Problem Set 7

PHYSICS 443
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1. As the function is odd, we can assume $A_m = 0 \; \forall \; m$ and only find B_m,

$$\begin{aligned}
B_m &= \frac{2}{T}\left[\int_0^{T/2} \sin(m\omega t)\,\mathrm{d}t - \int_{T/2}^{T} \sin(m\omega t)\,\mathrm{d}t\right] \\
&= -\frac{2}{m\omega T}\left[\cos(m\omega t)\Big|_0^{T/2} - \cos(m\omega t)\Big|_{T/2}^{T}\right] \\
&= -\frac{1}{m\pi}\left[\cos(m\pi) - 1 - 1 + \cos(m\pi)\right] \\
&= \frac{2}{m\pi}\left(1 - \cos m\pi\right)
\end{aligned}$$

   As this is only non-zero for odd m, the Fourier series is

$$f(t) = \frac{4}{\pi}\sum_{m=1,3,5\ldots} \frac{1}{m}\sin(m\omega t)$$

2. For the function

$$f(x) = e^{-ax}$$

   We can apply the Fourier transform for $x > 0$ and find

$$\begin{aligned}
F(k) &= \int_0^\infty e^{-ax}e^{-ikx}\,\mathrm{d}x \\
&= \int_0^\infty e^{(-a-ik)x}\,\mathrm{d}x \\
&= \frac{1}{a+ik}
\end{aligned}$$

3. If $\mathscr{F}\{g(x)\} = G(k)$, then $\mathscr{F}\{g(x-a)\} = G(k)e^{-ika}$

$$\begin{aligned}
\textit{Let } u &= x - a \\
\textit{Then } x &= u + a \\
\mathrm{d}x &= \mathrm{d}u \\
\mathscr{F}\{g(u)\} &= \int_{\mathbb{R}} g(u)e^{-ik(u+a)}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} g(u)e^{-ik(u+a)}\,\mathrm{d}x \\
&= e^{-ika}\underbrace{\int_{\mathbb{R}} g(u)e^{-iku}\,\mathrm{d}u}_{G(k)} \\
&= G(k)e^{-ika} \qquad \square
\end{aligned}$$

4. From the in-class derivation of the Fourier transform of a Gaussian cosine, the FWHM in the time domain is

$$\begin{aligned}\Delta t &= 2t_{1/2} = 2\sqrt{\alpha \ln 2} \\ &= 2\sqrt{\tau^2 \ln 2} = 33.3\,\mathrm{fs}\end{aligned}$$

   (a) In the frequency spectrum, if we assume a product of $8 \ln 2$,

$$\Delta\nu \Delta t = 8 \ln 2/(2\pi)$$
$$\boxed{\Delta\nu = 26.5\,\mathrm{THz}}$$

   (b) For the (absolute) wavelength,

$$\begin{aligned}\Delta\lambda &= \frac{\lambda^2 \Delta\nu}{c} \\ &= \frac{(800\,\mathrm{nm})^2\,(26.5\,\mathrm{THz})}{c}\end{aligned}$$
$$\boxed{\Delta\lambda = 56.5\,\mathrm{nm}}$$