Homework 9

Problem 1. (This problem and the next apply concept of triplet/singlet energy to other areas.) A proton and neutron are identical nucleons in all their strong-force aspects (although they differ electrically). Unlike electrons in atoms, nucleons interact with not only a spatial strong force, but a substantial spin-spin interaction energy, which should split singlets and triplets according to total S (n and p are spin-½, like electrons).

The preceding refers to ordinary, angular-momentum spin. Heisenberg suggested additionally replacing labels "neutron" and "proton" by $I_3 = \pm 1/2$ "orientations" of a so-called isospin, I, of identical nucleons. Isospin is is just like spin, only in a space of abstract directions 1,2,3 with $\mathbf{I} = (I_1, I_2, I_3)$. Even if the isospin states contribute nothing to the energy, because of the Pauli principle total isospin does correlate with energy (as does regular S in helium atoms).

(a) The deuteron is a bound two-nucleon state of a neutron and a proton. It has spin $S = 1$ and a symmetric spatial state under exchange. Dineutron and diproton states are unbound, but would have the same spatial ground state as the deuteron, as the next spatial state up has far higher energy. From the Pauli principle, how should total isospin I correlate with total spin S?

(b) Deduce the spin S of the *nn* and *pp* states, either using part (a) or just applying the Pauli principle directly to these pairs of identical particles. From your answer, does the nucleon spin-spin interaction favor aligned or opposite spins?

(c) Make an energy-level diagram that identifies isospin triplets and singlets as well as ordinary spin triplets and singlets of the two-nucleon system.

(c) Reverting to labels "n" and "p", the isospin theory tells you which states are *not* just product states like $|n\rangle|p\rangle = |np\rangle$. Write out all the states of part (c) as combinations of the products $|np\rangle$, $|pn\rangle$, $|nn\rangle$, and $|np\rangle$. (You'll see the deuteron state written as you have here in particle physics listings.)

Problem 2. Alongside their regular spin, quarks live in a "color space" in which they can "point" not up and down but in three directions called red, green and blue. You can think of these as three different m_{color} values for components along three symmetrically placed axes in a plane. All three together add to zero total color (white), analogous to how regular $m_s =$½ and $m_s = -$½ add to zero total spin. The spatial *color force* attracts different colors with a force of several tons per pair, and this means you simply can't ever find total color that is nonzero. Explain why this implies that the color wave function (analogous to spin functions like $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$) is

$$\text{Color function} \;=\; \sum_{ijk} \epsilon_{ijk} |i\rangle|j\rangle|k\rangle$$

where now i, j, k represent red, blue and green. Do you see why this explains the makeup of the world around you, at least as far as quarks go? (What are three-quark objects?)

Problem 3. Measurable quantities, (observables) are represented by Hermitian operators. This means $\hat{A}^\dagger = \hat{A}$ where the Hermitian conjugate $\hat{A}^\dagger$ of $\hat{A}$ is defined by

$$\langle\phi|\hat{A}|\psi\rangle \;=\; \int \phi^* \hat{A}\psi \, dx \;=\; \int (\hat{A}^\dagger\phi)^* \psi \, dx \tag{1}$$

In the following assume all wave functions are zero at $x = \pm\infty$ when integrating by parts (as they must be!).

(a) Check that $\hat{A} = x$ is Hermitian, so position is observable.

(b) Check that $\partial/\partial x$ is *not* Hermitian. What is $(\partial/\partial x)^\dagger$?

(c) Check that $\hat{p} = -i\hbar\,\partial/\partial x$ *is* Hermitian, so momentum is observable.

(d) Check that the Hamiltonian $\hat{H} = -(\hbar^2/2m)\partial^2/\partial x^2 + V(x)$ (where the Schrödinger equation is $\hat{H}\psi = E\psi$) is Hermitian, so energy is observable.

Problem 4. Continuing with Problem 3, the following formulas will be useful in the future.

(a) Check from Eq (1) that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

(b) Check from Eq (1) that $\langle\phi|\hat{A}^\dagger|\psi\rangle = \langle\psi|\hat{A}|\phi\rangle^*$. (Hint: take the complex conjugate of (1) with ϕ and ψ swapped.)

(c) Show that $(\hat{A}^\dagger)^\dagger = \hat{A}$. (Hint: Use part (b) repeatedly on $\langle\phi|(\hat{A}^\dagger)^\dagger|\psi\rangle$.)

(d) The state $|\psi'\rangle = \hat{A}|\psi\rangle$ is not normalized. Instead $|\psi'\rangle = C|\phi\rangle$ where $|\phi\rangle$ is normalized and C is a real number. From the integral $\int(\psi')^*\psi' dx$ show, using Eq (1), that $C^2 = \langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle$.

Problem 5. Suppose we have two Schrödinger wave functions $\hat{H}_0\phi_i = E_i\phi_i$ and $\hat{H}_0\phi_f = E_i\phi_f$, with $\hat{H}_0 = -(\hbar^2/2m)\nabla^2$. We can make up a state ψ out of

$$|\psi\rangle \;=\; a|\phi_i\rangle + b|\phi_f\rangle \tag{2}$$

(a) Show that the left side of the Schrödinger equation, $(\hat{H}_0 + \hat{V})\psi$, adding a *new* potential energy $\hat{V} = |\mathbf{r}\rangle V(\mathbf{r})\langle\mathbf{r}|$, is equivalent to a matrix multiplication:

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \hat{H}|\psi\rangle = \begin{bmatrix} E_i & \mathcal{M}_{fi} \\ \mathcal{M}_{fi}^* & E_f \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{3}$$

where

$$\mathcal{M}_{fi} \;=\; \int \phi_i^*(\mathbf{r})V(\mathbf{r})\phi_f(\mathbf{r})d\mathbf{r}. \tag{4}$$

Why does H_0 not appear in this integral? Why must a Hermition $\hat{V}$ (Problems 3 and 4) mean that this is a Hermitian matrix, $\mathcal{M}_{if} = \mathcal{M}_{fi}^*$?

(b) For $E_i = E_f$, the matrix problem gives us new non-equal energies and states (in terms of the old $|\phi_i\rangle$ and $|\phi_f\rangle$) by solving the matrix problem $\mathsf{H}\Psi = E\Psi$. You did this in the case of state splitting in He. Reviewing this, what were $V(\mathbf{r})$, $|\phi_i\rangle$, $|\phi_f\rangle$, and $\mathcal{M}_{fi}$ in that case? What were the old and new energies?

(c) In the case of a time-dependent $V(\mathbf{r}, t) = V(\mathbf{r})\cos\omega t$, then E_i and E_f have to differ by $\hbar\omega$, so that $E_f = E_i \pm \hbar\omega$, and whenever they do there is a transition rate

$$\text{Rate}\Big(|\phi_i\rangle \to |\phi_f\rangle\Big) \;=\; \text{factor} \times |\mathcal{M}_{fi}|^2 \,.$$

Why does this explain Einstein's assertion, before the preceding theory existed, that wherever you have absorption of light there is also an equal rate of stimulated emission? (I.e. equality of Einstein's B coefficients.) This is, as you might know, the basis of laser action.

Problem 6. In the next few problems you'll work abstractly with operators. (No wave functions!) Creation and annihilation operators $a^\dagger$ and a for a quantum system are *defined* by their commutation property

$$aa^\dagger - a^\dagger a \;=\; 1 \tag{5}$$

We also define the *number* operator $\hat{N} = a^\dagger a$. Each eigenvalue of $\hat{N}$ will be the number n of quantum particles or quantum excitations in the state $|n\rangle$, i.e. $\hat{N}|n\rangle = n|n\rangle$.

(a) Check that $\hat{N}$ is an observable, i.e. Hermitian (see Problem 4, parts (a) and (c)).

A quantum harmonic oscillator has energies $E_n = \hbar\omega(n+\frac{1}{2})$ where ω is the natural frequency of the oscillator. If we view the oscillator as a system whose states each have a number n of "particles" or excitations, the Hamiltonian is

$$\hat{H} = \hbar\omega(\hat{N} + \tfrac{1}{2}) \;=\; \hbar\omega\big(a^\dagger a + \tfrac{1}{2}\big) \tag{6}$$

(b) Show that an alternative writing of $\hat{H}$ is

$$\hat{H} = \tfrac{1}{2}\hbar\omega\big(a^\dagger a + aa^\dagger\big). \tag{7}$$

Problem 7. Check that the operators

$$a = \frac{1}{\sqrt{2}}\Big(\alpha\, x + \frac{1}{\alpha}\frac{d}{dx}\Big), \quad a^\dagger = \frac{1}{\sqrt{2}}\Big(\alpha\, x - \frac{1}{\alpha}\frac{d}{dx}\Big)$$

satisfy $aa^\dagger - a^\dagger a = 1$. Is $a^\dagger$ really the Hermitian conjugate of a? Then, taking $\hat{H}$ in terms of a and $a^\dagger$ from the previous problem and $\alpha = \sqrt{\omega/\hbar}$, show that these operators give give

$$\hat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \tfrac{1}{2}\omega^2 x^2 \tag{8}$$

which is indeed a harmonic oscillator with $m = 1$.

Problem 8. (a) Check, using only $aa^\dagger - a^\dagger a = 1$, that

$$\hat{N}a|n\rangle \;=\; (n{-}1)\, a|n\rangle \tag{9}$$
$$\hat{N}a^\dagger|n\rangle \;=\; (n{+}1)a^\dagger|n\rangle \tag{10}$$

where $\hat{N} = a^\dagger a$ is from the previous problem.

(b) Explain why part (a) implies that $a|n\rangle = C|n{-}1\rangle$ for some number C, and $a^\dagger|n\rangle = D|n{+}1\rangle$ for some number D. (Hint: Any state $|\psi\rangle$ having $\hat{N}|\psi\rangle = n|\psi\rangle$ *must* be proportional to the eigenstate $|n\rangle$.) Then use Problem 4, part (d) to show that, more specifically,

$$a|n\rangle \;=\; \sqrt{n}\,|n{-}1\rangle$$
$$a^\dagger|n\rangle \;=\; \sqrt{n{+}1}\,|n{+}1\rangle$$

Problem 9. Spontaneous emission can *only* be understood via quantum electrodynamics, which treats light as a quantum field. In quantum electrodynamics, one finds that for each plane-wave mode of an electromagnetic field within a volume V, with oscillation frequency is ω and whose wavelength is λ, the operators for the electric field and the magnetic field are

$$\hat{E}(\mathbf{r}) \;=\; i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\big(ae^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\big) \tag{11}$$

$$\hat{B}(\mathbf{r}) \;=\; \sqrt{\frac{\mu_0\hbar\omega}{2V}}\,\big(ae^{i\mathbf{k}\cdot\mathbf{r}} + a^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\big) \tag{12}$$

Here $|\mathbf{k}| = 2\pi/\lambda$, and a and $a^\dagger$ are the annihilation and creation operators for the photons contained in this mode.

From classical electodynamics the electric energy density is $(\epsilon_0/2)E^2(\mathbf{r})$ and the magnetic energy density is $(1/2\mu_0)B^2(\mathbf{r})$. Plug the quantum operators (11) and (12) into the total electromagnetic energy

$$\hat{H} \;=\; \int_V \left[\frac{\epsilon_0}{2}\hat{E}^2(\mathbf{r}) + \frac{1}{2\mu_0}\hat{B}^2(\mathbf{r})\right] dV$$

to see that this actually gives a fundamental quantum Hamiltonian for the photons. This should match, for example, the Hamiltonian for excitations of a harmonic oscillator in Problem 6.

Problem 10. Now we can actually derive both stimulated emission and spontaneous emission. As far as an atom is concerned, the electric field provides a potential energy $\hat{V}(\mathbf{r}) = -e\mathbf{r}\cdot\hat{\mathbf{E}}$ or, say, $-ez\hat{E}$ if $\mathbf{E}$ is in the z direction. From the previous problem we want to be using the correct quantum operator $\hat{E}$. But also in light of the previous problem, the atomic transitions $|\phi_i\rangle \to |\phi_f\rangle$ should specify photon states of the electromagnetic field too, in the mode with energy $\hbar\omega = E_f - E_i$, in the direction $\mathbf{k}$. The absorption of a photon should be (compare Problem 5)

$$\begin{aligned}
\text{Rate}\Big(|\phi_i\rangle|n\rangle \to |\phi_f\rangle|n{-}1\rangle\Big) \;&=\; \text{factor} \times |\mathcal{M}_{fi}|^2 \\
\mathcal{M}_{fi} \;&=\; \int \phi_i^*(\mathbf{r})\langle n{-}1|\hat{V}(\mathbf{r})|n\rangle\phi_f(\mathbf{r})d\mathbf{r} \\
&=\; \underbrace{\Big(-e\int \phi_i^*(\mathbf{r})z\phi_f(\mathbf{r})d\mathbf{r}\Big)}_{\mathcal{M}_{fi}^{\text{atom}}} \times \langle n{-}1|\hat{E}|n\rangle.
\end{aligned} \tag{13}$$

With n photons present in this mode, the rate of absorption from i to f is

$$\text{Absorption rate} \;=\; \text{factor} \times \left|\mathcal{M}_{fi}^{\text{atom}}\right|^2 \times \left|\langle n{-}1|\hat{E}(\mathbf{r})|n\rangle\right|^2 \tag{14}$$

whereas the rate of emission from f to i is

$$\text{Emission rate} \;=\; \text{factor} \times \left|\mathcal{M}_{if}^{\text{atom}}\right|^2 \times \left|\langle n{+}1|\hat{E}(\mathbf{r})|n\rangle\right|^2 \tag{15}$$

Since $\mathcal{M}_{if}^{\text{atom}} = (\mathcal{M}_{fi}^{\text{atom}})^*$ all factors are the *same* in Eqs (14) and (15).

(a) For an atom at $\mathbf{r} = 0$, insert the operator $\hat{E}(\mathbf{r})$ from Problem 9 into Eq (14), to show that when n photons are present with energy ω direction $\mathbf{k}$ as in this mode, we have

$$\begin{aligned}
\text{Absorption rate} \;&=\; \mathcal{B} \times n \\
\mathcal{B} \;&=\; \text{factor} \times \left|\mathcal{M}_{if}^{\text{atom}}\right|^2 \times \frac{\hbar\omega}{2\epsilon_0 V}
\end{aligned} \tag{16}$$

This $\mathcal{B}$ is the same as Einstein's "B" absorption coefficient, but expressed in terms of number of photons present instead of light intensity.

(b) Now insert the operator $\hat{E}(\mathbf{r})$ from Problem 9 into Eq (15), to show that when n photons are present with energy ω, and the direction $\mathbf{k}$ of this mode, there are two contributions to emission, and one of them is exactly the same as part (a). This is *induced emission.*

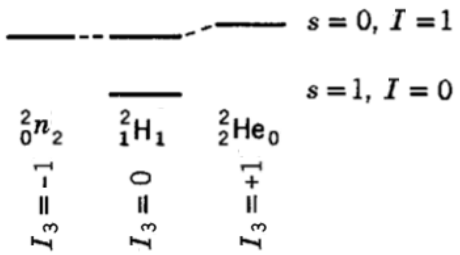
(c) Explain why the *other* contribution to emission is present even when no photons are present, and how this must be *spontaneous emission.* Explain why, since this is occuring into modes of energy $\hbar\omega$ in *all* directions, the total coefficient $\mathcal{A}$ of spontaneous emission will be different and larger than $\mathcal{B}$.

Homework 9 - SOLUTIONS

Answer to Problem 1. (a) From the isospin point of view all nucleons are identical. If the spatial state is symmetric under exchange, then Pauli antisymmetry must come from the product of spin state times isospin state. In either factor the singlet is antisymmetric and the triplet is symmetric. So we must have either spin singlet and isospin triplet, or the reverse. Thus S=0 must go with I=1, and S=1 must go with I=0.

(b) In the nn state (dineutron) both nucleons have $I_3 = -1/2$. Thus this must be the $I = 1$, or triplet isospin state. Similarly pp state must also belong to $I = 1$. Then by part (a), these must be $S = 0$ or spin singlet states. (We could also deduce this directly from Pauli for the identical-neutron nn and identical-proton pp states, given that the spatial state is symmetric, without use of isospin.) Now the deuteron was said to be $S = 1$. Since the deuteron has lower energy than the nn and pp states (since these are not bound) the spin-spin interaction must favor aligned spins.

(c) The diagram is as follows:



(d) Analogously to spin states as in He, the triplet (I=1) isospin states corresponding to I_3=$-1, 0, +1$ are

$$S{=}0: \begin{cases} |nn\rangle = |I{=}1, I_3{=}{-1}\rangle \\ \frac{1}{\sqrt{2}}\big(|np\rangle + |pn\rangle\big) = |I{=}1, I_3{=}0\rangle \\ |pp\rangle = |I{=}1, I_3{=}{+1}\rangle \end{cases}$$

Each of the above has only one spin orientation, $S_z = 0$. The singlet (I=0) isospin state, the deuteron, is

$$S{=}1: \quad \frac{1}{\sqrt{2}}\big(|np\rangle - |pn\rangle\big) = |I{=}0, I_3{=}0\rangle$$

The above state has three substates: $S_z = -1, 0, 1$. □

Answer to Problem 2. (Not provided here) □

Answer to Problem 3. For parts (a) through (d), examine the integral (1), integrating by parts and complex conjugating where necessary. □

Answer to Problem 4. (a) Following the hint,

$$\langle\phi|\hat{A}\hat{B}|\psi\rangle = \int(\hat{A}^\dagger\phi)^*\hat{B}\psi\,dx = \int(\hat{B}^\dagger\hat{A}^\dagger\phi)^*\psi\,dx$$

so by definition $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

(b) Following the hint,

$$\langle\psi|\hat{A}|\phi\rangle^* = \int(\hat{A}^\dagger\psi)\phi^*dx = \int\phi^*\hat{A}^\dagger\psi\,dx = \langle\phi|\hat{A}^\dagger|\psi\rangle.$$

(c) Following the hint, $\langle\phi|(\hat{A}^\dagger)^\dagger|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^* = \langle\phi|\hat{A}|\psi\rangle$. Thus $(\hat{A}^\dagger)^\dagger = \hat{A}$.

(d) On one hand

$$\int(\psi')^*\psi'dx = \int(C\phi)^*C\phi\,dx = C^2\int\phi^*\phi\,dx = C^2$$

since ϕ is normalized. On the other hand,

$$\begin{aligned}\int(\psi')^*\psi'dx &= \int(\hat{A}\psi)^*\hat{A}\psi\,dx \\ &= \int\psi^*\hat{A}^\dagger\hat{A}\psi\,dx \quad \text{(by definition of } \hat{A}^\dagger \text{ and part (c))} \\ &= \langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle.\end{aligned}$$

Comparing these results gives $C^2 = \langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle$. □

Answer to Problem 5. (Not provided here) □

Answer to Problem 6. (a) Since $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$, we have $\hat{N}^\dagger = (a^\dagger a)^\dagger = a^\dagger(a^\dagger)^\dagger = a^\dagger a = \hat{N}$. So $\hat{N}$ is Hermitian.

(b) Using $a^\dagger a - aa^\dagger = 1$,

$$\begin{aligned} a^\dagger a + aa^\dagger &= a^\dagger a + (1 + a^\dagger a) = 2a^\dagger a + 1 \\ \tfrac{1}{2}\hbar\omega\big(a^\dagger a + aa^\dagger\big) &= \tfrac{1}{2}\hbar\omega(2a^\dagger a + 1) = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \end{aligned}$$

□

Answer to Problem 7. Calculate

$$\begin{aligned} a^\dagger a &= \tfrac{1}{2}\Big(\alpha\,x - \frac{1}{\alpha}\frac{d}{dx}\Big)\Big(\alpha\,x + \frac{1}{\alpha}\frac{d}{dx}\Big) \\ &= \tfrac{1}{2}\Big(\alpha^2\,x^2 - \frac{1}{\alpha^2}\frac{d^2}{dx^2} - 1\Big) \\ aa^\dagger &= \tfrac{1}{2}\Big(\alpha^2\,x^2 - \frac{1}{\alpha^2}\frac{d^2}{dx^2} + 1\Big) \end{aligned}$$

From these we get

$$\begin{aligned} a^\dagger a - aa^\dagger &= 1 \\ \frac{\hbar\omega}{2}\big(a^\dagger a + aa^\dagger\big) &= \frac{\hbar\omega}{2}\Big[\Big(\frac{\omega}{\hbar}\Big)x^2 - \Big(\frac{\hbar}{\omega}\Big)\frac{d^2}{dx^2}\Big] \\ &= -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \tfrac{1}{2}\omega^2x^2 \end{aligned}$$

as claimed. □

Answer to Problem 8. (a) Using $\hat{N} = a^\dagger a$ and Eq (5),

$$\begin{aligned} \hat{N}a|n\rangle &= a^\dagger aa|n\rangle = (aa^\dagger - 1)a|n\rangle \\ &= a(a^\dagger a - 1)|n\rangle = a(\hat{N} - 1)|n\rangle = (n{-}1)\,a|n\rangle \end{aligned}$$

which is (9). Next,

$$\begin{aligned} \hat{N}a^\dagger|n\rangle &= a^\dagger aa^\dagger|n\rangle = a^\dagger(a^\dagger a + 1)|n\rangle \\ &= a^\dagger(\hat{N} + 1)|n\rangle = (n{+}1)a^\dagger|n\rangle \end{aligned}$$

which is (10).

(c) In part (b) we found that $\hat{N}a|n\rangle = (n{-}1)\,a|n\rangle$. Following the hint, it *must* be that $a|n\rangle = C|n{-}1\rangle$. Directly applying Problem 4, part (d), we get

$$\begin{aligned} C^2 &= \langle n|a^\dagger a|n\rangle = \langle n|\hat{N}|n\rangle = n\langle n|n\rangle = n \\ &\Rightarrow a|n\rangle = \sqrt{n}|n{-}1\rangle \end{aligned}$$

Also in part (b) we found that $\hat{N}a^\dagger|n\rangle = (n{+}1)\,a^\dagger|n\rangle$ Thus it *must* be that $a^\dagger|n\rangle = D|n{+}1\rangle$ for some number D. Again applying Problem 4, part (d), we get

$$\begin{aligned} D^2 &= \langle n|aa^\dagger|n\rangle = \langle n|(a^\dagger a + 1)|n\rangle = (n{+}1)\langle n|n\rangle \\ &\Rightarrow a^\dagger|n\rangle = \sqrt{n{+}1}|n{+}1\rangle. \end{aligned}$$

□

Answer to Problem 9. Plug in:

$$\int_V \left[\frac{\epsilon_0}{2}\left(-\frac{\hbar\omega}{2\epsilon_0 V}\right)\left(a^2 e^{2i\mathbf{k}\cdot\mathbf{r}} - aa^\dagger - a^\dagger a + (a^\dagger)^2 e^{-2i\mathbf{k}\cdot\mathbf{r}}\right)\right.$$

$$\left. + \frac{1}{2\mu_0}\left(\frac{\mu_0\hbar\omega}{2V}\right)\left(a^2 e^{2i\mathbf{k}\cdot\mathbf{r}} + aa^\dagger + a^\dagger a + (a^\dagger)^2 e^{-2i\mathbf{k}\cdot\mathbf{r}}\right)\right] dV$$

$$= \frac{\hbar\omega}{2V}\int_V (aa^\dagger + a^\dagger a)\, dV$$

$$= \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) \;=\; \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right).$$

This is precisely the expected quantum Hamiltonian, representing the photons in the field as excitations. □

Answer to Problem 10. (a) Let's locate our atom at $\mathbf{r} = 0$, where the operator $\hat{E}$ from Problem 9 becomes

$$\hat{E}(\mathbf{r}{=}0) \;=\; i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\left(a - a^\dagger\right)$$

At $\mathbf{r} = 0$, absorption is

$$\langle n{-}1|\hat{E}(\mathbf{r}{=}0)|n\rangle \;=\; i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\langle n{-}1|\left(a - a^\dagger\right)|n\rangle$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\langle n{-}1|\Big(\sqrt{n}\,|n{-}1\rangle - \sqrt{n{+}1}|n{+}1\rangle\Big)$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\Big(\sqrt{n}\,\langle n{-}1|n{-}1\rangle - \sqrt{n{+}1}\cancelto{0}{\langle n{-}1|n{+}1\rangle}\Big)$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\sqrt{n}$$

This means the rate of absorption is

$$\text{Absorption rate} \;=\; \text{factor} \times \left|\mathcal{M}_{fi}^{\text{atom}}\right|^2 \times \frac{\hbar\omega}{2\epsilon_0 V} \times n$$

$$= \mathcal{B} \times n$$

On the other hand, at $\mathbf{r} = 0$, emission is

$$\langle n{+}1|\hat{E}(\mathbf{r}{=}0)|n\rangle \;=\; i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\langle n{+}1|\left(a - a^\dagger\right)|n\rangle$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\langle n{+}1|\Big(\sqrt{n}\,|n{-}1\rangle - \sqrt{n{+}1}|n{+}1\rangle\Big)$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\Big(\sqrt{n}\,\cancelto{0}{\langle n{+}1|n{-}1\rangle} - \sqrt{n{+}1}\langle n{+}1|n{+}1\rangle\Big)$$

$$= i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\,\sqrt{n{+}1}$$

This means the rate of emission is

$$\text{Absorption rate} \;=\; \text{factor} \times \left|\mathcal{M}_{fi}^{\text{atom}}\right|^2 \times \frac{\hbar\omega}{2\epsilon_0 V} \times (n+1)$$

$$= \mathcal{B} \times (n+1)$$

$$= \mathcal{B} \times n(\text{into this mode}) + \mathcal{B}(\text{into all empty modes})$$

□