Homework 9

MATH 364
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5.1.8 A bakery, using flour and sugar, makes cakes, and pastries. Requirements and profits for making and selling a unit of each are as follows:

	Flour (lb)	Sugar (lb)	Profit ($)
Cake	10	15	40
Pastry	3	2	9

The bakery has available b_1 lb flour and b_2 lb of sugar. Assuming that all items can be sold, express the maximum profit attainable as a function of the ratio of b_1 to b_2.

Solution. We can let the decision variables be the number of cakes and pastries to make

$$\begin{aligned} \text{Let } x_1 &= \text{\# cakes to make,} \\ x_2 &= \text{\# pastries to make.} \end{aligned}$$

The objective function is the profit to maximize

$$\text{Profit } z = 40x_1 + 9x_2.$$

The constraints are given by the flour and sugar available,

$$10x_1 + 3x_2 \le b_1 \qquad \text{(flour)}$$
$$15x_1 + 2x_2 \le b_2 \qquad \text{(sugar)}$$

The linear program is given by

Primal Problem

$$\begin{aligned} \max \quad & z = 40x_1 + 9x_2 \\ \text{s.t.} \quad & 10x_1 + 3x_2 \le b_1 \\ & 15x_1 + 2x_2 \le b_2 \\ & x \ge 0 \\ & x \in \mathbb{R}^2 \end{aligned}$$

Using the conversion table from in-class, the dual problem is

Dual Problem

$$\begin{aligned} \min \quad & w = b_1y_1 + b_2y_2 \\ \text{s.t.} \quad & 10y_1 + 15y_2 \ge 40 \\ & 3y_1 + 2y_2 \ge 9 \\ & y \ge 0 \\ & y \in \mathbb{R}^2 \end{aligned}$$

Sorta following what we did in class, we can let $s = b_1/b_2$ and using the dual problem, we can express the maximum profit as a function of s,

Condition s	$s < 10/15$	$10/15 < s < 3/2$	$3/2 > s$
Optimal Point	$(4, 0)$	$(11/5, 6/5)$	$(2, 5/2)$
Optimal Objective	$4b_1$	$11b_1/5 + 6b_2/5$	$2b_1 + 5b_2/2$

5.3.3 Starting from the final tableau of Table 5.5, complete the problem of (5.3.1) if the objective function coefficient of

(a) x_3 is increased from 1 to 4.

Solution. As $c_3^* = 2$ and we're increasing x_3 by 3, the coefficient becomes negative. Using the tableau in Table 5.5, the new tableau becomes

x_1	x_2	x_3	x_4	x_5	x_6		
-2	0	$\boxed{5}$	1	2	-1	6	$\boxed{x_4}$
11	1	-18	0	-7	4	4	x_2
3	0	$\boxed{-1}$	0	2	1	106	z
$-2/5$	0	1	$1/5$	$2/5$	$-1/5$	$6/5$	x_3
$19/5$	1	0	$18/5$	$1/5$	$2/5$	$128/5$	x_2
$13/5$	0	0	$1/5$	$12/5$	$4/5$	$536/5$	z

As all the coefficients are non-negative, this is now optimal at the point

$$\boxed{x^* = (0, 128/5, 6/5, 0), z^* = 536/5.}$$

(b) x_4 is increased from 15 to 16½.

Solution. As x_4 was a basic variable, we're also now changing c_B, affecting r and z. This results in the tableau

x_1	x_2	x_3	x_4	x_5	x_6		
-2	0	5	$\boxed{1}$	2	-1	6	$\boxed{x_4}$
11	1	-18	0	-7	4	4	x_2
3	0	2	$\boxed{-3/2}$	2	1	115	z
-2	0	5	1	2	-1	6	x_4
11	1	-18	0	-7	$\boxed{4}$	4	$\boxed{x_2}$
0	0	$19/2$	0	5	$\boxed{-1/2}$	124	z
$3/4$	$1/4$	$1/2$	1	$1/4$	0	7	x_4
$11/4$	$1/4$	$-18/4$	0	$-7/4$	1	1	x_6
$11/8$	$1/8$	$29/4$	0	$33/8$	0	$249/2$	z

~~This is optimal at the point~~ $\cancel{\boxed{x^* = (0, 0, 0, 7), z^* = 249/2.}}$

I messed this up in the original z value on the RHS in the tableau. It's off by 9. So the optimal value is actually

$$\boxed{x^* = (0, 0, 0, 7), z^* = 231/2.}$$

(c) x_4 is decreased from 15 to 14 and the coefficient of x_3 is decreased from 1 to -2.

Solution. We'll need to enforce x_4 being in the basis and leave it out of the objective row in the tableau. The new tableau will be

$$\begin{array}{cccccc|cc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \\
\hline
-2 & 0 & 5 & 1 & 2 & -1 & 6 & x_4 \\
11 & 1 & -18 & 0 & -7 & 4 & 4 & x_2 \\
\hline
3 & 0 & 5 & \boxed{1} & 2 & 1 & 106 & z \\
\hline
-2 & 0 & 5 & 1 & 2 & -1 & 6 & x_4 \\
11 & 1 & -18 & 0 & -7 & 4 & 4 & x_2 \\
\hline
5 & 0 & 0 & 0 & 0 & 2 & 100 & z
\end{array}$$

The new optimal point is

$$\boxed{x^* = (0, 4, 0, 6), z^* = 100.}$$

5.5.2 Consider the linear program of Example 3.5.1 on page 87. Determine the maximum value of the objective function and a point at which this value is attained if

(a) b_2 is increased from 10 to 30 units, b_1 and b_3 remain unchanged.

Solution. The basic variables are the slack variables. From the final tableau, the submatrix of these variables is

$$A_B^{-1} = \begin{pmatrix} 1/5 & -2/5 & 0 \\ 1/5 & 3/5 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

The change in the objective value is, where c_B are the basic variable coefficients in the final tableau,

$$\begin{aligned}
\delta z &= c_B A_B^{-1}(\delta b) = c_B A_B^{-1} \begin{pmatrix} 0 & 20 & 0 \end{pmatrix}^T \\
&= -20. \\
-z^* &= -90.
\end{aligned}$$

(b) b_1, b_2, and b_3 are each decreased by 10 units from their original values.

Solution. Using the same method as (a),

$$\begin{aligned}
\delta b^* &= A_B^{-1} \begin{pmatrix} -10 \\ -10 \\ -10 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ -30 \end{pmatrix} \\
\delta z &= c_B(\delta b^*) = \begin{pmatrix} -2 & -3 & 0 \end{pmatrix} \begin{pmatrix} -2 & -8 & -30 \end{pmatrix}^T = 20 \\
-z^* &= 50.
\end{aligned}$$

5.6.7 The aluminum can company of Example 5.1.3 on page 166 has just signed a contract calling for the delivery of an additional 1,800 cases of the Type A can per month (with all other data as stated in the original example). Determine the revised optimal operating schedule and monthly costs, and the new marginal costs for the constraints.

Solution. The increase in requirements will affect the first constraint, so b_1 will increase by 1800.

The modified b^* can be determined using the existing tableau,

$$\begin{aligned} b^* = A_B^{-1} b = A_B^{-1} \begin{pmatrix} 2400 \\ 2800 \\ 600 \end{pmatrix} + A_B^{-1} \begin{pmatrix} 1800 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} 75 \\ 150 \\ 350 \end{pmatrix} + \begin{pmatrix} 3/16 & 0 & -5/8 \\ -1/8 & 0 & 3/4 \\ 3/8 & -1 & 15/4 \end{pmatrix} \begin{pmatrix} 1800 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} 825/2 \\ -75 \\ 1025 \end{pmatrix} \end{aligned}$$

These values result in $z^* = -50625$. The modified tableau with these values is

$9/8$	1	0	$-3/16$	0	$3/16$	0	$-5/8$	$825/2$	x_2
$\boxed{-3/4}$	0	1	$1/8$	0	$-1/8$	0	$3/4$	$\boxed{-75}$	x_3
$-23/4$	0	0	$-3/8$	1	$3/8$	-1	$15/4$	1025	x_5
$\boxed{185/4}$	0	0	$25/8$	0	$-25/8$	0	$-225/4$	-50625	$-z$
0	1	$3/2$	0	0	$5/16$	0	$1/2$	300	x_2
1	0	$-4/3$	$-1/6$	0	$1/6$	0	-1	100	x_1
0	0	$-23/3$	$-4/3$	1	$4/3$	-1	-2	1600	x_5
0	0	$185/3$	$65/6$	0	$-65/6$	0	-10	-55250	$-z$

(I'm not entirely sure if I can end the dual simplex method here, since there are still negatives in the last row.)

The new operating schedule is running all Process 3 for 66.6 hours and the monthly cost is \$55,250. The new marginal costs are \$65.6/case of Type A and \$10/lb of using recycled aluminum.