Homework 3

PHYSICS 465
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1. (a) The δ-function just acts as a comb, so taking the Fourier transform of the delta function results in

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= \int_{-\infty}^{\infty} \delta(t-0) e^{i\omega t}\,\mathrm{d}t \\ &= e^{i\omega t}\Big|_{t=0} \\ &= 1. \quad \square\end{aligned}$$

Next, we can use the inverse Fourier transform on $g(\omega) = 1$,

$$\mathcal{F}^{-1}\{1\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,\mathrm{d}\omega = \delta(t)$$

Since $\delta(t) \in \mathbb{R}$, the LHS must also be real and must be equal to its conjurgate,

$$\int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}\omega = 2\pi\delta(t). \quad \square$$

Finally, assuming the previous propositions, we can plug in a Fourier transform of an arbitrary function $f(t)$ into the inverse Fourier transform, and it should return $f(t)$, i.e. we can test

$$\mathcal{F}^{-1}\{\mathcal{F}[f(t)]\} \stackrel{?}{=} f(t).$$

$$\begin{aligned}f(t) &= \frac{1}{2\pi}\int \left\{\int f(t')e^{i\omega t'}\,\mathrm{d}t'\right\} e^{-i\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int f(t') \left\{\underbrace{\int e^{i\omega(t'-t)}\,\mathrm{d}\omega}_{2\pi\delta}\right\}\mathrm{d}t' && \text{reorganizing} \\ &= \frac{1}{2\pi}\int f(t') 2\pi\delta(t'-t)\,\mathrm{d}t' && \text{eval at } t'=t \\ &= f(t). \quad \square\end{aligned}$$

(b) Now, we are to show a similar proposition in x- and k-space,

$$\int |f(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int |F(k)|^2\,\mathrm{d}k\,.$$

Starting from the RHS, we can assume complex functions and break up the square

$$\begin{aligned}\frac{1}{2\pi}\int |F(k)|^2\,\mathrm{d}k &= \frac{1}{2\pi}\int F^*(k)F(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int \mathrm{d}k \left\{\int f^*(x)e^{ikx}\,\mathrm{d}x\right\}\left\{\int f(x')e^{-ikx'}\,\mathrm{d}x'\right\} \\ &= \frac{1}{2\pi}\int \mathrm{d}k \left\{\iint f^*(x)f(x')e^{ik(x-x')}\,\mathrm{d}x\,\mathrm{d}x'\right\}\end{aligned}$$

Taking the k-integral,

$$\begin{aligned}
&= \frac{1}{2\pi} \iint f^*(x) f(x') \, \mathrm{d}x \, \mathrm{d}x' \int e^{ik(x-x')} \, \mathrm{d}k \\
&= \frac{1}{2\pi} \iint f^*(x) f(x') \, \mathrm{d}x \, \mathrm{d}x' \, 2\pi \delta(x - x') \\
&= \int f^*(x) f(x) \, \mathrm{d}x = \int |f(x)|^2 \, \mathrm{d}x \, . \quad \square
\end{aligned}$$

(c) Equation (5) can be rewritten in 3D as

$$\int |f(\mathbf{r})|^2 \, \mathrm{d}^3\mathbf{r} = \frac{1}{(2\pi)^3} \int |F(\mathbf{k})|^2 \, \mathrm{d}^3\mathbf{r} \, .$$

Redoing part (b), we can start with the RHS once again, then begin substituting the Fourier transform,

$$\begin{aligned}
\frac{1}{(2\pi)^3} \int |F(\mathbf{k})|^2 \, \mathrm{d}^3\mathbf{k} &= \frac{1}{(2\pi)^3} \int F^*(\mathbf{k}) F(\mathbf{k}) \, \mathrm{d}^3\mathbf{k} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{r} \, f^*(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \int \mathrm{d}^3\mathbf{r}' \, f(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k} \iint f^*(\mathbf{r}) f(\mathbf{r}') e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}^3\mathbf{r}' \, .
\end{aligned}$$

Using the δ-substitution given in Equation (6) and combing on $\mathbf{r} = \mathbf{r}'$,

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \iint f^*(\mathbf{r}) f(\mathbf{r}') (2\pi)^3 \delta^3(\mathbf{r} - \mathbf{r}') \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}^3\mathbf{r}' \\
&= \int |f(\mathbf{r})|^2 \, \mathrm{d}^3\mathbf{r} \, . \quad \square
\end{aligned}$$

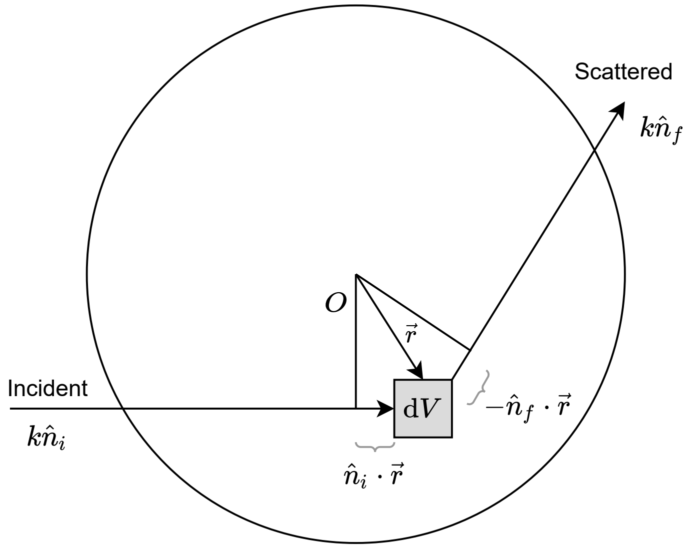
(d) Taking the Laplacian of the inverse Fourier transform,

$$\begin{aligned}
\nabla^2 f(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int F(\mathbf{k}) \nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{k} \\
&= \frac{1}{(2\pi)^3} \int F(\mathbf{k}) (i^2 k^2) e^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{k} \\
&= \frac{1}{(2\pi)^3} \int \underbrace{-k^2 F(\mathbf{k})}_{FT} e^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{k} \, . \quad \square
\end{aligned}$$

(e) From the relation given in Equation (7),

$$\begin{aligned}
\left[\nabla^2 V(\mathbf{r})\right]_{\mathbf{k}} &= -k^2 \left[V(\mathbf{r})\right]_{\mathbf{k}} \, , \\
\implies \left[V(\mathbf{r})\right]_{\mathbf{k}} &= \frac{1}{k^2 \epsilon_0} \left[\rho(\mathbf{r})\right]_{\mathbf{k}} = \frac{e}{k^2 \epsilon_0} \left[\delta^3(\mathbf{r})\right]_{\mathbf{k}} \\
&= \frac{e}{k^2 \epsilon_0} .
\end{aligned}$$

2. We can describe the system in this figure below:



The total extra phase is

$$e^{i\Delta\phi} = e^{ik\,\hat{\mathbf{n}}_i\cdot\mathbf{r} - ik\,\hat{\mathbf{n}}_f\cdot\mathbf{r}}$$

Then, combining the k's,

$$\begin{aligned} \text{Let } \boldsymbol{\kappa} &= \hat{\mathbf{k}}_f - \hat{\mathbf{k}}_i \\ e^{i\Delta\phi} &= e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}} \\ &= e^{-i\mathbf{q}\cdot\mathbf{r}/\hbar}, \end{aligned}$$

where the momentum transfer $\mathbf{q}$ is given by

$$\mathbf{q} = \hbar\boldsymbol{\kappa}.$$

3. If we assume spherically-symmetric charge distribution, i.e. $\rho(\mathbf{r}) = \rho(r)$, then the form factor is

$$ZeF(\boldsymbol{\kappa}) = \int \rho(r) e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{r}$$

Using the definition of the dot product in the exponential, and rotating the axis such that all the θ's align,

$$\begin{aligned} &= \iiint \rho(r) e^{i\kappa r\cos\theta}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,r^2\,\mathrm{d}r \\ &= 2\pi\int_0^\infty \rho(r) r^2 \left\{\int_0^\pi e^{-i\kappa r\cos\theta}\sin\theta\,\mathrm{d}\theta\right\}\mathrm{d}r \\ &= 2\pi\int_0^\infty \rho(r) r^2 \left\{2\sin(\kappa r)/\kappa r\right\}\mathrm{d}r && \text{WolframAlpha'd} \\ &= \frac{4\pi}{\kappa}\int_0^\infty \rho(r) r\sin(\kappa r)\,\mathrm{d}r\,. \quad \square \end{aligned}$$

4. From Problem 3, we can assume the charge density ρ is a constant, thus the form factor is

$$\begin{aligned} F(\boldsymbol{\kappa}) &= \frac{4\pi}{Ze\kappa}\int_0^\infty \rho(r) r \sin(\kappa r)\,\mathrm{d}r \\ &= \frac{4\pi}{Ze\kappa}\rho \int_0^a r\sin(\kappa r)\,\mathrm{d}r \\ &= \frac{4\pi}{Ze\kappa}\rho\left[\frac{\sin(\kappa a) - \kappa a\cos(\kappa a)}{\kappa^2}\right] \end{aligned}$$

Letting $\rho = Q/V = Ze/\frac{4}{3}\pi a^3$,

$$F(\kappa) = \frac{3}{a^3}\left[\frac{\sin(\kappa a) - \kappa a\cos(\kappa a)}{\kappa^3}\right].$$

Taking the limit as the momentum transfer goes to zero, then as $\kappa \propto q$,

$$\lim_{\kappa\to 0}\frac{\sin(\kappa a) - \kappa a\cos(\kappa a)}{\kappa^3} = \frac{a^3}{3}. \qquad \text{WolframAlpha}$$

5. From Problem 1(e), we found that

$$[V(\mathbf{r})]_k = e/k^2\epsilon_0.$$

From the Born approximation, then

$$\begin{aligned} \mathcal{M}(\mathbf{q}) &= [U(\mathbf{r})]_{\mathbf{k}=\mathbf{q}/\hbar} \\ &= \hbar^2 e/q^2\epsilon_0. \end{aligned}$$

The differential cross-section is then found directly as

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{\hbar^2 e}{q^2\epsilon_0}\right)^2 \\ &= \left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{\hbar^2 e}{4P^2\sin^2(\theta/2)\epsilon_0}\right)^2 \end{aligned}$$

Assuming classical motion, $T = p^2/2m \implies p^2 = 2mT$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{\hbar^2 e}{8mT\sin^2(\theta/2)\epsilon_0}\right)^2$$

Re-adding the forgotten Zze,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Zze^2}{16\pi\epsilon_0 T}\right)^2\csc^4(\theta/2). \quad \square$$