LAB #2

State Variable Modeling and Mutual Inductance

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Due Date: February 4, 2020

Lab Overview

In this lab, two experiments were done: one was to determine the mutual inductance of a transformer circuit, and the other was to use state variables to model a RLC circuit. The mutual inductance was found using a triangular wave input as well as an input phasor, and averaged to find the coupling coefficient. The RLC circuit was modeled using OrCAD initially and the model was confirmed experimentally. Next, a MATLAB model was created using state variables and it was found to be within 5% of the actual circuit response.

1 Mutually coupled transformer circuit

1.1 Purpose

The purpose of this experiment was to experimentally determine the mutual inductance M between two inductors in a transformer circuit. This was done using two methods: by measuring the sinusoidal response and the step response.

1.2 Theoretical background

Inductors can share a magnetic flux, when two are placed relatively nearby. This is often seen in transformers, which can step down or up a potential, using this coupling. In this experiment, the circuit shown in Figure 1 was assembled to experimentally determine the mutual inductance of two inductors, L_1 and L_2.
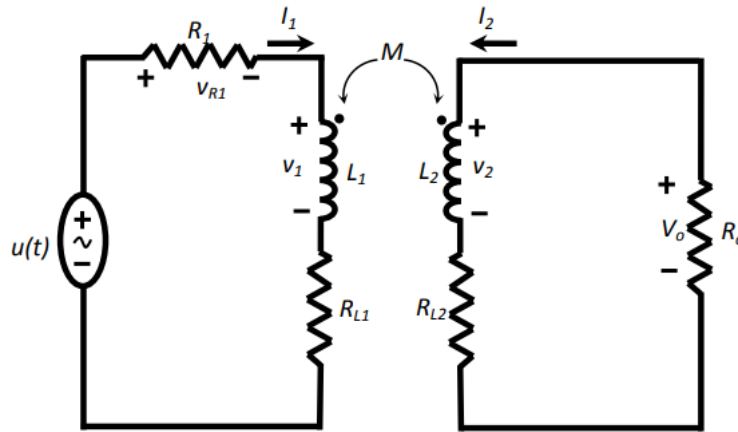

Figure 1: The circuit diagram used in this experiment.

Triangular wave: constant change in current

From here, we can generate a triangular wave from $u(t)$, with a constant ramping in voltage. If we set $R_o \to \infty$, it acts as an open circuit and no current flows in the second loop, I_2. In the time domain, for this coupled inductor pair, the voltages across each are given as

$$v_1(t) = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{1a}$$

$$v_2(t) = M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{1b}$$

Thus, if we measure only the current I_2 and the potential across v_o, we can determine the mutual inductance M. We additionally will make further assumptions: the resistance of coil 1 R_{L1} will be much less than R_1, and will be negligible in the calculations. Additionally, we will assume that V_{R1} will be equal to $u(t)$—as the voltage drop across L_1 will be minimal (this will be demonstrated in the sinusoidal response).

As a triangular wave is being used, we can assume that the current in the left loop, I_1, will follow

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = \frac{1}{R_1}\frac{\mathrm{d}V_{R1}}{\mathrm{d}t} \approx \frac{1}{R_1}\frac{\Delta V_{R1}}{\Delta t} \tag{2}$$

where ΔI_1 is the difference between the peak current values. This is expected to be constant as the voltage across L_1 was neglected from the earlier assumptions. For a $10\,\mathrm{V_{pp}}$ triangular pulse at $400\,\mathrm{Hz}$ across resistor R_1, the current differential (2) evaluates to

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} = 2 \times (400\,\mathrm{Hz}) \times \frac{10\,\mathrm{V_{pp}}}{10\,\mathrm{k\Omega}} = \pm 0.8\,\mathrm{A\cdot s^{-1}} \tag{3}$$

Equating this with (1b), if the voltage across the right loop v_2 is measured and the difference between the peak-to-peak voltage is calculated as ΔV_2, we can determine the mutual inductance as

$$M = \frac{\Delta V_2}{\frac{\mathrm{d}i_1}{\mathrm{d}t}} \tag{4}$$

Sinusoidal response

Next, to determine the sinusoidal response, we can characterize the inductor voltage drop using impedances. Once again, the I_2 becomes zero as the output resistance is modeled as infinite,

$$V_1 = j\omega L_1 I_1 + \cancel{j\omega M I_2} \tag{5a}$$
$$V_2 = j\omega M I_1 + \cancel{j\omega L_2 I_2} \tag{5b}$$

Next, we calculate the impedance of the left-hand loop using ideal values of

$$
\begin{aligned}
L_1 = L_2 &= 200\,\mathrm{mH} \\
R_{\mathrm{L1}} = R_{\mathrm{L2}} &= 50\,\Omega \\
R_1 &= 10\,\mathrm{k\Omega} \\
R_o &= \infty \\
u(t) &= 10\,\mathrm{V_{pp}}\underline{/0^\circ} && \omega = 400\,\mathrm{Hz} \\
V_1 &= V_m\underline{/\theta} && \omega = 400\,\mathrm{Hz}
\end{aligned}
$$

Then, the impedance of the left-hand loop is

$$Z_{\mathrm{LHS}} = Z_{\mathrm{R1}} + Z_{\mathrm{L}} \tag{6a}$$
$$= R_1 + j\omega L_1 + R_{\mathrm{L1}} \tag{6b}$$

However, relative to R_1, the impedance of the inductor Z_L is minuscule and can be neglected. The current will primarily be dictated by the magnitude of R_1, as $Z_{\mathrm{LHS}} \approx R_1$ for the ideal values. Then, the current I_1 can be calculated as

$$I_1 = \frac{u(t)}{Z_{\mathrm{LHS}}} = \frac{V_1}{R_1} \tag{7}$$

Combining (5b) and (7), the inductor coupling can be determined as

$$M = \frac{|V_2|}{\omega|I_1|} = \frac{V_2}{\omega V_1/R_1} \tag{8}$$

1.3 Procedure

The follow steps were carried out, as instructed by the lab assignment.

1. Two inductors were chosen (from the lab cabinets). Their inductance and resistances were recorded using the RLC meter.

2. The transformer circuit was constructed per Figure 1.

3. A $4\,\mathrm{V_{pp}}$ triangular pulse $400\,\mathrm{Hz}$ was generated using the function generator and attached to points $u(t)$ on the circuit diagram.

4. A tee was used to split the input signal into CH1 of the oscilloscope. CH2 was attached across inductor L_1 and CH3 was attached to the output voltage, V_2. All were set to 1X.

5. To reduce noise, CH3 was set to average every 4 samples. On the oscilloscope, `Acquire` $\rightarrow$ `Average` $\rightarrow$ CH3: 4 samples.

6. Resistor R_1 acted as a shunt to estimate the loop current. This was done by using the `Math` feature on the oscilloscope and setting it to subtract CH1 (input voltage) and CH2 (the inductor voltage), then the resistor current was estimated using this difference.

7. This was demonstrated to the TA and checked off.

8. Next, a $10\,\mathrm{V_{pp}}$ sinusoidal wave was generated and the current was estimated using the same method as Step 6. The oscilloscope outputs were recorded and saved.

9. The mutual inductance was estimated using (4) and (8).

1.4 Results and analysis

Experimental component values

The measured component values are shown in Table 1.

Table 1: Experimental and nominal component values.

Component	Nominal	Experimental	% Error (Tolerance)
$L_1{}^1$	–	$173\,\mathrm{mH}$	–
$L_2{}^1$	–	$175\,\mathrm{mH}$	–
R_{L1}	–	$72.83\,\Omega$	–
R_{L2}	–	$73.33\,\Omega$	–
R_1	$10\,\mathrm{k}\Omega$	$9.738\,\mathrm{k}\Omega$	2.6% (5%)

^{1} L_1 and L_2 were labeled as #13 and #6 respectively.

Triangular wave mutual inductance calculation

The instantaneous change in current I_1 was found by measuring the peak-to-peak voltage across R_1 using the oscilloscope, shown in Figure 2 This was applied using (2) and recorded. The peak-to-peak voltage across L_2 was measured, shown in Figure 2, and recorded as V_2. This value was halved in (9b), as the magnitude is needed. Using (4), the instantaneous change in current and mutual inductance can be found as

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = \frac{1}{R_1}\frac{\mathrm{d}V_{R1}}{\mathrm{d}t} = \frac{1}{9.738\,\mathrm{k}\Omega}\frac{9.84\,\mathrm{V}}{1.25\,\mathrm{ms}} = 0.808\,\mathrm{A}\cdot\mathrm{s}^{-1} \tag{9a}$$

$$M = \frac{V_2}{\frac{\mathrm{d}I_1}{\mathrm{d}t}} = \frac{49.2\,\mathrm{mV}/2}{0.808\,\mathrm{mA}\cdot\mathrm{s}^{-1}} = 30.45\,\mathrm{mH} \tag{9b}$$
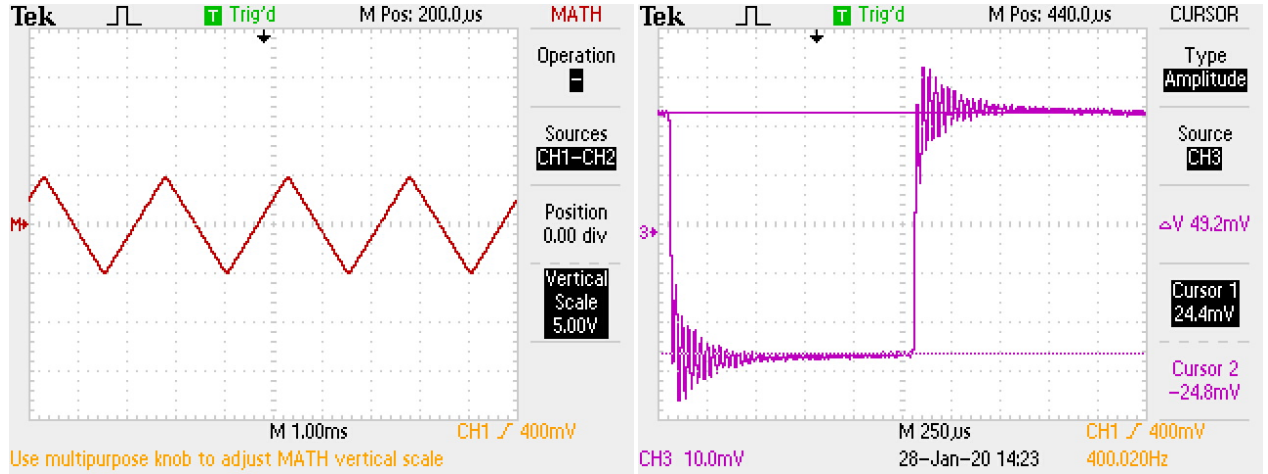
Figure 2: The voltage across the shunt resistor R_1 (left) and inductor L_2 (right).

Sinusoidal wave mutual inductance calculation

In the latter half of the experiment, a sine wave was used and the mutual inductance was measured. Shown in Figure 3, the voltage V_2 across inductor L_2 was measured,

$$
\begin{aligned}
V_2 &= 65.6\,\text{mV} && \text{(peak-to-peak)} \\
&= 32.8\,\text{mV} && \text{(zero-to-peak)}
\end{aligned}
\tag{10}
$$

The voltage V_{R1} across the shunt was measured and the magnitude of the current I_1 was calculated as

$$
\begin{aligned}
I_1 = \frac{V_{R1}}{R_1} &= \frac{9.84\,\text{V}}{9.738\,\text{k}\Omega} = 1.01\,\text{mA} && \text{(peak-to-peak)} \\
&= 0.51\,\text{mA} && \text{(zero-to-peak)}
\end{aligned}
\tag{11}
$$

From these, the mutual inductance was calculated using (8),

$$
\begin{aligned}
M &= \frac{32.8\,\text{mV}}{(800\pi\,\text{rad}/\sec)\,(0.51\,\text{mA})} \\
&= 25.60\,\text{mH}
\end{aligned}
\tag{12}
$$

Average mutual inductance and coupling coefficient

The average mutual inductance using the two methods was found as

$$
\begin{aligned}
\bar{M} &= \frac{30.45 + 25.60}{2} \\
&= 28.03\,\text{mH}
\end{aligned}
\tag{13}
$$

Using the inductor values from Table 1, the coupling coefficient can be estimated,

$$
\begin{aligned}
k = \frac{\bar{M}}{\sqrt{L_1 L_2}} &= \frac{28.03}{\sqrt{173 \times 175}} \qquad [\text{mH}/\text{mH}] \\
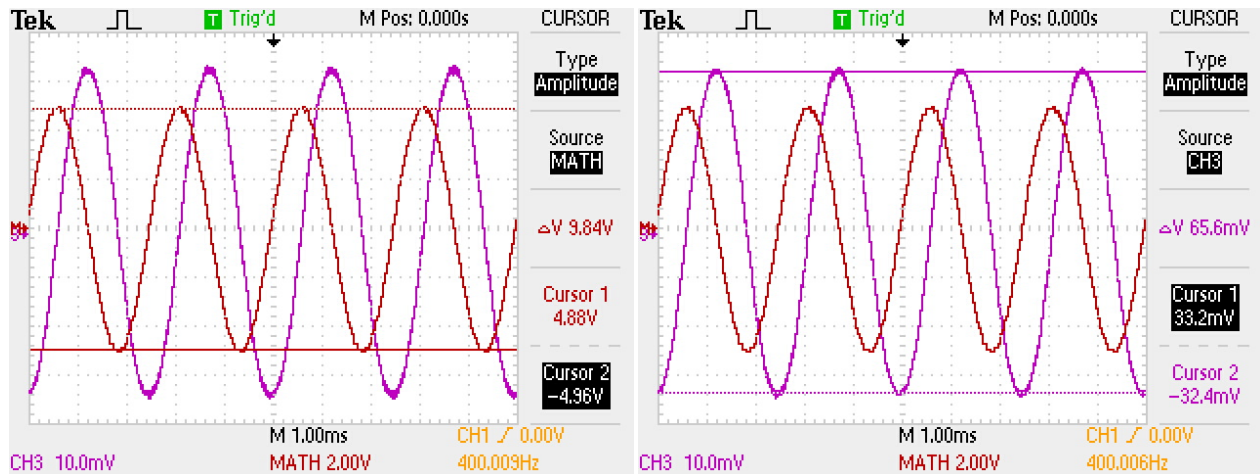&= 0.161
\end{aligned}
\tag{14}
$$

4

Figure 3: The voltage across the shunt resistor R_1 (left) and inductor L_2 (right).

Inductor dot notation

The dot notation of the circuit shown in Figure 5 is backwards in inductor L_2, and the dot should be placed at the negative terminal of V_2. This is evident in Figure 4 as the voltages across each inductor are half a cycle out of phase. Once inductor L_2 was reversed, the voltages became nearly in-phase.
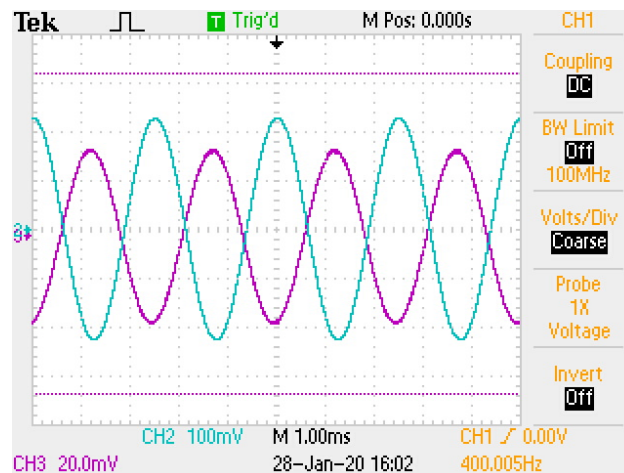

Figure 4: The voltage across inductor L_1 (blue) and L_2 (fuchsia).

1.5 Conclusion

This experiment found a mutual inductance using two methods: a triangular input response with constant current slope, and a sinusoidal phasor input and phasor response. There was a fairly large difference between the two and the average. Between the two calculated M values and the mean $\bar{M}$, there was 8.7% and 8.5% error using the triangular and sinusoidal response respectively. The difference between these two experiments could be due to several factors, but we believe the error came from the poor alignment of the inductors. During the experiment, it was noticed that barely nudging each transformer, even if moved a millimeter, would have a substantial effect on the output voltage of inductor L_2.

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2 State variable modeling of RLC circuits

2.1 Purpose

The purpose of this experiment was to solve for voltages and currents in a linear circuit using state variable models. In the experiment, we modeled an RLC circuit containing parallel RLC components.

2.2 Theoretical background

Circuits containing linear components can be modeled using the state variables, where capacitor voltages and inductor currents are set as continuous state variables. The derivatives of these state variables can be used to determine how other variables evolve over time in the circuit. Here, we use a state variable model to inspect the voltage across a capacitor and the current through an inductor, and determine the response of the system as a whole.

   We can model the circuit in Figure 5 using state variables. In this instance, the state variables are $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$. We can use KVL/KCL to determine the relation between x_1 and x_2 and their interaction in the circuit. From this, we can produce the matrices[1] to find solutions to the circuit,

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\mathbf{X}}} = \underbrace{\begin{bmatrix} -\frac{1}{C(R_s+R_1)} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2+R_L}{L} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} \frac{1}{C(R_s+R_1)} \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{(u(t) \times \mathbf{I}_2)}_{\mathbf{G}} \tag{15}$$

Next, we can simulate the circuit in OrCAD PSPICE. The circuit was modeled in OrCAD using the nominal values in Figure 5. A step response with pulse width of $5\,\text{ms}$ was used as it would allow the RLC circuit ample time to reach equilibrium. The simulation shows ringing occurring on both state variables before reaching equilibrium at $x_1 = 255.9\,\text{mV}$ and $x_2 = 82.6\,\text{mA}$. The full result of the simulation is shown in Figure 9 (Appendix).
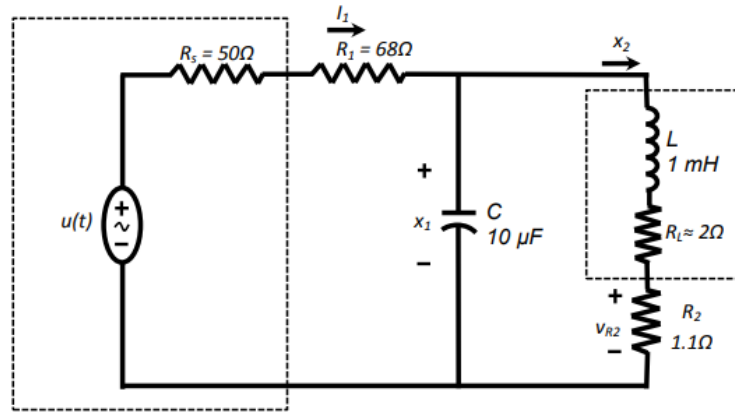

Figure 5: The RLC circuit used in this experiment.

2.3 Procedure

The follow steps were carried out, as instructed by the lab assignment.

1. The values for the components in Figure 5 were recorded.

2. A square wave of $10\,\text{V}_{\text{pp}}$ and 5 V offset was generated. This was equivalent to an input of $10u(t)$.

3. The frequency was set to $20\,\text{Hz}$ as it gave the best results.

4. The waveforms and points were captured and saved to a flash drive from the oscilloscope. In Excel, the trajectory was plotted.

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[1]The derivation for this is found in Appendix C of the lab handout.

2.4   Results and analysis

Experimental component values

The measured component values are shown in Table 2.

Table 2: Experimental and nominal component values.

Component	Nominal	Experimental	% Error (Tolerance)
L	$1\,\mathrm{mH}$	$1.146\,\mathrm{mH}$	4.1%
R_L	–	$2.23\,\Omega$	–
R_1	$68\,\Omega$	$67.42\,\Omega$	0.08% (10%)
R_2	$1.1\,\Omega$	$1.20\,\Omega$	9.1% (10%)
C	$10\,\mu\mathrm{F}$	$9.59\,\mu\mathrm{F}$	4.1% ([M] 20%)

Step response of the RLC circuit

The data from the step response was saved from the oscilloscope to a flash drive and loaded into Excel. Within Excel, the data from CH1 and CH2 were brought to the same worksheet and data was truncated at $t = 0$ (relative to the trigger) and the first 700 rows were used, about $2.8\,\mathrm{ms}$. The current was found using

$$I_L = \frac{V_{R2}}{R_2} = \frac{V_{\mathrm{CH2}}}{1.20\,\Omega}$$

Using Excel to tabulate the data, it was then saved as a `.csv` file and plotted using LaTeX, as shown in Figure 6 below.
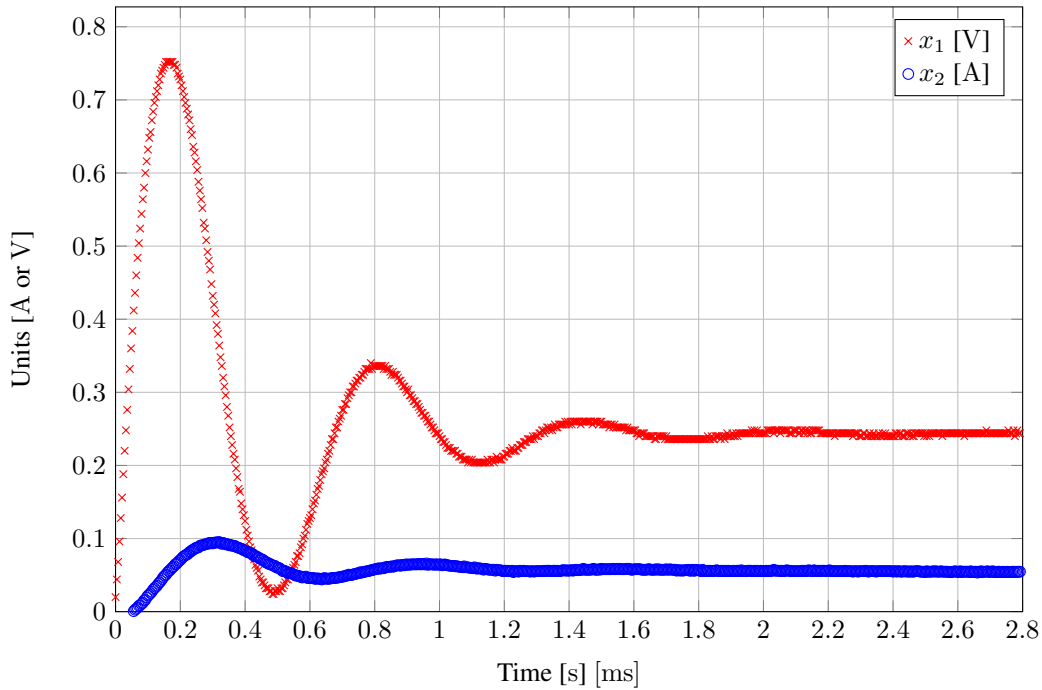

Figure 6: Plot of the two state variables, x_1 and x_2.

State trajectory

Using the data collected earlier, the state trajectory was plotted, x_1 vs x_2. This is shown in Figure 7. The plot stabilizes (at $t \to \infty$) near $(0.25\,\text{V}, 54\,\text{mA})$.
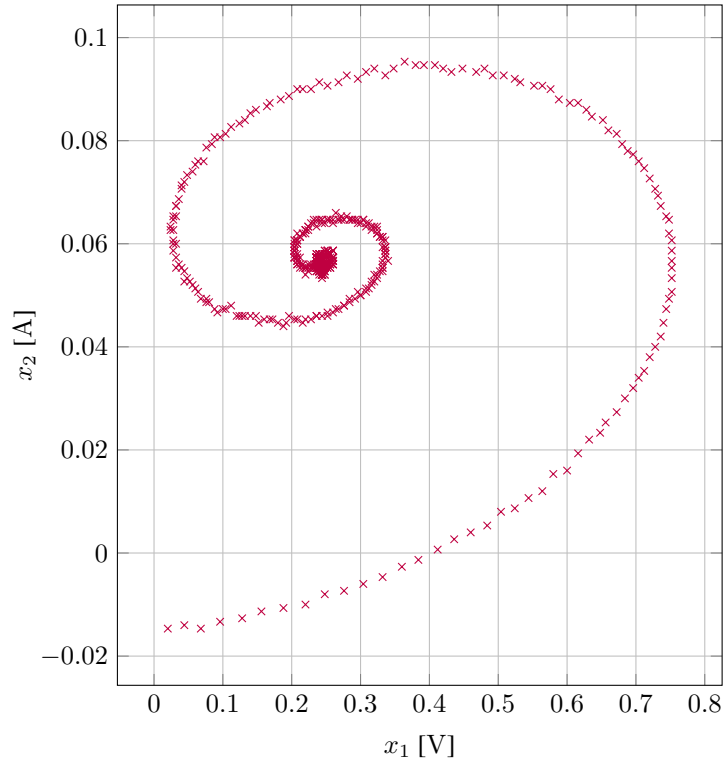

Figure 7: Plot of the two state variables, x_1 and x_2.

Post-Lab Exercise: MATLAB simulation

Using the state matrix, found in (15), we can use the `ss` and `step` functions within the Control Systems Toolbox. Plotting the three graphs results in Figure 8. When compared to the actual response, the peak capacitor voltage x_1 is slightly less: occurring at $(0.17\,\text{ms}, 0.92\,\text{V})$ in the simulation versus $(0.18\,\text{ms}, 0.76\,\text{V})$ on the actual waveform. A similar result is seen in the current as well, $(0.3\,\text{ms}, 0.12\,\text{A})$ in the simulation and $(0.3\,\text{ms}, 0.1\,\text{A})$. This error is likely due to the increased impedance from the inductor (which is unaccounted for in the simulation), as well as the probe impedance. However, it is worth noting that the timing of the peaks are quite similar, with under a 5% error in each ringing harmonic.
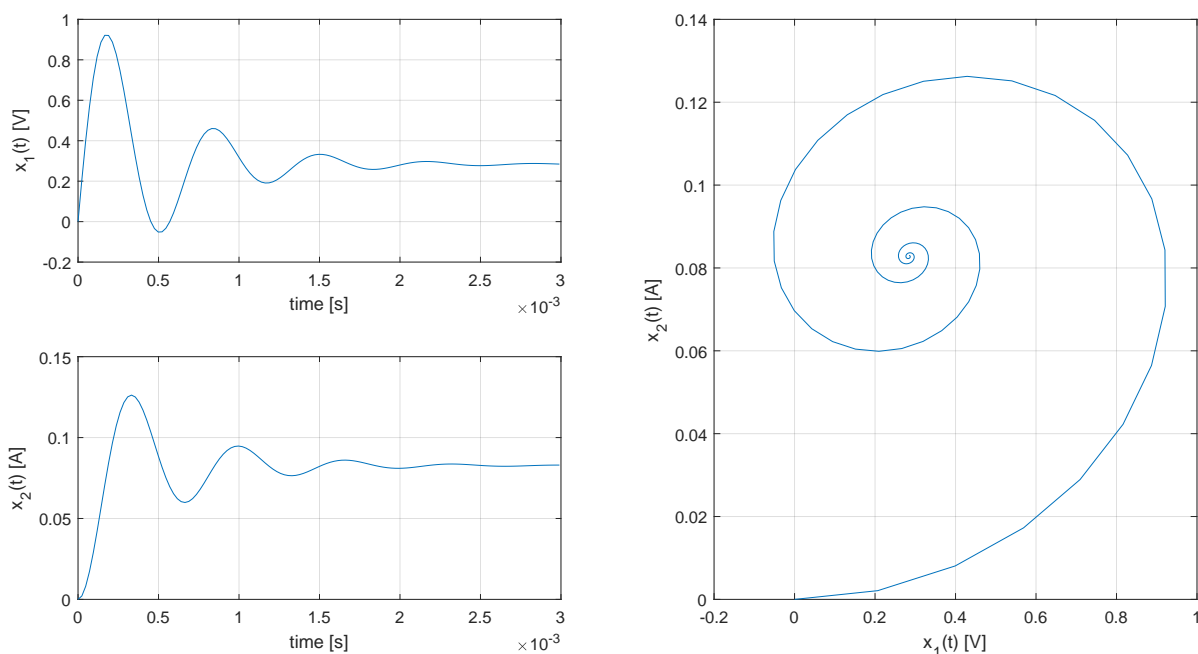

Figure 8: Output of the MATLAB simulation of the RLC step response.

2.5   Conclusion

This experiment confirmed the state variable model, as well as the OrCAD model. The MATLAB simulation can be a valuable tool to solve for the step responses of linear circuits and can be more effective at determining behavior without using a full SPICE simulation.
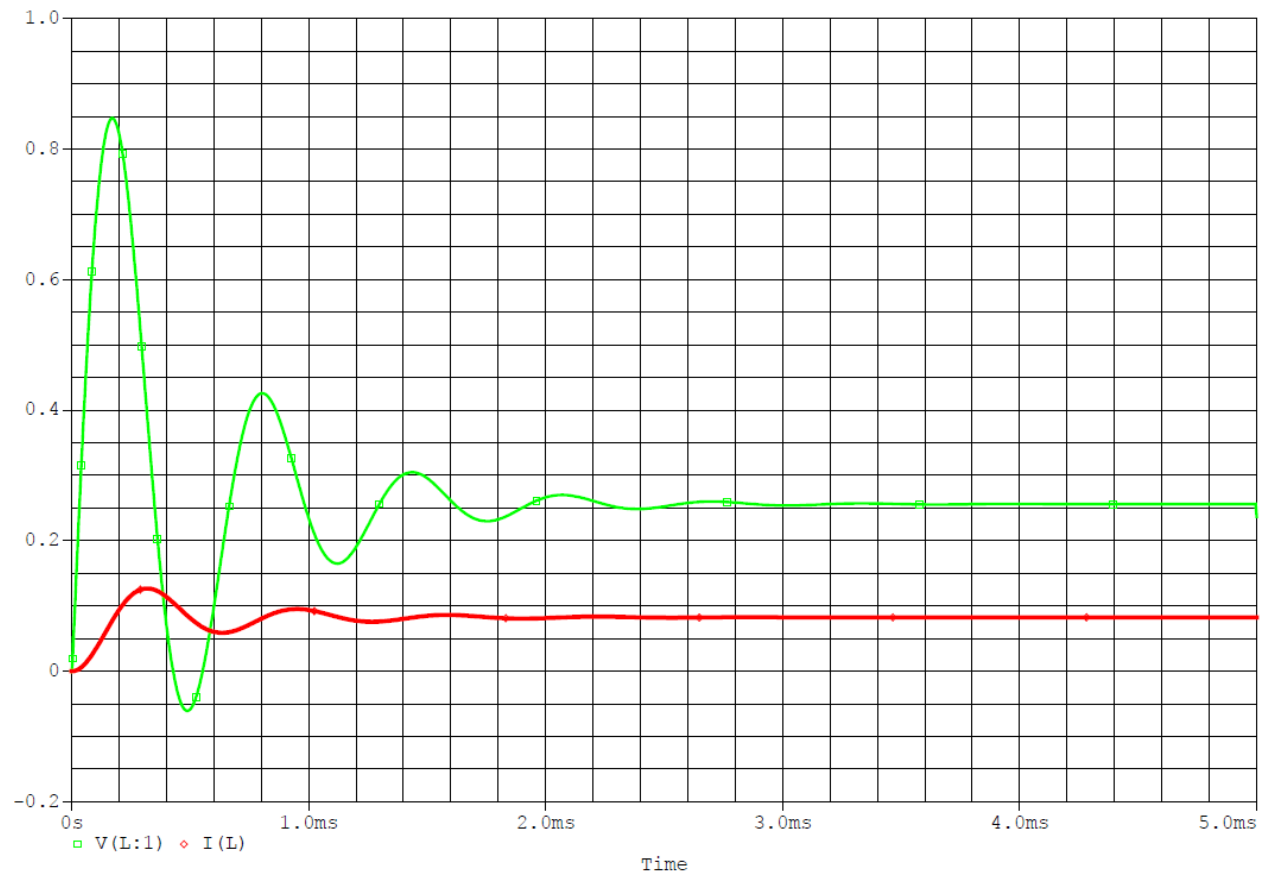
Appendix

Figure 9: Output of the OrCAD PSPICE step-response simulation.

```matlab
L   = 1.146e-3;
R_L = 2.23;


R_1 = 67.42;
R_2 = 1.20;
C   = 9.59e-6;


R_S = 50;

A = [-1/(C*(R_S+R_1)) -1/C;
      1/L                 -((R_2 + R_L)/L) ];

B = [ 1/(C*(R_S+R_1));
      0                  ];
C = [1 0;
     0 1];
D = [0;
     0];


sys     = ss(A,B,C,D);
[Y,T,X] = step(sys);

subplot(2, 2, 1);
plot(T, 10*X(:, 1));
xlabel('time [s]');
ylabel('x_1(t) [V]');
grid on;

subplot(2, 2, 3);
plot(T, 10*X(:, 2));
xlabel('time [s]');
ylabel('x_2(t) [A]');
grid on;

subplot(2, 2, [2, 4]);
plot(10*X(:,1), 10*X(:,2));
xlabel('x_1(t) [V]');
ylabel('x_2(t) [A]');
grid on;
```

Lab 2 Checklist

Name: Kevin Evans

Lab title and introduction
- Lab title, your name, date, and lab partner.
- Brief introduction (one to three sentences) explaining the purpose of this lab.

I. Mutual Inductance (50 pts total)

1. Diagram of transformer circuit (Figure 1).
2. Measured values for R_1, L_1, L_2, R_{L1}, R_{L2}.

MAH   3. **DEMO:** Have your TA or instructor initial this sheet, indicating that they have observed your circuit's operation.
- Circuit operates; responses to triangular and sinusoidal inputs obtained.
- Response to triangular input consistent with pre-lab analysis. Show results of pre-lab analysis to TA.
- Response to sinusoidal input consistent with pre-lab analysis. Show results of pre-lab analysis to TA.

4. *Triangular wave input:* Measured input and output voltage waveforms. Measurement-based estimate of M. Give pre-lab equation used to estimate M.
5. *Sinusoidal wave input:* Measured input and output voltage waveforms. Measurement-based estimate of M. Give pre-lab equation used to estimate M.
6. Calculated value of k, including equation used to compute k.
7. **DEMO:** Explain your dot convention to the TA or instructor. Have the TA initial this sheet, indicating that he/she agrees with your dot convention.
8. Post-lab exercise:
- MATLAB simulation of response to triangular wave input.
- MATLAB simulation of response to sinusoidal input.
- Comparison and discussion of measured vs. simulated circuit responses for both triangular and sinusoidal inputs.

II. State Variable Modeling of RLC circuit (50 pts total)

1. Circuit diagram (figure 4).
2. State variable model derived in pre-lab.

MAH   3. **DEMO:** Have your TA or instructor initial this sheet, indicating that he/she has observed your circuit's operation.
- Circuit operates; both states measured.
- TA approves state variable model derived in pre-lab.

4. Measured system states ($x_1(t)$ and $x_2(t)$) plotted vs. time.
5. Measured state trajectory plotted ($x_1(t)$ vs. $x_2(t)$).
6. Post-lab exercise:
- Simulated system states ($x_1(t)$ and $x_2(t)$) plotted vs. time.
- Simulated state trajectory plotted ($x_1(t)$ vs. $x_2(t)$).
- Comparison and discussion of measured vs. simulated step responses and state trajectories.