Mortar Element Method for the Two Dimensional Transverse Electric Boundary Integral Equation

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Outline

- Electromagnetic Background
- TM Scattering Analysis
- TE Scattering Analysis
- Mortar Element Method
- Conclusion

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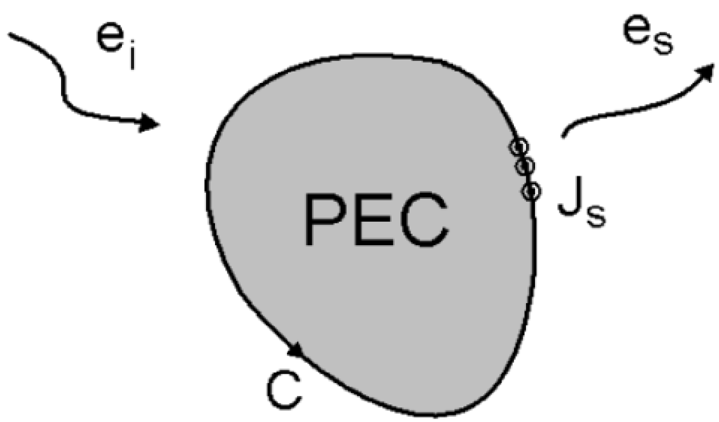

Figure 1: Scattering by a perfectly electrical conductor (PEC).

$$e^{tot}(\boldsymbol{\rho}) = [e^{i}(\boldsymbol{\rho}) + e^{s}(\boldsymbol{\rho})] = 0$$

Electromagnetic Background

- Maxwell Equations

$$\nabla \cdot \mathbf{d}(\boldsymbol{\rho}) = -\frac{1}{j\omega} \nabla \cdot \mathbf{j}(\boldsymbol{\rho})$$

$$\nabla \times \mathbf{e}(\boldsymbol{\rho}) = -j\omega \mathbf{b}(\boldsymbol{\rho})$$

TM TE

$$\nabla \cdot \mathbf{b}(\boldsymbol{\rho}) = 0$$

$$\nabla \times \mathbf{h}(\boldsymbol{\rho}) = j\omega \mathbf{d}(\boldsymbol{\rho}) + \mathbf{j}(\boldsymbol{\rho})$$

with the constitutive equations

$$\mathbf{d}(\boldsymbol{\rho}) = \varepsilon \mathbf{e}(\boldsymbol{\rho})$$

$$\mathbf{b}(\boldsymbol{\rho}) = \mu \mathbf{h}(\boldsymbol{\rho})$$

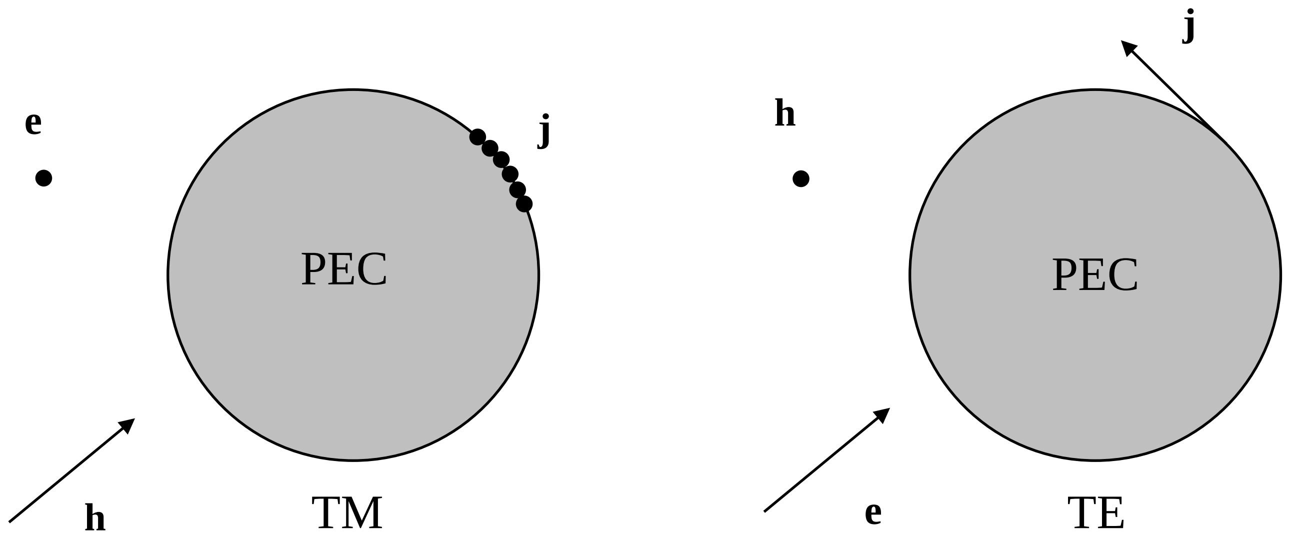

Figure 2: TM problem vs TE problem on the boundary of the 2D transverse.

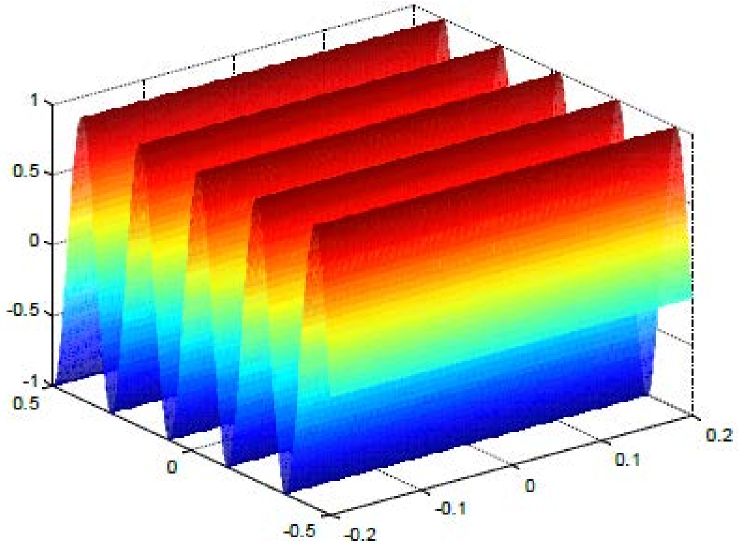

Figure 3: A uniform plane wave.

R=0.5m

$\lambda = 1.0\text{m}$

$c = 3\times 10^8\,\text{m/s}$

$\omega = 6\pi\times 10^8\,\text{rad/s}$

$\mu = 4\pi\times 10^{-7}\,\text{H/m}$

$\varepsilon = 8.85\times 10^{-12}\,\text{F/m}$

$\eta = 120\pi\Omega$

$E_0 = 1$

$H_0 = \sqrt{\dfrac{\mu}{\varepsilon}}$

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- Discretization of the Electric Boundary Integral Equation

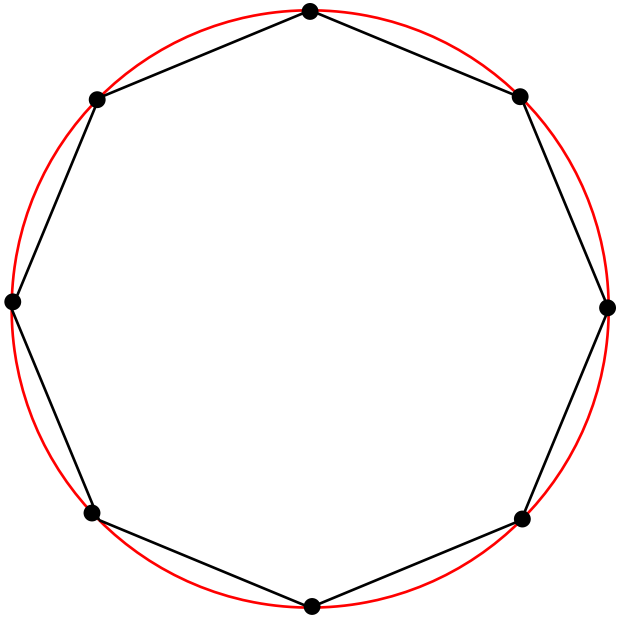


Figure 4: Approximation of the boundary of the scatterer by finite linear segments.

- Discretization of the Electric Boundary Integral Equation

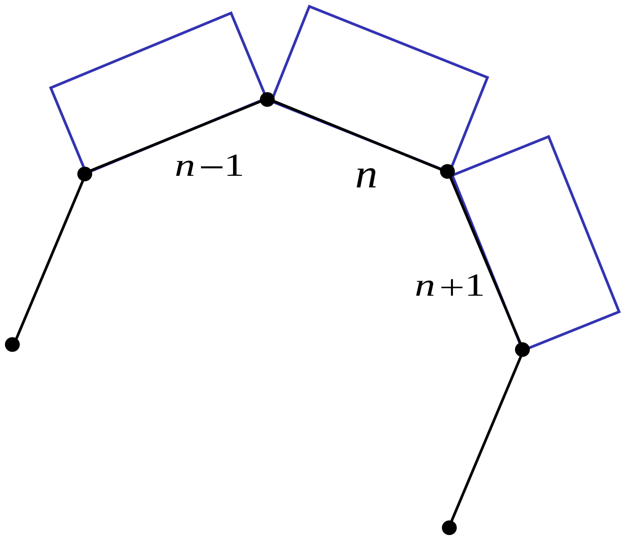


Figure 5: Basis functions of TM problem.

- Basis Function

$$f_n(\boldsymbol{\rho}) = \begin{cases} 1, & \forall \boldsymbol{\rho} \in \text{segment} \quad n \\ 0, & \text{elsewhere} \end{cases}$$

- Current Density

$$j(\boldsymbol{\rho}) = \sum_{n=1}^{N} x_n f_n(\boldsymbol{\rho})$$

- Electric Boundary Integral Equation

$$e^{s}(\boldsymbol{\rho}) = -j\omega\mu \sum_{n=1}^{N} x_n \int_C G(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\, f_n(\boldsymbol{\rho}')\,dc' = -e^{i}(\boldsymbol{\rho})$$

- Multiply and integrate one test term on both sides

$$-j\omega\mu \sum_{n=1}^{N} x_n \int_C \int_C f_m(\boldsymbol{\rho}) G(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\, f_n(\boldsymbol{\rho}')\,dc'dc = -\int_C f_m(\boldsymbol{\rho}) e^{i}(\boldsymbol{\rho})\,dc$$

- It can be expressed as a linear system

$$\mathbf{Zx} = \mathbf{L}$$

where

$$Z_{m,n} = -j\omega\mu \int_C \int_C f_m(\boldsymbol{\rho}) G(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\, f_n(\boldsymbol{\rho}')\,dc'dc$$

and

$$L_n = -\int_C f_n(\boldsymbol{\rho}) e^{i}(\boldsymbol{\rho})\,dc$$

TM Scattering Analysis

- Apply the numerical quadrature rule

$$\int_0^1 f(s)ds \approx \sum_{p=1}^{Q} w_p f(s_p)$$

- The integration can be expanded as

$$Z_{m,n} \approx -j\omega\mu \sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p w_q l_m l_n f_m(\boldsymbol{\rho}_m(s_p)) G(\left|\boldsymbol{\rho}_m(s_p) - \boldsymbol{\rho}'_n(s_q)\right|) f_n(\boldsymbol{\rho}'_n(s_q))$$

and

$$L_m = -\int_C f_m(\boldsymbol{\rho}) e_i(\boldsymbol{\rho}) dc = -\sum_{p=1}^{Q} w_p l_m f_m(\boldsymbol{\rho}_m(\mathrm{s}_p)) \mathrm{e}^i(\boldsymbol{\rho}_m(\mathrm{s}_p))$$

where

$$\boldsymbol{\rho}_m(s) = \boldsymbol{\rho}_{m,0} + s(\boldsymbol{\rho}_{m,1} - \boldsymbol{\rho}_{m,0})$$

TM Scattering Analysis

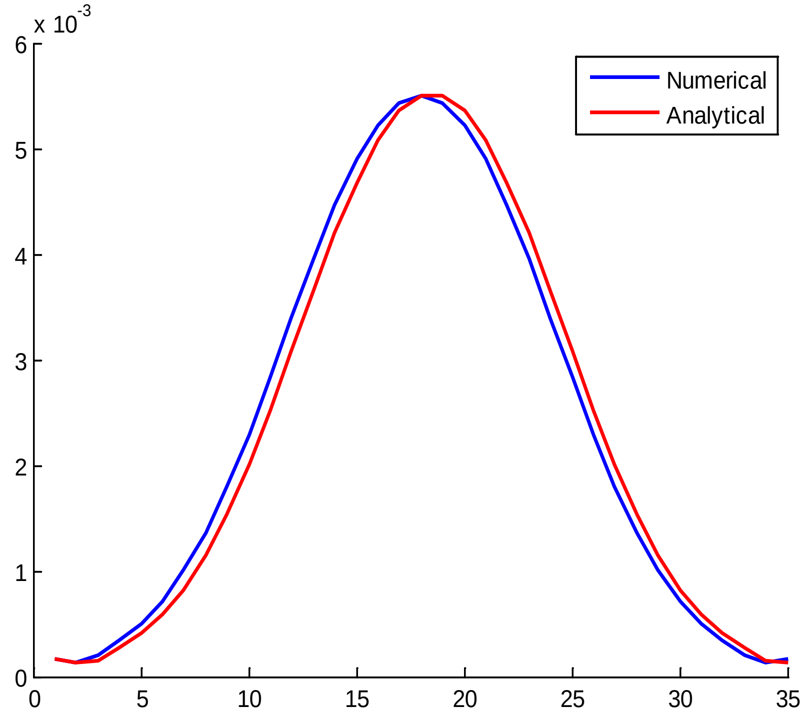
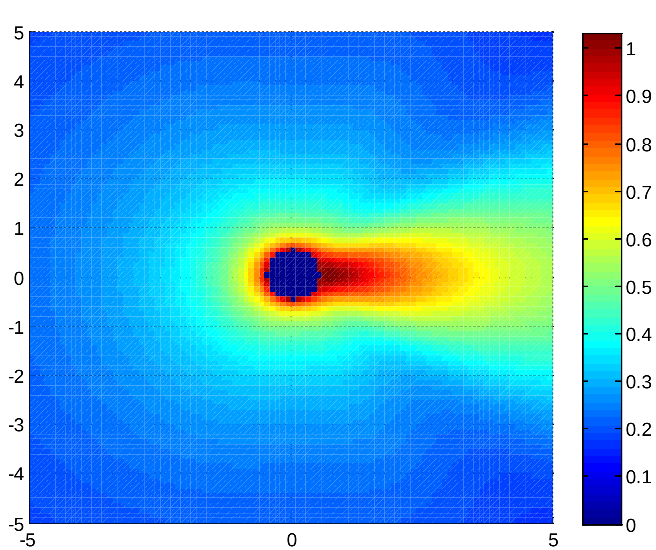


Figure 6: Comparison between the numerical and analytical solution to current density in TM case.

TM Scattering Analysis

- Expression of scattered field in TM mode

$$e^{s}(\boldsymbol{\rho}) = -j\omega\mu \sum_{m=1}^{N} x_m l_m \sum_{p=1}^{Q} w_p G\left(\left|\boldsymbol{\rho} - \boldsymbol{\rho}'_m(s_p)\right|\right) f_m(\boldsymbol{\rho}'_m(s_p))$$



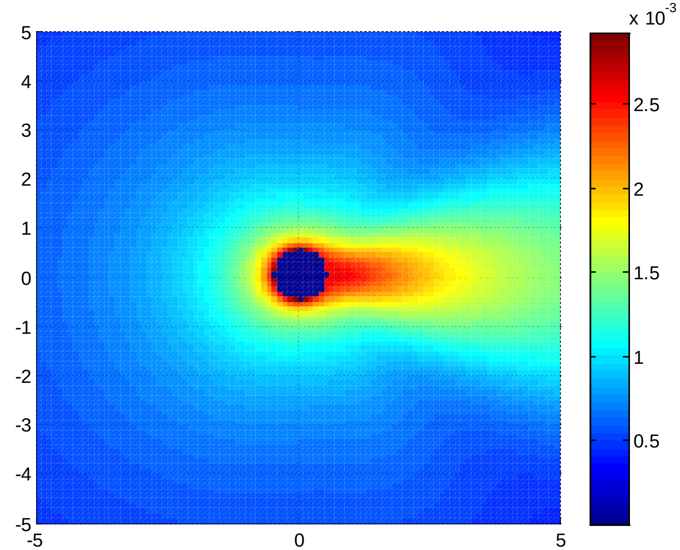


Figure 7: Magnitude of scattered electric field

Figure 8: Magnitude of scattered magnetic field

TE Scattering Analysis

- Discretization of the Electric Boundary Integral Equation

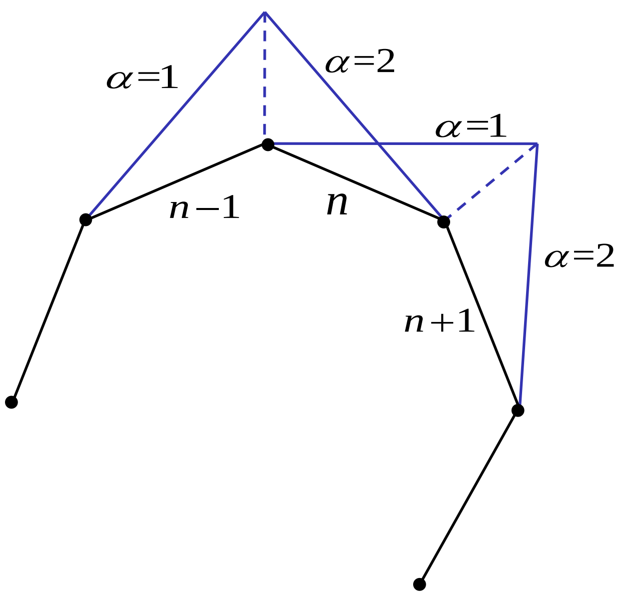


Figure 9: Basis functions of TE problem.

- Basis Function

$$f_n(\boldsymbol{\rho}_\alpha(l)) = \begin{cases} 1+\dfrac{l}{l_{n-1}}, & \alpha=1, n\neq N \Leftrightarrow \forall\boldsymbol{\rho}\in \text{segment} \quad n-1 \\ 1-\dfrac{l}{l_n}, & \alpha=2, \text{n}\neq \text{N} \Leftrightarrow \forall\boldsymbol{\rho}\in \text{segment} \quad n \\ 1-\dfrac{l}{l_1}, & \alpha=2, n=N \Leftrightarrow \forall\boldsymbol{\rho}\in \text{segment} \quad 1 \\ 0, & \text{elsewhere} \end{cases}$$

- Current Density

$$\left|\mathbf{j}(\boldsymbol{\rho})\right| = \sum_{\alpha=1}^{2}\sum_{n=1}^{N} x_n f_n(\boldsymbol{\rho}_\alpha)$$

TE Scattering Analysis

- Electric Boundary Integral Equation

$$\mathbf{e}^{s}(\boldsymbol{\rho})=\frac{\eta}{jk}\nabla\int_{C}G(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\partial_{s}j(\boldsymbol{\rho}')dc'-j\eta k\int_{C}G(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\mathbf{j}(\boldsymbol{\rho}')dc'=-\mathbf{e}^{i}(\boldsymbol{\rho})$$

- After multiplying and integrating, we can expand the above equation into

$$\begin{aligned}Z_{m,n}=j\frac{\eta}{k}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sum_{p=1}^{Q_1}\sum_{q=1}^{Q_2}[&w_p w_q l_{\alpha,m} l_{\beta,n}\partial_s(f_m(\boldsymbol{\rho}_{\alpha,m}(s_p)))G(|\boldsymbol{\rho}_{\alpha,m}(s_p)-\boldsymbol{\rho}'_{\beta,n}(s_q)|)\partial'_s(f_n(\boldsymbol{\rho}'_{\beta,n}(s_q)))\\&-k^2 w_p w_q l_{\alpha,m} l_{\beta,n} f_m(\boldsymbol{\rho}_{\alpha,m}(s_p))G(|\boldsymbol{\rho}_{\alpha,m}(s_p)-\boldsymbol{\rho}'_{\beta,n}(s_q)|)f_n(\boldsymbol{\rho}'_{\beta,n}(s_q))\hat{\mathbf{s}}(\boldsymbol{\rho}_{\alpha,m}(s_p))\cdot\hat{\mathbf{s}}'(\boldsymbol{\rho}'_{\beta,n}(s_q))]\end{aligned}$$

and

$$L_m=\sum_{\alpha=1}^{2}\sum_{p=1}^{Q}w_p l_m f_m(\boldsymbol{\rho}_{\alpha,m}(s_p))[\mathbf{e}^{i}(\boldsymbol{\rho}_{\alpha,m}(s_p))\cdot\hat{\mathbf{s}}(\boldsymbol{\rho}_{\alpha,m}(s_p))]$$

TE Scattering Analysis

where

$$\boldsymbol{\rho}_{\alpha,m} = \begin{cases} \boldsymbol{\rho}_{m-1}, & \alpha = 1 \\ \boldsymbol{\rho}_m, & \alpha = 2 \end{cases} \qquad \boldsymbol{\rho}'_{\beta,n} = \begin{cases} \boldsymbol{\rho}'_{n-1}, & \beta = 1 \\ \boldsymbol{\rho}'_n, & \beta = 2 \end{cases}$$

$$f_m(\boldsymbol{\rho}_{\alpha,m}(s_p)) = \begin{cases} s_p, & \alpha = 1 \\ 1 - s_p, & \alpha = 2 \end{cases} \qquad f_n(\boldsymbol{\rho}'_{\beta,\mathrm{n}}(s_q)) = \begin{cases} s_q, & \beta = 1 \\ 1 - s_q, & \beta = 2 \end{cases}$$

$$\partial_s(f_m(\boldsymbol{\rho}_{\alpha,m})) = \begin{cases} \dfrac{1}{l_{m-1}}, & \alpha = 1 \\ -\dfrac{1}{l_m}, & \alpha = 2 \end{cases} \qquad \partial'_s(f_n(\boldsymbol{\rho}'_{\beta,\mathrm{n}})) = \begin{cases} \dfrac{1}{l_{n-1}}, & \beta = 1 \\ -\dfrac{1}{l_n}, & \beta = 2 \end{cases}$$

$$l_{\alpha,m} = \begin{cases} l_{m-1}, & \alpha = 1 \\ l_m, & \alpha = 2 \end{cases} \qquad l_{\beta,n} = \begin{cases} l_{n-1}, & \beta = 1 \\ l_n, & \beta = 2 \end{cases}$$

TE Scattering Analysis

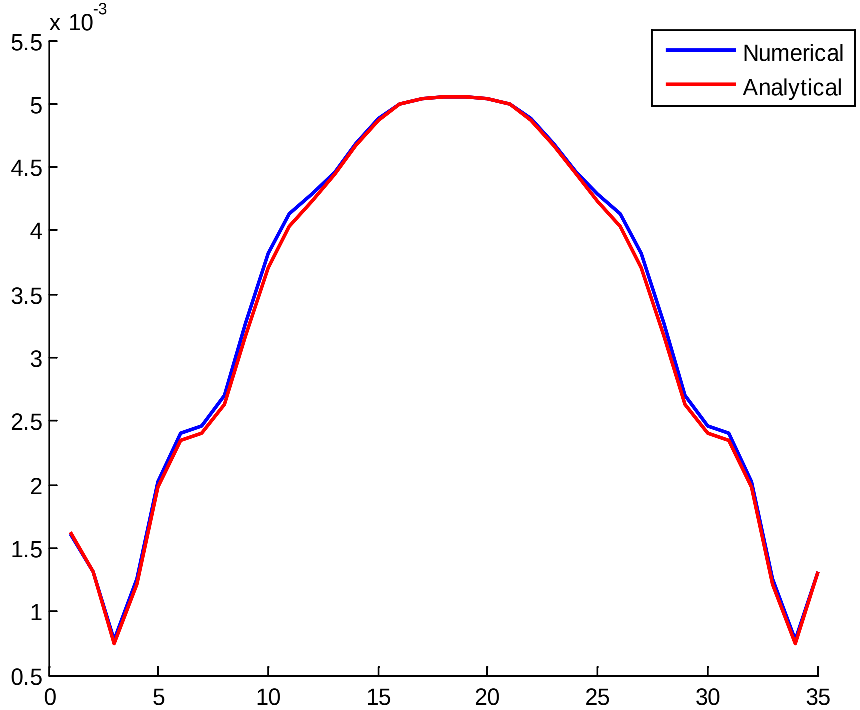


Figure 10: Comparison between the numerical and analytical solution to current density in TE case.

TE Scattering Analysis

- Expression of scattered field in TE mode

$$\mathbf{e}^s(\boldsymbol{\rho}) = \sum_{m=1}^{N} x_m \sum_{\alpha=1}^{2} \sum_{p=1}^{Q} [\frac{\eta}{jk} w_p l_{\alpha,m} (\nabla G(\left|\boldsymbol{\rho} - \boldsymbol{\rho}_{\alpha,m}(s_p)\right|))\partial_s(f_m(\boldsymbol{\rho}_{\alpha,m}(s_p)))$$
$$- j\eta k w_p l_{\alpha,m} G(\left|\boldsymbol{\rho} - \boldsymbol{\rho}_{\alpha,m}(s_p)\right|) f_m(\boldsymbol{\rho}_{\alpha,m}(s_p))\hat{\mathbf{s}}(f_m(\boldsymbol{\rho}_{\alpha,m}(s_p)))]$$

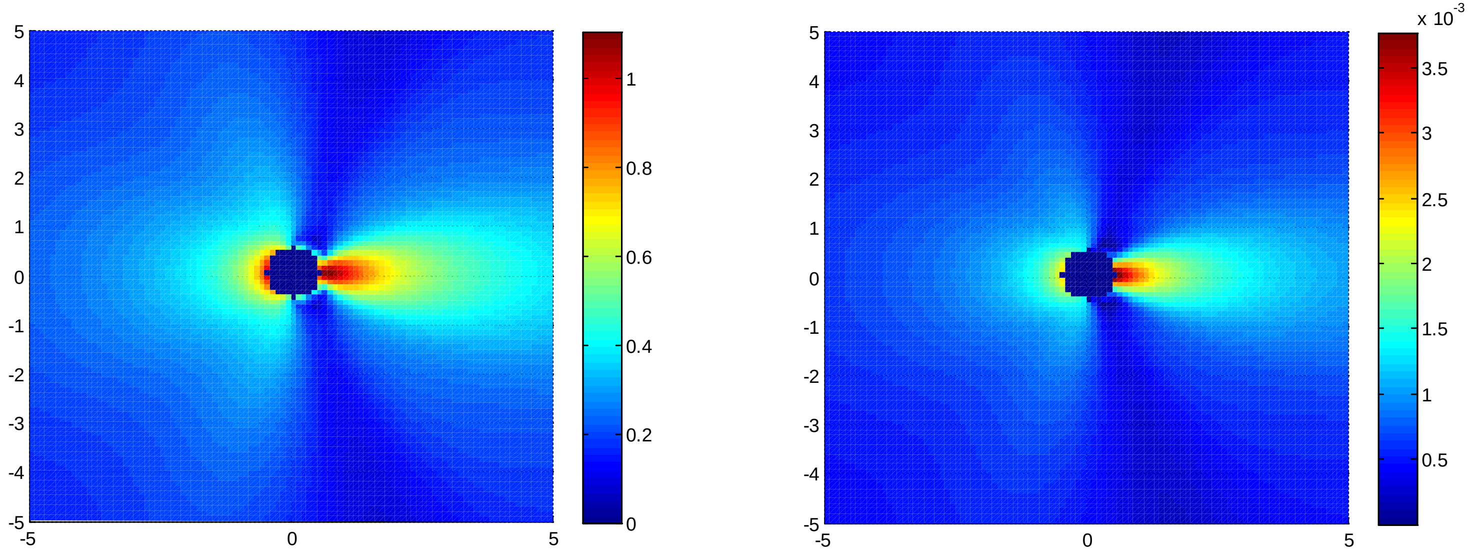


Figure 11: Magnitude of scattered electric field **Figure 12:** Magnitude of scattered magnetic field

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Mortar Element Method

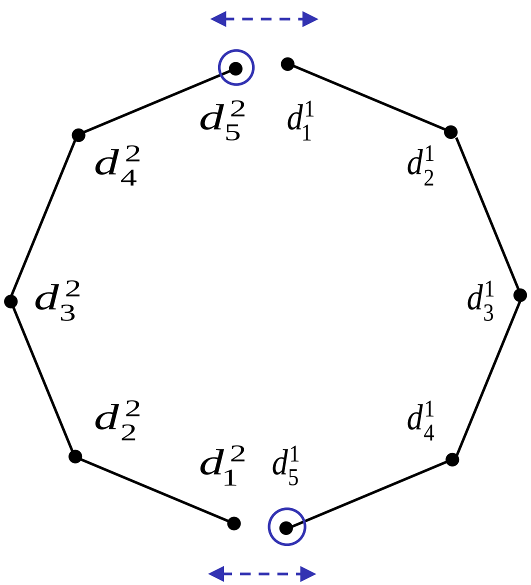

Figure 13: Example of choice of subcontours

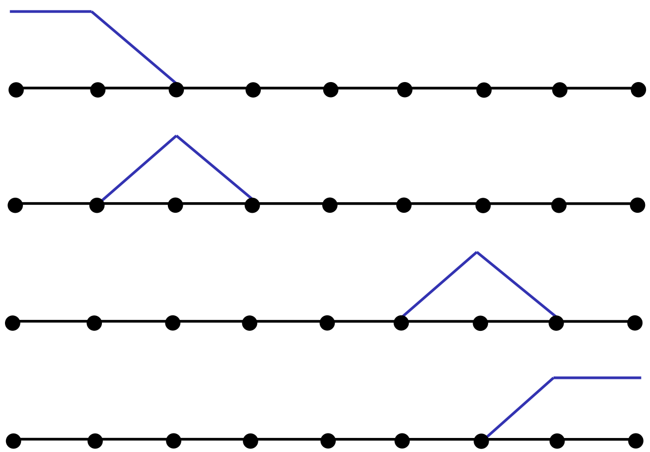

Figure 14: Basis functions in TE problem

Mortar Element Method

- Constraint of Current Continuity

$$\begin{bmatrix} -1 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & 1 & -1 & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & \ddots \\ & & & & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_{k_1-1}^1 \\ x_{k_1}^1 \\ x_1^2 \\ x_2^2 \\ \vdots \\ x_{k_M-1}^M \\ x_{k_M}^M \end{bmatrix} = \mathbf{0}$$

In short,

$$\mathbf{Bx} = \mathbf{0}$$

- Equivalent expression of linear system

$$\begin{cases} \mathbf{Zx} = \mathbf{L} \\ \mathbf{Bx} = \mathbf{0} \end{cases}$$

- For the same purpose, we construct a target function

$$\min_{\mathbf{x}} J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{Zx} - \mathbf{x}^T \mathbf{L}$$

$$\text{s.t.} \quad \mathbf{Bx} = \mathbf{0}$$

- Apply Lagrange Multiplier, we define the Lagrangian to be

$$I(\mathbf{x}, \boldsymbol{\lambda}) = J(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{Bx}$$

- Set the partial derivative to be zero

$$\begin{bmatrix} \mathbf{Z} & -\mathbf{B}^T \\ -\mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{L} \\ \mathbf{0} \end{bmatrix}$$

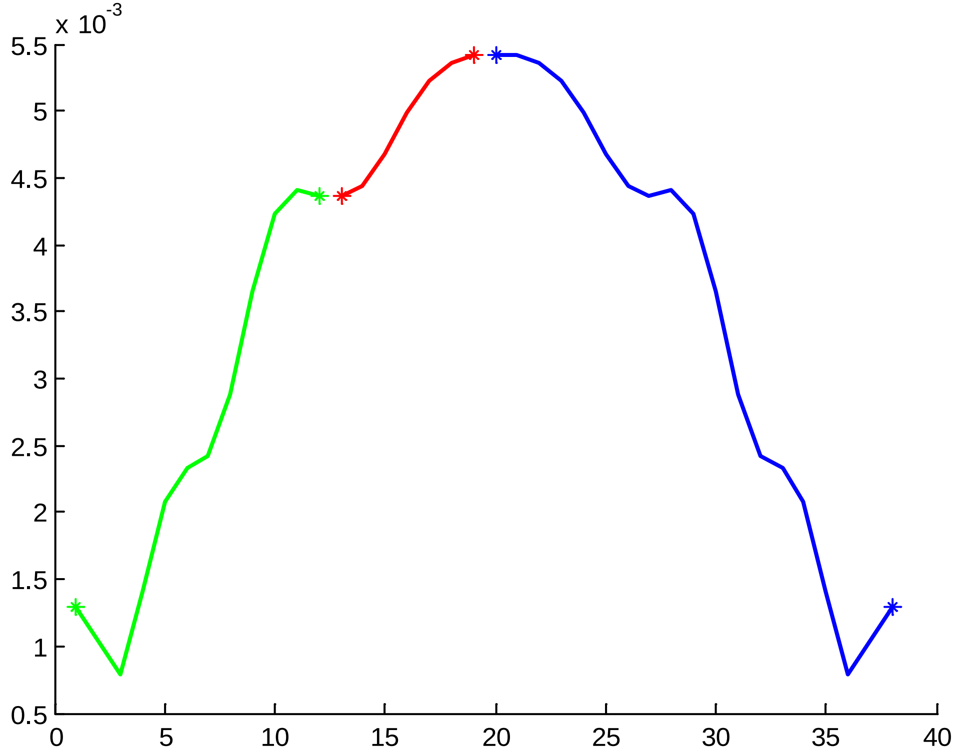

Figure 15: Solution to current density
on the boundary of the 2D transverse in TE problem.

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Conclusion

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- Comparison among different methods

Methods	Advantages	Disadvantages
Analytical Method	■ Simple Expression ■ Fast speed	■ Regular boundary
Numerical Method	■ Complicated boundary (e.g. piecewise boundary)	■ Slow speed
Mortar Element Method	■ Fast speed ■ Complicated boundary (e.g. piecewise boundary)	■ Increased error ■ Hardware cost (e.g. GPUs)

References

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Thank You