Homework 12

PHYSICS 450
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1. Study Chapter 4.4.

2. (a) To normalize, we should satisfy

$$\begin{aligned}
1 &= \chi^\dagger \chi \\
&= A^2 \begin{pmatrix} -3i \\ 4 \end{pmatrix} \begin{pmatrix} 3i & 4 \end{pmatrix} \\
&= A^2 \left[-3i(3i) + 4(4)\right] = 25A^2 \\
&\boxed{A = 1/5.}
\end{aligned}$$

   (b) For $S_x = \frac{\hbar}{2}\sigma_x$, its expectation is

$$\begin{aligned}
\langle S_x \rangle = \chi^\dagger S_x \chi &= \frac{\hbar A^2}{2} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \end{pmatrix} = \boxed{0.}
\end{aligned}$$

   Similarly for $S_y = \frac{\hbar}{2}\sigma_y$ and $S_z = \frac{\hbar}{2}\sigma_z$,

$$\begin{aligned}
\langle S_y \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} -4i \\ -3 \end{pmatrix} \\
&= \frac{\hbar}{50} \left(12i^2 - 12\right) = -\frac{24\hbar}{50} \\
&= \boxed{-\frac{12\hbar}{25}.} \\
\langle S_z \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ -4 \end{pmatrix} \\
&= \frac{\hbar}{50} \left(-9i^2 - 16\right) \\
&= \boxed{-\frac{7}{50}\hbar}
\end{aligned}$$

   (c) To find the uncertainties, we should first find the expectation of the squared operators. Using WolframAlpha, the square of each Pauli matrix is the identity,

$$S_x{}^2 = S_y{}^2 = S_z{}^2 = \frac{\hbar}{2}\,\mathbb{I}_2.$$

So, the expectations are also all the same,

$$\begin{aligned}\left\langle {S_x}^2 \right\rangle &= \left\langle {S_y}^2 \right\rangle = \left\langle {S_z}^2 \right\rangle \\ &= \frac{\hbar^2}{4} \quad \text{by normalization}\end{aligned}$$

The uncertainties are then

$$\begin{aligned}\sigma_{S_x} &= \left\langle {S_x}^2 \right\rangle - \left\langle S_x \right\rangle^2 \\ &= \frac{\hbar^2}{4} - 0 = \hbar^2/2. \\ \sigma_{S_y} &= \frac{\hbar^2}{4} - \frac{144\hbar^2}{25^2} = \frac{49}{2500}\hbar^2. \\ \sigma_{S_z} &= \frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2 = \frac{1201}{2500}\hbar^2.\end{aligned}$$

(d) For the three permutations,

$$\begin{aligned}\sigma_x\sigma_y &= \frac{7\sqrt{2}}{100}\hbar^2 \geq \frac{7}{50}\hbar/2 && \checkmark \\ \sigma_y\sigma_z &= \frac{58849}{6250000}\hbar^2 \geq 0 && \checkmark \\ \sigma_x\sigma_z &= \frac{49}{5000}\hbar^2 \geq 12/25\hbar && ?\end{aligned}$$

3. For a generalized spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, all those expectations are

$$\begin{aligned}\left\langle S_x \right\rangle &= \frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{\hbar}{2}\left(a^*b + b^*a\right). \\ \left\langle S_y \right\rangle &= \frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{\hbar}{2}\left(-a^*b + b^*a\right). \\ \left\langle S_z \right\rangle &= \frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{\hbar}{2}\left(a^*a - b^*b\right).\end{aligned}$$

Then as

$$\begin{aligned}{S_x}^2 = {S_y}^2 = {S_z}^2 &= \frac{\hbar^2}{4}\mathbb{I}_2, \\ \implies \left\langle S_x^2 \right\rangle &= \left\langle S_y^2 \right\rangle = \left\langle S_z^2 \right\rangle \\ &= \frac{\hbar^2}{4}\left(a^*a + b^*b\right) = \frac{\hbar}{4}. \quad \text{(normalized)}\end{aligned}$$

So, the sum of these is

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{3}{4}\hbar.$$

Using the eigenvalue definition of $\langle S^2 \rangle$, these are equal,

$$\hbar^2 s(s+1) = \hbar^2 \frac{1}{2}\frac{3}{2} = \frac{3}{4}\hbar.$$

4. (a) For S_y,

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

To find the eigenvalues, we need to first find when the determinate is zero,

$$\begin{aligned} \begin{vmatrix} -\lambda & -\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} &= 0. \\ \lambda^2 - \hbar^2/4 &= 0 \\ \lambda &= \pm\hbar/2. \end{aligned}$$

Plugging this into the eigenvalue problem to find the associated eigenvectors,

$$\begin{aligned} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} &= \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \\ \implies -i\beta &= \pm\alpha \\ i\alpha &= \pm\beta. \end{aligned}$$

From inspection, the eigenvectors are

$$\chi_{\pm}^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}.$$

(b) If we measure S_y on a generalized spinor χ, we will get either $\pm\hbar/2$. The corresponding probabilities are

$$\begin{aligned} P^{\pm} &= \langle \chi^{\pm} | \chi \rangle^2 \\ &= \frac{1}{2}\left| \begin{pmatrix} 1 & \mp i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \right|^2 \\ &= \frac{1}{2}|a \mp ib|^2. \end{aligned}$$

These add up to 1 (on WolframAlpha).

(c) By Problem 3, we would get the expectation, $\hbar/4$ always.