Homework 7

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1. (a) The wavefunction for $x < 0$ can be described as a free particle with a left (incident) and right (reflected) direction,

$$\Psi_I(x) = Ae^{ikx} + Be^{-ikx}.$$

At $x > 0$, it must be a real exponential as $E < V$. It should only be a negative exponential, as a positive one would explode at infinity,

$$\begin{aligned} \Psi_{II}(x) &= Ce^{-\ell x}, \\ \text{where} \quad k &= \frac{\sqrt{2mE}}{\hbar} \\ \ell &= \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \end{aligned}$$

From the continuity at $x = 0$,

$$\begin{aligned} A + B &= C \\ ik(A - B) &= -\ell C. \end{aligned}$$

Dividing these, the reflection coefficient can be found as

$$\begin{aligned} ik(A - B) &= -\ell(A + B) \\ A(ik + \ell) &= (ik - \ell)B \\ R &\equiv \frac{|B|^2}{|A|^2} \\ &= \frac{|ik + \ell|^2}{|ik - \ell|^2} = \frac{-k^2 + \ell^2}{-k^2 + \ell^2} = 1. \end{aligned}$$

The wave is completely reflected at the barrier.

(b) For $E > V_0$, the wavefunction over the barrier can take the form of a free particle moving right,

$$\begin{aligned} \Psi_{II}(x) &= Ce^{i\ell x} \\ \text{where} \quad \ell &= \frac{\sqrt{2m(E - V_0)}}{\hbar}. \end{aligned}$$

For continuity at $x = 0$ in $\Psi(x)$ and $\Psi'(x)$,

$$\begin{aligned} A + B &= C \\ ik(A - B) &= i\ell C \end{aligned}$$

Dividing these again, we see that

$$\begin{aligned} ik(A - B) &= i\ell(A + B) \\ i(k - \ell)A &= i(k + \ell)B \\ R = \frac{|B|^2}{|A|^2} &= \frac{|k - \ell|^2}{|k + \ell|^2} < 1. \end{aligned}$$

(c) This problem says to use (2.99) as a hint, but I'm guessing it's actually meaning to use (2.98)? In any case, we know that:

$$v_I = \sqrt{E/2m}$$
$$v_{II} = \sqrt{(E - V_0)/2m}.$$

I have no idea where to go from here, but to match the form of (2.175), we're going to need to take the ratio

$$\frac{v_{II}}{v_I} = \sqrt{\frac{E - V_0}{E}}$$

Then we can just tack it onto T?

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|F|^2}{|A|^2}$$

I don't understand this problem and I hope solutions get posted...

(d) From part (a) and (b),

$$\begin{aligned} E &= \frac{\hbar^2 k^2}{2m} \\ E - V_0 &= \frac{\ell^2 \hbar^2}{2m} \\ \implies \sqrt{\frac{E - V_0}{E}} &= \frac{\ell}{k} \end{aligned}$$

For $E > V_0$ and starting from the continuity of part (b) and the equation from (c),

$$\begin{aligned} A + B &= C \\ ik(A - B) &= -i\ell C \implies A - B = -\frac{\ell}{k} C \\ \implies 2A &= \left(1 - \frac{\ell}{k}\right) C \\ T &= \sqrt{\frac{E - V_0}{E}} \frac{|C|^2}{|A|^2} \\ &= \frac{4\ell}{k|1 - \ell/k|^2} = \frac{4k\ell}{(k - \ell)^2}. \end{aligned}$$

Checking $T + R = 1$,

$$\begin{aligned} T + R &= \frac{4k\ell}{(k - \ell)^2} + \frac{(k - \ell)^2}{(k + \ell)^2} \\ &= \text{not 1?} \end{aligned}$$

There's an algebraic mistake and I think I should've used $(k + \ell)^2$ in the denominator of the first term.

$$\begin{aligned} &= \frac{4k\ell}{(k + \ell)^2} + \frac{(k - \ell)^2}{(k + \ell)^2} \\ &= \frac{4k\ell + (k - \ell)^2}{(k + \ell)^2} = 1. \quad \square \end{aligned}$$