Homework 2

PHYSICS 450
September 8, 2021

Kevin Evans
ID: 11571810

---

1. ✓ Read through Chapter 1.

2. **Proposition.** If $\psi_1(\mathbf{r}, t)$ and $\psi_2(\mathbf{r}, t)$ are solutions to the Schrödinger equation (SE), then $\alpha\psi_1 + \beta\psi_2$ is also a solution, where $\alpha, \beta \in \mathbb{C}$.

   Proof. If we apply $\Psi = \alpha\psi_1 + \beta\psi_2$ to the SE, then

   $$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi \\ i\hbar\left(\alpha\frac{\partial\psi_1}{\partial t} + \beta\frac{\partial\psi_2}{\partial t}\right) &= -\frac{\hbar^2}{2m}\left[\alpha(\nabla^2 + V)\psi_1 + \beta(\nabla^2 + V)\psi_2\right] \end{aligned}$$

   We can then separate this to two equations in terms of ψ_1 and ψ_2,

   $$\begin{aligned} i\hbar\alpha\frac{\partial\psi_1}{\partial t} &= \alpha\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi_1 \\ i\hbar\beta\frac{\partial\psi_2}{\partial t} &= \beta\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi_2 \end{aligned}$$

   Since these two equations are satisfied, we've shown the linear sum of the two is also a solution to the SE. □

3. For an arbitrary wavefunction

   $$\psi(\mathbf{r}, t) = A(\mathbf{r}, t)e^{i\chi(\mathbf{r},t)},$$

   the quantum current density is given by

   $$\begin{aligned} \mathbf{j} &\equiv \frac{i\hbar}{2m}\left(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi\right) \\ &= \frac{i\hbar}{2m}\left[Ae^{i\chi}\boldsymbol{\nabla}\left(Ae^{-i\chi}\right) - Ae^{-i\chi}\boldsymbol{\nabla}\left(Ae^{i\chi}\right)\right] \\ &= \frac{i\hbar}{2m}\Big\{Ae^{i\chi}\left[\boldsymbol{\nabla}(A)e^{-i\chi} + Ae^{-i\chi}\left(-i\boldsymbol{\nabla}\chi\right)\right] \\ &\qquad - Ae^{-i\chi}\left[\boldsymbol{\nabla}(A)e^{i\chi} + Ae^{i\chi}\left(i\boldsymbol{\nabla}\chi\right)\right]\Big\} \\ &= \frac{i\hbar}{2m}\left(A\boldsymbol{\nabla}A - iA^2\boldsymbol{\nabla}\chi - A\boldsymbol{\nabla}A - iA^2\boldsymbol{\nabla}\chi\right) \\ &= \frac{\hbar A^2\boldsymbol{\nabla}\chi}{m} \end{aligned}$$

4. For the wavefunction

$$\psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

It is normalized as

$$\begin{aligned}\int_{\mathbb{R}} \psi^* \psi \,\mathrm{d}x &= A^2 \left(e^{i\omega t}e^{-i\omega t}\right) \int_{\mathbb{R}} e^{-2\lambda|x|}\,\mathrm{d}x = 1 \\ &= A^2 \left(\int_{-\infty}^{0} e^{2\lambda x}\,\mathrm{d}x + \int_{0}^{\infty} e^{-2\lambda x}\,\mathrm{d}x\right)\end{aligned}$$

Using WolframAlpha,

$$\begin{aligned} &= A^2 \left(\frac{1}{2\lambda} + \frac{1}{2\lambda}\right) \\ A &= \sqrt{\lambda}\end{aligned}$$

Note: I'm not sure if we're solving for A or creating a new normalization constant. If it's a new constant, we can set it to

$$\alpha = \sqrt{\frac{\lambda}{A^2}}.$$

5. For the wavefunction

$$\psi(x,t) = Ae^{-a\left(mx^2/\hbar + it\right)}$$

Before normalizing it, it's clear that the imaginary part will equal 1 when $\psi^*\psi$ is taken and can be ignored, so

$$\int_{\mathbb{R}} \psi^* \psi \,\mathrm{d}x = A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\,\mathrm{d}x = 1$$

This is a Gaussian integral and evaluates as

$$\begin{aligned} 1 &= A^2 \sqrt{\frac{\pi}{2am}} \\ A &= \left(\frac{2am}{\pi}\right)^{1/4}\end{aligned}$$

If we're using a new constant, it'll be

$$\alpha = \frac{1}{A}\left(\frac{2am}{\pi}\right)^{1/4}$$

6. (a) The probability on the range (a, b) is given by

$$P_{ab} = \int_a^b \psi^* \psi \, \mathrm{d}x \, .$$

Taking the time derivative,

$$\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \psi^* \psi \, \mathrm{d}x \, .$$

Applying the chain rule within the integral and applying the SE [eq. (1.23)],

$$\begin{aligned}\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} &= \int_a^b \frac{\mathrm{d}\psi^*}{\mathrm{d}t}\psi + \psi^* \frac{\mathrm{d}\psi}{\mathrm{d}t} \, \mathrm{d}x \\ &= \int_a^b \left(-\frac{i\hbar}{2m} \frac{\mathrm{d}^2\psi^*}{\mathrm{d}x^2} + \frac{i}{\hbar} V \psi^* \right) \psi + \psi^* \left(\frac{i\hbar}{2m} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \frac{i}{\hbar} V \psi \right) \mathrm{d}x \, .\end{aligned}$$

The potential terms cancel and we're left with

$$\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} = \frac{i\hbar}{2m} \int_a^b -\frac{\mathrm{d}^2\psi^*}{\mathrm{d}x^2}\psi + \psi^* \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \, \mathrm{d}x$$

By using $J(x,t) = \frac{i\hbar}{2m}\left(\psi \frac{\mathrm{d}\psi^*}{\mathrm{d}x} - \psi^* \frac{\mathrm{d}\psi}{\mathrm{d}x}\right)$,

$$\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} = \frac{i\hbar}{2m} \int_a^b \frac{\mathrm{d}J}{\mathrm{d}x} \, \mathrm{d}x = J(a,t) - J(b,t) \qquad \square$$

The units of J are energy/mass $\cdot$ time.

(b) From Problem 1.9, the wavefunction is given by

$$\Psi(x,t) = Ae^{-a\left[(mx^2/\hbar)+it\right]}$$

The probability current is

$$\begin{aligned}J &\equiv \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right) \\ &= \frac{i\hbar}{2m}\Bigg(Ae^{-a\left[(mx^2/\hbar)+it\right]} \frac{\mathrm{d}}{\mathrm{d}x} Ae^{-a\left[(mx^2/\hbar)-it\right]} \\ &\qquad - Ae^{-a\left[(mx^2/\hbar)-it\right]} \frac{\mathrm{d}}{\mathrm{d}x} Ae^{-a\left[(mx^2/\hbar)+it\right]} \Bigg)\end{aligned}$$

The complex phases $\pm it$ will cancel out and we're left with

$$\begin{aligned}J &= \frac{i\hbar A^2}{2m}\left(-(2amx/\hbar)e^{-2amx^2/\hbar} - \text{same thing?}\right) \\ &= 0\end{aligned}$$