Homework 10

PHYSICS 450 — Kevin Evans

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1. Using eq. (4.27), (4.28), and (4.32),

$$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}} P_0^0(\cos\theta) = \sqrt{\frac{1}{4\pi}}.$$

Checking to see if this is normalized,

$$\int_0^\pi \int_0^{2\pi} \frac{1}{4\pi} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = 1. \quad \checkmark$$

For the other spherical harmonic,

$$\begin{aligned} Y_2^1(\theta,\phi) &= \sqrt{\frac{5}{4\pi}\frac{1!}{3!}} e^{i\phi} \left(-3\sin\theta\cos\theta\right) \\ &= -\sqrt{\frac{45}{24\pi}} e^{i\phi} \sin\theta\cos\theta. \end{aligned}$$

Checking for normalization,

$$\frac{45}{24\pi} \int_0^\pi \int_0^{2\pi} \sin^3(\theta)\cos^2(\theta) \,\mathrm{d}\theta \,\mathrm{d}\phi = \frac{45}{24\pi}\frac{4}{15} 2\pi = 1. \quad \checkmark$$

Now, to check for orthogonality,

$$\begin{aligned} \left\langle Y_0^0 \middle| Y_2^1 \right\rangle &= -\sqrt{\frac{1}{4\pi}\frac{45}{24\pi}} \int_0^\pi \int_0^{2\pi} e^{i\phi} \sin^2\theta\cos\theta \mathrm{d}\theta\mathrm{d}\phi \\ &= \cdots \int \ldots \mathrm{d}\phi \underbrace{\int_0^{2\pi} \sin^2\theta\cos\theta\mathrm{d}\theta}_{\int_{-\pi}^{\pi} \cos^2\theta\sin\theta = 0 \text{ (odd)}} = 0. \end{aligned}$$

The two functions are orthogonal.

2. From (4.32),

$$\begin{aligned} Y_\ell^\ell &= \sqrt{\frac{2\ell+1}{4\pi}\frac{0!}{(2\ell)!}} e^{i\ell\phi} P_\ell^\ell(\cos\theta) \\ &= \sqrt{\frac{2\ell+1}{8\pi\ell!}} e^{i\ell\phi} P_\ell^\ell(\cos\theta) \end{aligned}$$

For P_ℓ^ℓ, from (4.27) and using the Rodrigues formula (4.28),

$$\begin{aligned} P_\ell^\ell(x) &= (-1)^\ell \left(1-x^2\right)^{m/2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell P_\ell(x) \\ &= (-1)^\ell \left(1-x^2\right)^{\ell/2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \left(x^2-1\right)^\ell \\ &= \frac{(-1)^\ell (1-x^2)^{\ell/2}}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2\ell} (x^2-1)^\ell \end{aligned}$$

Borrowing a hint from classmates: assuming $x^2 \gg 1$, we can approximate this as

$$\begin{aligned}
P_\ell^\ell &= \frac{(-1)^\ell (1-x^2)^{\ell/2}}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2\ell} (x^2)^\ell \\
&= \frac{(-1)^\ell (1-x^2)^{\ell/2}}{2^\ell \ell!} (2\ell)! \\
&= \frac{(-1)^\ell (1-x^2)^{\ell/2}}{2^{\ell+1}} \\
&\boxed{Y_\ell^\ell = \frac{(-1)^\ell (1-\cos^2\theta)^{\ell/2}}{2^{\ell+1}} \sqrt{\frac{2\ell+1}{8\pi\ell!}} e^{i\ell\phi}.}
\end{aligned}$$

To check if this satisfies the angular equation (4.18), it must satisfy

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)\sin^2\theta Y.$$

Using Y_ℓ^ℓ, and with the help of WolframAlpha to calculate the derivatives, the RHS becomes

$$\begin{aligned}
\text{constants} \times \sin\theta \frac{\partial}{\partial\theta}\Big(\ell\cos\theta(\sin^2(\theta))^{\ell/2}\Big) + i^2\ell^2 Y &= \text{constants} \times \ell \sin^2\theta \left(\ell\cot^2\theta - 1\right)(\sin^2\theta)^{\ell/2} - \ell^2 Y \\
&= \left[\ell\sin^2\theta(\cot^2\theta - 1) - \ell^2\right] Y \\
&= -\ell(\ell+1)\sin^2\theta Y.
\end{aligned}$$

This matches the LHS.

The other spherical harmonic is given by Table 4.3,

$$Y_3^2(\theta,\phi) = \sqrt{\frac{105}{32\pi}} \sin^2\theta\cos\theta e^{2i\phi}.$$

The RHS of the angular equation (4.18) becomes,

$$\begin{aligned}
\left(\sqrt{\frac{105}{32\pi}} e^{2i\phi}\right) \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\sin^2\theta\cos\theta\right) + 4i^2 Y &= \left(\sqrt{\frac{105}{32\pi}} e^{2i\phi}\right) \left[4\cos^2(\theta) - 2\sin^2(\theta)\right] \left(\sin^2\theta\cos\theta\right) - 4Y \\
&= \left[4\cos^2(\theta) - 2\sin^2(\theta) - 4\right] Y \\
&= -12\sin^2\theta Y.
\end{aligned}$$

This correctly matches the LHS of (4.18).

3. The raising operator is given by (4.130),

$$L_+ = \hbar e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right)$$

Applying this to Y_2^1,

$$\begin{aligned} L_+ Y_2^1 &= \hbar e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right) \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \right) \\ &= -\hbar e^{i\phi} \sqrt{\frac{15}{8\pi}} \left(\cos(2\theta) e^{i\phi} + i \cot\theta \sin\theta \cos\theta i e^{i\phi} \right) \\ &= \hbar e^{2i\phi} \sqrt{\frac{15}{8\pi}} \sin^2(\theta) \end{aligned}$$

The normalization is given by the A_ℓ^m coefficient from (4.121),

$$A_2^1 = \hbar \sqrt{(2+1+1)} = 2\hbar$$

$$\boxed{Y_2^2 = \hbar^2 e^{2i\phi} \sqrt{\frac{30}{4\pi}} \sin^2(\theta).}$$

4. Study Chapter 4.1.