Problem Set 4

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1. From the frequency dependent index of refraction,

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \left(\frac{f_j}{\omega_j^2 - \omega^2} \right)$$

We can apply the relation $\omega = 2\pi c/\lambda$ and rewrite n as

$$\begin{aligned} n^2 &= 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{4\pi^2 c^2/\lambda_j^2 - 4\pi^2 c^2/\lambda^2} \\ &= 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j \lambda_j^2 \lambda^2 / 4\pi^2 c^2}{\lambda^2 - \lambda_j^2} \end{aligned}$$

Condensing the coefficient of each term in the summation as $A_j = Ne^2 f_j \lambda_j^2 / 4\pi^2 c^2 m\epsilon_0$,

$$n^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_j^2} \quad \square$$

2. From Problem 4.4 of Hecht,

(a) The terms can roughly be described as

$$\begin{aligned} m_e \ddot{x} &: \text{The force experienced by an electron} \\ m_e \gamma \dot{x} &: \text{The resistive/friction term} \\ m_e \omega_0^2 x &: \text{The Hooke's law/spring restoring force term} \\ q_e E(t) &: \text{A driving force due to the electron charge within the electric field} \end{aligned}$$

(b) With a solution of form $x = x_0 e^{i(\omega t - \alpha)}$,

$$\begin{aligned} \dot{x} &= i x_0 \omega e^{i(\omega t - \alpha)} = i\omega x \\ \ddot{x} &= -x_0 \omega^2 e^{i(\omega t - \alpha)} = -\omega^2 x \end{aligned}$$

Applying this to the driven and damped oscillator DE,

$$\begin{aligned} m_e \ddot{x} + m_e \gamma \dot{x} + m_e \omega_0^2 x &= q_e E(t) \\ -m_e \omega^2 x + m_e \gamma i \omega x + m_e \omega_0^2 &= q_e E(t) \\ m_e x \left(\omega_0^2 - \omega^2 + i\omega\gamma \right) &= q_e E(t) \\ x_0 &= \frac{q_e E(t)}{m_e} \left[\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma} \right] \frac{1}{e^{i\omega t} e^{-i\omega t}} \\ &= \frac{q_e E_0}{m_e} \frac{e^{i\alpha}}{\omega_0^2 - \omega^2 + i\omega\gamma} \end{aligned}$$

Since $x_0 \in \mathbb{R}$, the modulus will be taken (and as $\left| e^{i\alpha} \right| = 1$)

$$|x_0| = \frac{q_e E_0}{m_e} \frac{1}{\left[\left(\omega_0^2 - \omega^2 \right)^2 + \omega^2 \gamma^2 \right]^{1/2}}$$

(c) From (b), we can use x_0 and find the phase angle by using the real and imaginary components,

$$\begin{aligned}
x_0 &= \frac{q_e E_0}{m_e} \frac{e^{i\alpha}}{\omega_0^2 - \omega^2 + i\omega\gamma} \\
\mathrm{Re}\{x_0\} &= \frac{q_e E_0}{m_e} \frac{\cos\alpha}{\omega_0^2 - \omega^2} \\
\mathrm{Im}\{x_0\} &= \frac{q_e E_0}{m_e} \frac{\sin\alpha}{\omega\gamma} \\
\tan\alpha &= \frac{\sin\alpha}{\cos\alpha} = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \\
\alpha &= \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)
\end{aligned}$$

3. From the trial solution derived in-class, the complex wavenumber was shown as

$$k^2 = \frac{\omega^2}{c^2} - \frac{i\omega\mu_0\sigma_0}{1 + i\omega\tau}$$

Then from the relations $k^2 = n^2\omega^2/c^2$ and $\sigma_0 = \omega_p^2\tau/\mu_0 c^2$,

$$\begin{aligned}
n^2 &= \frac{c^2}{\omega^2}\left[\frac{\omega^2}{c^2} - \frac{i\omega\mu_0\sigma_0}{1 + i\omega\tau}\right] \\
&= 1 - \frac{i\omega_p^2\tau}{\omega + i\omega^2\tau}
\end{aligned}$$

For $\omega_p = 10 \times 10^{15}\,\mathrm{rad}\cdot\mathrm{s}^{-1}$ and $\tau = 10 \times 10^{-13}\,\mathrm{s}$,

$$\begin{aligned}
n(\omega_p) &= \sqrt{1 - \frac{i\left(10^{15}\right)^2\left(10^{-13}\right)}{10^{15} + i\left(10^{15}\right)^2 10^{-13}}} \approx 0.071 - i0.070 \\
n(2\omega_p) &\approx 0.866 - i0.000722 \\
n(\omega_p/2) &\approx 0.023 - i1.732
\end{aligned}$$

4. The skin depth is given as $\delta = \frac{1}{\alpha} = \sqrt{2/\omega\mu_0\sigma_0}$.

(a) For $\lambda = 600\,\mathrm{nm}$,

$$\begin{aligned}
d &= \sqrt{\frac{2}{\omega\mu_0\sigma_0}} = \sqrt{\frac{\lambda}{\pi c\mu_0\sigma_0}} \\
&= \sqrt{\frac{600\,\mathrm{nm}}{\pi c\mu_0\left(6 \times 10^7\,\mathrm{S}\cdot\mathrm{m}^{-1}\right)}} \\
&\approx 2.9\,\mathrm{nm}
\end{aligned}$$

(b) For $\lambda = 0.6\,\mathrm{cm}$,

$$\begin{aligned}
d &= \sqrt{\frac{0.6\,\mathrm{cm}}{\pi c\mu_0\left(6 \times 10^7\,\mathrm{S}\cdot\mathrm{m}^{-1}\right)}} \\
&\approx 290\,\mathrm{nm}
\end{aligned}$$