Homework 8

PHYSICS 450
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1. Starting from (2.58) and following the proof on p. 42,

$$\begin{aligned}
\hat{H} &= \hbar\omega\left(\hat{a}_{\pm}\hat{a}_{\mp} \pm \frac{1}{2}\right), \\
\hat{H}(\hat{a}_-\psi) &= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_-\psi \\
&= \hbar\omega\hat{a}_-\left(\hat{a}_+\hat{a}_- - \frac{1}{2}\right)\psi
\end{aligned}$$

As $[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1 \implies \hat{a}_+\hat{a}_- = \hat{a}_-\hat{a}_+ - 1$,

$$\begin{aligned}
&= \hat{a}_-\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ - 1 - \frac{1}{2}\right)\psi\right] \\
&= \hat{a}_-\left[\underbrace{\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)}_{\hat{H}} - \hbar\omega\right]\psi \\
\hat{H}(\hat{a}_-\psi) &= \hat{a}_-\left(E - \hbar\omega\right)\psi \quad \square
\end{aligned}$$

Therefore $E - \hbar\omega$ is a solution to the Hamiltonian of the lowering operator on a wavefunction.

On the ground state, the energy is $E_0 = \hbar\omega/2$ and lowering it results in a negative energy

$$E_{-1} = -\hbar\omega/2.$$

2. From the analytical solution (2.86),

$$\begin{aligned}
\psi(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\xi^2/2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}.
\end{aligned}$$

Then applying the raising operator $\hat{a}_+$,

$$\begin{aligned}
\hat{a}_+\psi(x) &= \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega x\right)\psi(x) \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(m\omega x e^{-m\omega x^2/2\hbar} + m\omega x e^{-m\omega x^2/2\hbar}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}\left(2\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar}.
\end{aligned}$$

3. Study Chapter 3.5.