# Homework 10

MATH 301
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1. **Proposition.** *For a positive integer $n$,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

*Proof.* This will be proven with induction. For the base case, we will check $n = 1$.

(1) For $n = 1$,

$$\frac{1(1+1)}{2} = 1$$

(2) If we suppose the proposition is true for $k$, we will show it is also true for $k+1$,

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\left[(k+1)+1\right]}{2}\end{aligned}$$

This shows the proposition is also true for $k+1$.

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2. **Proposition.** *For any nonnegative integer $n$,*

$$2^{n+1} > \sum_{i=0}^{n} 2^i$$

*Proof.* This will be proven with induction. For the base case, we will check $n = 0$.

(1) For $n = 0$, it holds true that

$$2^{0+1} > 2^0$$

(2) If we suppose the proposition is true for $k$, we will show it is also true for $k + 1$. On the LHS, the $k + 1$ term is

$$2^{k+2} = 2\left(2^{k+1}\right)$$

From the proposition,

$$\begin{aligned}
2(2^{k+1}) &> 2\sum_{i=0}^{k} 2^i && \text{(assuming the hypothesis)} \\
2(2^{k+1}) &> \sum_{i=0}^{k} 2^{i+1} && \text{(moving the 2 into the indexed term)} \\
2(2^{k+1}) &> \sum_{i=0}^{k+1} 2^i && \text{(changing the indices)}
\end{aligned}$$

Substituting the original expression, it follows by induction that

$$2^{k+2} > \sum_{i=0}^{k+1} 2^i$$

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3. **Proposition.** *For any positive integer $n$, there exists a sequence $b_i \in \{0, 1\}$ such that $b_k = 1$ and*

$$n = \sum_{i=0}^{k} b_i 2^i$$

*Proof.* This will be shown using strong induction. For the base case, we will check $n = 1$.

(1) For $n = 1$, it holds true that $1 = (1)2^0$.

(2) Suppose $m \geq 1$ and every integer on between 1 and $m$ can be written as the sum of powers of two. If we consider $m + 1$, we can divide this into two cases: one where $m + 1$ is even and one where $m + 1$ is odd.

Case 1: $m + 1$ is even. Then $m + 1$ can be expressed as $m + 1 = 2a$, where $a \in \mathbb{Z}$. We know that $m + 1 \geq 2$ and thus $m + 1 > a$. Then, $1 \leq a \leq m$, and by the inductive hypothesis, $a = \sum_{i=0}^{l} c_i 2^i$ where $l \in \mathbb{Z}$.

Multiplying this by 2 to find $m + 1$,

$$\begin{aligned} m + 1 &= 2a = 2\left(\sum_{i=0}^{l} b_i 2^i\right) \\ m + 1 &= \sum_{i=0}^{l+1} b_i 2^i \end{aligned}$$

Therefore $m + 1$ can be written as the sum of powers of two.

Case 2: $m + 1$ is odd. If $m + 1$ is odd, then $m$ must be even. This means that in the sum representing $m$, the last coefficient $b_0 = 0$. Then $m + 1$ is the same sum but now with $b_0 = 1$,

$$\begin{aligned} m &= b_0 2^0 + b_1 2^1 + \ldots \\ &= 0 \times 2^0 + b_1 2^1 + \ldots \\ m + 1 &= 1 \times 2^0 + b_1 2^1 + \ldots \end{aligned}$$

Therefore $m + 1$ can be written as the sum of powers of two.

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4. **Proposition.** *The representation of a positive integer as a sum of powers of* $2$ *is unique.*

*Proof.* It will be shown by strong induction that the representation of a positive integer as a sum of powers of 2 is unique. For the base case, we will check $n = 1$.

(1) For $n = 1$, there is a single way to represent this as $1 = (1)2^0$.

(2) Suppose $m \geq 1$ and every integer between $1$ and $m$ can be written as a sum of powers of two. If we consider $m + 1$ and its representation as the sum of powers of 2, then assume there are actually two different ways of representing it,

$$m + 1 = \sum_{i=0}^{k} b_i 2^i = \sum_{i=0}^{\ell} c_i 2^i$$

where $b_k = 1, c_\ell = 1, b_i, c_i \in \{0, 1\}$.

(a) If we suppose that one sum has more terms than the other, perhaps $k > \ell$. Then there would exist a term $2^{\ell+1}$ (or greater) in $\sum_{i=0}^{k} b_i 2^i$. However, by Problem 2 (which finds $2^{n+1} > \sum_{i=0}^{n} 2^i$), this would lead to a contradiction: it means

$$\sum_{i=0}^{k} b_i 2^i > \sum_{i=0}^{\ell} c_i 2^i$$

which cannot be right, as we have stated these two values are equal to $m + 1$. Therefore, it must be true that $k = \ell$ for $m + 1$.

(b) Next, if we factor out a 2 from each side,

$$2\left(\sum_{i=1}^{k} b_i 2^{i-1}\right) + b_0 = 2\left(\sum_{i=1}^{\ell} c_i 2^{i-1}\right) + c_0$$

Then $b_0$ and $c_0$ must be equal, as the modulus $2$ of both sides must be equal. If this operation is continued $k$ times, in each iteration, $b_i = c_i$ for all values. Therefore, the coefficients $b_i = c_i$ for all $i$.

Since there is a single way of representing a number by the powers of 2, it is unique.

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