Problem Set 6

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1. ***Kinetic energy of electron gas.*** Show that the kinetic energy of a three-dimensional gas of N free electrons at $0\,\mathrm{K}$ is
$$U_0 = \frac{3}{5}N\epsilon_F.$$

Solution. From (12), the energy of the kth state is given by (12),

$$\epsilon_K = \frac{\hbar^2}{2m}k^2.$$

The average kinetic energy is then

$$\begin{aligned} u_0 &= \langle \epsilon_K \rangle \\ &= \frac{\int \epsilon_K f(k)\,\mathrm{d}^3\mathbf{k}}{\int f(k)\,\mathrm{d}^3\mathbf{k}}. \end{aligned}$$

As it's fully populated from 0 to k_F (so $f(k)$ is a step function) and as the non-radial components divide out, the energy per free electron is

$$\begin{aligned} u_0 &= \frac{\hbar^2}{2m}\frac{\int_0^{k_F} k^4\,\mathrm{d}k}{\int_0^{k_F} k^2\,\mathrm{d}k} \\ &= \frac{\hbar^2}{2m}\frac{k_F^5/5}{k^3/3} \\ &= \frac{3\hbar^2 k_F^2}{10m} = \frac{3}{5}\epsilon_F. \end{aligned}$$

For N free electrons, we can just multiply by N,

$$U_0 = Nu_0 = \frac{3}{5}N\epsilon_F. \quad \square$$

2. ***Pressure and bulk modulus of an electron gas.***

(a) Derive a relation connecting the pressure and volume of an electron gas at $0\,\mathrm{K}$. Hint: Use the result of Problem 1 and the relation between ϵ_F and the electron concentration. The result may be written as $p = \frac{2}{3}(U_0/V)$. Hint: You can use the general relation $p = -\,\partial U/\partial V$ at constant entropy and at absolute zero, all processes are at constant entropy.

Solution. From Problem 1 and from the definition of the Fermi energy (17),

$$\begin{aligned}
U_0 &= \frac{3}{5}N\epsilon_F = \frac{3}{5}N\frac{\hbar^2}{2m}\left(3\pi^2 N\right)^{2/3}V^{-2/3}.\\
p &= -\frac{\partial U_0}{\partial V}\bigg|_S\\
&= -(\dots)\left(-\frac{2}{3}\right)V^{-5/3}.\\
&= \frac{2}{3}(\dots)V^{-2/3}V^{-3/3}\\
&= \frac{2}{3}\frac{U_0}{V}. \quad \square
\end{aligned}$$

(b) Show that the bulk modulus $B = -V(\,\partial p/\partial V\,)$ of an electron gas at $0\,\mathrm{K}$ is

$$B = 5p/3 = 10U_0/9V.$$

Solution. From part (a), the bulk modulus is

$$\begin{aligned}
B &= -V\frac{\partial p}{\partial V} = -V\frac{2}{3}\left(\frac{1}{V}\frac{\partial U_0}{\partial V} + U_0\frac{\partial}{\partial V}V^{-1}\right)\\
&= -\frac{2V}{3}\left(-\frac{p}{V} - \frac{U_0}{V^2}\right)\\
&= \frac{2p}{3} + \underbrace{\frac{2}{3}\frac{U_0}{V}}_{p}\\
&= \frac{5p}{3}. \quad \square
\end{aligned}$$

(c) Estimate for potassium using Table 1, the value of the electron gas contribution to B.

Solution. For potassium, we can estimate the bulk modulus as

$$\begin{aligned}
B &= \frac{10U_0}{9V} = \frac{10}{9}\frac{N}{V}\frac{3}{5}\epsilon_F\\
&= \frac{2}{3}\left(1.40\times 10^{22}\,\mathrm{cm}^{-3}\right)(2.12\,\mathrm{eV})\\
&\approx 2\times 10^{22}\,\mathrm{eV}\cdot\mathrm{cm}^{-3}. \qquad (\approx 3.1\,\mathrm{GPa})
\end{aligned}$$

3. ***Chemical potential in two dimensions.*** Show that the chemical potential of a Fermi gas in two dimensions is given by
$$\mu(T) = k_B T \ln[\exp\bigl(\pi n\hbar^2/mk_B T\bigr) - 1],$$
for n electrons per unit area. Note: The density of orbitals of a free electron gas in two dimensions is independent of energy,
$$D(\epsilon) = m/\pi\hbar^2,$$
per unit area of specimen.

Solution. From the Fermi-Dirac distribution,
$$f(\epsilon) = \frac{1}{\exp[(\epsilon-\mu)/k_B T] + 1}, \tag{6.5}$$
and using the provided density of energy ϵ, the electron area density is
$$\begin{aligned} n &= \int D(\epsilon) f(\epsilon)\,\mathrm{d}\epsilon \\ &= \frac{m}{\pi\hbar^2}\int_0^\infty \frac{1}{\exp[(\epsilon-\mu)/k_B T]+1}\,\mathrm{d}\epsilon \\ &= \frac{m}{\pi\hbar^2} k_B T \ln[\exp(\mu/k_B T) + 1]. && \text{(WolframAlpha)} \end{aligned}$$
Solving for $\mu(T)$,
$$\begin{aligned} \ln[\ldots] &= \frac{n\pi\hbar^2}{mk_B T} \\ \exp(\ldots) &= \exp\left(\frac{n\pi\hbar^2}{mk_B T}\right) - 1 \\ \mu(T) &= k_B T \ln[\exp\bigl(\pi n\hbar^2/mk_B T\bigr) - 1]. \quad \square \end{aligned}$$

4. ***Liquid He³.*** The atom He^3 has spin $\frac{1}{2}$ and is a fermion. The density of liquid He^3 is $0.081\,\mathrm{g\cdot cm^{-3}}$ near absolute zero. Calculate the Fermi energy ϵ_F and the Fermi temperature T_F.

Solution. Using the definition of the Fermi energy,
$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3},$$
and using the mass of Helium-3 ($3.016\,\mathrm{u}$),
$$\begin{aligned} \epsilon_F &= \frac{(1.054\times10^{-34}\,\mathrm{J\cdot s})^2}{2(3.016\,\mathrm{u}\times1.66\times10^{-24}\,\mathrm{g})}\left(3\pi^2\times\frac{0.081\,\mathrm{g\cdot cm^{-3}}}{3\,\mathrm{g\cdot mol^{-1}}}\right)^{2/3} \\ &= 9.57\times10^{-39}\,\mathrm{J}. \end{aligned}$$
The associated Fermi temperature is
$$\begin{aligned} T_F &= \epsilon_F/k_B \\ &= 7\times10^{-16}\,\mathrm{K}. \end{aligned}$$