Homework 9

PHYSICS 342
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1. The vector potential can be written
$$\mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin(\omega(t - r/c)) \left[\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}\right]$$
To check if the potentials satisfy the Lorenz gauge condition, we can find the divergence of the vector potential,
$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{A} &= -\frac{\mu_0 p_0 \omega}{4\pi} \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left[r \sin(\omega(t - r/c)) \cos\theta\right] - \frac{\sin(\omega(t - r/c))}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left[\sin^2(\theta)\right] \right\} \\
&= -\frac{\mu_0 p_0 \omega}{4\pi} \left\{ \frac{\cos\theta}{r^2} \left[\sin(\omega(t - r/c)) - \frac{r\omega \cos(\omega(t - r/c))}{c}\right] - \frac{2\sin(\omega(t - r/c))\cos\theta}{r^2} \right\} \\
&= -\frac{\mu_0 p_0 \omega}{4\pi} \left\{ \frac{\cos\theta}{r^2} \left[-\frac{r\omega \cos(\omega(t - r/c))}{c}\right] - \frac{\sin(\omega(t - r/c))\cos\theta}{r^2} \right\}
\end{aligned}$$
Next, taking the time derivative of the scalar potential
$$\frac{\partial V}{\partial t} = \frac{p_0 \cos\theta \omega}{4\pi\epsilon_0 r} \left[-\frac{\omega}{c} \cos(\dots) + \frac{1}{r} \sin(\dots)\right]$$
Rearranging, it becomes clear that the potentials obey the Lorenz gauge condition,
$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}.$$

2. Starting from the power radiated via the Poynting vector and the current flowing in the "wire", eq. (11.22) and (11.15),
$$\begin{aligned}
\langle P \rangle &= \frac{\mu_0 p_0^2 \omega^4}{12\pi c} = \left\langle I^2 \right\rangle_T R \\
R &= \frac{\langle P \rangle}{\langle I^2 \rangle_T} \\
&= \frac{\mu_0 (q_0 d)^2 \omega^4}{12\pi c} \frac{2}{q_0^2 \omega^2} \\
&= \frac{\mu_0 d^2 \omega^2}{6\pi c} \\
&= \frac{\mu_0 d^2 \pi c}{3\lambda^2} \propto (d/\lambda)^2
\end{aligned}$$
For a $1\,\mathrm{m}$ antenna operating at $\lambda = 30\,\mathrm{cm}$, the resistance would be $4.4\,\mathrm{k\Omega}$. This is quite high for an antenna.

3. From (11.40), the average power is
$$\begin{aligned}
\langle P \rangle &= \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} = I^2 R \\
R &= \frac{\mu_0 \omega^4}{12\pi c^3} (\pi b^2 I_0)^2 \frac{1}{\langle I^2 \rangle} \\
&= \frac{\mu_0 \omega^4 b^4}{6c^3}
\end{aligned}$$
...as the time average of $\left\langle I^2 \right\rangle = I_0/2$.

4. Starting from (11.33),

$$\mathbf{A} = \frac{\mu_0 m_0}{4\pi} \frac{\sin\theta}{r} \left\{ \frac{1}{r} \cos\left[\omega(t - r/c)\right] - \frac{\omega}{c} \sin\left[\omega(t - r/c)\right] \right\} \hat{\boldsymbol{\phi}}$$

Assuming there is no net charge, the scalar potential is zero. The electric field is then

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} \\
&= -\frac{\mu_0 m_0}{4\pi} \frac{\sin\theta\omega}{r} \left\{ -\frac{1}{r} \sin\left[\omega(t - r/c)\right] - \frac{\omega}{c} \cos\left[\omega(t - r/c)\right] \right\} \hat{\boldsymbol{\phi}} \\
&= \frac{\mu_0 m_0}{4\pi} \frac{\omega \sin\theta}{r} \left\{ \frac{1}{r} \sin\left[\omega(t - r/c)\right] + \frac{\omega}{c} \cos\left[\omega(t - r/c)\right] \right\} \hat{\boldsymbol{\phi}}
\end{aligned}$$

The magnetic field is the curl of the vector potential. The non-zero terms are

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \sin\theta A_\theta \, \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} r A_\phi \, \hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 m_0}{4\pi} \left[\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\sin^2(\theta)}{r} \{\ldots\} \right) \hat{\mathbf{r}} - \frac{\sin\theta}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \cos(\ldots) - \frac{\omega}{c} \sin(\ldots) \right) \hat{\boldsymbol{\theta}} \right] \\
&= \frac{\mu_0 m_0}{4\pi} \left[\frac{2\cos\theta}{r} \{\ldots\} \, \hat{\mathbf{r}} - \frac{\sin\theta}{r} \left(\frac{c^2 \cos\left[\omega(t - r/c)\right] + (r^2\omega^2 - cr\omega) \sin\left[\omega(t - r/c)\right]}{c^2 r^2} \right) \hat{\boldsymbol{\theta}} \right]
\end{aligned}$$

The Poynting vector is found using

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

I've given up on this problem.

5. (a) Not sure if this is valid in this case, but if we reuse the setup from p. 480 and take the power radiating through a giant spherical surface, the total radiated power is

$$P_{\text{rad}} \cong \frac{\mu_0}{6\pi c} \ddot{p}^2$$

where the capacitor is approximately a dipole,

$$\begin{aligned}
p(t) &= Q_0 d e^{-t/RC} \\
\ddot{p}(t) &= \frac{Q_0 d}{(RC)^2} e^{-t/RC}
\end{aligned}$$

Integrating to find the total radiated energy,

$$\begin{aligned}
E_{\text{rad}} &= \int P(t) \, \mathrm{d}t \\
&= \frac{\mu_0}{6\pi c} \left(\frac{Q_0 d}{(RC)^2} \right)^2 \int_0^\infty e^{-t/RC} \, \mathrm{d}t \\
&= \frac{\mu_0}{6\pi c} \left(\frac{Q_0 d}{(RC)^2} \right)^2 RC \\
&= \frac{\mu_0 Q_0^2 d^2}{6\pi c R^3 C^3}
\end{aligned}$$

As a fraction of its initial energy,

$$\frac{E_{\text{rad}}}{E_{\text{initial}}} = \frac{\mu_0 d^2}{3\pi c R^3 C^2}$$

(b) Plugging all the values in, the fraction is

$$\frac{E_{\text{rad}}}{E_{\text{initial}}} \approx 4.44 \times 10^{-9}$$

which is pretty tiny.