Homework 5

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1. Read Chapters 2.2 and 1.6.

2. (a) The system is now in ψ_1.

   (b) We'll either measure b_1 or b_2. As the system was in ψ_1, we can just project the ψ_1 onto the new eigenstates, i.e. the probabilities of being in ϕ_1 and ϕ_2 are respectively

$$P_1 = \langle \phi_1 | \psi_1 \rangle^2 = 9/25$$
$$P_2 = \langle \phi_2 | \psi_1 \rangle^2 = 16/25.$$

   (c) Since we have now measured B, it's now in a superposition state with the probabilities above. To find the new state, we'll need to rearrange ϕ in terms of ψ

$$5\psi_1 = 3\phi_1 + 4\phi_2$$
$$5\psi_2 = 4\phi_1 - 3\phi_2$$

$$\implies 20\psi_1 = 12\phi_1 + 16\phi_2$$
$$15\psi_2 = 12\phi_1 - 9\phi_2$$

$$\implies 25\phi_2 = 20\psi_1 - 15\psi_2$$
$$\boxed{\phi_2 = (4\psi_1 - 3\psi_2)/5}$$
$$5\psi_1 = 3\phi_1 + (16/5)\psi_1 - (12/5)\psi_2$$
$$\boxed{\phi_1 = (3\psi_1 + 4\psi_2)/5.}$$

   So the superposition state looks something like

$$\frac{9}{25}\phi_1 + \frac{16}{25}\phi_2.$$

   This means the probability of getting a_1 after measuring A again is

$$\left\langle \psi_1 \middle| \frac{9}{25}\phi_1 + \frac{16}{25}\phi_2 \right\rangle^2 = \left(\frac{9}{25}\frac{3}{5} + \frac{16}{25}\frac{4}{5}\right)^2$$
$$\approx 0.53.$$

3. For the nth state of an infinite square well, its wavefunction is

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}.$$

   The expectation of the position $\langle x \rangle$ is

$$\langle x \rangle = \int \Psi^* x \Psi \, \mathrm{d}x$$

As the phase is imaginary and also doesn't depend on x,

$$\begin{aligned}\langle x\rangle &= \frac{2}{a}\int_0^a x\sin^2\!\left(\frac{n\pi}{a}x\right)\mathrm{d}x \\ &= \frac{2}{a}\frac{a^2}{4} \qquad \text{(WolframAlpha)} \\ &= \frac{a}{2}.\end{aligned}$$

This makes sense, as we'd intuitively expect this to be in the middle.

For $\langle x^2\rangle$,

$$\begin{aligned}\langle x^2\rangle &= \int \Psi^* x^2 \Psi\,\mathrm{d}x \\ &= \frac{2}{a}\int_0^a x^2\sin^2\!\left(\frac{n\pi}{a}x\right)\mathrm{d}x \\ &= \frac{2}{a}\left[\frac{a^3}{24\pi^3 n^3}\left(4\pi^3n^3 - 6\pi n\right)\right] \qquad \text{(WolframAlpha)} \\ &= a^2\left(\frac{1}{3} - \frac{1}{2\pi^2 n^2}\right).\end{aligned}$$

For the momentum $\langle p\rangle$,

$$\begin{aligned}\langle p\rangle &= \int \Psi^* i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\Psi\,\mathrm{d}x \\ &= -i\hbar\int_0^a \Psi^*\frac{\mathrm{d}\Psi}{\mathrm{d}x}\,\mathrm{d}x \\ &= -\frac{2i\hbar}{a}\frac{n\pi}{a}\int_0^a \sin\!\left(\frac{n\pi}{a}x\right)\cos\!\left(\frac{n\pi}{a}x\right)\mathrm{d}x \\ &= \cdots\int_0^a \sin\left(\frac{2n\pi}{a}x\right)\mathrm{d}x \\ &= 0,\ \text{because it's always over full cycles.}\end{aligned}$$

For the momentum squared, $\langle p^2\rangle$,

$$\begin{aligned}\langle p^2\rangle &= -\hbar^2\int_0^a \Psi^*\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2}\,\mathrm{d}x \\ &= -\frac{2\hbar^2}{a}\left(-\frac{n^2\pi^2}{a^2}\right)\int_0^a \sin^2\!\left(\frac{n\pi}{a}x\right)\mathrm{d}x \\ &= \frac{\hbar^2\pi^2n^2}{a^2}. \qquad \text{(WolframAlpha)}\end{aligned}$$

Now, for the corresponding deviations,

$$\begin{aligned}
{\sigma_x}^2 &= \left\langle x^2 \right\rangle - \langle x \rangle^2 \\
&= a^2 \left(\frac{1}{3} - \frac{1}{2\pi^2 n^2} \right) - \frac{a^2}{4} \\
&= \frac{a^2}{12} \left(1 - \frac{6}{\pi^2 n^2} \right). \\
{\sigma_p}^2 &= \left\langle p^2 \right\rangle - \langle p \rangle^2 \\
&= \frac{\hbar^2 \pi^2 n^2}{a^2}.
\end{aligned}$$

Using the uncertainty principle,

$$\begin{aligned}
\sigma_x(n)\sigma_p(n) &= \sqrt{\frac{\hbar^2 \pi^2 n^2 a^2}{12 a^2} \left(1 - \frac{6}{\pi^2 n^2} \right)} \\
&= \frac{\hbar}{2} \pi n \sqrt{\frac{1}{3} \left(1 - \frac{6}{\pi^2 n^2} \right)}.
\end{aligned}$$

This satisfies the uncertainty principle as it's greater than $\hbar/2$. The closest state to the uncertainty limit is $n = 1$,

$$\sigma_x \sigma_p \bigg|_{n=1} = \frac{\hbar}{2} \pi \sqrt{\frac{1}{3} \left(1 - \frac{6}{\pi^2} \right)} \approx \frac{\hbar}{2} \times 1.136.$$

4. (a) To normalize this, we can use the orthogonality of the eigenstates with

$$\begin{aligned}
1 &= A^2 \int \left(\psi_1^* + \psi_2^* \right) \left(\psi_1 + \psi_2 \right) \mathrm{d}x \\
&= A^2 \int \left| \psi_1 \right|^2 + \left| \psi_2 \right|^2 \mathrm{d}x\,.
\end{aligned}$$

Assuming the eigenstates are already normalized,

$$A = \frac{1}{\sqrt{2}}.$$

(b) Using (2.31) and adding in the time dependence,

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}} \left[\psi_1(x) e^{-i(\pi^2\hbar/2ma^2)t} + \psi_2(x) e^{-i(4\pi^2\hbar/2ma^2)t} \right] \\
&= \frac{1}{\sqrt{2}} \left[\psi_1(x) e^{-i\omega t} + \psi_2(x) e^{-4i\omega t} \right] \\
&= \frac{1}{\sqrt{a}} \left[\sin(\pi x/a) e^{-i\omega t} + \sin(2\pi x/a) e^{-4i\omega t} \right].
\end{aligned}$$

For the probability density,

$$\begin{aligned}
\left| \Psi(x,t) \right|^2 &= \frac{1}{a} \left(\sin(\pi x/a) e^{i\omega t} + \sin(2\pi x/a) e^{4i\omega t} \right) \left(\sin(\pi x/a) e^{-i\omega t} + \sin(2\pi x/a) e^{-4i\omega t} \right) \\
&= \frac{1}{a} \left[\sin^2(\pi x/a) + \sin^2(2\pi x/a) + \sin(\pi x/a)\sin(2\pi x/a) \left(e^{3i\omega t} + e^{-3i\omega t} \right) \right] \\
&= \frac{1}{a} \left[\sin^2(\pi x/a) + \sin^2(2\pi x/a) + 2\sin(\pi x/a)\sin(2\pi x/a)\cos(3\omega t) \right].
\end{aligned}$$

(c) The expectation of x is

$$\langle x\rangle = \frac{1}{a}\int x\left[\sin^2(\pi x/a) + \sin^2(2\pi x/a) + 2\sin(\pi x/a)\sin(2\pi x/a)\cos(3\omega t)\right] \mathrm{d}x$$

Using WolframAlpha to evaluate each term,

$$\begin{aligned}\langle x\rangle &= \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{8a^2}{9\pi}\cos(3\omega t)\right] \\ &= a\left(\frac{1}{2} - \frac{8}{9\pi}\cos(3\omega t)\right).\end{aligned}$$

(d) Using the hint and (1.33),

$$\begin{aligned}\langle p\rangle &= m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} \\ &= \frac{8ma\omega}{3\pi}\sin(3\omega t).\end{aligned}$$

(e) We can only get either eigenvalue E_1 or E_2 here. The coefficients and probabilities are equal as $1/2$ (as the normalization is $1/\sqrt{2}$). Taking the expectation of the H,

$$\begin{aligned}\langle H\rangle &= \frac{1}{2}\int (\psi_1^* + \psi_2^*)\hat{H}(\psi_1 + \psi_2)\,\mathrm{d}x \\ &= \frac{E_1 + E_2}{2} = \left(\frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2}\right) \\ &= \frac{5\pi^2\hbar^2}{2ma^2}.\end{aligned}$$

5. Tacking on the additional phase to ψ_2,

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i(\pi^2\hbar/2ma^2)t} + \psi_2(x)e^{i\phi}e^{-i(4\pi^2\hbar/2ma^2)t}\right] \\ &= \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\omega t} + \psi_2(x)e^{i\phi}e^{-4i\omega t}\right] \\ &= \frac{1}{\sqrt{a}}\left[\sin(\pi x/a)e^{-i\omega t} + \sin(2\pi x/a)e^{i\phi}e^{-4i\omega t}\right]. \\ |\Psi(x,t)|^2 &= \frac{1}{a}\left(\sin(\pi x/a)e^{i\omega t} + \sin(2\pi x/a)e^{4i\omega t}e^{i\phi}\right)\left(\sin(\pi x/a)e^{-i\omega t} + \sin(2\pi x/a)e^{-4i\omega t}e^{-i\phi}\right) \\ &= \frac{1}{a}\left[\sin^2(\pi x/a) + \sin^2(2\pi x/a) + \sin(\pi x/a)\sin(2\pi x/a)\left(e^{3i\omega t + i\phi} + e^{-3i\omega t - i\phi}\right)\right] \\ &= \frac{1}{a}\left[\sin^2(\pi x/a) + \sin^2(2\pi x/a) + 2\sin(\pi x/a)\sin(2\pi x/a)\cos(3\omega t + \phi)\right]. \\ \langle x\rangle &= \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{8a^2}{9\pi}\cos(3\omega t + \phi)\right] \\ &= a\left(\frac{1}{2} - \frac{8}{9\pi}\cos(3\omega t + \phi)\right).\end{aligned}$$

For $\phi = \pi/2$, we're just shifted a quarter wave and the cosine becomes a sine. For $\phi = \pi$, it's just shifting it a half wave, so the cosine becomes a negative cosine.