Homework 10

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1. (a) Not allowed, as $\Delta\ell = 2$.
   (b) Not allowed, as $\Delta j = 2$.
   (c) Allowed, as $\Delta\ell = 1$.
   (d) Allowed.
   (e) Not allowed, as $\Delta j = 2$.
2. (a) For a dampened classical oscillator,

$$0\ddot{x} + \gamma\dot{x} + \omega_0^2 x$$

The solution can be approximated as

$$x(t) \approx x_0 e^{-(\gamma/2)t} \cos\omega_0 t$$

Taking the Fourier transform, we can find the amplitude in the frequency domain as

$$\begin{aligned} A(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x_0 e{-}(\gamma/2)t \cos\omega_0 t e^{-j\omega t} \,\mathrm{d}t \\ &= \frac{x_0}{\sqrt{8\pi}} \left[\frac{1}{j(\omega_0 - \omega) + \gamma/2} + \frac{1}{j(\omega_0 + \omega) + \gamma/2} \right] \end{aligned}$$

   (b)

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1. (a) For the 3 $^2\mathrm{P}_{3/2}$ state, the energy is 3.4×10^{-19} J. The total emitted energy is found by integrating power with respect to time,

$$\begin{aligned} P &= P_0 \int_0^\infty e^{-t/\tau} \,\mathrm{d}t = P_0 \tau \\ P_0 &= (10^8 \times 3.4 \times 10^{19}\,\mathrm{J}) \times 1.6 \times 10^{-8}\,\mathrm{s} \\ &= 2.1 \times 10^{-3}\,\mathrm{W} \end{aligned}$$

   (b) From the problem, the angular distribution is

$$I(\theta) = I_0 \sin^2(\theta)$$

Integrating this,

$$\begin{aligned} W &= 2\pi W_0 \int \sin^2(\theta) \,\mathrm{d}\theta \\ &= \pi^2 W_0 \\ W_0 &= W_{\text{total}}/\pi^2 \end{aligned}$$

2. (a) For hydrogen, the molar mass is $1\,\mathrm{g/mol}$. The Doppler width is then

$$\begin{aligned}\delta\nu_D &= 7.16\times10^{-7}\left(2.47\times10^{15}\,\mathrm{Hz}\right)(300/1)^{1/2}\\ &= 30.6\,\mathrm{GHz}\end{aligned}$$

(b) The collimation ratio (this is Fraunhofer diffraction, right?) is

$$\epsilon = \frac{b}{2d} = 1/200$$

From (a), the reduced Doppler width is then

$$\delta_{\mathrm{D-beam}} = \delta_D \sin\epsilon = 150\,\mathrm{MHz}$$

(c) For $\tau(2p) = 1.2\,\mathrm{ns}$, the natural linewidth is

$$\delta\nu_N = \frac{1}{2\pi\tau} = 132\,\mathrm{MHz}$$

which is on the same order of magnitude as (b).

(d) Yes, as the hyperfine splitting is just the 21 cm line, $1400\,\mathrm{MHz}$.

11. (a) For the 21 cm line, the Einstein coefficient $A_{ik} = 2.9\times10^{-15}\,\mathrm{s}^{-1}$, the natural line width is given by

$$\delta\nu_v = \frac{A_{ik}}{2\pi} = 5\times10^{-16}\,\mathrm{s}^{-1}$$

The Doppler width is given by (7.72b),

$$\begin{aligned}\delta\nu_D &= 7.16\times10^{-7}\times\nu_0\sqrt{T/M}\\ &= 7.16\times10^{-7}\times\frac{c}{21\,\mathrm{cm}}\sqrt{10/1}\\ &= 3234\,\mathrm{s}^{-1}\end{aligned}$$

The collision broadening is given by

$$\begin{aligned}\delta\nu_{\mathrm{coll}} &= \frac{n\sigma}{2\pi}\sqrt{\frac{8kT}{\pi m}}\\ &= 7.3\times10^{-20}\,\mathrm{s}^{-1}\end{aligned}$$

As for the Lyman α line, $A_{ik} = 10\times10^{9}\,\mathrm{s}^{-1}$ and the natural line width is

$$\begin{aligned}\delta\nu_n &= 1.6\times10^{8}\,\mathrm{s}^{-1}\\ \delta\nu_D &= 5.6\times10^{9}\,\mathrm{s}^{-1}\\ \delta\nu_{\mathrm{coll}} &= 7.3\times10^{-13}\,\mathrm{s}^{-1}\end{aligned}$$

(b) I don't understand this problem at all. At $10\,\mathrm{K}$, the ratio of the populations is

$$\frac{N(F=1)}{N(F=0)} = 3^{-h\nu/kT} \approx 3\cdot 0.994$$

$$\begin{aligned}\Delta N &= N(F=0) - \frac{1}{3}N(F=1)\\ &= 0.006N(F=0)\end{aligned}$$

The absorption coefficient is neglible as

$$\begin{aligned}\alpha &= \Delta N \cdot \sigma_{\text{abs}} \\ &= 5.4 \times 10^{-6}\end{aligned}$$

For the Lyman α-line,

$$\alpha = \sigma N = 1 \times 10^{-15} 10 \times 10^{6} = 10 \times 10^{-9}$$

(c) The natural linewidth and Doppler linewidths can be calculated as

$$\begin{aligned}\delta\nu_N &= \frac{1}{2\pi\tau} = \frac{1}{2\pi \times 20\,\mathrm{ms}} \approx 8\,\mathrm{Hz} \\ \delta\nu_D &= 7.16 \times 10^{-7} \frac{c}{\lambda} \sqrt{T/M} \\ &= 274\,\mathrm{MHz}\end{aligned}$$

The pressure broadening can be calculated with mean velocity

$$\begin{aligned}\bar{v} &= \sqrt{\frac{8kT}{\pi m}} = 630\,\mathrm{m} \cdot \mathrm{s}^{-1} \\ \delta\nu_{\text{trans}} &= \frac{1}{2\pi(0.01/630)} \approx 100\,\mathrm{kHz}\end{aligned}$$

12. The matrix element (for $Z = 1$) is

$$\begin{aligned}M_{ik} &= e \int \psi(2s) \mathbf{r} \psi(1s)\,\mathrm{d}\tau \\ &= \frac{1}{4\pi\sqrt{2}a_0^3} \int (2 - r/a_0)\, e^{-r/2a_0} \mathbf{r} e^{-r/a_0} \mathrm{d}\tau \\ &= (\dots) \int_0^{2\pi} \cos\phi\,\mathrm{d}\phi = 0 \quad \square\end{aligned}$$

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1. (a) As the statistical weights $g_i = 2J_i + 1$, then the relative population ratio is

$$\frac{N_i}{N_k} = 3e^{-h\nu/kT} = 6.6 \times 10^{-42}$$

(b) The relative absorption of the incident wave is

$$\begin{aligned}A &= \frac{I_0 - I_t}{I_0} \\ &= \frac{I_0 - I_0 e^{-\alpha L}}{I_0} \\ &\approx \alpha L = N_k \sigma_{ki} L\end{aligned}$$

(c)