Homework 4

PHYSICS 450
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1. If we consider the product of the hermitian operators $\hat{x} \cdot \hat{p}$,

$$\begin{aligned}
\langle \psi | \hat{x}\hat{p}\psi \rangle &= \int \psi^* x \frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x} \psi \, \mathrm{d}x \\
&= -\int \frac{\hbar}{i} \psi \frac{\mathrm{d}}{\mathrm{d}x} x \psi^* \\
&= \int \left(\psi^* \hat{p}\hat{x}\psi \, \mathrm{d}x \right)^* \\
&= \langle \psi | \hat{p}\hat{x}\psi \rangle^*
\end{aligned}$$

This is only hermitian if $\hat{x}\hat{p} = \hat{p}x$.

For the operator $\frac{1}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)$, its expectation is

$$\frac{1}{2}\big(\langle \psi | \hat{x}\hat{p}\psi \rangle + \langle \psi | \hat{p}\hat{x}\psi \rangle \big).$$

From the first part of the problem, we can see that this is equivalent to

$$\frac{1}{2}\left(\langle \psi | \hat{p}\hat{x}\psi \rangle^* + \langle \psi | \hat{x}\hat{p}\psi \rangle^* \right) = \frac{1}{2} \left\langle \psi \middle| \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\psi \right\rangle^*$$

Therefore, this is indeed hermitian.

2. (a) Let $\hat{A}$ and $\hat{B}$ be both hermitian operators, then the sum of these are hermitian,

$$\begin{aligned}
\left\langle f \middle| (\hat{A} + \hat{B}) g \right\rangle &= \left\langle f \middle| \hat{A} g \right\rangle + \left\langle f \middle| \hat{B} g \right\rangle \\
&= \left\langle \hat{A} f \middle| g \right\rangle + \left\langle \hat{B} f \middle| g \right\rangle \\
&= \left\langle (\hat{A} + \hat{B}) f \middle| g \right\rangle \qquad \square
\end{aligned}$$

(b) If $\hat{Q}$ is hermitian and $\alpha \in \mathbb{C}$, their product is hermitian when

$$\begin{aligned}
&(\alpha\hat{Q})^\dagger = \alpha^* \hat{Q}^\dagger \\
&\implies \alpha = \alpha^* \\
&\implies \alpha \in \mathbb{R}
\end{aligned}$$

(c) For two hermitian operators $\hat{A}$ and $\hat{B}$, the expectation of the product is

$$\left\langle f \middle| \hat{A}\hat{B}g \right\rangle = \left\langle \hat{B}\hat{A}f \middle| g \right\rangle$$

For the product to be hermitian, $\hat{A}\hat{B} = \hat{B}\hat{A}$. This was also shown in Problem 1.

(d) The position operator $\hat{x}$ is hermitian as

$$\langle f|\hat{x}g\rangle = \int f^* x g \,\mathrm{d}x$$

As $x \in \mathbb{R}$, $x^* = x$,

$$\langle f|\hat{x}g\rangle = \int (fx)^* g \,\mathrm{d}x = \langle \hat{x}f|g\rangle$$

Because $\hat{x} = \hat{x}^\dagger$, $\hat{x}$ is hermitian.

The Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

Because $\hat{p}$ is hermitian, $\hat{p}^2$ is also hermitian. From linearity, we can conclude that the Hamiltonian operator is hermitian.

3. (a) The hermitian conjurgate of x is x as it's real.
For i, the hermitian conjurgate is just the complex conjurgate $-i$.
For $\frac{\mathrm{d}}{\mathrm{d}x}$, by integration by parts,

$$\begin{aligned}\left\langle f \middle| \frac{\mathrm{d}}{\mathrm{d}x} g \right\rangle &= \int f^* \frac{\mathrm{d}g}{\mathrm{d}x} \,\mathrm{d}x = 0 - \int g \frac{\mathrm{d}}{\mathrm{d}x} f^* \,\mathrm{d}x \\ &= -\left\langle \frac{\mathrm{d}f^*}{\mathrm{d}x} \middle| g \right\rangle\end{aligned}$$

The adjoint is then

$$\frac{\mathrm{d}}{\mathrm{d}x}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x}$$

(b) For the product of two operators $\hat{Q}\hat{R}$, its adjoint is

$$\begin{aligned}\left(\hat{Q}\hat{R}\right)^\dagger \implies \left\langle f \middle| Q^\dagger R^\dagger g \right\rangle &= \left\langle \hat{Q}f \middle| \hat{R}^\dagger g \right\rangle \\ &= \left\langle \hat{R}\hat{Q}f \middle| g \right\rangle\end{aligned}$$

(c) For the raising operator $\hat{a}_+$, the adjoint is

$$\hat{a}_+^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega x\right) = \hat{a}_-$$

4. (a) The expectation of an anti-hermitian operator $\hat{Q}$ is

$$\begin{aligned}\left\langle f \middle| \hat{Q} g \right\rangle &= \left\langle \hat{Q}^\dagger f \middle| g \right\rangle = \left\langle -\hat{Q} f \middle| g \right\rangle \\ &= -\left\langle f \middle| \hat{Q} g \right\rangle^*\end{aligned}$$

The condition $\left\langle \hat{Q} \right\rangle^* = -\left\langle \hat{Q} \right\rangle$ is only true for imaginary numbers.

(b) The eigenvalues of an operator are its expectation, so for the eigenvalue $\hat{Q}f = qf$, it must be true that $q^* = -q$, therefore its eigenvalue q is imaginary.

(c) If we consider two eigenstates of an anti-hermitian operator $\hat{Q}$,

$$\begin{aligned}\hat{Q}\phi_n &= a_n\phi_n \\ \hat{Q}\phi_m &= a_m\phi_m,\end{aligned}$$

and consider the inner product

$$\left\langle \phi_n \middle| \hat{Q}\phi_m \right\rangle = \langle \phi_n | a_m \phi_m \rangle = a_m \langle \phi_n | \phi_m \rangle$$

and by the anti-hermitian nature of the original inner product,

$$\left\langle \phi_n \middle| \hat{Q}\phi_m \right\rangle = \left\langle -\hat{Q}\phi_n \middle| \phi_m \right\rangle = -a_n \langle \phi_n | \phi_m \rangle \,.$$

Equating these expressions together (as they're the same thing),

$$0 = (a_m + a_n)\langle \phi_n | \phi_m \rangle$$

This is only true when ϕ_n and ϕ_m are orthogonal.

(d) For the commutator of two hermitian operators,

$$\begin{aligned}\left[\hat{A}, \hat{B}\right] &= \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) \\ \implies \left\langle f \middle| \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) g \right\rangle &= \left\langle f \middle| \hat{A}\hat{B} g \right\rangle - \left\langle f \middle| \hat{B}\hat{A} g \right\rangle \\ &= \left\langle \hat{B}^\dagger \hat{A}^\dagger f \middle| g \right\rangle - \left\langle \hat{A}^\dagger \hat{B}^\dagger f \middle| g \right\rangle \\ &= -\langle f | [A, B] g \rangle\end{aligned}$$

For anti-hermitian operators, we can follow the same general steps until:

$$\begin{aligned}\ldots &= -\left\langle \hat{B}^\dagger \hat{A}^\dagger f \middle| g \right\rangle + \left\langle \hat{A}^\dagger \hat{B}^\dagger f \middle| g \right\rangle \\ &= \langle f | [A, B] g \rangle\end{aligned}$$

The commutator of two anti-hermitian operators is hermitian.

(e) If we let $\hat{Q} = \hat{A} + \hat{B}$, then we can write

$$\hat{A} = \frac{1}{2}\left(\hat{Q} + \hat{Q}^\dagger\right)$$
$$\hat{B} = \hat{Q} - \hat{A}\frac{1}{2}\left(\hat{Q} - \hat{Q}^\dagger\right)$$