Module-3

1) a) **Binomial Distribution:-**

Binomial Distribution is discoved by James bernouli in the year 1700 and it is opp. A random variable 'x' has a binomial distribution if it assumes only non-negative values & its probability density function is given by

$$p(x=r) = p(r) = \begin{cases} {}^{n}c_{r}\, p^{r} q^{n-r} & ; r=0,1,2,\ldots n \\ 0 & ; \text{otherwise} \end{cases}$$

whor n = total no. of trials
r = no. of success
p = probability of success in a single trial
${}^{n}c_{r}$ = binomial co-efficient.

conditions:-

1. Trials are repeated identical conditions for a fixed no. of times(n)
2. There are only 2 possible outcomes that is success and failure.
3. The probability of sucess in each trial remains constant and does not change from trial to trial.

b) Given

$$p=\frac{1}{2}, \quad q=\frac{1}{2} \quad n=9 \quad r=5$$

$$\therefore p(x=5) = {}^{9}c_{5}\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{9-5}$$

$$= {}^{9}c_{5}\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{4}$$

$$= 0.204$$

2) a) **Derivations of BD:-**

let n is the no. of total trials in which r of them are success & remaining $n-r$ are failure. Total no. of ways to arrange r in n trials

$$= {}^{n}c_{r} = \frac{n!}{r!\,(n-r)!} \longrightarrow (1)$$

Now for the probability of a specific sequence of r success & $(n-r)$ failure is the product of the probability of success and failure

$\therefore p(\text{specific sequence}) = p^r (1-p)^{n-r} \rightarrow ②$

from ① & ② combining all possible arrangements the total probability

$$p(x=r) = {}^nC_r\, p^r (1-p)^{n-r}$$

$$p(x=r) = {}^nC_r\, p^r q^{n-r}$$

b) Given,

$$n = 5$$

$$p = 20\% = \frac{20}{100} = 0.2$$

$$q = 1 - 0.2 = 0.8$$

i) none is defective:-

$$r=0 \Rightarrow {}^5C_0 (0.2)^0 (0.8)^5 = 0.327$$

ii) $p(1 < x < 4) = p(2) + p(3)$

$$= {}^5C_2 (0.2)^2 (0.8)^3 + {}^5C_3 (0.2)^3 (0.2)^2$$

$$= 0.256$$

3) let p probability that the IC chips will have thick enough coating

$$p = 70\% = \frac{70}{100} = 0.7 \;;\; q = 1 - 0.7 = 0.3 \;;\; n = 15$$

i) Atleast 12 will have thick enough coating

i.e $r \geq 12$

$$p(r \geq 12) = p(13) + p(14) + p(15) + p(12)$$

$$= {}^{15}C_{12} (0.7)^{12} (0.3)^3 + {}^{15}C_{13} (0.7)^{13} (0.3)^2 + {}^{15}C_{14} (0.7)^{14} (0.3)$$
$$+ {}^{15}C_{15} (0.7)^{15} (0.3)^0$$

$$= 0.170 + 0.091 + 0.030 + 0.0047$$

$$= 0.2957$$

ii) Atmost 6 will have thick coating:-

$$p(r \leq 6) = p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$$

$$= {}^{15}C_0 (0.7)^0 (0.3)^{15} + {}^{15}C_1 (0.7)^1 (0.3)^{14} + {}^{15}C_2 (0.7)^2 (0.3)^{13} +$$
$${}^{15}C_3 (0.7)^3 (0.3)^{12} + {}^{15}C_4 (0.7)^4 (0.3)^{11} + {}^{15}C_5 (0.7)^5 (0.3)^{10}$$
$$+ {}^{15}C_6 (0.7)^6 (0.3)^9$$

$$= 1.43 \times 10^{-8} + 3.515 \times 10^{-7} + - - - -$$

$$= 0.015$$

iii) Exactly 10 will have thick enough coating

$$p(x=10) = {}^{15}C_{10} (0.7)^{10} (0.3)^5 = 0.206$$

4) Given,

$n=6$, $p=\frac{1}{2}$, $q=\frac{1}{2}$

i) Exactly 2 times

$$p(x=2) = {}^6C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{6-2} = {}^6C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^4 = 0.234$$

ii) $p(x>4) = p(5) + p(6)$

$$= {}^6C_5\, p^5 q^1 + {}^6C_6\, p^6 q^0$$

$$= 0.1093$$

iii) Atleast one $p(x \geq 1) = p(1)+p(2)+p(3)+p(4)+p(5)+p(6)$

(or)

$$= 1 - p(x=0)$$

$$= 1 - {}^6C_6 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^6$$

$$= 0.9375 \;/\; 0.984375$$

5) a) Mean:-

Since Mean $E(x) = \mu = \Sigma x \cdot p(x)$

$$\mu = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x(x-1)!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

let $x-1=y$

$$\mu = \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y+1}}{y!}$$

$$= \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda \lambda^y}{y!}$$

$$= \lambda \sum_{y=0}^{\infty} \frac{e^{\lambda} \lambda^y}{y!}$$

$$\mu = \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$\mu = \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$\mu = \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{\mu = \lambda}$$

ii) Variance:

variance $v(x) = E(x^2) - [E(x)]^2$

$= \sum x^2 p(x) - \lambda^2$

$= \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda} \lambda^x}{x!} - \lambda^2 \Rightarrow \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda} \lambda^x}{x(x-1)!} - \lambda^2$

$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2 \Rightarrow \sum_{x=1}^{\infty} [(x-1)+1] \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$

$= \sum_{x=1}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^x}{(x-1)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$

$= \sum_{x=1}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^x}{(x-1)(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$

$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$

let $x-2 = y$

$x-1 = z$

$= \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^{y+2}}{y!} + \sum_{z=0}^{\infty} \frac{\lambda^{z+1} e^{-\lambda}}{z!} - \lambda^2$

$= \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^z}{z!} - \lambda^2$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda} - \lambda^2$

$= \lambda^2 + \lambda - \lambda^2$

$\boxed{v(x) = \lambda}$

b) Given,

$p(x=0) = p(x=1)$

by def of poission distribution

$p(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$\frac{e^{-\lambda} \lambda^0}{0!} = \frac{e^{-\lambda} \lambda^1}{1!}$

$\boxed{\lambda = 1}$

$p(x=0) = \frac{e^{-\lambda} \lambda^0}{0!} = \frac{e^{-1} (1)^0}{0!} = 0.3678$

$p(X=0) = p(x=1)$ then

$p(x=1) = 0.3678$

By recurrence Relation,

$$p(x+1) = \frac{\lambda}{x+1}\, p(x)$$

for $x=1$

$$p(2) = \frac{\lambda}{2}\, p(1) = \frac{1}{2}(0.3678) = 0.1839$$

for $x=2$ $\quad p(3) = \frac{\lambda}{3}\, p(2) = \frac{0.1839}{3} = 0.0613$

for $x=3$ $\quad p(4) = \frac{1}{4}(p(3)) = \frac{0.061}{4} = 0.0152$

for $x=4$ $\quad p(5) = \frac{1}{5}\, p(4) = \frac{0.0152}{5} = 0.0030$

6) By def of poission distribution

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

i) $p(x \leq 4) = p(0) + p(1) + p(2) + p(3) + p(4)$

$$= \frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(+1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!} + \frac{e^{-1.5}(1.5)^3}{3!} + \frac{e^{-1.5}(1.5)^4}{4!}$$

$$= 0.223 + 0.334 + 0.25 + 0.1255 + 0.0047$$

$$= 0.9805$$

ii) $p(x=4) = \frac{e^{-1.5}(1.5)^4}{4!} = 0.047$

iii) $p(x \geq 4) = 1 - P[x < 4]$

$$= 1 - [p(0) + p(1) + p(2) + p(3)]$$

$$= 1 - [0.223 + 0.334 + 0.25 + 0.1255]$$

$$= 0.066$$

7) Mean of Normal Distribution:-

By def of nrml distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-b}{\sigma}\right)^2}$$

Since mean $\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-b}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-1/2\left(\frac{x-b}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + b)\, e^{-\frac{z^2}{2}} \sigma\, dx$$

$$= \frac{1}{2\sqrt{\pi}} \left[\int_{-\infty}^{\infty} (\sigma z + b) e^{-z^2/2} \, dz \right]$$

$$= \frac{1}{2\sqrt{\pi}} \left[\int_{-\infty}^{\infty} (\sigma z + b) e^{-z^2/2} \, dz \right]$$

$$= \frac{1}{2\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \sigma z \, e^{-z^2/2} \, dz + b \int_{-\infty}^{\infty} e^{-z^2/2} \, dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + b\,2 \int_{0}^{\infty} e^{-z^2/2} \, dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \, 2b \int_{0}^{\infty} e^{-z^2/2} \, dz \qquad \boxed{\mu = b}$$

$$= \frac{1}{\sqrt{2\pi}} \, 2b \sqrt{\frac{\pi}{2}} \quad \left(\because \int_{0}^{\infty} e^{-z^2/2} \, dz = \sqrt{\frac{\pi}{2}} \right)$$