Thesis Outline

PHYSICS 490
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Abstract

- Purpose: measure how chaotic the GPE is in 1D
- Methods: created Python simulations, modeled the GPE and calculated Lyapunov exponents as it evolves in time
- Results: the GPE has positive Lyapunov exponents in 1D. There seems to be a dependence on the energy of the system with the exponent
- Conclusion: further testing is needed to see how chaos arises, and how chaos is characterized in higher dimensions

1 Introduction

- Bose-Einstein condensates (BECs) are formed by bosonic gases (at low densities) are cooled near absolute zero, resulting in occupation of the lowest quantum state leading to various quantum effects
- The GPE is a nonlinear Schrödinger equation that models Bose Einstein condensates. If we used the Schrödinger equation directly, each equation would have to include a term for interaction, leading to a system of $O(N^2)$ equations. The GPE is instead a multiparticle wavefunction, which uses a mean-field (Hartree-Fock) approximation.
- Chaos is the exponential separation in distance and is characterized by positive Lyapunov exponents. Instead of using a Euclidian distance, we can find a new distance function for separation in Hilbert space (the realm of these wavefunctions)
- It is unclear how chaos arises in classical physics, considering that quantum physics (by the Schrödinger equation) is fundamentally linear (and linearity cannot lead to a positive Lyapunov exponent [need to find a citation]).

2 Background

- BECs are modeled by the GPE, which is a Hartree-Fock approximation—basically treating particle interaction as a mean-field—resulting in the g term below
- The GPE is a nonlinear Schrodinger equation,

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g|\Psi|^2 \right) \Psi$$

  where the g-term is the coupling constant describing particle interaction. Positive g is from repulsive interactions, which is what you'd expect in an actual BEC

- We can see how the PDE evolves in time by discretizing space and using a solver (like Runge-Kutta) to evolve it in time
- Lyapunov exponents characterize chaos
  - Lorenz system is a classic example with positive exponents
  - The trajectories of 3 planets can be considered chaotic
  - The Lyapunov exponent λ is defined for a separation vector $\mathbf{Z}$ as

$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t}|\delta\mathbf{Z}_0|.$$

  - In the GPE, we're looking at the multiparticle wavefunction and we can look at "differences" in trajectories in different ways (differences in phase or amplitudes, etc.)
  - Since wavefunctions exist in L^2, we need to find a new distance function instead of Euclidian distances. In Hilbert space, we can define a difference function as

$$d^{(2)}(\psi_1, \psi_2, t) = \frac{1}{2}\left\langle \psi_1 - \psi_2 | \psi_1 - \psi_2 \right\rangle .$$

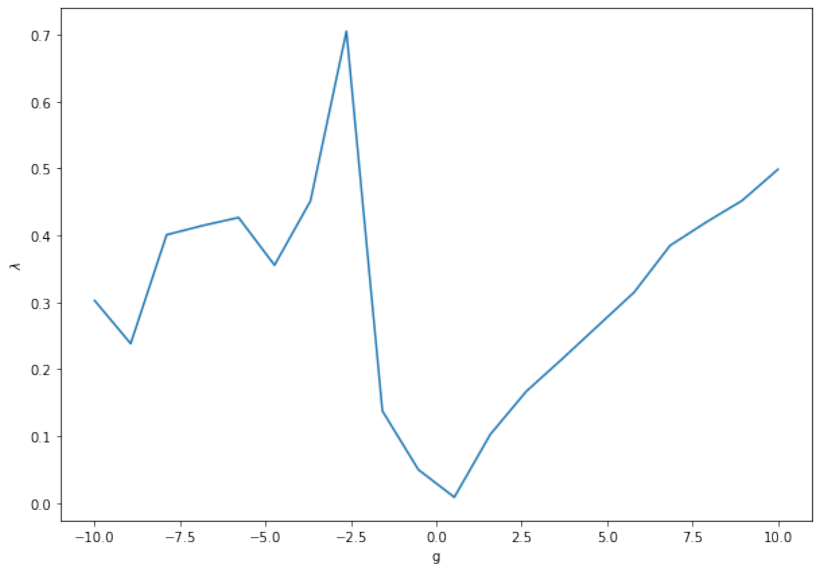
- Turbulence is hard to define, but we can see turbulence by looking at the energy vs k plot and searching for a $-5/3$-power dissipation ("energy cascade")
- There are also several potential sources of error that can lead to false positive exponents, like floating point and rounding errors, as well as errors in the Taylor expansion of functions
- Python is an easy-to-use and performant language for modeling physical phenomena. Can use libraries like numpy and scipy to evolve the GPE in time

3 Procedures

- Simulations were creating using Python with Numpy
- The GPE was approximated using a finite (spatial) difference technique initially
- Also used spectral methods
- An initial state ψ_0 was generated by imprinting turbulence and allowing the wavefunction to evolve in time.
- Used an adaptive Runge-Kutta via `solve_ivp` to evolve the wavefunction in time
- Separately, wrote a program showing linearity between `solve_ivp` parameters and the Lyapunov exponent for the linear ($g = 0$), showing tolerances of 10^{-12} is enough

4 Findings

- Positive Lyapunov exponents were found in the case of 1D
- Turbulent BECs are chaotic, where the BECs are imprinted with a random phase
- The Lyapunov exponent is proportional to the coupling constant of the GPE (for $g > 0$):



- BECs with more vorticies seem to be less chaotic? Need to talk more with my advisor about this

5 Summary and Conclusions

- The GPE exhibits chaos for $g > 0$, with Lyapunov exponents proportional to g
- Likely not due to floating point or `solve_ivp` atol/rtol errors
- Further testing is needed to see if chaos can be shown in higher dimensions (or perhaps this is just an artifact of 1D)

Appendices

- Code samples?
- Could include additional plots

References

Essential ones:

- https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.62.2065
- https://journals.aps.org/pra/abstract/10.1103/PhysRevA.83.043611
- https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.71.2683
- https://arxiv.org/pdf/1605.09580.pdf
- Pethick/Smith