Homework 7

PHYSICS 342
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1. (a) For the electric field,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \begin{cases} -\frac{\mu_0 k s}{2}\,\hat{\boldsymbol{\phi}} & s \le R \\ -\frac{\mu_0 k R^2}{2s}\,\hat{\boldsymbol{\phi}} & s \ge R \end{cases}$$

For the magnetic field,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \begin{cases} \mu_0 k t\,\hat{\mathbf{z}} & s \le R \\ 0 & s \ge R \end{cases}$$

(b) The charge distribution can be found using

$$-\frac{\rho}{\epsilon_0} = \cancel{\nabla^2 V} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nabla} \cdot \mathbf{A}$$

$$\rho = 0 \text{ as there's no } \phi \text{ dependence on the } \phi \text{ component}$$

From Ampere's law,

$$\begin{aligned} \mu_0 \mathbf{J} &= \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ &= 0? \end{aligned}$$

These can't both be zero, so I'm guessing there's an arithmetic mistake somewhere?

2. (a) The magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = 0$$

The electric field is

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} \\ &= \frac{q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} \end{aligned}$$

By Gauss' law,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \rho &= \frac{q}{4\pi\epsilon_0}(4\pi\delta)\epsilon_0 \\ &= q\delta \end{aligned}$$

The current can be found using Ampere's law,

$$\begin{aligned} \mu_0 \mathbf{J} &= \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ &= 0 \end{aligned}$$

(b) Using $\lambda = -(1/4\pi\epsilon_0)(qt/r)$,

$$\begin{aligned}
\mathbf{A}' &= \mathbf{A} + \nabla\lambda \\
&= -\frac{qt}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{1}{r^2}\right) = 0 \\
V' &= V - \frac{\partial\lambda}{\partial t} \\
&= \frac{q}{4\pi\epsilon_0 r}
\end{aligned}$$

3. (i) Lorenz gauge due to the t dependence.
   (ii) Could be both?
   (iii) Neither? This can't be the Coulomb gauge as $\nabla \cdot \mathbf{A} \neq 0$ and it can't be in the Lorenz gauge as it can't satisfy (10.12).

4. We can divide this problem to two parts: the semicircle and the ends. For the semicircle, the distance $\mathscr{r}$ is constant and we can just integrate along the swept angle,

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi}\int \frac{I(t_r)}{\mathscr{r}}\,\mathrm{d}\vec{\ell}' \\
&= \frac{\mu_0 I_0}{4\pi}\int_{-\pi/2}^{\pi/2} (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})\,\mathrm{d}\phi \\
&= \frac{\mu_0 I_0}{2\pi}\,\hat{\mathbf{x}}
\end{aligned}$$

For the two lines, the point only "sees" up to ct and is multiplied twice as the lines are symmetric, so the vector field is

$$\begin{aligned}
\mathbf{A} &= 2 \times \frac{\mu_0 I_0}{4\pi}\,\hat{\mathbf{x}}\int_R^{ct} x^{-1}\,\mathrm{d}x \\
&= \frac{\mu_0 I_0}{2\pi}\ln(ct/R)\,\hat{\mathbf{x}}
\end{aligned}$$

The total vector field is then

$$\mathbf{A}(t) = \begin{cases} 0 & t \leq 0 \\ \frac{\mu_0 I_0}{2\pi}\left(\ln(ct/R) + \frac{1}{R}\right)\,\hat{\mathbf{x}} & t > 0? \end{cases}$$

(I'm not sure if it's only non-zero after $t = 0$ or if it would be after $t = R/c$.)

5. Starting from (10.36), $\dot{\mathbf{J}} = 0$ and

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r'}, t)}{\mathscr{r}^2} \,\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r'}, t_r)}{c\mathscr{r}} \,\hat{\boldsymbol{\mathscr{r}}} \right] \mathrm{d}\tau' \\
&= \frac{1}{4\pi\epsilon_0} \int \left[\frac{c\rho(\mathbf{r'}, t)}{c\mathscr{r}^2} \,\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r'}, t_r)\,\mathscr{r}}{c\mathscr{r}^2} \,\hat{\boldsymbol{\mathscr{r}}} \right] \mathrm{d}\tau' \\
&= \frac{1}{4\pi\epsilon_0} \int \left[\frac{c\rho(\mathbf{r'}, t) + \dot{\rho}(\mathbf{r'}, t_r)\,\mathscr{r}}{c\mathscr{r}^2} \right] \,\hat{\boldsymbol{\mathscr{r}}} \,\mathrm{d}\tau' \\
&= \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r'}, t) + (t_r - t)\dot{\rho}(\mathbf{r'}, t_r)\,\mathscr{r}}{\mathscr{r}^2} \right] \,\hat{\boldsymbol{\mathscr{r}}} \,\mathrm{d}\tau' \\
&= \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r'}, t) + (t - t_r)\dot{\rho}(\mathbf{r'}, t_r)\,\mathscr{r}}{\mathscr{r}^2} \right] \,\hat{\boldsymbol{\mathscr{r}}} \,\mathrm{d}\tau'
\end{aligned}
$$

By the ρ given in the problem,

$$
\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}', t)}{\mathscr{r}^2} - \frac{t_r \dot{\rho}(\mathbf{r}', t_r)}{\mathscr{r}^2} \right] \,\hat{\boldsymbol{\mathscr{r}}} \,\mathrm{d}\tau'
$$

Not really sure what to do with that extra t_r term here...