Homework 11

PHYSICS 341 | Kevin Evans
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1. (a) Since the current is uniformly distributed, we can assume the current density to be

$$J = \frac{I}{\pi a^2}$$

At a radius s, the current is then

$$I_{\text{enc}}(s) = \frac{Is^2}{a^2}$$

The azimuthal magnetic field is then

$$\begin{aligned} B &= \frac{I_{\text{enc}}}{2\pi s \mu_0} \\ &= \begin{cases} \dfrac{\mu_0 I s}{2\pi a^2} & 0 < s < a \\ \dfrac{\mu_0 I}{2\pi s} & s \geq a \end{cases} \end{aligned}$$

(b) For current density proportional to s^2, we can assume $J = ks^2$ where $k \in \mathbb{R}^+$, and integrating this to find the enclosed current,

$$\begin{aligned} I_{\text{enc}}(s) &= \oint_0^s \mathbf{J} \cdot \mathrm{d}\mathbf{a} \\ &= k \int_0^{2\pi} \mathrm{d}\phi \int_0^s s^3 \, \mathrm{d}s \\ &= \frac{k\pi s^4}{2} \end{aligned}$$

The azimuthal magnetic field is

$$B = \begin{cases} \frac{\mu_0 k \pi s^3}{4\pi} & 0 < s < a \\ \frac{\mu_0 k \pi a^4}{2\pi s} & s \geq a \end{cases}$$

2. Within the first solenoid, both solenoids will affect the magnetic field and will be uniform. Between the two solenoids, only the larger one will affect the magnetic field. Outside both, there is no magnetic field. This leaves

$$\mathbf{B} = \begin{cases} \mu_0 (n_2 - n_1) I \, \hat{\mathbf{z}} & 0 < s < a \\ \mu_0 n_2 I \, \hat{\mathbf{z}} & a < s < b \\ 0 & s > b \end{cases}$$

3. Using (5.66) and from the discussion in-class, the $\hat{\mathbf{y}}$-component will cancel. Then, x can be parameterized by the angle θ as $x = R\sin(\theta)$,

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 I}{4\pi} \int \frac{1}{R} \, \mathrm{d}\mathbf{l}' \\ &= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{R\cos\theta}{R} \, \hat{\mathbf{x}} \\ &= -\frac{\mu_0 I}{2\pi} \, \hat{\mathbf{x}} \end{aligned}$$

4. (a) The magnetic dipole moment is

$$\begin{aligned}\mathbf{m} &\equiv I\,\mathrm{d}a \\ &= 4Ia^2\,\hat{\mathbf{z}} \\ &= 4Ia^2\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)\end{aligned}$$

(b) The dipole term of the magnetic potential is

$$\begin{aligned}\mathbf{A}_{\text{dip}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\,\hat{\mathbf{r}}}{r^2} \\ &= \frac{\mu_0 I a^2\sin\theta}{\pi r^2}\,\hat{\boldsymbol{\phi}}\end{aligned}$$

The magnetic field becomes

$$\begin{aligned}\mathbf{B}_{\text{dip}}(\mathbf{r}) &= \boldsymbol{\nabla}\times\mathbf{A} \\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0 I a^2\sin^2(\theta)}{\pi r^2}\right)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I a^2\sin\theta}{\pi r}\right)\,\hat{\boldsymbol{\theta}} \\ &= \frac{2\mu_0 I a^2}{\pi r^3}\left(\cos\theta\,\hat{\mathbf{r}} + \frac{\sin\theta}{r}\,\hat{\boldsymbol{\theta}}\right)\end{aligned}$$

5. The magnetic moment is defined as

$$\mathbf{m} \equiv I\int \mathrm{d}\mathbf{a}$$

where the current I is

$$\begin{aligned}I &= \int_S \mathbf{J}\cdot\mathrm{d}\mathbf{a} \\ &= \int_0^R\int_{-L/2}^{L/2}\frac{Qs\omega}{\pi R^2 L}\,\mathrm{d}z\,\mathrm{d}s\;\hat{\boldsymbol{\phi}} \\ &= \frac{Q\omega}{2\pi} \\ \mathbf{m} &= \frac{Q\omega RL}{2\pi}\,\hat{\boldsymbol{\phi}}\end{aligned}$$