Homework 2

PHYSICS 341
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1. (a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$$\begin{aligned}\int_P \mathbf{v} \cdot \mathrm{d}\vec{\ell} &= \int_0^1 xy\,\mathrm{d}x\Big|_{y=0} + \int_0^1 yz\,\mathrm{d}y\Big|_{z=0} + \int_0^1 zx\,\mathrm{d}z\Big|_{x=1} \\ &= \frac{z^2}{2}\Big|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

$$\begin{aligned}\int_P \mathbf{v} \cdot \mathrm{d}\vec{\ell} &= \int_0^1 zx\,\mathrm{d}z\Big|_{x=0} + \int_0^1 yz\,\mathrm{d}y\Big|_{z=1} + \int_0^1 xy\,\mathrm{d}x\Big|_{y=1} \\ &= \frac{1}{2}\left(y^2 + x^2\right)_{x,y=0}^{x,y=1} \\ &= 1\end{aligned}$$

(c) If we parameterize the function by t from $0 \to 1$,

$$\begin{aligned}\mathbf{v} &= t^2\left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\right) \\ \mathrm{d}\vec{\ell} &= \mathrm{d}t\left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\right) \\ \int_P \mathbf{v} \cdot \mathrm{d}\vec{\ell} &= 3\int_0^1 t^2\,\mathrm{d}t = t^3\Big|_0^1 \\ &= 1\end{aligned}$$

2. Applying the divergence theorem, as it's a closed surface,

$$\begin{aligned}\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} &= \int_V (\boldsymbol{\nabla} \cdot \mathbf{v})\,\mathrm{d}\tau \\ &= \int_0^1 \int_0^1 \int_0^1 (x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^1 \int_0^1 \left[\frac{x^2}{2} + yx + zx\right]_0^1 \mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z\right)\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + zy\right]_0^1 \mathrm{d}z = \int_0^1 (1+z)\,\mathrm{d}z \\ &= \left[z + \frac{z^2}{2}\right]_0^1 \\ &= \frac{3}{2}\end{aligned}$$

3. For the function

$$f(r,\theta,\phi) = r\left(\cos\theta + \sin\theta\cos\phi\right)$$

The gradient and Laplacian are found as

$$\begin{aligned}
\boldsymbol{\nabla} f &= \frac{\partial f}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} \\
&= \left(\cos\theta + \sin\theta\cos\phi\right)\,\hat{\boldsymbol{r}} + \frac{1}{\cancel{r}}\cancel{r}\left(-\sin\theta + \cos\phi\cos\theta\right)\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\left(-r\sin\theta\sin\phi\right)\,\hat{\boldsymbol{\phi}} \\
&= \left(\cos\theta + \sin\theta\cos\phi\right)\,\hat{\boldsymbol{r}} + \left(\cos\phi\cos\theta - \sin\theta\right)\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\
\nabla^2 f &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\cos\theta + \sin\theta\cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\cos\phi\cos\theta - \sin\theta\right)\right] \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-\sin\phi\right) \\
&= \frac{1}{r^2}\left(2r\right)\left(\cos\theta + \sin\theta\cos\phi\right) + \frac{1}{r\sin\theta}\left[\cos^2(\theta)\cos\phi - \sin^2(\theta)\cos\phi - 2\sin\theta\cos\theta\right] \\
&\quad - \frac{1}{r\sin\theta}\cos\phi \\
&= \frac{2}{r}\left(\cos\theta + \sin\theta\cos\phi\right) + \frac{1}{r\sin\theta}\left[\cos(2\theta)\cos\phi - \sin(2\theta)\right] - \frac{1}{r\sin\theta}\cos\phi \\
&= \frac{1}{r\sin\theta}\left[\underbrace{2\sin\theta\cos\theta}_{\sin(2\theta)} + \underbrace{2\sin^2(\theta)\cos\phi + \cos(2\theta)\cos\phi}_{\cos\phi} - \sin(2\theta) - \cos\phi\right] \\
&= 0
\end{aligned}$$

Converting to Cartesian first,

$$\begin{aligned}
f(x,y,z) &= x + z \\
\nabla^2 f(x,y,z) &= 0 && \text{(all first-order)}
\end{aligned}$$

4. For $\mathbf{v} = z\cos\phi\,\hat{\boldsymbol{s}} + s\sin\phi\,\hat{\boldsymbol{\phi}} + 2s\,\hat{\boldsymbol{z}}$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left[sz\cos\phi\right] + \frac{1}{s}\frac{\partial}{\partial\phi}s\sin\phi + \frac{\partial}{\partial z}2s \\
&= \frac{z\cos\phi}{s} + \cos\phi = \cos\phi\left(1 + \frac{z}{s}\right) \\
\boldsymbol{\nabla}\times\mathbf{v} &= \left(\frac{1}{s}(0) - 0\right)\,\hat{\boldsymbol{s}} + \left(\cos\phi - 2\right)\,\hat{\boldsymbol{\phi}} + \frac{1}{s}\left(2s\sin\phi + z\sin\phi\right)\,\hat{\boldsymbol{z}} \\
&= \left(\cos\phi - 2\right)\,\hat{\boldsymbol{\phi}} + \sin\phi\left(2 + \frac{z}{s}\right)\,\hat{\boldsymbol{z}}
\end{aligned}$$

5. (a) Statement:

$$\int_V (\nabla T)\,\mathrm{d}\tau = \oint_S T\,\mathrm{d}\mathbf{a}$$

Proof.

$$\begin{aligned}
\text{Let } \mathbf{v} &= \mathbf{c}T && \text{where } \mathbf{c} \text{ is a constant vector} \\
\int \nabla\cdot[\mathbf{v}]\,\mathrm{d}\tau &= \int \nabla\cdot[T\mathbf{c}]\,\mathrm{d}\tau && (1) \\
&= \int [T(\nabla\cdot\mathbf{c}) + \mathbf{c}\cdot(\nabla T)]\,\mathrm{d}\tau && \text{Product rule} \\
&= \int \mathbf{c}\cdot(\nabla T)\,\mathrm{d}\tau && \text{As } \nabla\cdot\mathbf{c} = 0 \\
&= \mathbf{c}\cdot\int \nabla T\,\mathrm{d}\tau && \text{Moving the constant out}
\end{aligned}$$

If we apply the divergence theorem to the original equation (1),

$$\begin{aligned}
\int \nabla\cdot[T\mathbf{c}]\,\mathrm{d}\tau &= \oint (\mathbf{c}T)\cdot\mathrm{d}\mathbf{a} \\
&= \mathbf{c}\cdot\left[\oint T\,\mathrm{d}\mathbf{a}\right] && \text{Moving the constant out}
\end{aligned}$$

If we equate these two results, then from inspection:

$$\int \nabla T\,\mathrm{d}\tau = \oint T\,\mathrm{d}\mathbf{a} \quad \square$$

(b) Statement:

$$\int_V (\nabla\times\mathbf{v})\,\mathrm{d}\tau = -\oint_S \mathbf{v}\times\mathrm{d}\mathbf{a}$$

Proof.

$$\begin{aligned}
\text{Let } \mathbf{A} &= \mathbf{v}\times\mathbf{c} && \mathbf{c} \text{ is const} \\
\int_V (\nabla\times\mathbf{A})\,\mathrm{d}\tau &= \int_V \nabla\times[\mathbf{v}\times\mathbf{c}]\,\mathrm{d}\tau && (2) \\
&= \int_V \mathbf{c}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{c})\,\mathrm{d}\tau && \text{Product rule} \\
&= \int_V \mathbf{c}\cdot(\nabla\times\mathbf{v})\,\mathrm{d}\tau && \text{As } \nabla\times\mathbf{c} = 0 \\
&= \mathbf{c}\cdot\int_V (\nabla\times\mathbf{v})\,\mathrm{d}\tau
\end{aligned}$$

Applying the divergence theorem to the original intergral (2),

$$\begin{aligned}
\int_V \nabla\times[\mathbf{v}\times\mathbf{c}]\,\mathrm{d}\tau &= \oint (\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\mathbf{a} \\
&= \mathbf{c}\cdot\oint_S (\mathrm{d}\mathbf{a}\times\mathbf{v}) && \text{Triple product} \\
&= \mathbf{c}\cdot\left[-\oint_S \mathbf{v}\times\mathrm{d}\mathbf{a}\right] && \text{Swapping cross product order} \\
\int_V (\nabla\times\mathbf{v})\,\mathrm{d}\tau &= -\oint_S \mathbf{v}\times\mathrm{d}\mathbf{a} \quad \square
\end{aligned}$$

(c) Statement:

$$\int_V \left[T\nabla^2 U + (\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} U)\right] \mathrm{d}\tau = \oint_S (T\boldsymbol{\nabla} U)\cdot \mathrm{d}\mathbf{a}$$

Proof.

$$\begin{aligned}
\text{Let } \mathbf{v} &= T\boldsymbol{\nabla} U \\
\int_V \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{d}\tau &= \int_V \boldsymbol{\nabla}\cdot[T\boldsymbol{\nabla} U] \\
&= \int_V \left[T\nabla^2 U + (\boldsymbol{\nabla} U)\cdot(\boldsymbol{\nabla} T)\right]\mathrm{d}\tau \qquad \text{Product rule}
\end{aligned}$$

Applying the divergence theorem to the original statement,

$$\int_V \boldsymbol{\nabla}\cdot[T\boldsymbol{\nabla} U]\,\mathrm{d}\tau = \oint_S (T\boldsymbol{\nabla} U)\cdot\mathrm{d}\mathbf{a}$$

Then equating the two results,

$$\int_V \left[T\nabla^2 U + (\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} U)\right]\mathrm{d}\tau = \oint_S (T\boldsymbol{\nabla} U)\cdot\mathrm{d}\mathbf{a} \quad \square$$

(d) Statement:

$$\int_V \left(T\nabla^2 U - U\nabla^2 T\right)\mathrm{d}\tau = \oint_S (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{a}$$

Proof. From (c), if we now let two vectors

$$\begin{aligned}
\mathbf{v} &= T\boldsymbol{\nabla} U \\
\mathbf{w} &= U\boldsymbol{\nabla} T
\end{aligned}$$

And we take the vector differences and apply the divergence theorem, the shared dot product $(\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} U)$ will cancel out. We are then left with

$$\begin{aligned}
\int_V \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathrm{d}\tau - \int_V \boldsymbol{\nabla}\cdot\mathbf{w}\,\mathrm{d}\tau &= \int_V \left(T\nabla^2 U - U\nabla^2 T\right)\mathrm{d}\tau \\
&= \oint_S (T\boldsymbol{\nabla} U)\cdot\mathrm{d}\mathbf{a} - \oint_S (U\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{a} \\
\int_V \left(T\nabla^2 U - U\nabla^2 T\right)\mathrm{d}\tau &= \oint_S (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{a} \quad \square
\end{aligned}$$

(e) Statement:

$$\int_S \boldsymbol{\nabla} T \times \mathrm{d}\mathbf{a} = -\oint_P T\,\mathrm{d}\boldsymbol{\ell}$$

Proof.

$$\begin{aligned}
\text{Let } \mathbf{A} &= \mathbf{c}T && \mathbf{c} \text{ is a constant vector} \\
\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} &= \int_S (\boldsymbol{\nabla} \times [\mathbf{c}T]) \cdot \mathrm{d}\mathbf{a} \\
&= \int_S [T\,(\boldsymbol{\nabla} \times \mathbf{c}) - \mathbf{c} \times (\boldsymbol{\nabla} T)] \cdot \mathrm{d}\mathbf{a} && \text{Product rule} \\
&= -\int_S (\mathbf{c} \times \boldsymbol{\nabla} T) \cdot \mathrm{d}\mathbf{a} && \boldsymbol{\nabla} \times \mathbf{c} = 0 \\
&= \mathbf{c} \cdot \left[-\int_S \boldsymbol{\nabla} T \times \mathrm{d}\mathbf{a}\right] && \text{Triple product}
\end{aligned}$$

Applying Stokes' theorem to the original integral,

$$\int_S (\boldsymbol{\nabla} \times [\mathbf{c}T]) \cdot \mathrm{d}\mathbf{a} = \oint_P (\mathbf{c}T) \cdot \mathrm{d}\boldsymbol{\ell} = \mathbf{c} \cdot \left[\oint_P T\,\mathrm{d}\boldsymbol{\ell}\right]$$

Equating these results and changing sign,

$$\int_S \boldsymbol{\nabla} T \times \mathrm{d}\mathbf{a} = -\oint_P T\,\mathrm{d}\boldsymbol{\ell}$$