Alternate Playfair's Axiom

Kevin Hinterlong
Modern Geometries

February 16, 2016

The following axioms appear to be equivalent:

Playfair's Axiom

In a plane, given a line and a point not on it, exactly one line parallel to the given line can be drawn through the point.

Proclus' Axiom

If a straight line intersects one of two parallel lines, it will intersect the other also.

Layout for Proof

In order to prove this, we must show that:

- Playfair's axiom $\implies$ Proclus' axiom
- Proclus' axiom $\implies$ Playfair's axiom

We must assume one, and prove the other to show that these imply each other and are equivalent statements.

Prove Proclus' axiom

Playfair's Axiom $\implies$ Proclus' Axiom

- Suppose that the Playfair Axiom holds.

Prove Proclus' axiom

Playfair's Axiom $\implies$ Proclus' Axiom

- Suppose that the Playfair Axiom holds.
- Let $\ell_1 \parallel \ell_2$ and suppose m is a new line intersecting ℓ_1 at point P.

Prove Proclus' axiom

Playfair's Axiom $\implies$ Proclus' Axiom

- Suppose that the Playfair Axiom holds.
- Let $\ell_1 \parallel \ell_2$ and suppose m is a new line intersecting ℓ_1 at point P.
- As ℓ_2 is parallel to ℓ_1 and m is a line through P, we have that either $m = \ell_1$ or $m \nparallel \ell_2$ according to Playfair's Axiom.

Prove Proclus' axiom

Playfair's Axiom $\implies$ Proclus' Axiom

- Suppose that the Playfair Axiom holds.
- Let $\ell_1 \parallel \ell_2$ and suppose m is a new line intersecting ℓ_1 at point P.
- As ℓ_2 is parallel to ℓ_1 and m is a line through P, we have that either $m = \ell_1$ or $m \nparallel \ell_2$ according to Playfair's Axiom.
    - But we assumed that $m \neq \ell_1$, and so we must have $m \nparallel \ell_2$.

Prove Proclus' axiom

Playfair's Axiom $\implies$ Proclus' Axiom

- Suppose that the Playfair Axiom holds.
- Let $\ell_1 \parallel \ell_2$ and suppose m is a new line intersecting ℓ_1 at point P.
- As ℓ_2 is parallel to ℓ_1 and m is a line through P, we have that either $m = \ell_1$ or $m \nparallel \ell_2$ according to Playfair's Axiom.
  - But we assumed that $m \neq \ell_1$, and so we must have $m \nparallel \ell_2$.
- By definition of parallel lines m must intersect ℓ_2.

Therefore Proclus' Axiom holds.

Prove Playfair's axiom

Proclus' Axiom $\implies$ Playfair's Axiom

- Suppose that the Proclus Axiom holds.

Prove Playfair's axiom

Proclus' Axiom $\implies$ Playfair's Axiom

- Suppose that the Proclus Axiom holds.
- Let m be a line with point P not on m and assume to the contrary that there are distinct lines ℓ_1 and ℓ_2 through P and parallel to m.

Prove Playfair's axiom

Proclus' Axiom $\implies$ Playfair's Axiom

- Suppose that the Proclus Axiom holds.
- Let m be a line with point P not on m and assume to the contrary that there are distinct lines ℓ_1 and ℓ_2 through P and parallel to m.
- Now P is on ℓ_1 and ℓ_2 with $\ell_2 \parallel m$ implies by the Proclus Axiom that ℓ_1 intersects m in a point Q.

Prove Playfair's axiom

Proclus' Axiom $\implies$ Playfair's Axiom

- Suppose that the Proclus Axiom holds.
- Let m be a line with point P not on m and assume to the contrary that there are distinct lines ℓ_1 and ℓ_2 through P and parallel to m.
- Now P is on ℓ_1 and ℓ_2 with $\ell_2 \parallel m$ implies by the Proclus Axiom that ℓ_1 intersects m in a point Q.
  - But this contradicts the assumption that $\ell_1 \parallel m$.

Prove Playfair's axiom

Proclus' Axiom $\implies$ Playfair's Axiom

- Suppose that the Proclus Axiom holds.
- Let m be a line with point P not on m and assume to the contrary that there are distinct lines ℓ_1 and ℓ_2 through P and parallel to m.
- Now P is on ℓ_1 and ℓ_2 with $\ell_2 \parallel m$ implies by the Proclus Axiom that ℓ_1 intersects m in a point Q.
    - But this contradicts the assumption that $\ell_1 \parallel m$.
- Thus we must have that there is at most one line parallel to m and through P.

Therefore Playfair's Axiom holds.

Conclusion

Both Axiom's are equivalent

Because we have shown that both:

- Playfair's axiom $\implies$ Proclus' axiom
- Proclus' axiom $\implies$ Playfair's axiom

We can now say these axioms are equivalent