Homework 8

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1. *Disproof.* This statement is false, as $\exists\, x, y \in \mathbb{Z} : |x + y| \neq |x| + |y|$.

   Counterexample: Suppose $x = -1$ and $y = 1$, then $|x + y| = 0$, but $|x| + |y| = 1$. ■

2. *Disproof.* This statement is false, as if any two of the integers is zero and the other integer is non-zero and odd, then the products will always be even.

   Counterexample: Suppose $a, c = 0$ and $b = 1$. Then

   $$ab = bc = ac = 0$$

   Therefore although the products have the same parity, the constituents do not. ■

3. **Proposition.** *Every odd integer is the sum of three odd integers.*

   Proof. Suppose there are three odd integers o_1, o_2, and o_3. These can be expressed of the form

   $$o_i = 2n_i + 1$$

   where $n_i \in \mathbb{Z}$. If we sum these three odd integers, it results another odd integer,

   $$\begin{aligned} o_1 + o_2 + o_3 &= 2n_1 + 2n_2 + 2n_3 + 3 \\ &= 2\Bigg(\underbrace{n_1 + n_2 + n_3 + 1}_{\in\mathbb{Z}}\Bigg) + 1 \end{aligned}$$

   Therefore, every odd integer can be expressed as a sum of three odd integers. ■

4. *Disproof.* This statement is false, as if there are shared elements between the sets, the LHS will count the intersection once, whereas they will be doublecounted on the RHS.

   Counterexample: Suppose $A = \{1\}$, $B = \{1\}$. Then $A \cup B = \{1\}$ and $|A \cup B| = 1$. However, $|A| + |B| = 2$. And $1 \neq 2$. ■

5. **Proposition.** *If $a, b, c \in \mathbb{Z}$, then at least one of $a - b$, $a + c$, and $b - c$ is even.*

   Proof. Suppose the proposition is *false*. Then $a - b$, $a + c$ and $b - c$ are all odd. However, if we subtract $b - c$ from $a - b$, it results in an even number

   $$\begin{aligned} (a - b) - (b - c) &= (2k_1 + 1) - (2k_2 + 1) && k_i \in \mathbb{Z} \\ &= 2\,(k_1 - k_2) \end{aligned}$$

   However, this leads to a contradiction because $(a - b) = (b - c) = a + c$, and $a + c$ cannot be both even and odd. Thus the original proposition is true. ■

6. *Disproof.* The statement is false.

   Counterexample: Let $a = 6$, $b = 3$, $c = 4$. Then $6 \mid 12$, but $6 \nmid 3$ and $6 \nmid 4$. ■

7. *Disproof.* The statement is false.

   Counterexample: Let $a = b = 0$. Then $a + b = ab = 0$, and $0 \not< 0$. ■

8. *Disproof.* The statement is false. If $x = 0$, then y can be any integer and the statement will hold true. Counterexample: Let $x = 0$, $y = 1$. Then $|x + y| = |x - y|$, but $y \neq 0$. ■