CHAOS IN BOSE-EINSTEIN CONDENSATES

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Abstract

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The Gross-Pitaevskii equation (GPE) is a nonlinear Schrödinger equation used in modeling Bose-Einstein condensates (BECs). Using Lyapunov exponents, chaos was characterized in the GPE in the one-dimensional case. The GPE was simulated using Python using both spectral and finite-difference methods in space and an adaptive Runge-Kutta solver was used to evolve the equation in time. When a turbulent state is perturbed, positive Lyapunov exponents were found. There was a proportionality found between positive Lyapunov exponents and the nonlinear coupling constant of the GPE, g. Further research can be done to analyze this proportionality.

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Dedication

This thesis is dedicated to my pet dog.

Chapter One

Introduction

The Schrödinger equation is a linear differential equation that describes how quantum mechanical objects evolve in time and space. Any quantum mechanical object can be described by its corresponding wavefunction, governing its probability density. This complex wavefunction exists in a mathematical space known as Hilbert space, differing from a Euclidian space by its inner product and infinite dimensionality [1].

Bose-Einstein condensates (BECs) are formed by bosonic gases at low densities near absolute zero temperatures, resulting in the occupation of the lowest quantum state. The collapse to the lowest quantum state is due to the indistinguishability and bosonic nature of these particles [2]. This additional state of nature was predicted by Einstein and Bose in the early twentieth century [3]. This state of matter is a purely quantum phenomena and are a recent research interest since its experimental realization of rubidium-87 by Cornell and Wieman in 1995 [4]. Although we can model BECs using the Schrödinger equation, problems lay when the particle count is increased.

Systems of many bodies can be difficult to model as the wavefunction grows exponentially to account for Coulombic interaction. Several approximations can be used to reduce the modeling complexity. For a many-body Schrödinger equation, a mean-field pseudopotential approximation can be employed to reduce the complexity from exponential to a constant complexity, as well as using a Hartree-Fock approximation [5]. This resulting nonlinear

Schrödinger equation is known as the Gross-Pitaevskii equation (GPE), capable of accurately modeling BECs routinely produced experimentally [5].

Chaos is the apparent disorder and irregular motion of a dynamical system—more formally, the exponential divergence of a trajectory in time [6]. Chaos can be characterized through Lyapunov exponents, where the sign of the exponent denotes the chaos in a system. A positive maximal exponent characterizes chaos, as it implies an exponential diverging growth. A zero maximal exponent is found when no chaos is present in a system.

An example of chaotic motion can be seen in the the Lorenz attractor, resulting in positive Lyapunov exponents [7]. The Lorenz system is composed of a set of interdependent, nonlinear differential equations, initially used to model atmospheric convection by Edward Lorenz [7]. As two points with slightly different initial conditions evolve in time in this system, they may have wildly varying trajectories. Using the distance between these two points in time, their Lyapunov exponent can be calculated. This process can be repeated for long durations in time and for several different initial conditions. From this set of Lyapunov exponents, the maximal exponent can be taken to represent the chaos within this system. As there are no classical trajectories for calculating Lyapunov exponents in Hilbert space, there are several potential metrics that emulate these trajectories and allow a difference to be found. We will use the L^2 norm to measure distances in the Hilbert space.

The correspondence principle states that quantum mechanics should reproduce classical mechanics in the limit of higher states, as stated by Bohr [8]. Yet, here lies a puzzling conundrum: the world of classical mechanics is chaotic, but the Schrodinger equation is fundamentally linear and cannot exhibit chaos. This is an unsolved problem in physics and several theories exist [9]. Although this paper does not discuss the source of quantum chaos, it will instead look towards the effects of GPE parameters on quantum chaos in BECs.

Previous works have demonstrated chaos in Bose-Einstein condensates through positive Lyapunov exponents [10–12]. This work aims to replicate these results in one-dimension using both spectral and finite-difference methods, as well as discusses the effect of GPE parameters on the Lyapunov exponent. Additionally, this paper discusses numeric errors in both the finite difference method and the adaptive Runge-Kutta solver.

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