Homework 6

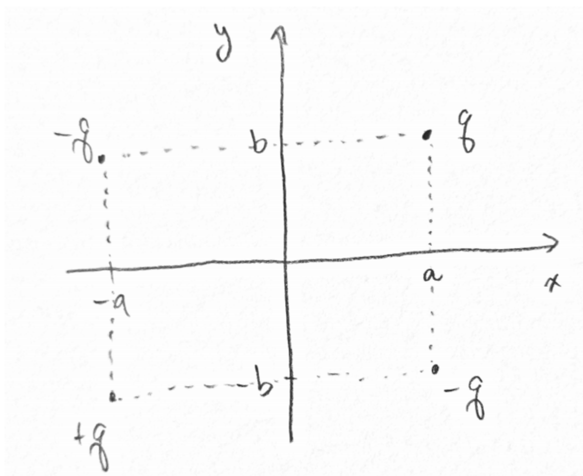
PHYSICS 341 Kevin Evans
October 14, 2020 ID: 11571810

---

1. From the method of images, the arrangement is equivalent to a $-q$ charge at $-d$ and a q charge at $-2d$. Summing these,

$$\begin{aligned}
\mathbf{F} &= q \sum_i \mathbf{E}_i \\
&= \frac{q^2}{4\pi\epsilon_0} \left[\frac{1}{d^2} - \frac{1}{(2d)^2} + \frac{1}{(3d)^2} \right] \hat{\mathbf{z}} \\
&= \frac{31 q^2}{144 \pi \epsilon_0 d^2} \hat{\mathbf{z}}
\end{aligned}$$

2. There would be 3 other charges:



Summing the three point charge potentials,

$$\begin{aligned}
V(x, y) = \frac{q}{4\pi\epsilon_0} \Bigg[&\frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2}} \\
&- \frac{1}{\sqrt{(x+a)^2 + (y-b)^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2}} \Bigg]
\end{aligned}$$

Summing the forces on q,

$$\begin{aligned}
\mathbf{F} &= \frac{q^2}{4\pi\epsilon_0} \left[-\frac{1}{4a^2} \hat{\mathbf{x}} - \frac{1}{4b^2} \hat{\mathbf{y}} + \frac{1}{\left(\sqrt{4a^2 + 4b^2}\right)^2} \left(\frac{2a}{\sqrt{4a^2 + 4b^2}} \hat{\mathbf{x}} + \frac{2b}{\sqrt{4a^2 + 4b^2}} \hat{\mathbf{y}} \right) \right] \\
&= \frac{q^2}{4\pi\epsilon_0} \left[\left(\frac{2a}{(4a^2 + 4b^2)^{3/2}} - \frac{1}{4a^2} \right) \hat{\mathbf{x}} + \left(\frac{2b}{(4a^2 + 4b^2)^{3/2}} - \frac{1}{4b^2} \right) \hat{\mathbf{y}} \right]
\end{aligned}$$

To bring q from infinity, the work required

$$\begin{aligned}
W &= q\sum_i V_i \\
&= \frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+a)^2+(y+b)^2}}\right. \\
&\qquad \left.-\frac{1}{\sqrt{(x+a)^2+(y-b)^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2}}\right]_{x=a,y=b} \\
&= \frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{4a^2+4b^2}}-\frac{1}{\sqrt{4a^2}}-\frac{1}{\sqrt{4b^2}}\right]
\end{aligned}$$

The method of images seems to work with angles where $360°/n$, where $n = 1, 2, 3...$?

3. (a) Since the potential must be periodic-ish in x, the assumed form of the potential is

$$V(x,y) = \left(Ae^{ky}+Be^{-ky}\right)(C\sin kx + D\cos kx)$$

Since $V(0,y) = 0 \implies D = 0$, and since it's zero at $x = a$, the wavenumber can be found

$$V(x,y) = \left(Ae^{ky}+Be^{-ky}\right)C\sin\left(\frac{n\pi}{a}x\right)$$

Since $V(x,0) = 0$, then $A = -B$ and we can replace the exponentials with a $\sinh$ and use a single constant C',

$$V(x,y) = C'\sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

Since any linear combination is a solution, the general solution has the form

$$V(x,y) = \sum_n C_n\sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

Applying Fourier's trick at the $y = b$ boundary to find the coefficients,

$$\begin{aligned}
\int_0^a V_0(x)\sin\left(\frac{n'\pi x}{a}\right)\mathrm{d}x &= \sum_n C_n\int_0^a \sinh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n'\pi}{a}x\right)\mathrm{d}x \\
&= C_n\sinh\left(\frac{n\pi}{a}b\right)\frac{a}{2} \\
C_n &= \frac{2}{a}\left(\frac{1}{\sinh(n\pi b/a)}\right)\int_0^a V_0(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x
\end{aligned}$$

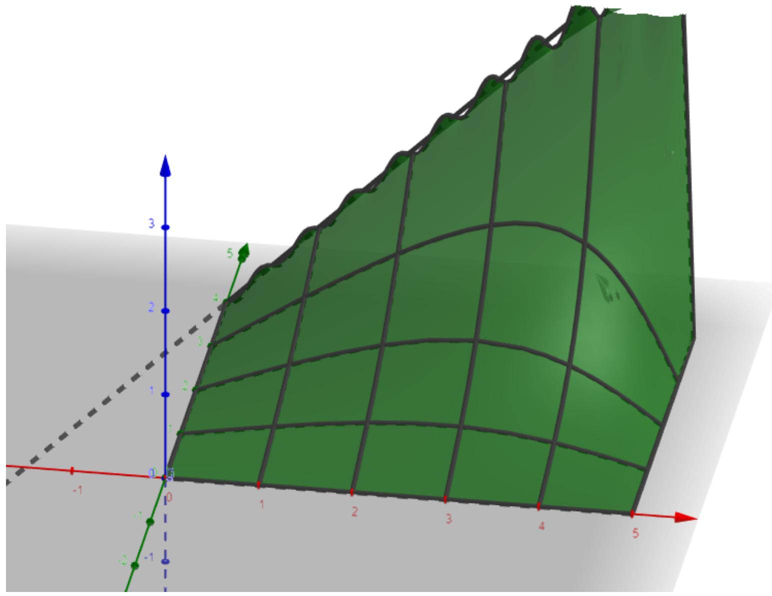
(b) For $V_0(x) = \beta x$,

$$\begin{aligned}
C_n &= \frac{2\beta}{a}\left(\frac{1}{\sinh(n\pi b/a)}\right)\int_0^a \sin\left(\frac{n\pi x}{a}\right)x\,\mathrm{d}x \\
&= \frac{2\beta}{a}\left(\frac{1}{\sinh(n\pi b/a)}\right)\left(-\frac{a^2\cos(\pi n)}{\pi n}\right) && \leftarrow \text{Used WolframAlpha} \\
&= -\frac{2\beta a}{\pi n}\frac{\cos(\pi n)}{\sinh(n\pi b/a)}
\end{aligned}$$

Using this C_n in $V(x,y)$,

$$V(x,y) = -\sum_n \frac{2\beta a}{\pi n}\frac{\cos(\pi n)}{\sinh(n\pi b/a)}\sinh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

Plotting this to check the result and it seems to resemble the boundary conditions (for $a = 5$, $b = 4$, $\beta = 1$, $N = 20$):



4. From the geometry, it seems like the x and y directions will have a periodic term, and the z direction will have an exponential. The potential will be in the form of

$$V(x,y,z) = (A\sin kx + B\cos kx)\,(C\sin ly + D\cos ly)\,\left(Ee^{mz} + Fe^{-mz}\right)$$

Applying the x boundary conditions, $V(0,y,z) = V(a,y,z) = 0$, then $B = 0$ and k can be determined. Similarly for y, $D = 0$ and l can be found. In z, $E = -F$ at $z = 0$ and additionally, m can be written in terms of k and l as well (arising from the separable DE). The form becomes

$$V(x,y,z) = C'\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{k^2+l^2}z\right)$$

Applying the orthogonality of sines,

$$V(x,y,z) = \sum_n\sum_m C_{n,m}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{k^2+l^2}z\right)$$

At the $z = c$ boundary,

$$V(x,y,c) = \sum_n\sum_m C_{n,m}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{k^2+l^2}c\right)$$

$$V_0\int_0^a\int_0^b \sin\left(\frac{n'\pi}{a}x\right)\sin\left(\frac{m'\pi}{b}y\right)\mathrm{d}y\,\mathrm{d}x = \int_0^a\int_0^b\sum_n\sum_m C_{n,m}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{k^2+l^2}c\right)$$
$$\times\sin\left(\frac{n'\pi}{a}x\right)\sin\left(\frac{m'\pi}{b}y\right)\mathrm{d}y\,\mathrm{d}x$$

Filtering out only the $n = n'$ and $m = m'$ cases (Fourier's trick) and evaluating the integrals, for odd n and m,

$$\begin{aligned} C_{n,m} &= \frac{4V_0}{ab\sinh\left(\sqrt{k^2+l^2}c\right)}\left(\frac{2a}{\pi n}\right)\left(\frac{2b}{\pi m}\right) \\ &= \frac{16V_0}{\pi^2 nm\sinh\left(\sqrt{k^2+l^2}c\right)} \end{aligned}$$

Putting this all together in the general equation and substituting k and l into the $\sinh$,

$$\begin{aligned} V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots} &\frac{1}{nm\sinh\left(\pi\sqrt{(n/a)^2+(m/b)^2}c\right)} \\ &\times\sinh\left(\pi\sqrt{(n/a)^2+(m/b)^2}z\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right) \end{aligned}$$

5. On the ground plane at $y = a$, the normal direction is $\hat{y}$, the induced charge is found through

$$\begin{aligned} \sigma(x) &= -\epsilon_0\left.\frac{\partial V(x,y)}{\partial y}\right|_{y=a} \\ &= -\frac{4V_0}{\pi\epsilon_0}\sum_{n=1,3\ldots}\left.\frac{\partial}{\partial y}\frac{1}{n}\frac{\sinh(n\pi x/a)}{\sinh(n\pi b/a)}\sin(n\pi y/b)\right|_{y=a} \\ &= -\frac{4V_0}{\epsilon_0 b}\sum_{n=1,3\ldots}\frac{\sinh(n\pi x/a)}{\sinh(n\pi b/a)}\cos(n\pi a/b) \end{aligned}$$