Introduction

Because of our deep love and appreciation for Chain Rule that was sparked during BC Calculus 1, we would like to carry it's beauty and versatility over to the world of BC Calculus 2. We decided to reverse the process of Chain Rule in order to create an Anti-Chain rule to compute anti-derivatives. For this project, our goal was to find when our conjecture, Anti-Chain Rule, could be used in order to calculate an anti-derivative. Specifically we are saying that Anti-Chain Rule is

$$\int f(g(x))\,\mathrm{d}x = \frac{F(g(x))}{g'(x)} + C \qquad \text{where } f(x) = F'(x) \text{ and } C \text{ is a constant}$$

and we will attempt to generalize the conditions that must be true for our conjecture to be valid.

Process

In order to generalize the conditions, we will represent the composition of functions with $F(x)$ and $g(x)$. Our first generalization was found by looking at the fraction from Anti-Chain Rule

$$\frac{F(g(x))}{g'(x)}$$

Computing the anti-derivative with this method leads us to the conclusion that $g(x)$ has a constant, nonzero slope because $g'(x) \neq 0$.

From the prompt for this assignment we see that this conjecture does not work when $g(x)$ is a second degree polynomial. So in order to test the scope of Anti-Chain Rule, we will check with an arbitrary function $f(x) = x^n$ with $g(x) = x^a$. In order to test this, we will check the anti-derivative computed with Anti-Chain Rule and compare it to the real anti-derivative. From Anti-Chain Rule, we conjecture that the anti-derivative will be

$$\begin{aligned}\frac{\frac{(x^a)^{n+1}}{n+1}}{a \times x^{a-1}} + C &= \frac{x^{a\times n+a}}{a \times x^{a-1} \times (n+1)} + C \qquad \text{where } a,\, n \in \mathbb{Z}\\ &= \frac{x^{a\times n+a} \times x^{-(a-1)}}{a \times (n+1)} + C\\ &= \frac{x^{a\times n+1}}{a \times (n+1)} + C\end{aligned}$$

Now let's check it using Anti-Power Rule, which we can trust while anti-differentiating.

$$\int (x^a)^n\,\mathrm{d}x = \int x^{a\times n}\,\mathrm{d}x$$

$$= \frac{x^{a\times n+1}}{a\times n+1} + C$$

We can further solve for when the two denominators are equal to each other in order to find the powers of x that allow this method to work for polynomials.

$$\cancel{a \times n +} a = \cancel{a \times n +} 1$$

This leads us to conclude that $g(x)$ must be a linear function with a degree of 1. We can also see that this causes $g''(x)$ to be equal to 0, and it shows that since the slope is nonzero and any constants will be reduced to 0 through differentiation, $g(x)$ must be of the form $mx + b$ where $m \neq 0$ for this to work.

Now that we know when Anti-Chain Rule works for polynomials, we can generalize our results to prove the scopes of Anti-Chain Rule. Working backwards, we can check when the derivative of Anti-Chain Rule with respect to x will be equal to $f(g(x))$.

$$f(g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{F(g(x))}{g'(x)} + C\right)$$

Through quotient rule, we can see that the derivative with respect to x is

$$f(g(x)) = \frac{g'(x)^2 \times f(g(x)) - g''(x) \times F(g(x))}{g'(x)^2}$$

However, we can see that in order for them to be equal

$$g''(x) = 0 \qquad\qquad g'(x) = C$$

$$f(g(x)) = \frac{f(g(x)) \cancel{\times g'(x)^2} - 0}{\cancel{g'(x)^2}}$$

Results

From our work with polynomial functions, we were able to predict the scope of Anti-Chain Rule and we used them for guidance while generalizing the scope of Anti-Chain Rule in any case.

From our investigations, we have determined that Anti-Chain Rule can compute anti-derivatives of $f(g(x))$ if $g(x)$ is a function with a constant nonzero slope, and zero concavity; $g(x) = mx + b$ where $m \neq 0$. Unfortunately, the scope of this method of integration is very limited and we will need to use a different approach further in the course to integrate complex compositions of functions.