Homework 12

MATH 301
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1. (a) Reflexive: as xR_1x for all $x \in A$.
   Symmetric as $aR_1b \implies bR_1a$.
   Transitive as aR_1b and bR_1a implies aR_1a.
   (b) Not reflexive $(a, a) \notin R_2$.
   Not symmetric as $(a, b) \in R_2$, but $(b, a) \notin R_2$.
   Not transitive as $(a, b), (b, c) \in R_2$, but $(a, c) \notin R_2$.
   (c) Not reflexive, $(a, a) \notin R_3$.
   Not symmetric as $(a, b) \in R_3$, but $(b, a) \notin R_2$.
   Transitive as there's no non-transitive pairs.

2. Not reflexive, as $\exists a \in A$ where $a \notin \varnothing$.

   Symmetric, as there is not a pair that is non-symmetric in $\varnothing$.

   Transitive, by the same logic as it being symmetric.

3. *Disproof.* There exists a relation that is both symmetric and transitive, but is not reflexive.

   For example, a set $A = \{a, b, c\}$ and relation on A as $R = \{(a, a), (b, b), (a, b), (b, a)\}$. The relation R is symmetric and transitive, but is not reflexive as $(c, c) \notin R$.

4. **Proposition.** *The relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$ is an equivalence relation.*

   Proof. (a) Reflexive. For $(x, x) \in \mathbb{R}^2$, $x - x = 0$ and $0 \in \mathbb{Z}$.

   (b) Symmetric. For $(x, y) \in \mathbb{R}^2$, as $x - y \in \mathbb{Z}$, then $y - x = -(x - y) \in \mathbb{Z}$ too.

   (c) Transitive. For $(x, y), (y, z) \in \mathbb{R}^2$, subtracting the two relations from another,

   $$\begin{aligned} x - y &\in \mathbb{Z} \\ y - z &\in \mathbb{Z} \end{aligned}$$

   Then $x - z \in \mathbb{Z}$ as subtraction is closed for integers, therefore $(x, z) \in R$.

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5. (a) $R_1 = \{(x, y) \in \mathbb{Z}^2 : 2^2 \times 2^{|x-y|} - 1 \text{ is prime}\}$
   (b) $R_2 = \{(x, y) \in \mathbb{Z}^2, 2^2 \times 2^{x-y} - 1 \text{ is prime}\}$
   (c) R_3 can be the not-equal operator on $\mathbb{Z}$, $\neq$.

6. Two classes, just by listing out the related elements: $\dfrac{[a] \mid [d]}{[b] \mid [c]\ [e]}$

7. **Proposition.** *If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.*

   Proof. Let $T = R \cap S$ and element $t \in T$.

   (a) As t is also an element of R and S, tRt and tSt. This implies tTt, thus T is reflexive.

   (b) Suppose there exists element $u \in A$ where $(t, u) \in R$ and $(t, u) \in S$. Since R and S are equivalence relations, they are each symmetric and $(u, t) \in R$ and $(u, t) \in S$. Thus both $(t, u), (u, t) \in T$ and T is symmetric.

   (c) Additionally, suppose there exists elements $u, v \in A$ where each ordered pair belongs to R and S: $(t, u) \wedge (u, v) \implies (t, v)$. Since this is true for R and S, it must also exist within the intersection of R and S, $(t, u) \in T \wedge (u, v) \in T \implies (t, v) \in T$; thus T is transitive.

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8. *Disproof.* Let $A = \{a, b, c\}$. $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$, $S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$, i.e. where each relation has a different symmetric pair. For $R \cup S$ to be an equivalence relation it must be transitive. However, $(a, b), (b, c) \in (R \cup S)$ but $(a, c) \notin (R \cup S)$.