Homework 7

PHYSICS 341
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1. On the inside of the sphere, we can assume the potential to have form (as $B = 0$ to prevent $V \to \infty$). Outside the sphere $A = 0$, allowing the voltage to tend toward zero at infinity.

$$V(r,\theta) = \begin{cases} \displaystyle\sum_l A_l r^l P_l(\cos\theta) & \text{inside} \\ \displaystyle\sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta) & \text{outside} \end{cases}$$

The given voltage can be written in terms of Legendre polynomials for $l = 0$ and $l = 2$,

$$\begin{aligned} V_0(\theta) &= k\cos 2\theta \\ &= \frac{4k}{3}\left(\frac{3\cos^2(\theta) - 1}{2} - \frac{1}{4}\right) \\ &= \frac{4k}{3}\left(P_2(\cos\theta) - \frac{1}{4}P_0(\cos\theta)\right) \end{aligned}$$

Inside the sphere, the coefficients can be found as

$$\begin{aligned} A_l &= \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)\sin\theta\,\mathrm{d}\theta \\ A_0 &= \frac{-k}{6}\int_0^\pi \sin(\theta)\,\mathrm{d}\theta = -\frac{2k}{6} \\ A_2 &= \frac{5}{2R^2}\int_0^\pi \frac{4k}{3}\left[P_2(\cos(\theta))\right]^2\sin(\theta)\,\mathrm{d}\theta = \frac{4k}{3R^2} \end{aligned}$$

Plugging in these A_l values and expanding the sum, inside the sphere, the potential is

$$\boxed{V_{\text{in}}(r,\theta) = -\frac{2k}{6} + \frac{4kr^2}{3R^2}\left(\frac{3\cos^2(\theta) - 1}{2}\right)}$$

Outside the sphere, we can apply the relation (3.75) from Griffith's, $B_l = -A_l R^{2l+1}$

$$\begin{aligned} B_0 &= \frac{2Rk}{6} \\ B_2 &= \frac{-4R^3k}{3} \end{aligned}$$

$$\boxed{V_{\text{out}}(r,\theta) = \frac{2Rk}{6r} - \frac{4R^3k}{3r^3}\left(\frac{3\cos^2(\theta) - 1}{2}\right)}$$

Since the normal derivatives are discontinuous at $r = R$ by the surface charge, i.e.

$$\begin{aligned}
\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\bigg|_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\left[\frac{4R^3k}{r^4}\left(\frac{3\cos^2(\theta)-1}{2}\right) - \frac{8kr}{3R^2}\left(\frac{3\cos^2(\theta)-1}{2}\right)\right]_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\sigma(\theta) &= \epsilon_0\left(\frac{8k}{3R} - \frac{4k}{R}\right)\left(\frac{3\cos^2(\theta)-1}{2}\right) \\
&\boxed{\sigma(\theta) = -\frac{4k\epsilon_0}{3R}\left(\frac{3\cos^2(\theta)-1}{2}\right)}
\end{aligned}$$

2. Following the steps on page 148 of Griffith's,

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}}\left[\int_0^{\pi/2} \sigma_0 P_l(\cos\theta)\sin(\theta)\,\mathrm{d}\theta + \int_{\pi/2}^{\pi} (-\sigma_0)\,P_l(\cos\theta)\sin(\theta)\,\mathrm{d}\theta\right]$$

The first 6 coefficients can be found as

$$\begin{aligned}
A_0 &= \frac{\sigma_0}{2\epsilon_0 R^{-1}}\left[\int_0^{\pi/2}\sin(\theta)\,\mathrm{d}\theta - \int_{\pi/2}^{\pi}\sin(\theta)\,\mathrm{d}\theta\right] = 0 \\
A_1 &= \frac{\sigma_0}{2\epsilon_0}\left[\int_0^{\pi/2}\cos(\theta)\sin(\theta)\,\mathrm{d}\theta - \int_{\pi/2}^{\pi}\cos(\theta)\sin(\theta)\,\mathrm{d}\theta\right] = \frac{\sigma_0}{2\epsilon_0} \\
A_2 &= 0 \\
A_3 &\approx -10.84\sigma_0/\epsilon_0 \qquad \text{...used a calculator for these} \\
A_4 &= 0 \\
A_5 &\approx -188.89\sigma_0/\epsilon_0
\end{aligned}$$

3. For the charge density

$$\rho(r,\theta) = k\left(\frac{R^2}{r^2} - 3\right)\cos\theta$$

The monopole potential is

$$\begin{aligned}
V_{\text{mono}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{k}{z}\int_V \left(\frac{R^2}{r^2} - 3\right)\cos\theta\,\mathrm{d}\tau \\
&= \frac{1}{4\pi\epsilon_0}\frac{k}{z}(\dots)\int_0^{\pi}\cos(\theta)\sin(\theta)\,\mathrm{d}\theta \\
&= 0 \quad \text{as the } \cos\theta\sin\theta \text{ integral evaluates to zero}
\end{aligned}$$

For the dipole potential,

$$\begin{aligned}
V_{\text{dipole}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0 r^2}\left[\int_V r'\cos\theta\rho(\mathbf{r}')\,\mathrm{d}\tau\right] \\
&= \frac{k}{4\pi\epsilon_0 r^2}\left[\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta\int_0^R\left(\frac{R^2}{r^2} - 3\right)r^2\,\mathrm{d}r\right] \\
&= \frac{k}{2\epsilon_0 r^2}\left(\frac{2}{3}\right)(0) \\
&= 0
\end{aligned}$$

For the quadrupole term,

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int_V (r')^2 \left(\frac{3}{2}\cos^2(\alpha) - \frac{1}{2}\right) \rho(\mathbf{r}')\, \mathrm{d}\tau'$$

As the observation points are along the z-axis, the angle $\alpha = \theta$,

$$\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{k}{4\pi\epsilon_0 r^3} \int_V (r')^2 \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right)\left(\frac{R^2}{r'^2} - 3\right) \cos\theta\, \mathrm{d}\tau' \\
&= \frac{k}{4\pi\epsilon_0 r^3} \left[\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \int_0^{R} \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right)\left(\frac{R^2}{r'^2} - 3\right) \cos(\theta)\sin(\theta)(r')^4\, \mathrm{d}r\, \mathrm{d}\theta\right] \\
&= 0 \quad \text{Due to the } \theta \text{ integral evaluating to zero}
\end{aligned}$$

4. The monopole term is found by summing all the charges near the origin,

$$V_{\text{mono}} = \frac{q}{4\pi\epsilon_0 r}$$

The dipole term can be found by first summing the dipole moments,

$$\begin{aligned}
\mathbf{p} &= \sum_i q_i r_i' \\
&= q\left((2-1)a\,\hat{\mathbf{z}} + 0\,\hat{\mathbf{x}}\right) = qa\,\hat{\mathbf{z}} \\
&= qa\,\hat{\mathbf{r}} \\
V_{\text{dipole}} &= \frac{qa}{4\pi\epsilon_0 r^2}
\end{aligned}$$

5. The monopole term is zero as the total net charge is zero.

For the dipole term,

$$\begin{aligned}
\mathbf{p} &= qa\left(-2\,\hat{\mathbf{z}} + (1-1)\,\hat{\mathbf{x}}\right) \\
&= -2qa\,\hat{\mathbf{z}} \\
V_{\text{dipole}} &= -\frac{2qa}{4\pi\epsilon_0 r^2}
\end{aligned}$$