Homework 6

PHYSICS 342
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1. (a) The reflection and transmission coefficients are given by (9.86) and (9.87) respectively as

$$\begin{aligned} R &= \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \\ &\approx 0.05 \\ T &= \left(\frac{4n_1 n_2}{(n_1 + n_2)^2}\right) \\ &\approx 0.95 \end{aligned}$$

(b) At Brewster's angle, $r = 0$, and from Fresnel's equations,

$$\begin{aligned} \alpha &= \beta \\ \implies n_1 \cos(\theta_T) &= n_2 \cos(\theta_I) \end{aligned}$$

By Snell's law,

$$\cos(\theta_I) = \frac{n_1}{n_2} \cos\left(\arcsin\left(\frac{n_1}{n_2} \sin\theta_I\right)\right)$$

Plugging this into WolframAlpha and solving for θ_I,

$$\begin{aligned} \theta_B &= \arccos\left(\frac{n_1}{\sqrt{n_1^2 + n_2^2}}\right) \\ &\approx 0.998\,\mathrm{rad} \approx 57.2\,\mathrm{deg} \end{aligned}$$

(c) At the crossover angle,

$$\begin{aligned} \alpha - \beta &= 2 \\ \frac{\cos(\theta_T)}{\cos(\theta_I)} &= 2 + \frac{n_2}{n_1} \\ \frac{\cos\left(\arcsin\left(\frac{n_1}{n_2}\sin(\theta_I)\right)\right)}{\cos(\theta_I)} &= \frac{2n_1 + n_2}{n_1} \end{aligned}$$

Using $n_1 = 1$ and using WolframAlpha, this simplifies to

$$\begin{aligned} \theta_C &= \operatorname{arcsec}\left(\sqrt{\frac{(n_2+1)^2(n_2+2n_2+1)^2}{n_2^2 - 1}}\right) \\ &\approx 1.35\,\mathrm{rad} \approx 77.4\,\mathrm{deg} \end{aligned}$$

2. (a) The characteristic time is

$$\begin{aligned} \tau &= \epsilon/\sigma \\ &= \frac{\epsilon_0 n^2}{1/\rho} \\ &= 2.42^2 \times (8.854 \times 10^{-12}\,\mathrm{F \cdot m^{-1}}) \times (1 \times 10^{11}\,\Omega \cdot \mathrm{m}) \\ &= 5.2\,\mathrm{s} \end{aligned}$$

(b) The imaginary wavenumber κ is given by

$$\begin{aligned}\kappa &= \frac{\omega}{c\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon_0\omega}\right)^2}-1\right]^{1/2} \\ &= 1.187\times 10^6\,\mathrm{m}^{-1}\end{aligned}$$

The skin depth is then

$$d = \frac{1}{\kappa} = 0.843\,\mu\mathrm{m}$$

(c) The real wavenumber k is given by

$$\begin{aligned}k &= \frac{\omega}{c\sqrt{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon_0\omega}\right)^2}+1\right]^{1/2} \\ &= 11\,866\,\mathrm{m}^{-1}\end{aligned}$$

The wavelength and propagation speed is given by (9.129),

$$\begin{aligned}\lambda &= \frac{2\pi}{k} = 529.5\,\mu\mathrm{m} \\ v &= \frac{\omega}{k} = 529.5\,\mathrm{m}\cdot\mathrm{s}^{-1}\end{aligned}$$

In vacuum, the wavelength is

$$\begin{aligned}\lambda_0 &= c/f = 300\,\mathrm{m} \\ v_0 &= c\end{aligned}$$

3. Letting $\gamma_j = 0$, the wavenumber becomes real

$$k = \frac{\omega}{c}\left[1+\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2}\right]$$

Taking the derivative with respect to ω,

$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{1}{c}\left[1+\frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left(\frac{\omega_j^2+\omega^2}{(\omega_j^2-\omega^2)^2}\right)\right]$$

The velocity is then just the multiplicative inverse of that

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c\left[1+\frac{Nq^2}{2m\epsilon_0}\sum_j f_j\left(\frac{\omega_j^2+\omega^2}{(\omega_j^2-\omega^2)^2}\right)\right]^{-1}$$

Since the sum above is always positive, then $1+\sum_j(\dots)$ will always be greater than 1. As it's in the denominator, $v_g < c$.

4. Using (9.188), the associated frequencies for TE mode mn are

$$\omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2}$$

By iterating over potential modes, we can find frequencies that are $< 1.5 \times 10^{10}$ Hz,

m	n	f_{mn}
1	0	3.3×10^9 Hz
2	0	6.7×10^9 Hz
3	0	1×10^{10} Hz
4	0	1.3×10^{10} Hz
1	1	1.1×10^{10} Hz
2	1	1.2×10^{10} Hz
3	1	1.4×10^{10} Hz
0	1	1×10^{10} Hz

To excite a single mode, these combinations are possible: $mn = \{10, 20, 30, 40, 01\}$.

5. The time averaged Poynting vector is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathbf{E} \times \mathbf{B}^*$$

where B_z is given by (9.186) and $E_z = 0$. Since the Poynting vector is directed in z, this resolves to

$$\langle S_z \rangle = \frac{1}{2\mu_0}\left(E_x B_y - E_y B_x\right)$$

Applying the forms written in (9.180), this becomes

$$\begin{aligned}
&= -\frac{1}{2\mu_0}\left(\frac{i^2\omega k}{[(\omega/c)^2 - k^2]^2}\right)\left[\frac{\partial B_z}{\partial y}\frac{\partial B_z}{\partial y} + \frac{\partial B_z}{\partial x}\frac{\partial B_z}{\partial x}\right] \\
&= \frac{\omega k B_0^2}{2\mu_0[(\omega/c)^2 - k^2]^2}\left[\frac{n^2\pi^2}{b^2}\cos^2(m\pi x/a)\sin^2(n\pi y/b)\right. \\
&\qquad \left. + \frac{m^2\pi^2}{a^2}\sin^2(m\pi x/a)\cos^2(n\pi y/a)\right]
\end{aligned}$$

Integrating this mess over the cross-sectional area gives

$$\iint \langle S_z \rangle \,\mathrm{d}x\,\mathrm{d}y = \frac{\omega k B_0^2 ab\pi^2}{8\mu_0[(\omega/c)^2 - k^2]^2}\left(\frac{n^2}{b^2} + \frac{m^2}{a^2}\right)$$

For the average energy density,

$$\begin{aligned}
\langle u\rangle &= \frac{1}{4}\left[\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right] \\
&= \frac{1}{4}\left[\epsilon_0\left(E_x^2+E_y^2\right) + \frac{1}{\mu_0}\left(B_x^2+B_y^2+B_z^2\right)\right] \\
&= \frac{1}{4[(\omega/c)^2-k^2]^2}\left\{\epsilon_0\omega^2\left[\left(\frac{\partial B_z}{\partial y}\right)^2 + \left(\frac{\partial B_z}{\partial x}\right)^2\right] + \frac{k^2}{\mu_0}\left[\left(\frac{\partial B_z}{\partial x}\right)^2 + \left(\frac{\partial B_z}{\partial y}\right)^2 + B_z^2\right]\right\} \\
&= \frac{B_0^2}{4[(\omega/c)^2-k^2]^2}\Bigg[(\epsilon_0\omega^2 + k^2/\mu_0)\Bigg(\frac{m^2\pi^2}{b^2}\sin^2(m\pi x/a)\cos^2(n\pi y/b) \\
&\qquad + \frac{n^2\pi^2}{a^2}\cos^2(m\pi x/a)\sin^2(n\pi y/b)\Bigg) + (k^2/\mu_0)\cos(m\pi x/a)\cos(n\pi y/b)\Bigg]
\end{aligned}$$

Integrating this over the area results in

$$\iint \langle u\rangle \,\mathrm{d}x\,\mathrm{d}y = \frac{B_0^2}{4[(\omega/c)^2-k^2]^2}\left[\frac{\pi^2 ab(\epsilon_0\omega^2+k^2/\mu_0)}{4}\left(\frac{n^2}{b}+\frac{m^2}{a}\right)\right]$$

Putting it all together,

$$\begin{aligned}
\frac{\int \langle S_z\rangle \,\mathrm{d}a}{\int \langle u\rangle \,\mathrm{d}a} &= \frac{16\omega k}{8\mu_0(\epsilon_0\omega^2 + k^2/\mu_0)} \\
&= \frac{2\omega k}{\mu_0\epsilon_0\omega^2 + k^2} = \frac{2\omega k}{(\omega/c)^2+k^2}
\end{aligned}$$

I think I might have an error somewhere because this doesn't seem to reduce?