Homework 8

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4.2.1 Determine the dual of each of the following linear programming problems.

(d) Minimize $6x_1 + 12x_2 - 18x_3$
subject to

$$\begin{aligned} x_1 - 3x_2 + \ \ 6x_3 &= 30 \\ 2x_1 + 8x_2 - 16x_3 &= 70 \end{aligned}$$

$x_1, x_2 \geq 0, x_3$ unrestricted

Solution. Using the conversion table from in-class, we have a maximization problem using the B's from the original matrix. Then we can transpose the constraints matrix A and convert the variables to constraints,

Maximize $30y_1 + 7y_2$
subject to

$$\begin{array}{rcrcr} y_1 & + & 2y_2 & \leq & 6 \\ -3y_1 & + & 8y_2 & \leq & 12 \\ 6y_1 & - & 16y_2 & = & -18 \end{array}$$

y_1, y_2 urs

(e) Maximize $x_1 - 7x_2 + 3x_3$
subject to

$$\begin{array}{rcrcrl} & & 2x_2 & + & 5x_3 & = 20 \\ 8x_1 & & & - & 3x_3 & = 40 \\ & & x_2 & + & 4x_3 & \geq 60 \end{array}$$

$x_1, x_3 \geq 0, x_2$ unrestricted

Solution. Again, using the conversion table from in-class, we have

Minimize $20y_1 + 40y_2 + 60y_3$
subject to

$$\begin{array}{rcrcrcr} & & 8y_2 & & & \geq & 1 \\ 2y_1 & & & + & 1y_3 & = & -7 \\ 5y_1 & - & 3y_2 & + & 4y_3 & \geq & 3 \end{array}$$

y_1, y_2 unrestricted, $y_3 \leq 0$

4.4.6 Consider the problem of

$$\begin{array}{lrcrcrcrcr}
\min & \multicolumn{9}{l}{z = 13x_1 + 15x_2 + 12x_3 + 8x_4} \\
\text{s.t.} & 4x_1 & + & 8x_2 & - & 5x_3 & + & 3x_4 & = & 32 \\
 & 3x_1 & - & 2x_2 & + & 6x_3 & - & x_4 & \geq & 3 \\
 & \multicolumn{9}{l}{x \geq 0}
\end{array}$$

(a) Determine which of the following points are feasible solutions to this min problem: $(9, 0, 2, 2)$, $(4, 1, -1, 1)$, $(5, 1, 1, 3)$.

Solution. Checking against the three constraints,

- $(9, 0, 2, 2)$ is feasible as it abides by the constraints;
- $(4, 1, -1, 1)$ is not feasible, as $x \geq 0$;
- $(5, 1, 1, 3)$ is feasible as it abides by the constraints.

(b) Evaluate the function z at those points in part (a) that are feasible solutions to the problem.

Solution. For the two feasible points found in (a), the objective function value is

- $z(9, 0, 2, 2) = 157$,
- $z(5, 1, 1, 3) = 116$.

(c) Write out the dual to the min problem.

Solution. Using the conversion table from class,

$$\begin{array}{rl}
\max & w = 32y_1 + 3y_2 \\
\text{s.t.} & 4y_1 + 3y_2 \leq 13 \\
 & 8y_1 - 2y_2 \leq 15 \\
 & -5y_1 + 6y_2 \leq 12 \\
 & 3y_1 - 1y_2 \leq 8 \\
 & y_1 \text{ urs}, y_2 \geq 0
\end{array}$$

(d) Determine which of the following points are feasible solutions to this dual problem: $(-1, 1)$, $(0, 2)$, $(1, 3)$.

Solution.

- $(-1, 1)$ is feasible;
- $(0, 2)$ is feasible;
- $(1, 3)$ is not feasible (fails the third constraint).

(e) Evaluate the dual objective function at those points in part (d) that are feasible solutions to the problem.

Solution.

- $w(-1, 1) = -32 + 3 = -29$;
- $w(0, 2) = 6$.

(f) Using only the information above, what can you say about the minimum value of z?

Solution. It is somewhere between -29 and 116 (by Theorem 4.4.1).

4.5.2 Consider the linear program

$$\begin{array}{rrcrcrcrl}
\max & 2x_1 + 2x_2 \\
\text{s.t.} & x_1 & & & + & x_3 & + & x_4 & \leq 1 \\
& & & x_2 & + & x_3 & - & x_4 & \leq 1 \\
& x_1 & + & x_2 & + & 2x_3 & & & \leq 3 \\
& x \geq 0
\end{array}$$

(a) Determine the dual problem.

Solution. Using the conversion table from in-class, the dual of the problem is

$$\begin{array}{rrcrcrl}
\min & 1y_1 + 1y_2 + 3y_3 \\
\text{s.t.} & y_1 & & & + & y_3 & \geq 2 \\
& & & y_2 & + & y_3 & \geq 2 \\
& y_1 & + & y_2 & + & 2y_3 & \geq 0 \\
& y_1 & - & y_2 & & & \geq 0 \\
& y \geq 0
\end{array}$$

(b) Show that $X^* = (1,1,0,0)$ and $Y^* = (1,1,1)$ are feasible solutions to the original and dual problems, respectively.

Solution. For $X^* = (1,1,0,0)$,

$$1 + 0 + 0 \leq 1 \qquad \checkmark$$
$$1 + 0 - 0 \leq 1 \qquad \checkmark$$
$$1 + 1 + 2(0) \leq 3 \qquad \checkmark$$

And for $Y^* = (1,1,1)$,

$$1 + 1 \geq 2 \qquad \checkmark$$
$$1 + 1 \geq 2 \qquad \checkmark$$
$$1 + 1 + 2 \geq 2 \qquad \checkmark$$
$$1 - 1 \geq 0 \qquad \checkmark$$

(c) Show that for this pair of solutions, for each j, $x_j^* > 0$ implies that the slack in the corresponding dual constraint is zero.

Solution. Looking at the indices $j = 1, 2$, $x_j^* = 1, 1$, the corresponding slacks for $Y^* = (1,1,1)$ are: $2 - (1+1) = 0$ and $2 - (1+1) = 0$, respectively.

(d) Show that Y^* is not an optimal solution to the dual.

Solution. For $X^* = (1,1,0,0)$, the corresponding slack is $(0,0,1)$. However, since the last index of $Y^* = (1,1,1)$ is not zero, this solution cannot be optimal.

(e) Does this contradict the Complementary Slack Theorem?

Solution. No, because both the slack in the primal and dual constraints must either be zero. In this case, we only have it in one direction, i.e. X^* is an optimal solution to the dual, but Y^* is not.