Problem Set 7

PHYSICS 463
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6.6 ***Frequency dependence of the electrical conductivity.*** Use the equation $m(\mathrm{d}v/\mathrm{d}t + v/\tau) = -eE$ for the electron drift velocity v to show that the conductivity at frequency ω is

$$\sigma(\omega) = \sigma(0)\left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right),$$

where $\sigma(0) = ne^2\tau/m$.

Solution. From the Drude model discussion in class, we can let

$$\begin{aligned} E &= E_0 e^{-i\omega t} \\ v &= v_D e^{-i\omega t}, \end{aligned}$$

where $v_0 \in \mathbb{C}$ for a phase offset. Then applying the damping equation above, the relation between v and E is

$$\begin{aligned} (-i\omega)v_0 e^{-i\omega t} + \frac{1}{\tau}v_0 e^{-i\omega t} &= \frac{e}{m}E_0 e^{-i\omega t} \\ v_D &= \frac{eE_0}{m}\frac{1}{1/\tau - i\omega} = \frac{eE_0\tau}{m}\frac{1}{1 - i\omega\tau} \\ &= \frac{eE_0\tau}{m}\left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right). \end{aligned}$$

Then, by the definition of current density,

$$\begin{aligned} J &= -nev_D = \sigma E \\ &= \underbrace{\frac{ne^2E_0\tau}{m}}_{\text{dc cond. } \sigma_0}\left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right) \\ \sigma &= \sigma(0)\left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right). \quad \square \end{aligned}$$

6.9 ***Static magnetoconductivity tensor.*** For the drift velocity theory of (51), show that the static current density can be written in matrix form as

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0}{1+(\omega_c\tau)^2} \begin{pmatrix} 1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & 1+(\omega_c\tau)^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}.$$

In the high magnetic field limit of $\omega_c\tau \gg 1$, show that

$$\sigma_{yx} = nec/B = -\omega_{xy}.$$

In this limit, $\sigma_{xx} = 0$, to order $1/\omega_c\tau$. The quantity σ_{yx} is called the Hall conductivity.

Solution. From Ohm's law, $\mathbf{j} = nq\mathbf{v}$ and using (52),

$$\begin{aligned} j_x &= nqv_x = ne\left(-\frac{e\tau}{m}E_x - \omega_c\tau v_y\right); \\ j_y &= nqv_y = ne\left(-\frac{e\tau}{m}E_y + \omega_c\tau v_x\right); \\ j_z &= nqv_z = ne\left(-\frac{e\tau}{m}E_z\right). \end{aligned}$$

Since we're dealing with the static current case where

$$\mathbf{j} = ne^2\tau\mathbf{E}/m,$$

we can make some substitutions,

$$\begin{aligned} j_x &= -\frac{ne^2\tau}{m}E_x - \omega_c\tau nev_y \\ &= -\frac{ne^2\tau}{m}E_x - \omega_c\tau j_y \\ &= -\frac{ne^2\tau}{m}E_x - \omega_c\tau\left(ne^2\tau/m\right)E_y; \\ j_y &= \frac{ne^2\tau}{m}E_y + \omega_c\tau(ne^2\tau/m)E_x; \\ j_z &= \text{(unchanged)}. \end{aligned}$$

Putting this in matrix form,

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{ne\tau}{m} \begin{pmatrix} -1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}?$$

In the high magnetic field limit of $\omega_c\tau \gg 1$, we can omit the $+1$ terms, and the conductivity looks something like

$$\sigma = \frac{\sigma_0}{(\omega_c\tau)^2} \begin{pmatrix} 0 & -\omega_c\tau & 0 \\ \omega_c\tau & 0 & 0 \\ 0 & 0 & (\omega_c\tau)^2 \end{pmatrix}.$$

The yx element is then

$$\begin{aligned}\omega_{yx} &= \frac{\sigma_0}{(\omega_c\tau)^2}\omega_c\tau \\ &= \sigma_0/\omega_c\tau \\ &= \frac{ne^2}{m\omega_c} = \frac{ne^2mc}{meB} \\ &= nec/B. \quad \square\end{aligned}$$

6.10 ***Maximum surface resistance.*** Consider a square sheet of side L, thickness d, and electrical resistivity ρ. The resistance measured between opposite edges of the sheet is called the surface resistance: $R_{sq} = \rho L/Ld = \rho/d$, which is independent of the area L^2 of the sheet. (R_{sq} is called the resistance per square and is expressed in ohms per square, because ρ/d has the dimensions of ohms.) If we express ρ by (44), then $R_{sq} = m/nde^2\tau$.

Suppose now that the minimum value of the collision time is determined by scattering from the surface of the sheet, so that $\tau \approx d/v_F$, where v_F is the Fermi velocity. Thus the maximum surface resistivity is $R_{sq} \approx mv_F/nd^2e^2$.

Show for a monatomic metal sheet one atom in thickness that $R_{sq} \approx \hbar/e^2 = 4.1\,\mathrm{k}\Omega$.

Solution. We can assume for a one atom thick sheet, the density is

$$n = \frac{N}{V} = \frac{1}{d^3},$$

and the maximum surface resistance is

$$R_{sq} = \frac{mv_Fd}{e^2}.$$

If we assume $\lambda \approx d$,

$$\begin{aligned} p = mv_F &= \hbar k \approx \hbar/\lambda \\ \implies mv_Fd &\approx \hbar.\end{aligned}$$

Then the resistance is

$$R_{sq} = \hbar/e^2 \approx 4.1\,\mathrm{k}\Omega.$$