Homework 5

PHYSICS 342 | Kevin Evans
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1. (a) By the product rule,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \underbrace{\exp(\ldots)\nabla\cdot\mathbf{E}_0}_{0} + \mathbf{E}_0\cdot(\nabla\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)])\\ &= \mathbf{E}_0\cdot(i\mathbf{k}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)])\\ &= i\mathbf{k}\cdot\mathbf{E}\end{aligned}$$

(b) From the other product rule,

$$\begin{aligned}\nabla\times\mathbf{E} &= \exp(\ldots)(\nabla\times\mathbf{E}_0) - \mathbf{E}_0\times(\nabla\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)])\\ &= -\mathbf{E}_0\times(i\mathbf{k}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)])\\ &= i\mathbf{k}\times\mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)] = i\mathbf{k}\times\mathbf{E}\end{aligned}$$

2. (a) From the discussion in-class, the only change would be the condition where

$$T\left(\left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-}\right) = m\frac{\partial^2 f}{\partial t^2}$$

(b) Using the waveforms of (9.25),

$$\tilde{f}_z(t) = \begin{cases}\tilde{A}_I e^{i(k_1 z-\omega t)} + \tilde{A}_R e^{i(-k_1 z-\omega t)} & z<0\\ \tilde{A}_T e^{i(k_2 z-\omega t)} & z>0\end{cases}$$

Then applying the boundary condition from (a) at $z=0^+$ and $z=0^-$,

$$iT\left(k_2\tilde{A}_T - \tilde{A}_I k_1 + \tilde{A}_R k_1\right) = -m\left(\tilde{A}_T\omega^2\right) = -m\omega^2\left(\tilde{A}_I+\tilde{A}_R\right)$$

And as the wave is continuous, as in (9.26),

$$\begin{aligned}\tilde{A}_I + \tilde{A}_R &= \tilde{A}_T\\ \implies iT\left[k_2 A_T - A_I k_1 + k_1(A_T - A_I)\right] &= -mA_T\omega^2\\ \tilde{A}_T\left(i(k_1+k_2)T + m\omega^2\right) &= 2ik_1 T\tilde{A}_I\\ \tilde{A}_T &= \frac{2ik_1T}{i(k_1+k_2)T+m\omega^2}\tilde{A}_I\end{aligned}$$

For the reflected amplitude, we can just apply the continuous boundary condition,

$$\tilde{A}_R = \left(\frac{2ik_1T}{i(k_1+k_2)T+m\omega^2} - 1\right)\tilde{A}_I$$

Using WolframAlpha to simplify, for the magnitude and phase,

$$
\begin{aligned}
A_T &= \sqrt{A_T^* A_T} = \frac{2k_1 T}{\sqrt{(k_1+k_2)T + m\omega^2}} A_I \\
A_I &= \sqrt{\frac{(2k_1 T)^2}{(k_1+k_2)T + m\omega^2} + 1} \\
\delta_T &= \arctan\Big(\operatorname{Im}\Big\{\tilde{A}_T\Big\} / \operatorname{Re}\Big\{\tilde{A}_T\Big\}\Big) \\
&= \arctan\left(\frac{m\omega^2}{(k_1+k_2)T}\right) \\
\delta_I &= \arctan\left(\frac{2k_1 T m\omega^2}{T^2(k_1-k_2)^2 - (m\omega^2)^2}\right)
\end{aligned}
$$

3. If the components are unequal in magnitude and phase, then the wave can be described like

$$
\begin{aligned}
\tilde{A} &= \left(\tilde{A}_v\,\hat{\mathbf{x}} + \tilde{A}_h\,\hat{\mathbf{y}}\right) e^{i(kz-\omega t)} \\
&= \left(\frac{1}{2}A_h\,\hat{\mathbf{x}} + A_h e^{i\pi/2}\,\hat{\mathbf{y}}\right) e^{i(kz-\omega t)}
\end{aligned}
$$

Taking the real part of the wave, it reduces to

$$
A(z,t) = \left[\left(\frac{A_h}{2}\right)^2 \cos^2(kz-\omega t)\,\hat{\mathbf{x}} + {A_h}^2 \sin^2(kz-\omega t)\,\hat{\mathbf{y}}\right]^{1/2}
$$

...which is the equation for an ellipse?

4. Equating the radiation force to the gravitational force,

$$
\begin{aligned}
F_{\text{rad}} &= P_{\text{rad}} A = m g_{\text{Mars}} \\
A &= \frac{m c g_{\text{Mars}}}{2I} \\
&= \frac{\left(1\,\text{kg} \times 3.7\,\text{m}\cdot\text{s}^{-2}\right) \times 3 \times 10^8\,\text{m}\cdot\text{s}^{-1}}{2 \times 590\,\text{W}\cdot\text{m}^{-2}} \\
&= 940\,678\,\text{m}^2
\end{aligned}
$$

5. The electric and magnetic fields for circularly polarized light are

$$
\begin{aligned}
E_x &= E_0 \cos(kz-\omega t) \\
E_y &= E_0 \sin(kz-\omega t) \\
\implies E^2 &= {E_0}^2 \\
B_x &= \frac{E_0}{c} \sin(kz-\omega t) \\
B_y &= \frac{E_0}{c} \cos(kz-\omega t) \\
\implies B^2 &= \frac{{E_0}^2}{c^2}
\end{aligned}
$$

The elements of the Maxwell stress tensor are

$$
\begin{aligned}
T_{xx} &= \epsilon_0 {E_0}^2 \left(\cos^2(\theta) - \frac{1}{2}\right) + \frac{{E_0}^2}{\mu_0 c^2}\left(\sin^2(\theta) - \frac{1}{2}\right) = 0 \\
T_{yy} &= \epsilon_0 {E_0}^2 \left(\sin^2(\theta) - \frac{1}{2}\right) + \frac{{E_0}^2}{\mu_0 c^2}\left(\cos^2(\theta) - \frac{1}{2}\right) = 0 \\
T_{zz} &= -\frac{\epsilon_0 {E_0}^2}{2} - \frac{{E_0}^2}{2\mu_0 c^2} = -\epsilon_0 {E_0}^2 \\
T_{xy} &= T_{yx} = \epsilon_0 {E_0}^2 \cos\theta \sin\theta + \frac{{E_0}^2}{\mu_0 c^2} \sin\theta \cos\theta \\
&= 2\epsilon_0 {E_0}^2 \sin\theta \cos\theta? \\
T_{xz} &= T_{zx} = T_{yz} = T_{zy} = 0
\end{aligned}
$$

where $\theta = kz - \omega t$. The T_{xy} and T_{yx} should've been zero, probably, since the wave is traveling in the z direction. Assuming those components were actually zero, then it's equal to the energy density, $u = \epsilon_0 E^2$.