Homework 4

PHYSICS 341
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1. For a sphere with charge density $\rho = kr^2$, the charge enclosed within a Gaussian surface is

$$\begin{aligned}
q_{\text{enc}} &= \int_0^r \rho(r')\,\mathrm{d}\tau \\
&= 4k\pi \int_0^r r'^4\,\mathrm{d}r' \\
&= \frac{4k\pi r^5}{5}
\end{aligned}$$

From Gauss' law, since the electric field is outward and uniform in the $\hat{\mathbf{r}}$ direction as the charge density only depends on r,

$$\begin{aligned}
\mathbf{E} \cdot \int_S \mathrm{d}\mathbf{a} &= \frac{q_{\text{enc}}}{\epsilon_0} \\
\mathbf{E} &= \frac{4k\pi r^5}{5\epsilon_0} \left(\frac{4}{3}\pi r^3\right)^{-1} \hat{\mathbf{r}} \\
&= \frac{3r^2}{5\epsilon_0}\,\hat{\mathbf{r}}
\end{aligned}$$

2. i. Within the inner cylinder, the enclosed charge of a Gaussian surface is

$$\begin{aligned}
q_{\text{enc}} &= \int \rho(s')\,\mathrm{d}\tau \\
&= 2\pi k L \int_0^s s'^2\,\mathrm{d}s' \\
&= \frac{2\pi k L s^3}{3}
\end{aligned}$$

As the electric field will point cylindrically outward in $\hat{\mathbf{s}}$ and from Gauss' law,

$$\begin{aligned}
\mathbf{E} &= \frac{2\pi k L s^3}{3\epsilon_0} \frac{1}{2\pi s L}\,\hat{\mathbf{s}} \\
&= \frac{ks^2}{3\epsilon_0}\,\hat{\mathbf{s}}
\end{aligned}$$

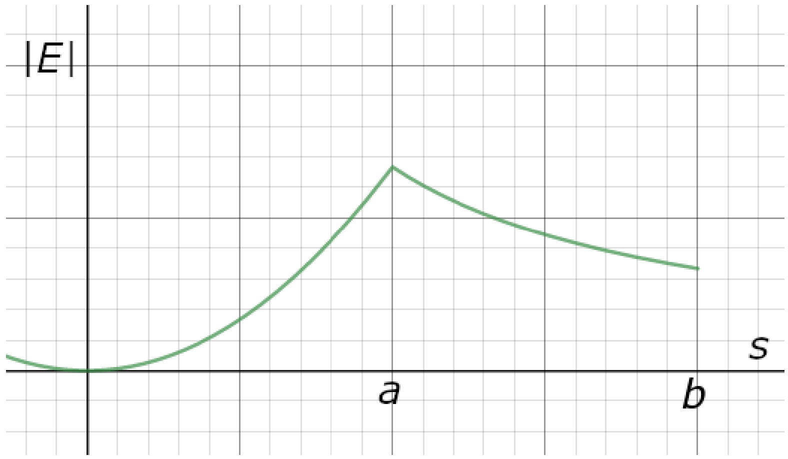
ii. Between the cylinders, the enclosed charge will be the entirety of the inner cylinder,

$$q_{\text{enc}} = \frac{2\pi k L a^3}{3}$$

From a similar approach as above,

$$\mathbf{E} = \frac{ka^3}{3\epsilon_0 s}$$

iii. Outside the cylinders, the enclosed charge is zero and there will be no electric field, $\mathbf{E} = 0$.



3. Integrating the electric field along s,

$$V = -\frac{k}{3\epsilon_0}\left[\int_0^a s^2\,\mathrm{d}s + a^3\int_a^b s^{-1}\,\mathrm{d}s\right]$$
$$= -\frac{k}{3\epsilon_0}\left[\frac{a^3}{3} + a^3\ln\left(\frac{b}{a}\right)\right]$$
$$= -\frac{ka^3}{3\epsilon_0}\left[\frac{1}{3} + \ln\left(\frac{b}{a}\right)\right]$$

4. The distance between a segment of the charged ring and the observation point is given as

$$|\mathscr{r}| = \left(z^2 + R^2\right)^{1/2}$$

Using the distance above, the potential is

$$V(z) = \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi}\left(z^2+R^2\right)^{-1/2} R\,\mathrm{d}\phi$$
$$= \frac{\lambda R}{2\epsilon_0\left(z^2+R^2\right)}$$

From this potential, the electric field can be determined using the gradient of V,

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\frac{\partial}{\partial z}V(z)$$
$$= -\frac{\lambda R}{2\epsilon_0}\left(-\frac{1}{2\left(z^2+R^2\right)^{3/2}}\right)(2z)\ \hat{\mathbf{z}}$$
$$= \frac{\lambda R z}{2\epsilon_0\left(z^2+R^2\right)^{3/2}}\,\hat{\mathbf{z}}$$

5. From the potential
$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r}$$
The electric field $\mathbf{E}(\mathbf{r})$ is found using the negative gradient
$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\left[A\frac{e^{-\lambda r}}{r}\right] \\
&= -A\frac{\partial}{\partial r}\frac{e^{-\lambda r}}{r}\,\hat{\mathbf{r}} = A\left[\frac{\lambda e^{-\lambda r}}{r} + \frac{e^{-\lambda r}}{r^2}\right]\hat{\mathbf{r}} \\
&= \frac{Ae^{-\lambda r}}{r}\left(\lambda + \frac{1}{r}\right)\hat{\mathbf{r}}
\end{aligned}$$
The charge density is given by Gauss's law,
$$\begin{aligned}
\rho(r) &= \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\epsilon_0}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) \\
&= \frac{A\epsilon_0}{r^2}\boldsymbol{\nabla}\cdot\left[e^{-\lambda r}\left(\lambda r + 1\right)\left(\frac{\hat{\mathbf{r}}}{r^2}\right)\right]
\end{aligned}$$
Applying the product rule and delta properties,
$$\begin{aligned}
&= \frac{A\epsilon_0}{r^2}\left[4\pi e^{-\lambda r}\left(\lambda r + 1\right)\delta(r) + \frac{\mathbf{r}}{r^2}\cdot\boldsymbol{\nabla}\left[e^{-\lambda r}\left(\lambda r + 1\right)\right]\right] \\
&= \frac{A\epsilon_0}{r^2}\left[4\pi e^{-\lambda r}\left(\lambda r + 1\right)\delta(r) + \frac{1}{r^2}\left(\lambda e^{-\lambda r} - \lambda e^{-\lambda r}\left(\lambda r + 1\right)\right)\right] \\
&= \frac{A\epsilon_0}{r^2}\left(e^{-\lambda r}\right)\left[4\pi(\lambda r + 1)\delta(r) - \frac{\lambda^2}{r}\right]
\end{aligned}$$