Homework 8

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1. The neutral atom will have a dipole moment
$$\mathbf{p} = \alpha \mathbf{E}$$
If the electric field is from a point charge r from the atom, then the force is given by
$$\begin{aligned}\mathbf{F} &= (\mathbf{p}\cdot\boldsymbol{\nabla})\,\mathbf{E}\\ &= (\alpha\mathbf{E}\cdot\boldsymbol{\nabla})\,\mathbf{E}\\ &= \alpha E\left(\frac{\partial E_r}{\partial r}\right)\hat{\mathbf{r}}\\ &= \alpha\frac{q}{4\pi\epsilon r^2}\left(\frac{-2q}{4\pi\epsilon r^3}\right)\hat{\mathbf{r}}\\ &\boxed{\mathbf{F} = -\frac{\alpha q^2}{8\pi^2\epsilon^2 r^5}\,\hat{\mathbf{r}}}\end{aligned}$$

2. (a) From Gauss's law and using WolframAlpha to integrate,
$$\begin{aligned}E\left(4\pi r^2\right) &= \frac{4\pi}{\epsilon_0}\int_0^r\left(\frac{q}{\pi a^3}e^{-2r/a}\right)r^2\,\mathrm{d}r\\ &= \frac{4q}{a^3\epsilon_0}\left(\frac{a}{4}\right)\left[a^2 - e^{-2r/a}\left(a^2+2ar+2r^2\right)\right]\\ \mathbf{E}(r) &= \frac{q}{4\pi\epsilon_0 a^2 r^2}\left[a^2 - e^{-2r/a}\left(a^2+2ar+2r^2\right)\right]\hat{\mathbf{r}}\end{aligned}$$
(b) Expanding the exponential as $e^x = \sum_n x^n/n!$,
$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0 a^2 r^2}\left[a^2 - \left(1 - \frac{2r}{a} + \frac{4r^2}{2a^2} - \frac{8r^3}{6a^3} + \cdots\right)\left(a^2+2ar+2r^2\right)\right]\hat{\mathbf{r}}\\ &= \frac{q}{4\pi\epsilon_0 a^2 r^2}\left[\frac{4r^3}{3a^3} + \cdots\right] \text{ (used WolframAlpha to simplify)}\\ &= \frac{qr}{3\pi\epsilon_0 a^5}\end{aligned}$$
Matching the terms to (4.1),
$$\begin{aligned}\alpha &= 3\pi\epsilon_0 a^5\\ &= 3.45\times 10^{-62}\,\mathrm{C}\cdot\mathrm{m}\cdot\mathrm{N}^{-1}\end{aligned}$$
Pretty sure the a^5 is wrong here as a^3 gives a relatively accurate value.

3. This is pretty hand-wavy: work due to a torque over an angle can be generalized from work due to a force over a distance,
$$\begin{aligned}U &= \int \mathbf{N}\cdot\left(\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}\right)\\ &= \int pE\sin\theta\,\mathrm{d}\theta = pE\int\sin\theta\,\mathrm{d}\theta\\ &= pE\cos\theta = \mathbf{p}\cdot\mathbf{E}\end{aligned}$$
Since the work is done by the force, the result would have a negative sign attached (I think?).

4. (a) For the inner surface at $r = a$, the normal is pointing inward, so the bound surface charge is

$$\begin{aligned}\sigma_b\bigg|_{r=a} &= \mathbf{P}\cdot(-\,\hat{\mathbf{r}})\\ &= -\frac{k}{a^2}\end{aligned}$$

On the outer surface, the normal is pointing outward,

$$\sigma_b\bigg|_{r=b} = \frac{k}{b^2}$$

The bound volume charge is given by the divergence of the polarization,

$$\begin{aligned}\rho_b &\equiv -\boldsymbol{\nabla}\cdot\mathbf{P}\\ &= 0 \text{ as the } r^2 \text{ cancels out}\end{aligned}$$

(b) i. Within $r < a$, there is no enclosed charge, so $\mathbf{E} = 0$.

ii. From Gauss's law, the charge distribution is uniform through r,

$$\begin{aligned}E_r\left(4\pi r^2\right) &= \frac{1}{\epsilon_0}\left[\sigma_a A_a\right]\\ &= \frac{1}{\epsilon_0}\left[\sigma_a 4\pi a^2\right]\\ \mathbf{E} &= -\frac{k}{\epsilon_0 r^2}\,\hat{\mathbf{r}}\end{aligned}$$

iii. The bound surface charges σ_a and σ_b would cancel, so $\mathbf{E} = 0$.

5. Converting this to Cartesian and noting that $\hat{\mathbf{r}}\cdot\,\hat{\mathbf{n}}$ is positive on all sides, then for a single side perhaps in $+\,\hat{\mathbf{x}}$,

$$\begin{aligned}\sigma_b &\equiv \mathbf{P}\cdot\,\hat{\mathbf{n}} = kr\,\hat{\mathbf{r}}\cdot\,\hat{\mathbf{n}}\\ &= k\left(x^2+y^2+z^2\right)^{1/2}\frac{x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}}{\left(x^2+y^2+z^2\right)^{1/2}}\cdot(\,\hat{\mathbf{x}})\\ &= kx = \frac{ka}{2}\end{aligned}$$

For all six sides, the total charge is

$$Q_\sigma = \sigma_b A = \frac{6ka}{2}a^2 = 3ka^3$$

For the bound charges,

$$\begin{aligned}\rho_b &\equiv -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{k}{r^2}\frac{\partial}{\partial r}r^3 = -3k\\ Q_\rho &= -3ka^3\end{aligned}$$

The total bound charges are equal to each other. Could I have just shown this to be true with the divergence theorem?