Homework 9

PHYSICS 341 Kevin Evans
November 4, 2020 ID: 11571810

---

1. (a) The bound surface charge density is $\mp k$ at a and b. The bound volume charge is taken from the divergence,

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = -k/s$$

Within the dielectric, the total bound charge at radius s is

$$\begin{aligned} Q_b &= -k(2\pi a l) + \int_V \rho_b \,\mathrm{d}\tau \\ &= -2\pi k a l + 2\pi l \int_a^s -k \,\mathrm{d}s \\ &= -2\pi k l \,(a + s - a) \\ &= -2\pi k l s \end{aligned}$$

From Gauss's law,

$$\begin{aligned} E(2\pi s l) &= -2\pi l k s/\epsilon_0 \\ \mathbf{E} &= -\frac{k}{\epsilon_0}\,\hat{\mathbf{s}} \end{aligned}$$

(b) Since there is no free charge, $\mathbf{D} = 0$. Then,

$$\begin{aligned} \mathbf{E} &= -\frac{1}{\epsilon_0}\mathbf{P} \\ &= -\frac{k}{\epsilon_0}\,\hat{\mathbf{s}} \end{aligned}$$

2. (a) In each dielectric, the displacement field points downward and the only free charge is from the plates,

$$\mathbf{D} = -\sigma\,\hat{\mathbf{z}}$$

(b) Since it's a linear dielectric,

$$\begin{aligned} \mathbf{E}_1 &= \frac{1}{\epsilon}\mathbf{D} = -\frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{z}} \\ \mathbf{E}_2 &= -\frac{\sigma}{3\epsilon_0}\,\hat{\mathbf{z}} \end{aligned}$$

(c) From $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$,

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{D} - \epsilon_0\mathbf{E}_1 \\ &= \left(-\sigma + \frac{\sigma}{2}\right)\hat{\mathbf{z}} \\ &= -\frac{\sigma}{2}\,\hat{\mathbf{z}} \\ \mathbf{P}_2 &= (-\sigma + \sigma/3)\,\hat{\mathbf{z}} \\ &= -\frac{2\sigma}{3}\,\hat{\mathbf{z}} \end{aligned}$$

(d) The potential difference can be found by integrating the electric fields along a path between the two plates,

$$\begin{aligned} V &= \int_{0\to a} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \\ &= \int_0^{3a/4} (-\sigma/3\epsilon_0)\,\mathrm{d}z + \int_{3a/4}^{a} (-\sigma/2\epsilon_0)\,\mathrm{d}z \\ &= \frac{-\sigma}{\epsilon_0}\left[\frac{1}{3}(3a/4) + \frac{1}{2}(a - 3a/4)\right] \\ &= -\frac{3\sigma a}{8\epsilon_0} \end{aligned}$$

(e) The bound charges can be found from the polarization. In each dielectric, the upper bound charges will be negative, and the lower will have positive,

$$\begin{aligned} \sigma_b &\equiv \mathbf{P} \cdot \hat{\mathbf{n}} \\ \sigma_{b1} &= \pm\frac{\sigma}{2} \\ \sigma_{b2} &= \pm\frac{2\sigma}{3} \end{aligned}$$

For the volume charges, they are all zero as the polarization is uniform throughout the dielectrics

$$\rho_b \equiv -\boldsymbol{\nabla} \cdot \mathbf{P} = 0$$

(f) On each surface, the total surface charge density is $\sigma_{\text{tot}} = \sigma_b + \sigma_f$. From Gauss's law, for the dielectric layers, the electric fields are

$$\begin{aligned} \int \mathbf{E}_1 \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_0}\int \sigma_{\text{tot}}\,\mathrm{d}a \\ \mathbf{E}_1 &= \frac{1}{\epsilon_0}\left[\sigma - \frac{\sigma}{2}\right](-\hat{\mathbf{z}}) \\ &= -\frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{z}} \\ \mathbf{E}_2 &= \frac{1}{\epsilon_0}\left[-\sigma + \frac{2\sigma}{3}\right](-\hat{\mathbf{z}}) = \frac{\sigma}{3\epsilon_0}\,\hat{\mathbf{z}}? \end{aligned}$$

3. From Gauss's law, within the sphere

$$\begin{aligned} D(4\pi r^2) &= \int_0^r \rho_f\,\mathrm{d}\tau \\ &= 4\pi k \int_0^r r^3\,\mathrm{d}r \\ &= \pi k r^4 \\ \mathbf{D} &= \frac{kr^2}{4}\,\hat{\mathbf{r}} \end{aligned}$$

The electric field then follows

$$\mathbf{E} = \begin{cases} \frac{kr^2}{4\epsilon_r\epsilon_0}\,\hat{\mathbf{r}} & r < R \\ \frac{kR^4}{4\epsilon_0 r^2}\,\hat{\mathbf{r}} & r > R \end{cases}$$

And the potential at the center from infinity,

$$\begin{aligned}
V &= -\int_\infty^0 \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\left[\frac{kR^4}{4\epsilon_0}\int_\infty^R r^{-2}\,\mathrm{d}r + \frac{k}{4\epsilon_r\epsilon_0}\int_R^0 r^2\,\mathrm{d}r\right] \\
&= -\left[-\frac{kR^4}{4\epsilon_0}\left(R^{-1}-0\right) + \frac{k}{12\epsilon_r\epsilon_0}\left(0-R^3\right)\right] \\
&= \frac{kR^3}{4\epsilon_0}\left(1+\frac{1}{3\epsilon_r}\right)
\end{aligned}$$

4. From the in-class discussion, if we let $\mathbf{E} = E_0\,\hat{\mathbf{z}}$ and apply the boundary conditions:

$$\begin{aligned}
V(r\le a) &= 0 \\
V_{\text{in}}(r=b) &= V_{\text{out}}(r=b) \\
V_{\text{out}}(r\to\infty) &= -E_0 r\cos\theta \\
\epsilon_0\frac{\partial V_{\text{out}}}{\partial r} - \epsilon\frac{\partial V_{\text{in}}}{\partial r} &= 0
\end{aligned}$$

In the dielectric layer, we can assume the general form of the potential of

$$V_{\text{in}}(r,\theta) = \sum_l \left[A_{\text{in,l}} r^l + \frac{B_{\text{in,l}}}{r^{l+1}}\right] P_l(\cos\theta)$$

At $r=a$, this potential must be zero (from the first BC) and it's found that

$$B_{\text{in,l}} = A_{\text{in,l}} a^{2l+1}$$

Outside of the sphere, we can assume the potential to have the form

$$V_{\text{out}}(r\to\infty,\theta) = -E_0 r\cos\theta = (-E_0) r P_1(\cos\theta)$$

Only the $l=1$ term remains for the A_l values (but the B_l values can still be non-zero?)

$$V_{\text{out}}(r) = -E_0 r\cos\theta + \sum_l \frac{B_{\text{out,l}}}{r^{l+1}} P_l(\cos\theta)$$

For the BC at $r=b$, the potentials are continuous and we can find the relation between B_{out} and A_{in}

$$\begin{aligned}
V_{\text{in}}(b,\theta) &= V_{\text{out}}(b,\theta) \\
\sum_l \left[A_{\text{in}} b^l + \frac{A_{\text{in}} a^{2l+1}}{b^{l+1}}\right] P_l(\cos\theta) &= -E_0 b\cos\theta + \sum_l \frac{B_{\text{out}}}{b^{l+1}} P_l(\cos\theta) \\
A_{\text{in}} b + \frac{A_{\text{in}} a^3}{b^2} &= -E_0 b\cos\theta + \frac{B_{\text{out}}}{b^2} \\
B_{\text{out}} &= A_{\text{in}}\left(b^3+a^3\right) + E_0 b^3
\end{aligned}$$

From the last BC, we can expect the normal derivatives of the potential to be discontinuous by the free charge (evaluated at $r = b$). But since there's no free charge, we can equate

$$\begin{aligned}
\epsilon_0 \frac{\partial V_{\text{out}}}{\partial r} &= \epsilon \frac{\partial V_{\text{in}}}{\partial r} \\
\epsilon_0 \cos\theta \left[-E_0 - 2B_{\text{out}} r^{-3}\right]_b &= \epsilon \cos\theta \left[A_{\text{in}} - 2A_{\text{in}} a^3 r^{-3}\right] \\
\epsilon_0 \left[-E_0 - 2\left(A_{\text{in}}\left(b^3 + a^3\right) + E_0 b^3\right) r^{-3}\right]_b &= \epsilon \left[A_{\text{in}} - 2A_{\text{in}} a^3 r^{-3}\right]_b \\
A_{\text{in}} \left[\epsilon - 2\epsilon a^3 b^{-3} + 2\epsilon_0 (b^3 + a^3) b^{-3}\right] &= -3\epsilon_0 E_0 \\
A_{\text{in}} \epsilon_0 \left[\epsilon_r + \frac{2a^3}{b^3}(1 - \epsilon_r)\right] &= -3\epsilon_0 E_0 \\
A_{\text{in}} &= -\frac{3E_0 b^3}{\epsilon_r b^3 + 2a^3(1 - \epsilon_r)}
\end{aligned}$$

Putting this all together, the potential (with only the $l = 1$ term) and electric field inside will be

$$\begin{aligned}
V_{\text{in}} &= -\frac{3E_0 b^3}{\epsilon_r b^3 + 2a^3(1 - \epsilon_r)} \left[r + \frac{a^3}{r^2}\right] \cos\theta \\
\mathbf{E} &= -\boldsymbol{\nabla} V \\
&= \frac{3E_0 b^3}{\epsilon_r b^3 + 2a^3(1 - \epsilon_r)} \left[\left(1 - 2\frac{a^3}{r^3}\right) \cos\theta \,\hat{\mathbf{r}} - \left(1 + \frac{a^3}{r^3}\right) \sin\theta \,\hat{\boldsymbol{\theta}}\right]
\end{aligned}$$

5. From the charged sphere, its electric field is only non-zero outside of a, and it's

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

Through the dielectric and outside of it, the displacement is

$$\begin{aligned}
\mathbf{D} &= \epsilon \mathbf{E} \\
&= \begin{cases} \frac{(1+\chi_e)Q}{4\pi r^2}\,\hat{\mathbf{r}} & a < r < b \\ \epsilon_0 \mathbf{E} & r > b \end{cases}
\end{aligned}$$

Integrating over all space,

$$\begin{aligned}
W &= \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \,\mathrm{d}\tau \\
&= \frac{1}{2} \left[\frac{(1 + \chi_e)Q^2}{16\pi^2\epsilon_0} (4\pi) \int_a^b r^{-4} \,\mathrm{d}r + \frac{Q^2}{16\pi^2\epsilon_0} (4\pi) \int_b^\infty r^{-4} \,\mathrm{d}r\right] \\
&= \frac{1}{2} \left(\frac{Q^2}{4\pi\epsilon_0}\right) \left(\frac{1}{3}\right) \left[(1 + \chi_e) \left(\frac{1}{a^3} - \frac{1}{b^3}\right) + \frac{1}{b^3}\right] \\
&= \frac{Q^2}{24\pi\epsilon_0} \left(\frac{1 + \chi_e}{a^3} - \frac{\chi_e}{b^3}\right)
\end{aligned}$$