Homework 8

PHYSICS 342
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1. Not sure if it's better to do this problem in Cartesian or cylindrical coordinates, so... in Cartesian coordinates, the trajectory is a circle on the xy plane

$$\mathbf{w} = a\cos(\omega t)\,\hat{\mathbf{x}} + a\sin(\omega t)\,\hat{\mathbf{y}}$$

The Liénard-Wiechert scalar potential is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}$$

However the displacement vector $\boldsymbol{\mathscr{r}}$ is always perpendicular to $\hat{\boldsymbol{\phi}}$, so the dot product part is zero. Then we are left with

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\frac{q}{\left(z^2 + a^2(\cos^2(\omega t_r) + \sin^2(\omega t_r))\right)^{1/2}} \\ &= \frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}} \end{aligned}$$

From eq. (10.47), the vector potential is

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mathbf{v}}{c^2}V \\ &= \frac{a\omega}{c^2}\frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}}\,\hat{\boldsymbol{\phi}} \end{aligned}$$

2. Given the trajectory

$$\mathbf{w}(t) = \sqrt{b^2 + (ct)^2}\,\hat{\mathbf{x}}$$

The retarded time can be determined as

$$\begin{aligned} |\mathbf{r} - \mathbf{w}(t_r)| &= c(t - t_r) \\ \sqrt{x^2 + b^2 + (ct_r)^2} &= c(t - t_r) \end{aligned}$$

Using WolframAlpha to do algebra and solve for t_r,

$$t_r = -\frac{b^2 + (ct)^2 - x^2}{2c^2 t}$$

3. From eq. (10.72), the electric field is described by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$$

As $\boldsymbol{\mathscr{r}} \parallel \mathbf{u}$ on $\hat{\mathbf{x}}$, we can reduce this expression to

$$\begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2 u^2}\left[(c^2 - v^2)\right]\,\hat{\mathbf{x}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2(c-v)^2}\left[(c+v)(c-v)\right]\,\hat{\mathbf{x}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{(c+v)}{\mathscr{r}^2(c-v)}\,\hat{\mathbf{x}} \quad \square \end{aligned}$$

For the magnetic field, it must be zero as $\boldsymbol{\mathscr{r}} \parallel \hat{\mathbf{x}} \implies \hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} = 0$.

4. (a) We can rewrite a little bit of charge as $\mathrm{d}q = \lambda\,\mathrm{d}x$ and $\sin\theta = d/R$, then

$$\mathbf{E} = \int \frac{\lambda\,\mathrm{d}x}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2\sin^2(\theta)/c^2)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$$

As $\cos\theta = x/R \implies \mathrm{d}x = -R\sin\theta\,\mathrm{d}\theta$ and since it is symmetric in x,

$$\begin{aligned}
&= -\frac{\lambda}{4\pi\epsilon_0} \int \frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}} \frac{1}{R^2} R\sin\theta\,\mathrm{d}\theta\ \hat{\mathbf{s}} \\
&= -\frac{\lambda}{4\pi\epsilon_0} \int \frac{1 - v^2/c^2}{(1 - v^2\sin\theta/c^2)^{3/2}} \frac{\sin\theta}{d} \sin\theta\,\mathrm{d}\theta\ \hat{\mathbf{s}} \\
&= -\frac{\lambda}{4\pi\epsilon_0 d} \int_0^\pi \frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}} \sin^2(\theta)\,\mathrm{d}\theta\ \hat{\mathbf{s}} \\
&= ?
\end{aligned}$$

Can't seem to solve this easily and WolframAlpha isn't able to solve this either...

(b) The magnetic field is

$$\begin{aligned}
\mathbf{B} &= \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \\
&= -\frac{\lambda}{4\pi\epsilon_0 d}\hat{\boldsymbol{\theta}} \int_0^\pi (\dots)\,\mathrm{d}\theta
\end{aligned}$$

5. As it's moving with a constant angular velocity ω, then

$$\begin{aligned}
\mathbf{\mathscr{r}} &= a\,\hat{\mathbf{s}} \\
\mathbf{u} &= c\hat{\mathbf{\mathscr{r}}} - \mathbf{v} = c\,\hat{\mathbf{s}} - \omega a\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

From eq. (10.72), the electric field is

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\mathbf{r}\cdot\mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \mathbf{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})\right] \\
&= \frac{q}{4\pi\epsilon_0} \frac{a}{(ac)^3} \left[(c^2 - v^2)(c\,\hat{\mathbf{s}} - \omega a\,\hat{\boldsymbol{\phi}}) + a\,\hat{\mathbf{s}} \times \left(-\frac{(a\omega)^3}{a}\,\hat{\mathbf{z}}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0} \frac{a}{(ac)^3} \left[(c^2 - \omega^2 a^2)(c\,\hat{\mathbf{s}} - \omega a\,\hat{\boldsymbol{\phi}}) + (\omega a)^3\,\hat{\boldsymbol{\phi}}\right] \\
&= \frac{q}{4\pi\epsilon_0} \left[\hat{\mathbf{s}} + \frac{2a^2\omega^3 - c^2\omega}{ac^3}\,\hat{\boldsymbol{\phi}}\right]
\end{aligned}$$

The magnetic field is given by eq. (10.73),

$$\begin{aligned}
\mathbf{B}(\mathbf{r}, t) &= \frac{1}{c}\mathbf{\mathscr{r}} \times \mathbf{E} \\
&= \frac{q\omega}{4\pi\epsilon_0 c^4}\left[2a^3 - c^2\right]\,\hat{\mathbf{z}}
\end{aligned}$$

For a current I going around a loop of circumference $2\pi a$, the moving charge equivalent is $q\omega = 2\pi I$. The magnetic field can then be written as

$$\mathbf{B} = \frac{I}{2\epsilon_0 c^4}\left(2a^3 - c^2\right)\,\hat{\mathbf{z}}$$