Homework 7

MATH 301 Kevin Evans
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1. **Proposition.** *Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.*

   (a) *Proof.* Suppose for the sake of contradiction, there is a $n \in \mathbb{Z}$ where n^2 is odd and n is even. As n^2 is odd, then
   $$n^2 = 1 \pmod 2$$
   However, if n is even, then we can write this as $2m$ for some $m \in \mathbb{Z}$. Squaring this expression of n,
   $$\begin{aligned}(2m)^2 &= 2\left(2m^2\right)\\ &= 0 \pmod 2\end{aligned}$$
   This is a contradiction, as n^2 is both even and odd. ■

   (b) *Proof.* For the contrapositive, we will show if n is even, then n^2 is even. Suppose n is even, then it is expressed as twice an integer m,
   $$\begin{aligned}n &= 2m\\ n^2 &= 2(\underbrace{2m^2}_{\in\mathbb{Z}})\end{aligned}$$
   Squaring this, it is still twice another integer. Therefore n^2 is also even and we have shown the contrapositive to be true. ■

   (c) *Proof.* Suppose n^2 is odd, where $n \in \mathbb{Z}$. As it is odd, it can be expressed as
   $$\begin{aligned}n^2 &= 2m + 1\\ n^2 - 1 &= 2m\\ (n-1)(n+1) &= 2m\end{aligned}$$
   On the LHS, both groups must have the same parity and each group with $n \pm 1$ must have even parity. Therefore, n must be odd as $n - 1$ and $n + 1$, and their product will be even. ■

2. **Proposition.** *If $x, y \in \mathbb{Z}$, then $x^2 - 4y - 2 \neq 0$.*

   Proof. For the sake of contradiction, suppose $x, y \in \mathbb{Z}$ and $x^2 - 4y - 2 = 0$. Then solving for x^2,

   $$x^2 = 4y - 2 = 2(y+1)$$

   As x^2 is even, it follows that x is also even. We can express $x = 2z$ for some integer z. Using this expression for x in the original equation,

   $$\begin{aligned} 0 &= x^2 - 4y - 2 \\ &= 4z^2 - 4y - 2 \\ 2 &= 4z^2 - 4y \\ 1 &= 2\left(2z^2 - 2\right) \end{aligned}$$

   As 1 cannot be twice another integer, the original assumptions must be incorrect and the proposition must be true. ■

3. (a) **Proposition.** *There exists a prime number p such that $p + 4$ and $p + 6$ are also prime numbers.*

   Proof. There exists a prime number $p = 7$ where both $p+4 = 11$ and $p+6 = 13$ are primes. ■

   (b) **Proposition.** *There exists prime numbers p and q such that $p + q = 128$.*

   Proof. There exists prime numbers $p = 109$ and $q = 19$ such that $p + q = 128$. ■

4. **Proposition.** *For integers a and $b > 0$, there exists unique integers q and r such that $a = bq + r$, where $0 \leq r < b$.*

   Proof. Suppose there exists two sets of integers q_i and r_i such that

   $$\begin{aligned} a &= bq_1 + r_1 \\ a &= bq_2 + r_2 \end{aligned}$$

   and $0 \leq r_i < b$. Then, if we subtract the two expressions,

   $$\begin{aligned} 0 &= b(q_1 - q_2) + (r_1 - r_2) \\ \frac{r_1 - r_2}{b} &= q_1 - q_2 \end{aligned}$$

   Because of the original conditions on r_i, it must hold true $|r_1 - r_2|/b < 1$. As $q_1 - q_2$ must be an integer as well, it can only be 0. Therefore $q_1 = q_2$ (and $r_1 = r_2$ as well). ■

5. **Proposition.** *Suppose $a, b, p \in \mathbb{Z}$ and p is a prime number. If $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Proof. Suppose $a, b, p \in \mathbb{Z}$ and p is a prime number and $p \mid ab$.
Case 1: $p \mid a$.
Case 2: $p \nmid a$. Then $\gcd(p, a) = 1$ and by Bezout's theorem,

$$1 = n_1 p + n_2 a \qquad n_1, n_2 \in \mathbb{Z}$$

Multiplying by b and since $p \mid ba$, we can replace $n_2 ba$ as some integer n_3 multiple of p,

$$\begin{aligned} b &= n_1 bp + n_2 ba = n_1 bp + n_3 p \\ &= (n_1 b + n_3)\, p \end{aligned}$$

Therefore, $p \mid b$. ■