Homework 13

MATH 301
December 3, 2020
Kevin Evans
ID: 11571810

---

1. Set f is a function as since all elements of the domain correspond to something on the codomain.

   Set g is not a function as there are elements (like $x = 0$) which limit the range of y to a smaller bit of the codomain.

2. The function is not injective: $p_1 = (-3, -3)$ and $p_2 = (3, 0)$ both are in the domain $\mathbb{Z}^2$, however $f(p_1) = f(p_2) = 3$.

   The function is surjective. Since $\gcd(3, -4) = 1$ and using Bezout's theorem, any integer can be represented by a linear combination of 3 and -4 (as any integer is a multiple of 1).

3. The function is not injective, since there exists $f(0, 0) = f(2, 1)$.

   The function is not surjective, as odd numbers cannot be represented by the sum of two even numbers. Not sure if these need a formal proof or not.

4. **Proposition.** *The function* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$, *defined* $f(x) = \frac{5x+1}{x-2}$ *is bijective.*

   Proof. To show f is bijective, it will be shown to be both injective and surjective.
   Suppose $a, a' \in \mathbb{R}$ and $f(a) = f(a')$. Then

$$\begin{aligned} \frac{5a+1}{a-2} &= \frac{5a'+1}{a'-2} \\ 5 + 11/(a-2) &= 5 + 11/(a'-2) \\ a - 2 &= a' - 2 \\ a &= a' \end{aligned}$$

   Therefore $a = a'$ and f is injective.
   Suppose $b \in \mathbb{R} - \{5\}$. Then

$$\begin{aligned} b &= \frac{5x+1}{x-2} \\ b(x-2) &= 5x+1 \\ x &= \frac{2b+1}{b-5} \end{aligned}$$

   Therefore $x \in \mathbb{R}$ for $b \in \mathbb{R} - \{5\}$ and f is surjective. Since f is both injective and surjective, it is bijective. ∎

5. Suppose $x, y, m, n \in \mathbb{Z}$ and $f(x, y) = f(m, n)$. Then

$$\begin{aligned} x + y &= m + n \\ 2x + y &= 2m + n \end{aligned}$$

   Subtracting these two, it's clear that $x = m$. From that, we can determine that $y = n$. Therefore f is injective.

   Suppose $(a, b) \in \mathbb{Z}^2$. Then

$$\begin{aligned} (a, b) &= (m + n, 2m + n) \\ a &= m + n \\ b &= 2m + n \end{aligned}$$

   Subtracting these two equations, $m, n \in \mathbb{Z}$,

$$\begin{aligned} m &= b - a \\ n &= a - m \end{aligned}$$

   Therefore the function is surjective.