Homework 2

PHYSICS 450
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1. Read Chapter 2.1, 2.4.

2. From the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

If we assume a separable solution and plug it into the time-dependent SE,

$$\begin{aligned}\Psi(x,t) &= \psi(x)\phi(t) \\ i\hbar\psi\frac{\partial\phi}{\partial t} &= -\frac{\hbar^2}{2m}\phi\frac{\partial^2\psi}{\partial x^2} + V\phi\psi\end{aligned}$$

Dividing by Ψ, we can see the two sides must equal a constant E,

$$\begin{aligned}\frac{i\hbar}{\phi}\frac{\partial\phi}{\partial t} &= -\frac{\hbar^2}{2m\psi}\frac{\partial^2\psi}{\partial x^2} + V = E \\ \implies &- \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi \\ \implies &i\hbar\frac{\partial\phi}{\partial t} = E\phi\end{aligned}$$

These equations are solvable by integrating

$$\begin{aligned}i\hbar\frac{\partial\phi}{\partial t} &= E\phi \\ \frac{i\hbar}{\phi}\,\mathrm{d}\phi &= E\,\mathrm{d}t \\ i\hbar\ln\phi &= Et + k_0 \\ \phi(t) &= k_1 e^{-iEt/\hbar}\end{aligned}$$

Assuming $V = 0$, the spatial equation is solvable with

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

By inspection, this is a sinusoid,

$$\begin{aligned}\psi(x) &= e^{i\hbar x/\sqrt{2mE}} \\ &= e^{ikx}, \quad \text{where } k = \frac{\hbar}{\sqrt{2mE}}\end{aligned}$$

Bringing this all together, the wavefunction with both parts is

$$\begin{aligned}\Psi(x,t) &= \psi(x)\phi(t) \\ &= e^{i(kx-\omega t)}, \qquad \text{where } k = \hbar/\sqrt{2mE} \\ &\qquad\qquad\qquad\qquad \omega = E/\hbar\end{aligned}$$

3. No it is not quantized, since there is an infinite spectrum of energies. If it were bounded in a square well, then the energies could be quantized.

4. (a) From class, we derived

$$a(t) = a_0\sqrt{1 + \left(\frac{\hbar t}{m{a_0}^2}\right)^2}.$$

For $m = 1\,\mathrm{g}$ and $a_0 = 1\,\mu\mathrm{m}$, it doubles in width when

$$\begin{aligned}\sqrt{1 + \left(\frac{\hbar t}{m{a_0}^2}\right)^2} &= 2\\ \frac{\hbar t}{m{a_0}^2} &= \sqrt{3}\\ t &= \frac{\sqrt{3}m{a_0}^2}{\hbar}\\ &\approx 1.64 \times 10^{19}\,\mathrm{s} \approx 38 \times \text{age of the universe.}\end{aligned}$$

(b) For $m = m_e$,

$$t \approx 14.96\,\mathrm{ns}.$$

After one second, it's huge

$$\begin{aligned}a(t) &= a_0\sqrt{1 + \left(\frac{\hbar t}{m_e{a_0}^2}\right)^2}\\ &\approx 115.8\,\mathrm{m}.\end{aligned}$$

5. Considering the Gaussian pdf

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

(a) It is normalized with

$$\begin{aligned}1 &= A\int_{\mathbb{R}} e^{-\lambda(x-a)^2}\,\mathrm{d}x = A\sqrt{\frac{\pi}{\lambda}}\\ \implies A &= \sqrt{\frac{\lambda}{\pi}}.\end{aligned}$$

(b) The expectation of x is

$$\begin{aligned}\langle x\rangle &= \int_{\mathbb{R}} x\rho(x)\,\mathrm{d}x\\ &= A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}\,\mathrm{d}x\,.\end{aligned}$$

Let $z = x - a$. Then we can write the integral as

$$\begin{aligned}\langle x\rangle &= A\int (z+a)e^{-\lambda z^2}\,\mathrm{d}z\\ &= A\left(\int ze^{-\lambda z^2}\,\mathrm{d}z + a\int e^{-\lambda z}\,\mathrm{d}z\right)\end{aligned}$$

Let $u = -\lambda z^2$, then $\mathrm{d}u = -2\lambda z \,\mathrm{d}z$,

$$\begin{aligned}
\langle x \rangle &= A\left(-\frac{1}{2\lambda}\int e^u \,\mathrm{d}u + a\sqrt{\frac{\pi}{\lambda}}\right) \\
&= A\left(-\frac{1}{2\lambda}\, e^{-2\lambda(x-a)^2}\Big|_{z=-\infty}^{\infty} + a\sqrt{\frac{\pi}{\lambda}}\right) \\
&= Aa\sqrt{\frac{\pi}{\lambda}} = a?
\end{aligned}$$

For the expectation of x^2, we can first let $u = x - a$, then $x = u + a$, and

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 \rho(x) \,\mathrm{d}x \\
&= A\int (u^2 + a^2 + 2au) e^{-\lambda u^2} \,\mathrm{d}x
\end{aligned}$$

The last term is an odd function, so it evaluates to zero and we're left with

$$\langle x^2 \rangle = A\int u^2 e^{-\lambda u^2} \,\mathrm{d}u + a^2 A \int e^{-\lambda u^2} \,\mathrm{d}u$$

Using some WolframAlpha to integrate

$$\begin{aligned}
\langle x^2 \rangle &= A\left(\frac{\sqrt{\pi}}{2\lambda^{3/2}} + a^2\sqrt{\frac{\pi}{\lambda}}\right) \\
&= \frac{1}{2\sqrt{\lambda}} + a^2
\end{aligned}$$

(c) The variance is given by

$$\begin{aligned}
\sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
&= a^2 - a + \frac{1}{2\sqrt{\lambda}}
\end{aligned}$$

6. Read 3.1 and 3.2.