Homework 3

PHYSICS 341
September 16, 2020

Kevin Evans
ID: 11571810

---

1. (a) Since the δ function is at $x = 3$ and the upper bound of the integral is 2, the integral evaluates to zero,

$$\int_0^2 (2x+3)\,\delta(x-3)\,\mathrm{d}x = 0$$

(b) The δ function is at zero and evaluate to 1.

$$\int_{-2}^{2} \left(x^2+x+1\right)\delta(x)\,\mathrm{d}x = \left[x^2+x+1\right]_{x=0} = 1$$

(c) Using the scaling property,

$$\begin{aligned}\int_{-1}^{1} 9\left(x+1\right)^2\delta(3x)\,\mathrm{d}x &= \int_{-1}^{1} 3\left(x+1\right)^2\delta(x)\,\mathrm{d}x \\ &= \left[3\left(x+1\right)^2\right]_{x=0} = 3\end{aligned}$$

(d) The δ function is located at $x = 0$ and will evaluate to zero,

$$\int_{-\pi}^{\pi} \sin(x)\delta(x)\,\mathrm{d}x = \left[\sin x\right]_{x=0} = 0$$

2. (a) For $\mathbf{v} = yz\,\hat{\mathbf{x}} + xz\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \\
&= (x - x)\ \hat{\mathbf{x}} + (y - y)\ \hat{\mathbf{y}} + (z - z)\ \hat{\mathbf{z}} = 0
\end{aligned}$$

(b) For the scalar potential V, we can just inspect and integrate each differential and find

$$\begin{aligned}
\mathbf{v} &= -\boldsymbol{\nabla} V \\
yz\,\hat{\mathbf{x}} + xz\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}} &= -\left(\frac{\partial V}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\,\hat{\mathbf{z}}\right) \\
V &= -xyz
\end{aligned}$$

The vector potential $\mathbf{A}$ is found using

$$\begin{aligned}
\mathbf{v} &= \boldsymbol{\nabla} \times \mathbf{A} \\
v_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz \\
v_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = xz \\
v_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy
\end{aligned}$$

Letting the second differential equal zero in each v_i expression above, a vector potential is

$$\mathbf{A} = \frac{xz^2}{2}\,\hat{\mathbf{x}} + \frac{yx^2}{2}\,\hat{\mathbf{y}} + \frac{zy^2}{2}\,\hat{\mathbf{z}}$$

3. The displacement vector from each charge is

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} \pm \frac{d}{2}\,\hat{\mathbf{x}} \\
|\boldsymbol{\mathscr{r}}| &= \left(z^2 + \frac{d^2}{4}\right)^{1/2}
\end{aligned}$$

The sum of the two electric fields add and point toward the $-\,\hat{\mathbf{x}}$ direction

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\boldsymbol{\mathscr{r}}_i|^2}\ \hat{\boldsymbol{\mathscr{r}}}_i \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{(z^2 + d^2/4)^{3/2}} \Bigg(\underbrace{z\,\hat{\mathbf{z}} - \frac{d}{2}\,\hat{\mathbf{x}}}_{\text{due to } +q} - \underbrace{z\,\hat{\mathbf{z}} - \frac{d}{2}\,\hat{\mathbf{x}}}_{-q} \Bigg) \\
&= -\frac{qd\,\hat{\mathbf{x}}}{4\pi\epsilon_0\,(z^2 + d^2/4)^{3/2}}
\end{aligned}$$

4. For a line segment, the tiny bit of charge would be $\mathrm{d}q = \lambda\,\mathrm{d}\ell'$ and the displacement vector

$$\begin{aligned} \boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = d\,\hat{\mathbf{x}} - y'\,\hat{\mathbf{y}} \\ |\boldsymbol{\mathscr{r}}| &= \left(d^2 + y'^2\right)^{1/2} \end{aligned}$$

The electric field at point P becomes

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_0^L \frac{1}{\left(d^2 + y'^2\right)^{3/2}} \left(d\,\hat{\mathbf{x}} - y'\,\hat{\mathbf{y}}\right) \left(\lambda\,\mathrm{d}\ell'\right)$$

Bringing out the λ (as it's uniform) and rearranging,

$$\begin{aligned} \mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0} \left[d\,\hat{\mathbf{x}} \int_0^L \left(d^2 + y'^2\right)^{-3/2} \mathrm{d}y' - \hat{\mathbf{y}} \int_0^L \frac{y'}{\left(d^2 + y'^2\right)^{3/2}}\,\mathrm{d}y' \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{L}{d\sqrt{d^2 + L^2}}\,\hat{\mathbf{x}} + \left(\frac{1}{\sqrt{d^2 + L^2}} - \frac{1}{d} \right) \hat{\mathbf{y}} \right] \quad \leftarrow \text{integration table} \end{aligned}$$

For the case $d \gg L$, the terms in the $\hat{\mathbf{y}}$ group would cancel and we would be left with something like a point charge at the origin

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{d\sqrt{d^2}}\,\hat{\mathbf{x}} = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{total}}}{d^2}\,\hat{\mathbf{x}}$$

5. From the symmetry of the ring, we can expect the only non-zero component of the electric field will be in the $\hat{\mathbf{z}}$-direction and reduce the $\boldsymbol{\mathscr{r}}$ vector expression, though its magnitude would still contain an R term.

$$\begin{aligned} \mathrm{d}q &= \lambda\,\mathrm{d}\ell' = \lambda s'\,\mathrm{d}\phi' = \lambda R\,\mathrm{d}\phi' \\ \boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' \approx z\,\hat{\mathbf{z}} \\ |\boldsymbol{\mathscr{r}}| &= \left(z^2 + R^2\right)^{1/2} \end{aligned}$$

Putting this all together, we would integrate fully around ϕ',

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda R z\,\mathrm{d}\phi'}{\left(z^2 + R^2\right)^{3/2}}\,\hat{\mathbf{z}} \\ &= \frac{\lambda R z}{2\epsilon_0 \left(z^2 + R^2\right)^{3/2}}\,\hat{\mathbf{z}} \end{aligned}$$

At $z \gg R$, the electric field approaches something resembling a point charge

$$\mathbf{E} = \frac{\lambda R z}{2\epsilon_0 \left(z^2\right)^{3/2}}\,\hat{\mathbf{z}} = \frac{\lambda R}{2\epsilon_0 z^2}\,\hat{\mathbf{z}}$$