Introduction

Often times, functions which we are trying to integrate are not suitable to be solved directly and therefore must be manipulated through the use of substitutions and other methods of integration. We have recently learned how to use trigonometric substitution, in which variables and expressions are redefined as trigonometric functions in order to simplify the integral. In this project we will integrate several functions using the trigonometric substitutions we are familiar with as well as hyperbolic trigonometric substitutions, which have different definitions and consequently have different properties and identities. Then we will display our findings in an easy-to-use, general guide for solving integrals with the use of hyperbolic trigonometric substitutions. Finally, we will discuss our results, covering both the fine details of using hyperbolic trigonometric substitution and why substitutions work.

Calculations

The three integrals which we will approach are as follows:

$$1.\ \int \frac{\mathrm{d}x}{1-x^2} \qquad 2.\ \int \frac{x^2}{\sqrt{4x^2-1}}\,\mathrm{d}x \qquad 3.\ \int \frac{\mathrm{d}x}{\sqrt{x^2+4x+13}}$$

Problem 1

The first problem which we will approach, $\int \frac{1}{1-x^2}\,\mathrm{d}x$, is a common example of the familiar trigonometric substitution. In order to find the indefinite integral of it, we will need to find an identity of the form $g(\theta)^2 = 1 - f(\theta)^2$. The easiest substitution we can make is $g(\theta) = \cos(\theta)$ and $f(\theta) = \sin(\theta)$. So we will substitute $x = \sin(\theta)$ and $\mathrm{d}x = \cos(\theta)\,\mathrm{d}\theta$. The range of θ is $\in (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ which is also true for all following trigonometric substitutions.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1-x^2} &= \int \frac{\cos(\theta)\,\mathrm{d}\theta}{1-\sin^2(\theta)} && \text{Substitute}\\
&= \int \sec(\theta)\,\mathrm{d}\theta && \text{Simplify}\\
&= \ln\left|\sec\left(\sin^{-1}(x)\right) + \tan\left(\sin^{-1}(x)\right)\right| + C && \text{Integrate and Substitute } \theta\\
&= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C && \text{Simplify}
\end{aligned}$$

In order to solve this integral using hyperbolic trigonometric substitution, we can force the denominator into a more convenient form in order to fit the identity $\sinh^2(\theta) = \cosh^2(\theta) -$

1. So we will make the substitution $x = \cosh(\theta)$ and $\mathrm{d}x = \sinh(\theta)\,\mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1-x^2} &= \int \frac{-\mathrm{d}x}{x^2-1} && \text{Rearrange} \\
&= \int \frac{-\sinh(\theta)\,\mathrm{d}\theta}{\cosh^2(\theta)-1} && \text{Substitute} \\
&= \int -\coth(\theta)\cdot\ \mathrm{csch}(\theta)\,\mathrm{d}\theta && \text{Simplify} \\
&= \mathrm{csch}(\theta) + C && \text{Recognize relationship}
\end{aligned}$$

Because of the relationship between the identities of trigonometric and hyperbolic trigonometric fucntions, we were able to recognize that just as $\frac{\mathrm{d}}{\mathrm{d}x}\csc(\theta) = -\cot(\theta)\cdot\csc(\theta)$, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{csch}(\theta) = -\coth(\theta)\cdot\ \mathrm{csch}(\theta)$.

Problem 2

The second problem which we will approach is $\int \frac{x^2}{\sqrt{4x^2-1}}\,\mathrm{d}x$. For this problem we will need to make a substitution for the identity $g(\theta)^2 = f(\theta)^2 - 1$. For our regular trigonometric substitution, we will use the identity $\tan^2(\theta) = \sec^2(\theta) - 1$. So we will make the substitution $2x = \sec(\theta)$ and $2\mathrm{d}x = \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4x^2-1}}\,\mathrm{d}x &= \int \frac{\frac{1}{4}\sec^2(\theta)\cdot\frac{1}{2}\sec(\theta)\tan(\theta)\,\mathrm{d}\theta}{\sqrt{\sec^2(\theta)-1}} && \text{Substitute} \\
&= \frac{1}{8}\int \sec^3(\theta)\,\mathrm{d}\theta && \text{Simplify}
\end{aligned}$$

At this point we would like to do integration by parts with $u = \sec(\theta)$ and $\mathrm{d}u = \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$; $\mathrm{d}v = \sec^2(\theta)\,\mathrm{d}\theta$ and $v = \tan(\theta)$

$$\begin{aligned}
\frac{1}{8}\int \sec^3(\theta)\,\mathrm{d}\theta &= \frac{1}{8}\left(\tan(\theta)\cdot\sec(\theta) - \int \tan^2(\theta)\sec(\theta)\,\mathrm{d}\theta\right) && \text{IBP} \\
&= \frac{1}{8}\left(\tan(\theta)\cdot\sec(\theta) - \int \sec^3(\theta)\,\mathrm{d}\theta + \int \sec(\theta)\,\mathrm{d}\theta\right) && \text{Expand} \\
\frac{2}{8}\int \sec^3\theta\,\mathrm{d}\theta &= \frac{1}{8}\left(\tan(\theta)\cdot\sec(\theta) + \int \sec(\theta)\,\mathrm{d}\theta\right) && \text{Solve} \\
\frac{1}{8}\int \sec^3\theta\,\mathrm{d}\theta &= \frac{1}{16}\left(\tan(\theta)\cdot\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) + C && \text{Simplify} \\
&= \frac{1}{16}\left(x\sqrt{4-\frac{1}{x^2}}\cdot 2x + \ln\left|2x + x\sqrt{4-\frac{1}{x^2}}\right|\right) + C && \text{Substitute}
\end{aligned}$$

In order to make a substitution on this integrand using hyperbolic trigonometric substitutions, we need to find an identity of the same general form as before. So we can make the substitution $2x = \cosh(\theta)$ and $2\mathrm{d}x = \sinh(\theta)$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{4x^2-1}}\,\mathrm{d}x &= \int \frac{\frac{1}{4}\cosh^2(\theta)\cdot 2\sinh(\theta)\,\mathrm{d}\theta}{\sqrt{\cosh^2(\theta)-1}} && \text{Substitute}\\ &= \frac{1}{2}\int \frac{\cosh^2(\theta)\cdot\sinh(\theta)\,\mathrm{d}\theta}{|\sinh(\theta)|} && \text{Simplify}\end{aligned}$$

Note: The sign is the same as the sign of x.

$$= \pm\frac{1}{2}\int \cosh^2(\theta)\,\mathrm{d}\theta \qquad \text{Simplify}$$

Integration by parts with $u = \cosh(\theta)$ and $\mathrm{d}u = \sinh(\theta)\,\mathrm{d}\theta$; $\mathrm{d}v = \cosh(\theta)\,\mathrm{d}\theta$ and $v = \sinh(\theta)$.

$$\begin{aligned}\int \cosh^2(\theta)\,\mathrm{d}\theta &= \sinh(\theta)\cosh(\theta) - \int \sin^2(\theta)\,\mathrm{d}\theta && \text{IBP}\\ &= \sinh(\theta)\cosh(\theta) - \int \left(\cosh^2(\theta)-1\right)\,\mathrm{d}\theta && \text{Substitute}\\ 2\int \cosh^2(\theta)\,\mathrm{d}\theta &= \sinh(\theta) + \int \mathrm{d}\theta && \text{Solve}\\ \int \cosh^2(\theta)\,\mathrm{d}\theta &= \frac{1}{2}\left(\sinh(\theta)\cosh(\theta) + \frac{\theta}{2}\right) + C && \text{Simplify}\\ \pm\frac{1}{2}\int \cosh^2(\theta)\,\mathrm{d}\theta &= \pm\frac{1}{4}\left(\sinh(\theta)\cosh(\theta) + \frac{\theta}{2}\right) + C && \text{Solve}\\ &= \pm\left(\frac{\sinh(\cosh^{-1}(2x))\cdot x}{2} + \frac{\cosh^{-1}(2x)}{8}\right) + C && \text{Simplify}\end{aligned}$$

Problem 3

The third problem which we will approach is $\int \frac{1}{\sqrt{x^2+4x+13}}\,\mathrm{d}x$. By recognizing that one can easily complete the square of the radicand to obtain an expression for which a substitution is clear, this integral will be much easier.

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+4x+13}} = \int \frac{\mathrm{d}x}{3\sqrt{\left(\frac{x+2}{3}\right)^2+1}} \qquad \text{Simplify}$$

Now we would like to make the substitution $\left(\frac{x+2}{3}\right)^2 = \tan(\theta)$ and $\mathrm{d}x = 3\sec^2(\theta)\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{\sec^2(\theta)\,\mathrm{d}\theta}{\sqrt{\tan^2(\theta)+1}} &= \int \sec(\theta)\,\mathrm{d}\theta && \text{Simplify}\\ &= \ln\left|\sec\left(\tan^{-1}\left(\frac{x+2}{3}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x+2}{3}\right)\right)\right| && \text{Substitute}\\ &= \ln\left|\sqrt{1+\left(\frac{x+2}{3}\right)^2} + \left(\frac{x+2}{3}\right)\right| + C && \text{Simplify}\end{aligned}$$

Now we must again find an identity which will be useful in manipulating our integrand with hyperbolic trigonometry. From the basic identity $\cosh^2(\theta) = \sinh^2(\theta) + 1$, we are able to make use of the substitution $\frac{x+2}{3} = \sinh(\theta)$ and $\mathrm{d}x = 3\cosh(\theta)\,\mathrm{d}\theta$

$$\begin{aligned}
\frac{1}{3}\int \frac{\mathrm{d}x}{\sqrt{\left(\frac{x+2}{3}\right)^2+1}} &= \int \frac{\cosh(\theta)\,\mathrm{d}\theta}{\sqrt{\cosh^2(\theta)}} && \text{Substitute} \\
&= \int \mathrm{d}\theta && \text{Simplify} \\
&= \sinh^{-1}\left(\frac{x+2}{3}\right) + C && \text{Solve for } \theta
\end{aligned}$$

General Approach to Solving

When you see integrals in the form of

$$\int \frac{\mathrm{d}x}{(x^2 \pm a^2)^n} \qquad \text{where } n \text{ is an integer}$$

You can make a trigonometric or hyperbolic trigonometric substitution. Below is a table of useful identities which should help students in their subsitutions

Form	Substitution	Identity	Becomes
$u^2 - a^2$	$u = a\cosh(\theta)$	$\cosh^2(\theta) - 1 = \sinh^2(\theta)$	$u^2 - a^2 \to a^2\sinh^2(\theta)$
$u^2 + a^2$	$u = a\sinh(\theta)$	$\sinh^2(\theta) + 1 = \cosh^2(\theta)$	$a^2 + u^2 \to a^2\cosh^2(\theta)$

Conclusion

In conclusion, hyperbolic trig substitution can be seen as an alternative to regular trig substitution. While many of the rules are same, the identities are slightly different than normal trigonometric identities, making them useful during some situations, as seen by our examples. This brings up the question of why this kind of substitution works. It can be confusing to look at first how can an integral without any trigonometric values suddenly equal an integral with trigonometric values in it. Since one knows the pattern of trigonometric identities, however, one can recognize that when faced with an integral that mirrors the trigonometric identity, one can easily substitute a trig function of an angle in place of x. It is, essentially, just a u-substitution. As we have seen, just as it is useful to substitute for an x-term when it is inside another function, one can make x a function of another variable in order to actually make the integral easier, which explains how trig substitution works. Hyperbolic trig substitution works in the same manner, by using identities and substituting in order to get those identities to become present in the integral. Any other sort of substitution that contains identities like that, thus simplifying the integral even though it seems one is making it more complicated, will likely work.