Homework 11

PHYSICS 304
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State vectors and Dirac brackets

1. Because $\langle e_n|e_m\rangle = \delta_{nm}$, when you take $\langle e_n|V\rangle$, it takes the coefficient of only the $|e_n\rangle$ component of V, as $\delta_{nm} = 1$ for $n = m$. The other components are zeroed out.

2. Starting from (6),

$$\begin{aligned}
\mathbb{1}\,|V\rangle &= \sum_n |e_n\rangle \langle e_n|V\rangle \\
&= \sum_n a_n |e_n\rangle \\
&= |V\rangle \quad \square
\end{aligned}$$

3. For an infinite square well of length L, the normalized wavefunction is given as

$$|\Psi_n\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad n \in \mathbb{Z}$$

If we take the inner product of two states, then

$$\langle \Psi_n|\Psi_m\rangle = \int_{\mathbb{R}} \Psi_n^*(x)\Psi_m(x)\,\mathrm{d}x$$

Recognizing that $\Psi(x) \in \mathbb{R}\ \forall\ n, m \in \mathbb{Z}$ and the wavefunctions must be zero outside the interval $[0, L]$, we can limit the integral bounds and remove the conjurgation,

$$\begin{aligned}
\langle \Psi_n|\Psi_m\rangle &= \int_0^L \Psi_n(x)\Psi_m(x)\,\mathrm{d}x \\
&= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\
&= \frac{1}{L}\int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]\mathrm{d}x
\end{aligned}$$

As $(n \pm m) \in \mathbb{Z}$ (closure under addition), then the integral must be zero for $m \neq n$ as we are integrating over equal parts. For $n = m$, the first cosine evaluates to L and the second evaluates to zero. Then,

$$\begin{aligned}
\langle \Psi_n|\Psi_m\rangle &= \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \\
&= \delta_{mn}, \text{ these states form an orthonormal basis.} \quad \square
\end{aligned}$$

4. From (8),

$$\langle \phi | \Psi \rangle = \int_{\mathbb{R}} \overline{\phi(x)} \Psi(x) \, \mathrm{d}x$$

If we take the conjurgation of the RHS, it can be shown

$$\begin{aligned} \overline{\int_{\mathbb{R}} \overline{\phi(x)} \Psi(x) \, \mathrm{d}x} &= \int_{\mathbb{R}} \phi(x) \overline{\Psi(x)} \, \overline{\mathrm{d}x} \\ &= \int_{\mathbb{R}} \overline{\Psi(x)} \phi(x) \, \mathrm{d}x && \text{As } \overline{\mathrm{d}x} = \mathrm{d}x \text{ since } x \in \mathbb{R} \\ &= \langle \Psi | \phi \rangle \quad \square \end{aligned}$$

5. If we let $\alpha = 2\pi/L$, then

$$\begin{aligned} \langle \phi_n | \phi_m \rangle &= \frac{1}{L} \int_{\mathbb{R}} e^{-i\alpha n x} e^{i\alpha m x} \, \mathrm{d}x \\ &= \frac{1}{L} \int_0^L e^{i\alpha x(m-n)} \, \mathrm{d}x \end{aligned}$$

For $m = n$,

$$\langle \phi_n | \phi_m \rangle = \frac{1}{L} \, x \bigg|_0^L = 1$$

For $m \neq n$, if we expand the exponential using Euler's formula, the integrals must evaluate to zero, since the evaluation subtracts integral multiples of an entire sinusodial cycle, i.e. it's zero since

$$\begin{aligned} \cos(2\pi k) &= 1, \\ \sin(2\pi k) &= 0 \qquad \forall\, k \in \mathbb{Z} \\ \langle \phi_n | \phi_m \rangle &= \delta_{mn} \quad \square \end{aligned}$$

6. (a) It's normalized as the probability for $0 < x < L/2$ is given as

$$P(x) = \Psi(x) * \Psi(x) = 2/L$$
$$1 = \frac{2}{L}x\Big|_0^{L/2} \text{ is true}$$

(b) By definition,

$$|\Psi(x)\rangle = \sum_n a_n |\phi_n\rangle$$

If we take $\langle\phi_m|\cdot\rangle$ on both sides, where $|\phi_m\rangle$ is an arbitrary element of the basis, then the RHS will reduce to a single element in the summation (as shown in Problem 5),

$$\langle\phi_m|\Psi\rangle = \sum_n a_n \langle\phi_m|\phi_n\rangle$$
$$\int_{\mathbb{R}} \phi_m^*(x)\Psi(x)\,\mathrm{d}x = \sum_n a_n\delta_{mn} = a_m$$

Reducing the bounds from $\mathbb{R} \to [0, L/2]$ and renaming $m \to n$,

$$a_n = \int_0^{L/2} \frac{1}{\sqrt{L}} e^{-2\pi i n x/L}\sqrt{\frac{2}{L}}\,\mathrm{d}x$$
$$= -\frac{i\left(1 - e^{-in\pi}\right)}{\sqrt{2}n\pi} \qquad \text{Evaluated using WolframAlpha}$$

Since $1 - e^{-in\pi} = 0$ for $n =$ even integers, the summation becomes

$$|\Psi(x)\rangle = \sum_{n=\text{odd}} -\frac{i\left(1 - e^{-in\pi}\right)}{\sqrt{2}n\pi}|\phi_n\rangle$$

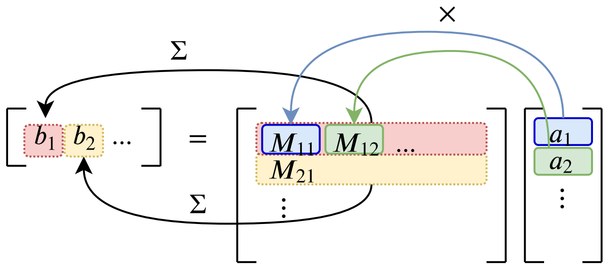
7. From the completeness,

$$\langle\Psi|\Psi\rangle = \langle\Psi|\mathbb{1}|\Psi\rangle = \langle\Psi|\sum_n a_n|\phi_n\rangle$$
$$= \sum_n a_n\langle\Psi|\phi_n\rangle$$
$$= \sum_n (a_n)^2 \quad \square$$

8. I think I must've messed up in Problem 6, so I'm going to just omit the exponential term and use

$$a_n = -\frac{i}{\sqrt{2}n\pi}$$
$$\hat{P}|\phi_n\rangle = \sum_n |a_n|^2 = \sum_n \frac{1}{2\pi^2 n^2}$$
$$= \frac{1}{2\pi^2} \times \frac{\pi^2}{4}$$
$$= \frac{1}{8} \leftarrow \text{Definitely not right...}$$

9. Assuming a and b are matrices as well, it iterates over the rows in a, takes the product of M_{nm} and a_m for every element in a. Then that row is summed and stored in b_n. Maybe?



10. From applying the completeness twice,

$$
\begin{aligned}
|\Phi\rangle &= \hat{A}\,|\Psi\rangle \\
&= \hat{A}\,[\mathbb{1}\,|\Psi\rangle] \\
&= \mathbb{1}\left[\hat{A}\sum_m a_m\,|\phi_m\rangle\right] \\
&= \sum_n \sum_m a_m\,|\phi_n\rangle\,\underbrace{\langle\phi_n|\,\hat{A}\,|\phi_m\rangle}_{A_{nm}} \\
\sum_n b_n\,|\phi_n\rangle &= \sum_n \sum_m a_m\,|\phi_n\rangle\,A_{nm} && \text{Equating the inner parts} \\
b_n &= \sum_m A_{nm} a_m
\end{aligned}
$$