3 such hexagons, then 6 such hexagons with $n = 5$, and so forth. This pattern of triangular numbers continues: with side length n we are able to construct T_{n-2} hexagons with radius 1, where $T_n = n(n+1)/2$ is the nth triangular number.

However, at $n = 6$, we are able to construct two new types of hexagons from the center point with "radius 2" (i.e. the vertices are two units away from the center, counting distances along the edges of the equilateral triangles). We can construct one of each type, so including the 10 hexagons with radius 1, we have 12 total hexagons, hence $H(6) = 12$. With $n = 7$, we are able to construct 3 of each type of hexagon with radius 2, and adding the 15 hexagons with radius 1, we have $15 + 2 \times 3 = 21$ total hexagons, hence $H(7) = 21$. Again, these radius 2 hexagons follow the same triangular number pattern: with side length n we are able to construct $2T_{n-5}$ hexagons with radius 2.

Then at $n = 9$, we are able to construct three new types of hexagons from the center point with "radius 3". This pattern also continues: when the side length n is a multiple of 3, we are able to construct $n/3$ new types of hexagons. These new hexagons follow the same triangular number pattern.

To give a sample calculation, in calculating $H(20)$, we can have up to radius 6 hexagons. The total number of hexagons we can construct is,

$$T_{18} + 2T_{15} + 3T_{12} + 4T_9 + 5T_6 + 6T_3 = 966$$

Using this, we can express our sum as a sum over triangular numbers:

$$\sum_{n=3}^{12345} H(n) = \sum_{n=1}^{12345-2} T_n + 2 \times \sum_{n=1}^{12345-5} T_n + 3 \times \sum_{n=1}^{12345-8} T_n + \cdots + 4115 \times \sum_{n=1}^{1} T_n$$

We will use the following formulas,

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \qquad \sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4} \qquad \sum_{n=1}^{N} n^4 = \frac{N(N+1)(2N+1)(3N^2+3N-1)}{30}$$

So we can write,

$$\sum_{n=1}^{N} T_n = \frac{N(N+1)(N+2)}{6}$$

This implies,

$$\begin{aligned}\sum_{n=3}^{12345} H(n) &= 4115\frac{1(1+1)(1+2)}{6} + 4114\frac{4(4+1)(4+2)}{6} + \cdots + \frac{12343(12343+1)(12343+2)}{6} \\ &= \frac{1}{6}\sum_{n=0}^{4114}(4115-n)(1+3n)(2+3n)(3+3n)\end{aligned}$$

Expanding this out and using the sums for n, n^2, n^3, n^4 above, we get the answer $\boxed{265695031399260211}$.

587 – Concave triangle

If the circle has unit radius, then the L-section has area $1 - \pi/4$. If the corner of the L-section is at the origin, then the straight line is given by the equation $y = nx$ for a particular value of n. The intersect of this straight line and the circle occurs at the x-value,

$$x_n = \frac{n + 1 - \sqrt{2n}}{n^2 + 1}$$

Then using calculus, the area of the concave triangle is,

$$\int_0^{x_n} \mathrm{d}x \left[(1 - \sqrt{1 - (x-1)^2}) - nx\right] = \frac{1}{2}\left[(1 - x_n)\sqrt{1 - (1 - x_n)^2} + \sin^{-1}(1 - x_n) - nx_n^2 + 2x_n - \frac{\pi}{2}\right]$$

We iterate through values of n until the area is small enough.