Homework 3

MATH 364
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Kevin Evans
ID: 11571810

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1. Find the largest circle contained in the polygon

$$P = \left\{x \in \mathbb{R}^2 \mid\ a_k x_1 + b_k x_2 \le c_k,\ k = 1, 2, \cdots, n; \text{with } a_k^2 + b_k^2 = 1\right\}.$$

Suppose $(x_1, x_2) \in P$, then the signed-distance from x to the line $a_k x_1 + b_k x_2 = c_k$ is $d_k(x) = (c_k - a_k x_1 - b_k x_2)$, where positive distances correspond to points in the corresponding halfspace and negative distances correspond to points outside the halfspace.

To find the largest circle, you must find the center point $x \in P$ that maximizes the signed-distance to the closest line defined by the polygon boundary. This maximal distance is the radius of the largest possible circle. That is, the goal is $\max_x(\min_k d_k(x))$. Formulate this problem as a linear program.

Solution. The decision variables are the location of the circle and its radius,

$$\begin{aligned} \text{Let } x &= \text{the location of the circle in } \mathbb{R}^2 \\ r &= \text{the radius of the circle.} \end{aligned}$$

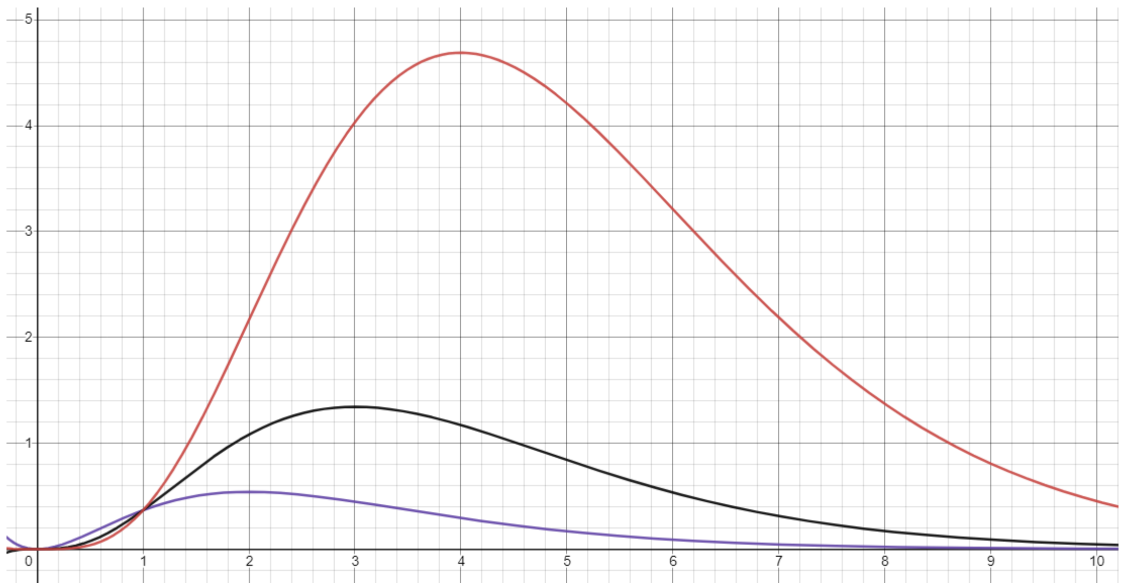
The objective function is the radius of the circle that we're maximizing, r.

The constraints are given by whether the points lay within the polygon or beyond it: for all k, $d_k \ge 0$ and ensuring that x is within the polygon, $x \in P$ (I'm not sure if this constraint makes sense).

$$\begin{aligned} \max_{x \in \mathbb{R}^2, r \in \mathbb{R}} \quad & z = r \\ \text{s.t.} \quad & d_k \ge 0 \qquad \text{for } k = 1, 2, \ldots, n \\ & x \in P? \\ \text{where} \quad & P = \left\{x \in \mathbb{R}^2 \mid\ a_k x_1 + b_k x_2 \le c_k,\ k = 1, 2, \cdots, n; \text{with } a_k^2 + b_k^2 = 1\right\} \\ & d_k(x) = (c_k - a_k x_1 - b_k x_2) \end{aligned}$$

2. Consider a population model for bacteria in a closed environment given by $p(t) = At^n e^{-t/\tau}$. The population at time t is $p(t)$, where A, n, and τ are constants.

   (a) Plot or very carefully sketch $p(t)$ for $n = 2, 3, 4$. Be sure and consider an appropriate range of values for $t \geq 0$ to illustrate plot features.



   (b) Using your plot features, explain why this model may be physically reasonable. How do the constants A, n, τ affect the population growth and decay?

   Solution. It seems reasonable as it has an exponential curve that initially increases, slows down, then decreases as overpopulation (resource depletion) occurs.

   - A affects the maximum population, i.e. the amplitude
   - n affects how quickly the maximum ramps up
   - τ affects the how long the entire lifecycle takes

   (c) Consider a data set of time-population pairs, $\{(t_k, p_k)\}_{k=1}^N$. Formulate a linear program which can be used to solve the problem of fitting this data to the model using the method of least deviation. The goal is to determine optimal coefficients A, n, τ. This cannot be done directly as outlined in class. Instead, formulate the problem using the logarithm of the model function.

   Solution. The decision variables are the parameters of the fitting function A, n, and τ.

   From the model function, we will take the log of it before applying the method of least deviation[1],

$$\ln\left(At^n e^{-t/\tau}\right) = A\left(n \ln t - t/\tau\right)$$

[1] Would it be possible to expand out the exponential instead of taking the log?

The objective function is the sum of the absolute residuals,

$$
\begin{aligned}
z &= \sum_{k=1}^{N} |\hat{y}_k - y_k| \\
&= \sum_{k=1}^{N} \left| A\left(n \ln t_k - t_k/\tau\right) - \ln(p_k) \right|
\end{aligned}
$$

The constraints will come once we convert the absolute function to a LP. The complete linear program is

$$
\begin{aligned}
\min_{\delta, A, n, \tau} \quad & \sum_{k=1}^{N} \delta_k \\
\text{s.t.} \quad & \delta_k \geq A(n \ln t_k - t_k/\tau) - \ln(p_k) \\
& \delta_k \geq -A(n \ln t_k - t_k/\tau) + \ln(p_k) \\
& \delta \in \mathbb{R}^N \\
& A, n, \tau \in \mathbb{R}
\end{aligned}
$$