Homework 9

PHYSICS 461
March 24, 2021

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1. Assuming the isospin function must also be antisymmetric (by the Pauli principle) and the proton and neutron are orthogonal states, then deuteron would be more favorable than diproton and dineutron.

   Deuteron would have a total isospin of 1 or 0. The diproton and dineutron would have isospin 1.

2.

3. (a) For $\hat{A} = x$,

$$\begin{aligned}\langle\phi|\, x \,|\psi\rangle &= \int \phi^* x \psi \, \mathrm{d}x \\ &= \int x \phi^* \psi \, \mathrm{d}x\end{aligned}$$

   (b) For $\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x}$,

$$\begin{aligned}\langle\phi|\, \frac{\mathrm{d}}{\mathrm{d}x} \,|\psi\rangle &= \int \phi^* \frac{\mathrm{d}\psi}{\mathrm{d}x} \, \mathrm{d}x \\ &= \phi^* \psi \Big|_{\pm\infty} - \int \psi \frac{\mathrm{d}\phi^*}{\mathrm{d}x} \, \mathrm{d}x \\ &= -\int \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^* \psi \, \mathrm{d}x\end{aligned}$$

   Therefore, $\frac{\mathrm{d}}{\mathrm{d}x}$ is antihermitian.

   (c) For $\hat{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$,

$$\begin{aligned}\langle\phi|\, \hat{p} \,|\psi\rangle &= \int \phi^* \left(-i\hbar \frac{\mathrm{d}\psi}{\mathrm{d}x}\right) \mathrm{d}x \\ &= -\int \left(-i\hbar\psi\right) \frac{\mathrm{d}\phi^*}{\mathrm{d}x} \, \mathrm{d}x \\ &= \int \left(-i\hbar \frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^* \psi \, \mathrm{d}x\end{aligned}$$

   The momentum operator is indeed Hermitian as $\hat{p} = \hat{p}^\dagger$.

   (d) For the Hamiltonian $\hat{H} = -(\hbar^2/2m)\frac{\partial^2}{\partial x^2} + V(x)$,

$$\begin{aligned}\langle\phi|\, \hat{H} \,|\psi\rangle &= \int \phi^* \left(-(\hbar^2/2m)\frac{\partial^2}{\partial x^2} + V(x)\right) \psi \, \mathrm{d}x \\ &= -\int \left(-(\hbar^2/2m)\frac{\partial^2}{\partial x^2} + V(x)\right)\end{aligned}$$

4. (a) By the anti-distributive property of the Hermitain adjoint, $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$.
   (b) If we have $\langle\psi|\, \hat{A}\, |\phi\rangle$ and conjugate that,
$$\begin{aligned}\Big(\langle\psi|\, \hat{A}\, |\phi\rangle\Big)^* &= |\psi\rangle\, \hat{A}^\dagger\, \langle\phi| \\ &= \langle\phi|\, \hat{A}^\dagger\, |\psi\rangle \quad \square\end{aligned}$$
   (c)

5. (a) Taking the Hamiltonian of a state,
$$\begin{aligned}\hat{H}\,|\psi\rangle &= a\hat{H}\,|\phi_i\rangle + b\hat{H}\,|\phi_f\rangle \\ &= a(\hat{H}_0 + \hat{V})\,|\phi_i\rangle + b(\hat{H}_0 + \hat{V})\,|\phi_f\rangle \\ &= \begin{bmatrix} E_i & M_{fi} \\ M_{if} & E_f \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\end{aligned}$$
   For the M_{fi} integral, there is no H_0 term as it is in the E_i/E_f components? And as $\langle\phi_i|\phi_f\rangle = 0$.

   (b)

6. (a) By Problem 4, it must be Hermitain as $\hat{N}^\dagger = \hat{N}$:
$$\hat{N}^\dagger = (a^\dagger a)^\dagger = a^\dagger a$$
   (b) The Hamiltonian can be written as
$$\begin{aligned}\hat{H} &= \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \\ &= \frac{1}{2}\hbar\omega\left(\hat{N} + \hat{N} + 1\right)\end{aligned}$$
   As $1 = aa^\dagger - a^\dagger a$,
$$= \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right)$$

7. Multiplying it out and letting $\hat{x}$ and $\hat{y}$ be the things in the parenthesis,
$$\begin{aligned}aa^\dagger - a^\dagger a &= \frac{1}{2}\left(\hat{x}^2 - \hat{x}\hat{y} + \hat{y}\hat{x} - \hat{y}^2\right) - \frac{1}{2}\left(\hat{x}^2 + \hat{x}\hat{y} - \hat{y}\hat{x} - \hat{y}^2\right) \\ &= \frac{1}{2} \times 2 \qquad \text{as the first derivatives are evaluated, resulting in 1 for each cross terms}\end{aligned}$$
   From eq. (7),
$$\begin{aligned}\hat{H} &= \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right) \\ &= \frac{1}{4}\hbar\omega\left(\alpha^2x^2 - \alpha x\frac{\mathrm{d}}{\mathrm{d}x} - 1 + \frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha^2x^2 + \alpha x\frac{\mathrm{d}}{\mathrm{d}x} + 1 + \frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \\ &= \frac{1}{2}\hbar\omega\left(\alpha^2x^2 + \frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \\ &= \frac{1}{2}\omega^2x^2 + \frac{\hbar\omega}{2}\frac{\hbar}{\omega}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ &= \frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\omega^2x^2\end{aligned}$$

Not sure where the negative is coming from...

8. (a) For eq. (9),

$$\hat{N}a\,|n\rangle = (a^\dagger a)a\,|n\rangle = (\underbrace{aa^\dagger}_{\hat{N}=n} - 1)a\,|n\rangle = (n-1)a\,|n\rangle$$

   Similarly for eq. (10),

$$\hat{N}a^\dagger\,|n\rangle = (1 + a^\dagger a)a^\dagger\,|n\rangle = (n+1)a^\dagger\,|n\rangle$$

   (b)

9. Using eq. (11) and (12) in the Hamiltonian,

$$\begin{aligned}\hat{H} &= \int_V \left[-\frac{\epsilon_0}{2}\frac{\hbar\omega}{2\epsilon_0 V}\left(ae^{ikr} - a^\dagger e^{-ikr}\right)^2 + \frac{1}{2\mu_0}\frac{\mu_0\hbar\omega}{2V}\left(ae^{ikr} + a^\dagger e^{-ikr}\right)\right] \mathrm{d}\tau \\ &= \frac{\hbar\omega}{4V}\int_V \left[2\left(a^2 e^{2ikr} + (a^\dagger)^2 e^{-2ikr}\right)\right] \mathrm{d}\tau\end{aligned}$$

   I assume this somehow integrates to V? So it'll leave

$$\hat{H} = \frac{\hbar\omega}{2}$$

10. (a) Using eq. (11) and (14),

$$\text{Rate} = \text{factor} \times |M_{fi}|^2 \times \left|\langle n-1|\,\frac{\hbar\omega}{2\epsilon_0 V}\left(a - a^\dagger\right)^2 |n\rangle\right|^2$$