Homework 1

PHYSICS 461
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1. (a) If they're both real, there's no phase shift and it's linearly polarized.
   (b) As long as they're equal, it's still linearly polarized.
   (c) There needs to be a phase shift of some multiple of $\pi/2$ to be circularly polarized.

2. (a) From the basis

$$\hat{\mathbf{e}}_\pm = \frac{1}{\sqrt{2}}\,(\,\hat{\mathbf{e}}_x \pm i\,\hat{\mathbf{e}}_y)$$

Identity. $\hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_\pm = 1$

Proof. For $\hat{\mathbf{e}}_+$ and ignoring the normalization constant, $\hat{\mathbf{e}}_+^* = \hat{\mathbf{e}}_x - i\,\hat{\mathbf{e}}_y$. Multiplying this out,

$$(\,\hat{\mathbf{e}}_x - i\,\hat{\mathbf{e}}_y)\,(\,\hat{\mathbf{e}}_x + i\,\hat{\mathbf{e}}_y) = \,\hat{\mathbf{e}}_x^2 - i^2\,\hat{\mathbf{e}}_y^2 = 2$$

For $\hat{\mathbf{e}}_-$, it's the same thing but the order of the products are flipped. Since multiplication is communicative, it must be equal. ■

Identity. $\hat{\mathbf{e}}_\pm^* \cdot \hat{\mathbf{e}}_\mp = 0$

Proof. Conjurgating $\hat{\mathbf{e}}_-$, it results in $\hat{\mathbf{e}}_+$, so we're left with

$$(\,\hat{\mathbf{e}}_x + i\,\hat{\mathbf{e}}_y)\,(\,\hat{\mathbf{e}}_x + i\,\hat{\mathbf{e}}_y) = \,\hat{\mathbf{e}}_x^2 + i^2\,\hat{\mathbf{e}}_y^2 = 0$$

And it's the same thing if we had conjurgated $\hat{\mathbf{e}}_+$ instead, as the product of the two negatives would be a positive. ■

Identity. $\hat{\mathbf{e}}_\pm^* \times \hat{\mathbf{e}}_\pm = \pm i\,\hat{\mathbf{z}}$

Proof. For $\hat{\mathbf{e}}_+$ and ignoring the normalization constant again,

$$\begin{aligned}
(\,\hat{\mathbf{e}}_x - i\,\hat{\mathbf{e}}_y) \times (\,\hat{\mathbf{e}}_x + i\,\hat{\mathbf{e}}_y) &= (\,\hat{\mathbf{e}}_x \times i\,\hat{\mathbf{e}}_y) + (-\,\hat{\mathbf{e}}_y \times \,\hat{\mathbf{e}}_x) \\
&= 2\,(\,\hat{\mathbf{e}}_x \times i\,\hat{\mathbf{e}}_y) \\
&= i\,\hat{\mathbf{z}} && \text{Renormalizing}
\end{aligned}$$

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Identity. $i\,\hat{\mathbf{z}} \times \hat{\mathbf{e}}_\pm = \pm\,\hat{\mathbf{e}}_\pm$

Proof. For either $\hat{\mathbf{e}}_\pm$ vector,

$$\begin{aligned}
i\,\hat{\mathbf{z}} \times (\,\hat{\mathbf{e}}_x \pm i\,\hat{\mathbf{e}}_y) &= (i\,\hat{\mathbf{z}} \times \,\hat{\mathbf{e}}_x) \pm (i\,\hat{\mathbf{z}} \times i\,\hat{\mathbf{e}}_y) \\
&= (i\,\hat{\mathbf{e}}_y) \mp (-\,\hat{\mathbf{e}}_x) \\
&= \pm\,(\,\hat{\mathbf{e}}_x \pm i\,\hat{\mathbf{e}}_y) = \pm\,\hat{\mathbf{e}}_\pm
\end{aligned}$$

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(b) Since $E_\pm = \mathbf{E}^* \cdot \hat{\mathbf{e}}_\pm$ (maybe?),

$$\begin{aligned}
E_+ &= \overbrace{(E_x\,\hat{\mathbf{e}}_x - E_y\,\hat{\mathbf{e}}_y)}^{\mathbf{E}^*} \cdot \hat{\mathbf{e}}_+ \\
&= E_x\,(\hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_+) - E_y\,(\hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_+) \\
&= \frac{1}{\sqrt{2}}\,(E_x + iE_y) \\
E_- &= E_x\,(\hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}}_-) - E_y\,(\hat{\mathbf{e}}_y \cdot \hat{\mathbf{e}}_-) \\
&= \frac{1}{\sqrt{2}}\,(E_x - iE_y)
\end{aligned}$$

3. The product of the two real functions gives

$$\begin{aligned}
A(t)B(t) &= \mathrm{Re}\big\{Ae^{-i\omega t}\big\}\,\mathrm{Re}\big\{Be^{-i\omega t}\big\} \\
&= \frac{1}{4}\left(Ae^{-i\omega t} + A^* e^{i\omega t}\right)\left(Be^{-i\omega t} + B^* e^{i\omega t}\right) \\
&= \frac{1}{4}\left[ABe^{-2i\omega t} + A^*B^*e^{2i\omega t} + AB^* + A^*B\right]
\end{aligned}$$

Taking the time average, the oscillatory component goes to zero, leaving a real-valued thing

$$\begin{aligned}
\langle A(t)B(t)\rangle &= \frac{1}{4}\,(AB^* + A^*B) \\
&= \frac{1}{2}\,\mathrm{Re}\{A^*B\}
\end{aligned}$$

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