Homework 4

MATH 364
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3.1.3 Put the following problems into standard form.

(f) Maximize $x_1 + 2x_2 + 4x_3$
subject to
$|4x_1 + 3x_2 - 7x_3| \leq x_1 + x_2 + x_3$
$x \geq 0$

Solution. First, we can change the objective from a maximization to a minimization and flipping the sign,

$$\min_x -x_1 - 2x_2 - 4x_3.$$

Next, the absolute value constraint can be split into two constraints,

$$\begin{aligned} 4x_1 + 3x_2 - 7x_3 &\leq x_1 + x_2 + x_3 \\ -4x_1 + -3x_2 + 7x_3 &\leq x_1 + x_2 + x_3. \end{aligned}$$

Then, we can add in slack variables x_4 and x_5, convert these into equalities,

$$\begin{aligned} 4x_1 + 3x_2 - 7x_3 + x_4 &= x_1 + x_2 + x_3 \\ -4x_1 + -3x_2 + 7x_3 + x_5 &= x_1 + x_2 + x_3. \end{aligned}$$

Subtracting out the extraneous terms,

$$\begin{aligned} 3x_1 + 2x_2 - 8x_3 + x_4 &= 0 \\ -5x_1 + -4x_2 + 6x_3 + x_5 &= 0. \end{aligned}$$

The linear program in standard form is

$$\begin{aligned} \min_x \quad & -x_1 - 2x_2 - 4x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 - 8x_3 + x_4 = 0 \\ & -5x_1 + -4x_2 + 6x_3 + x_5 = 0 \\ & x \geq 0 \end{aligned}$$

(g) Maximize $x_1 + 6x_2 + 12x_3$
subject to
$-x_1 - x_2 + x_4 \geq$ maximum of $7x_1 + 2x_2$ and $5x_2 + x_3 + x_4$
$x \geq 0$

Solution. We can convert the maximization into a minimization by flipping signs again,

$$\min_x -x_1 - 6x_2 - 12x_3.$$

The "maximum of..." constraint can be split into two separate constraints,

$$\begin{aligned} -x_1 - x_2 + x_4 &\geq 7x_1 + 2x_2 \\ -x_1 - x_2 + x_4 &\geq 5x_2 + x_3 + x_4. \end{aligned}$$

Adding in slack variables, we can convert this into equalities,

$$\begin{aligned} -x_1 - x_2 + x_4 + x_5 &= 7x_1 + 2x_2 \\ -x_1 - x_2 + x_4 + x_6 &= 5x_2 + x_3 + x_4. \end{aligned}$$

Making one side a constant, the equality constraints are

$$\begin{aligned} -8x_1 - 3x_2 + x_4 + x_5 &= 0 \\ -x_1 - 6x_2 - x_3 + x_6 &= 0. \end{aligned}$$

The linear program in standard form is

$$\boxed{\begin{aligned} \min_x \quad & -x_1 - 6x_2 - 12x_3 \\ \text{s.t.} \quad & -8x_1 - 3x_2 + x_4 + x_5 = 0 \\ & -x_1 - 6x_2 - x_3 + x_6 = 0 \\ & x \geq 0 \end{aligned}}$$

3.2.3 A system of equations is said to be *inconsistent* if the system has no solution. Show by using the pivot operation that the following systems are inconsistent. Is either of these systems equivalent to a system in canonical form?

(a) $x_1 + 2x_2 = 3$
$x_1 + 2x_2 = 4$

Solution. Putting this into matrix form, it's clear that because the coefficients of x_1 and x_2 are equal, there is no pivot operation to separate out a single variable,

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 4 \end{array}\right).$$

Because the coefficients are equal, the last row will be something like $(0, 0, -1)$, so this system is not solvable. ~~This system is also not in canonical form, as if we take either x_1 or x_2 as the basic variable, there is no feasible solution.~~ I'm not sure, since a canonical form requires an objective function.

(b) $x_1 + x_2 - 3x_3 = 7$
$-2x_1 + x_2 + 5x_3 = 2$
$3x_2 - x_3 = 15$

Solution. In matrix form, this becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ -2 & 1 & 5 & 2 \\ 0 & 3 & -1 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 3 & -1 & 16 \\ 0 & 3 & -1 & 15 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 3 & -1 & 16 \\ 0 & 3 & -1 & 15 \end{array}\right)$$

Because row 2 and row 3 have equal coefficients, the last row will be something like $(0, 0, 0, 1)$, which means this system is not solvable. ~~I think this could be in canonical form, if we take x_3 as the non-basic variable and set it to 0, we would have a basic solution.~~ Can this even be in canonical form if there is no objective function z?