# Homework 1

PHYSICS 342
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1. (a) From Gauss's Law, the electric field as a function of $r$ and $Q$ is

$$\begin{aligned}\int \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{Q}{\epsilon_0} \\ \mathbf{E}(r, Q) &= \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}\end{aligned}$$

   (b) From the definition of the potential,

$$\begin{aligned}V &= -\int_b^a \mathbf{E} \cdot \mathrm{d}\vec{\ell} \\ &= -\frac{Q}{4\pi\epsilon_0}\int_b^a r^{-2}\,\mathrm{d}r \\ &= \frac{Q}{4\pi\epsilon_0}\, r^{-1}\Big|_b^a \\ &= \frac{Q}{4\pi\epsilon_0}\left(a^{-1} - b^{-1}\right)\end{aligned}$$

   Solving for $Q$,

$$Q(V) = 4\pi\epsilon_0 V \left(a^{-1} - b^{-1}\right)^{-1}$$

   (c) The current is found using (a) and (b),

$$\begin{aligned}I &= \int \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \sigma \int \mathbf{E} \cdot \mathrm{d}\mathbf{a} \\ &= \iint \left(\frac{\sigma}{4\pi\epsilon_0} r^{-2}\right)\left[4\pi\epsilon_0 V \left(a^{-1} - b^{-1}\right)\right] \underbrace{r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}_{\mathrm{d}a} \\ &= \frac{4\pi\sigma V}{a^{-1} - b^{-1}}\end{aligned}$$

   (d) The resistance is given by $V/I$,

$$R = \frac{a^{-1} - b^{-1}}{4\pi\sigma}$$

2. From the voltage and as the current is constant,

$$\begin{aligned}V &= -\int \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int \frac{\mathbf{J}}{\sigma} \cdot \mathrm{d}\mathbf{l} \\ &= -\int \frac{\mathbf{I}}{\sigma A} \cdot \mathrm{d}\mathbf{l} = -\int_a^b \frac{I s^2}{k(2\pi s L)}\,\mathrm{d}s \\ &= \frac{I}{4k\pi L}(a^2 - b^2)\end{aligned}$$

   As the resistance is $V/I$,

$$R = \frac{a^2 - b^2}{4k\pi L}$$

3. Since the power dissipated by the load is given by

$$\begin{aligned} P &= I^2 R \\ &= \left(\frac{V^2}{(r+R)^2}\right) R \end{aligned}$$

The power is maximized when its derivative is 0. Omitting the voltage $V$,

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}R} &= \frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3} \\ &= \frac{r+R}{(r+R)^3} - \frac{2R}{(r+R)^3} \\ 0 &= \frac{r-R}{(r+R)^3} \end{aligned}$$

This occurs when $R = r$.

4. (a) The magnetic field from the wire is found using Ampere's law,

$$B = \frac{\mu_0 I}{2\pi s}$$

Changing $s \to x$, the flux is

$$\begin{aligned} \Phi &\equiv \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} \\ &= \frac{\mu_0 I}{2\pi} \int_0^l \int_d^{d+w} x^{-1} \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{\mu_0 I l}{2\pi} \ln\left(\frac{d+w}{d}\right) \end{aligned}$$

(b) The flux would change with $d = vt$,

$$\Phi = \frac{\mu_0 I l}{2\pi} \ln\left(1 + \frac{w}{vt}\right)$$

The induced emf is

$$\begin{aligned} \mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= -\frac{\mu_0 I l}{2\pi}\left(\frac{v}{vt+w} - \frac{1}{t}\right) \qquad \text{(WolframAlpha)} \end{aligned}$$

From Lenz's law, the induced current should flow counterclockwise.

5. (a) From Ampere's law, we can create an Amperian loop of length $L$,

$$\begin{aligned} \oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} &= \mu_0 I_{\text{enc}} \\ BL &= \mu_0 n L I(t) \\ B &= \mu_0 n I_0 \cos \omega t \end{aligned}$$

(b) The flux through the loop of radius $s$ is limited to the radius $a$ of the solenoid, as the magnetic field is zero outside of $a$. The flux is

$$\begin{aligned}\Phi &= \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} \\ &= (\mu_0 n I_0 \cos \omega t) \left(\pi a^2\right)\end{aligned}$$

The induced emf is then

$$\begin{aligned}\mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= \mu_0 n I_0 \pi a^2 \omega \sin \omega t\end{aligned}$$