Homework 11

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1. **Proposition.** *For a positive integer n,*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Proof. This will be proven with induction using a base case of $n=1$.

(1) For $n=1$, it is true that

$$\frac{1}{1(1+2)}=1-\frac{1}{1+1}$$

(2) Suppose it is true

$$S(k)=\sum_{i=1}^{k}\frac{1}{i(i+1)}=1-\frac{1}{k+1}$$

Then for the $k+1$ term,

$$\begin{aligned}
S(k+1)&=\sum_{i=1}^{k+1}\frac{1}{i(i+1)}\\
&=\sum_{i=1}^{k}\frac{1}{i(i+1)}+\frac{1}{(k+1)(k+2)}\\
&=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)} && \text{(ind. hyp.)}\\
&=1-\frac{k+2}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}\\
&=1-\frac{k+1}{(k+1)(k+2)}\\
&=1-\frac{1}{(k+1)+1}
\end{aligned}$$

We have shown $S(k+1)$ to be true and it follows that all $S(n)$ must be true.

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2. **Proposition.** *For every* $n \in \mathbb{Z}_{>0}$,

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

Proof. This will be proven with strong induction and a base case of $n = 1$.

(1) For $n = 1$, it is true that

$$1 \leq 1$$

(2) Suppose the true statement S_k:

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

If we consider the next statement in sequence, S_{k+1},

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$$

From the inductive hypothesis and by rearranging,

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

It is evident that for all positive k, as $1/(k+1)^2 < 1/(k+1)$,

$$-\frac{1}{k} + \frac{1}{(k+1)^2} \leq -\frac{1}{k+1}$$

And as the $\leq$ relation is transitive, the inequality of S_{k+1} is also true as

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$$

Therefore this proposition is true for all positive integers.

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3. **Proposition.** *For the Fibonacci sequence F_n,*

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Proof. This will be proven with induction and a base case of $n = 1$.

(1) For $n = 1$, this holds true as $1 = 2 - 1$.

(2) Suppose the proposition is true for a positive integer k,

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1$$

For the next k in sequence,

$$\begin{aligned} \sum_{i=1}^{k+1} F_i &= \sum_{i=1}^{k} F_i + F_{k+1} \\ &= (F_{k+2} - 1) + F_{k+1} && \text{(ind. hyp.)} \end{aligned}$$

By the definition the Fibonacci sequence, the RHS can be rewritten as the next in sequence as

$$\sum_{i=1}^{k+1} F_i = F_{(k+1)+1} - 1$$

Therefore this proposition is true for all F_n in the Fibonacci sequence.

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4. **Proposition.** *For all integers $n \geq 1$,*

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Proof. This will be proven with induction and a base case of $n = 1$ and $n = 2$.

(1) For $n = 1$, $F_n = 1$ and the RHS fraction is

$$\frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = 1$$

For $n = 2$, $F_n = 1$ and the RHS is

$$\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4\sqrt{5}} = 1$$

(2) Suppose the proposition is true for S_k. This means it is true that

$$F_s = \frac{\varphi^k - \psi^k}{\sqrt{5}}$$

Next, we will check this to be true for $k+1$. By the definition of the Fibonacci sequence,

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&= \frac{1}{\sqrt{5}}\left[\left(\varphi^k - \psi^k\right) + \left(\varphi^{k-1} - \psi^{k-1}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\varphi^k\left(\underbrace{1+\varphi^{-1}}_{\varphi}\right) - \psi^k\left(\underbrace{1+\psi^{k-1}}_{\psi}\right)\right] \\
&= \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}
\end{aligned}$$

As this is true for the $k+1$ term, it must be true for all $n \geq 1$.

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