# Homework 5

MATH 301
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1. **Proposition.** *Let $n$ be an integer. If $4$ divides $(n-1)$, then $4$ divides $(n^2-1)$.*

   *Proof.* Suppose $n$ is an integer and $4$ divides $(n-1)$. Then $n-1=4k$ for some $k \in \mathbb{Z}$. This can be rewritten as $n=4k+1$.
   Next, if we take $(n^2-1)$ and substitute this new expression for $n$,

   $$\begin{aligned} n^2-1 &= (4k+1)^2-1 \\ &= 16k^2+8k \\ &= 4\left(4k^2+2k\right) \end{aligned}$$

   If we let $m=4k^2+2k$, then $m \in \mathbb{Z}$, and we can write $n^2-1$ as

   $$n^2-1=4m$$

   Therefore $4$ divides $(n^2-1)$ if $4$ divides $(n-1)$. ■

B. Just gonna copy most of the proof from Problem 1...

   **Proposition.** *Let $n$ be an integer. If $4$ divides $(n-1)$, then $8$ divides $(n^2-1)$.*

   *Proof.* Suppose $n$ is an integer and $4$ divides $(n-1)$. Then $n-1=4k$ for some $k \in \mathbb{Z}$. This can be rewritten as $n=4k+1$.
   Next, if we take $(n^2-1)$ and substitute this new expression for $n$,

   $$\begin{aligned} n^2-1 &= (4k+1)^2-1 \\ &= 16k^2+8k \\ &= 8\left(2k^2+k\right) \end{aligned}$$

   If we let $m=2k^2+k$, then $m \in \mathbb{Z}$, and we can write $n^2-1$ as

   $$n^2-1=8m$$

   Therefore $8$ divides $(n^2-1)$ if $4$ divides $(n-1)$. ■

2. **Proposition.** *If $n \in \mathbb{Z}$, then $5n^2 + 3n + 1$ is odd.*

   *Proof.* We can divide this into two cases for $n$.
   Case 1: Suppose $n$ is even, then $n$ can be written as $2k$ where $k \in \mathbb{Z}$, then this expression can be substituted in the original statement,

   $$\begin{aligned} 5n^2 + 3n + 1 &= 5\left(2k\right)^2 + 3(2k) + 1 \\ &= 2m + 1 \end{aligned}$$

   where $m = 10k^2 + 3k$, then $m \in \mathbb{Z}$. Therefore, for an even $n$, the original expression is odd.

   Case 2: Suppose $n$ is odd, then $n$ can be written as $2p + 1$ where $p \in \mathbb{Z}$, then this can be substituted in the original expression,

   $$\begin{aligned} 5n^2 + 3n + 1 &= 5\left(2p + 1\right)^2 + 3(2p + 1) + 1 \\ &= 2q + 1 \end{aligned}$$

   where $q = 10p^2 + 13p + 8$, then $q \in \mathbb{Z}$. Therefore, the original expression is odd. ∎

3. **Proposition.** *Suppose $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.*

   *Proof.* Suppose $a, b, c \in \mathbb{Z}$, and also $a \mid b$ and $a \mid c$. Then we know that $b = k_1 a$ and $c = k_2 a$, for some $k_1, k_2 \in \mathbb{Z}$.
   Then by substitution, the sum $b + c$ can be written as

   $$\begin{aligned} b + c &= k_1 a + k_2 a \\ &= \left(k_1 + k_2\right) a \\ &= k_3 a \end{aligned}$$

   Since addition is closed on integers, $k_3 \in \mathbb{Z}$, $b + c$ can be written as an integer multiple of $a$.
   Therefore $a \mid (b + c)$. ∎

4. **Proposition.** *Let $x$ and $y$ be positive integers. If $\gcd(x, y) > 1$, then $x \mid y$ or $x$ is not prime.*

   *Proof.* Suppose $x$ and $y$ are positive integers and $\gcd(x, y) > 1$. Then there exists an integer that divides both $x$ and $y$, i.e. there are integers $n_1$ and $n_2$, with $k = \gcd(x, y)$, where

   $$\begin{aligned} x &= n_1 k \\ y &= n_2 k \end{aligned}$$

   We can split $x$ to two cases: $x$ is prime or $x$ is not prime.
   Case 1: $x$ is prime. If $x$ is prime, then $n_1 = 1$ as $k = x$. Then, $y = n_2 x$. Therefore $x \mid y$.
   Case 2: $x$ is not prime. (Does this case need a body?)

   Therefore, if $\gcd(x, y) > 1$, then $x \mid y$ or $x$ is not prime. ∎

5. **Proposition.** *Let $a$ be an integer. If there exists an integer $n$ such that $a \mid (4n+3)$ and $a \mid (2n+1)$, then $a = 1$ or $a = -1$.*

*Proof.* Let $a$ be an integer. Suppose there is an integer $n$ such that $a \mid (4n+3)$ and $a \mid (2n+1)$, then this may be written as multiples of $a$ where $k_1, k_2 \in \mathbb{Z}$

$$4n + 3 = k_1 a \tag{1}$$
$$2n + 1 = k_2 a \tag{2}$$

Subtracting the two expressions (1) and (2), then squaring,

$$\begin{aligned} 2n + 2 &= (k_1 - k_2)\, a \\ (2n+2)^2 &= (k_1 - k_2)^2\, a^2 \\ &= \left(k_1^2 - 2k_1k_2 + k_2^2\right) a^2 \end{aligned}$$

Moving the cross term to the other side,

$$(2n+2)^2 + 2k_1k_2a^2 = \left(k_1^2 + k_2^2\right) a^2 \tag{3}$$

Next, if we multiply the original expressions (1) and (2),

$$(4n+3)\,(2n+1) = k_1k_2a^2$$

Substituting this into (3), then expanding the LHS

$$\begin{aligned} (2n+2)^2 + 2\,(4n+3)\,(2n+1) &= \left(k_1^2 + k_2^2\right) a^2 \\ 20n^2 + 28n + 10 &= \left(k_1^2 + k_2^2\right) a^2 \end{aligned}$$

Guessing the squares,

$$\begin{aligned} (4n+3)^2 + (2n+1)^2 &= \left(k_1^2 + k_2^2\right) a^2 \\ \therefore a &= \pm 1 \end{aligned}$$

■