Incentivized Bandit Learning with Self-Reinforcing User Preferences

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Abstract

In this paper, we propose to study a new multi-armed bandit (MAB) online learning model that combines two real-world phenomena in many recommender systems in practice: (i) the service provider cannot pull the arms by itself and thus has to offer rewards to users to incentivize arm-pulling indirectly; and (ii) if users with specific arm preferences are well rewarded, they induce a "self-reinforcing" effect in the sense that they will attract more users of similar arm preferences. The goal of the service provider is to maximize the total rewards over a time horizon T with a low total payment. Our contributions in this paper are two-fold: (i) We propose a new MAB model that considers both users' self-reinforcing preference behaviors and incentives; and (ii) We leverage the properties of a multi-color Pólya urn model with nonlinear feedback to propose two policies termed "At-Least-n Greedy" and "UCB-List." We prove that both policies achieve $O(\log T)$ expected regrets with $O(\log T)$ expected payments over a time horizon T. We conduct extensive experiments to demonstrate the performances of these two policies and compare their robustness under various settings.

1 Introduction

In many online learning systems, there exists a self-reinforcing phenomenon, where the current user's behavior is influenced by the user behaviors in the past (Barabási and Albert, 1999; Chakrabarti et al., 2005; Ratkiewicz et al., 2010), or an item is getting increasingly more popular as it accumulates more positive feedbacks. For example, on a movie rental website, current customers tend to have more interest in Movie A that has 500 positive reviews, compared with Movie B that only has 10 positive reviews. On the other hand, the website owner, who aims to maximize the total profit in the long run, wants to identify the most profitable movies. The online learning problem under self-reinforcing preferences can be modeled by the multi-armed bandit (MAB) framework (Berry and Fristedt, 1985; Bubeck and Cesa-Bianchi, 2012). In the classic stochastic MAB setting, an agent observes and chooses to pull an arm (action). The chosen arm, as feedback, generates a random reward from an unknown distribution and is assumed to be independent and identically distributed (i.i.d.) over time. The agent, aiming to maximize total rewards over a given time horizon T, tries to design a strategy to perform arm selection in the presence of unknown arm reward distributions.

In the literature, existing works (Fiez et al., 2018; Shah et al., 2018) that incorporate the self-reinforcing preferences into the MAB model remain limited. Shah et al. (2018) showed that the self-reinforcing preferences might render the classic UCB (upper confidence bound) algorithm (Auer et al., 2002) sub-optimal, and proposed new optimal arm selection algorithms. However, in many platforms that utilize the MAB framework for online sequential decision making (e.g., recommender systems, healthcare, finance, and dynamic pricing, see Bouneffouf and Rish (2019)), the service providers cannot select arms directly. Rather, arms are pulled by users who are exhibiting self-reinforcing preferences. The service provider thus needs to incentivize users to select certain arms to maximize the total rewards, while avoiding incurring high incentive costs. Hence, the bandit models in Fiez et al. (2018); Shah et al. (2018) are no longer applicable even though the self-reinforcing preferences behavior is considered. On the other hand, there exist several works (Frazier et al., 2014; Mansour et al., 2015, 2016; Wang and Huang, 2018) that studied incentivized bandit under various settings and proposed efficient algorithms (more details in Section 2), but none of these works models users with self-reinforcing preferences. To our knowledge, our work is the first to jointly consider both incentives and self-reinforcing preferences in the MAB framework – two key features of many online learning systems in practice. As will be seen later, the combined effect of incentives and self-reinforcing preferences creates significant challenges in analyzing this new MAB model.

Specifically, in this paper, we propose a new MAB model to address the self-reinforcing preferences and incentivized arm selections. In this model, two fundamental trade-offs naturally emerge: On the one hand, sufficient exploration is required to find an optimal arm, which may result in pulling sub-optimal arms, while adequate exploitation is needed to stick with the arm that did well in the past, which may or may not be the best choice in the long run. On the other hand, the service provider needs to provide enough incentives to mitigate unfavorable self-reinforcing preferences, while in the meantime avoiding unnecessarily high compensations for users. As in most online learning problems, we use *regret* as a benchmark to evaluate the performance of our learning policy, which is defined as the performance gap between the proposed policy and an optimal policy in hindsight. The major challenges thus lie in three aspects: (a) During incentivized pulling, how could the service provider maintain a balance between exploration and exploitation to minimize regret? (b) How long should the service provider incentivize arm selections until the right self-reinforcing preferences are established toward an optimal arm, so that no further incentive is needed? (c) Is the self-reinforcing preferences strong and stable enough to sustain the sampling of an optimal arm over time without additional incentives? If yes, under what conditions could this happen?

In this work, we address the above challenges and questions by proposing two optimal policies for the incentivized MAB framework with self-reinforcing preferences. Our main contributions are as follows:

- We propose an incentivized bandit learning model with nonlinear feedback. Compared to the basic stochastic MAB model, this proposed model considers non-deterministic pulling from users who exhibit self-reinforcing preferences, and the service provider implements a bandit policy with incentives. To our knowledge, this is the first work that integrates both self-reinforcing preference and incentive in MAB.

- We show that no incentivized bandit policy achieves a sub-linear regret with a sub-linear total payment if the feedback function that models the self-reinforcing preferences does not satisfy certain conditions. The proof is inspired by a multi-color Pólya urn model, and we also show how to guide the self-reinforcing preferences toward a desired direction.

- We propose two bandit policies, namely At-Least-n Greedy and UCB-List, both of which are optimal in regret. Specifically, for the two policies, we analyze the upper bounds of the expected regret and the expected total payment over a fixed time horizon T. We show that both policies achieve $O(\log T)$ expected regrets, which meet the lower bound in Lai and Robbins (1985). Meanwhile, the expected total incentives for both policies are upper bounded by $O(\log T)$.

- We conduct extensive simulations to demonstrate the performance of both policies. Our results show that both policies are not only effective in performance but also robust under various settings.

2 Related Work

The self-reinforcing phenomenon has received increasing interest in several different fields recently. In the random network literature, previous works have studied the network evolution with "preferential attachment" (Barabási and Albert, 1999; Chakrabarti et al., 2005; Ratkiewicz et al., 2010). Also, a similar social behavior, referred to as *herding*, is studied in the Bayesian learning model literature (Bikhchandani et al., 1992; Smith and Sørensen, 2000; Acemoglu et al., 2011). For example, Acemoglu et al. (2011) first studied the conditions under which there exists a convergence in probability to the desired action as the size of a social network increases. More recently, Shah et al. (2018) incorporated positive externalities in user arrivals and proposed bandit algorithms to maximize the total reward. Then, Fiez et al. (2018) provided a more general model, where the service provider has limited information. Unlike previous works, the service providers in Shah et al. (2018); Fiez et al. (2018) have full control in determining which arm for users to pull. In contrast, the service provider in our model can only incentivize users to indirectly induce the preferences toward a desired direction, and which arm to be pulled is entirely dependent on the current user's random preference.

On the other hand, incentivized MAB has attracted growing attention in recent years (Kremer et al., 2014; Frazier et al., 2014; Mansour et al., 2015, 2016; Wang and Huang, 2018). To our knowledge, Frazier et al. (2014) first adopted incentive schemes into a Bayesian MAB setting. In their model, the service provider seeks to maximize time-discounted total rewards by incentivizing arm selections. Kremer et al. (2014) shares a similar motivation as Frazier et al. (2014). But in the model of Kremer et al. (2014), the service provider does not offer payments to the users. Instead, he decides the information to be revealed to users as incentives. Subsequently, Mansour et al. (2015) studied the case where the rewards are not discounted over time. More recently, Wang and Huang (2018) considered the non-Bayesian setting with non-discounted rewards. These models differ from our model in both the incentive schemes and user behaviors. Another line of research similar to incentivized bandit is bandit with budgets (Guha and Munagala, 2007; Goel et al., 2009; Combes et al., 2015; Xia et al., 2015), where the service provider takes actions with budget constraints. Guha and Munagala (2007) developed approximation algorithms for a large class of budgeted learning problems. Then, Goel et al. (2009) proposed index-based algorithms for this problem. The key difference from our work is that in these models, the budget constraints are pre-determined, and the service providers cannot take any further actions as soon as the budget constraints are violated. In contrast, the total payment in our model is evaluated only after the time horizon is finished, which implies that bounding the total payment is also part of our goals.

Although not cast in the MAB framework, the works on *urn models* (Khanin and Khanin, 2001; Drinea et al., 2002; Oliveira, 2009; Zhu, 2009) also share some relevant feedback settings to our model. Drinea et al. (2002) first proposed a class of processes called *balls and bins models with feedback*, which is a preferential attachment model for large networks. They then proved the convergence results of the model with various feedback functions. Later, Khanin and Khanin (2001) improved the convergence result by showing monopoly (to be defined later) happens with probability one under a class of feedback functions included in Drinea et al. (2002). Our proposed model is inspired by the ideas of feedback from Oliveira (2009), in which the author discussed a natural evolution of the balls and bins process with nonlinear feedback. However, our model is focused on MAB regret minimization, which is completely different from the goals considered in these works.

3 Modeling and Notations

In this paper, we denote the set of arms offered by the service provider as $A = \{1, \ldots, m\}$. Each arm a follows a Bernoulli reward distribution D_a with an unknown mean $\mu_a > 0$. The process runs for T rounds. At each time step $t \in \{1, \ldots, T\}$, a user arrives and chooses an arm $I(t)$ to pull. After pulling arm $I(t)$, the service provider receives a random reward $X(t) \sim D_{I(t)}$. For $t \geq 1$, we denote the number of times that an arm a is pulled up to time t as $T_a(t) \coloneqq \sum_{i=1}^{t} \mathbb{1}_{\{I(i)=a\}}$, and denote the total reward generated by arm a up to time t as $S_a(t) \coloneqq \sum_{i=1}^{t} X(i) \cdot \mathbb{1}_{\{I(i)=a\}}$. For each arm $a \in A$, we let $T_a(0) = 0$ and $S_a(0) = 0$. We assume that there exists a unique best arm $a^* \in A$ such that $a^* = \arg\max_a \mu_a$.

1) Preference and bias modeling: In our model, the user behavior is stochastic. Specifically, in each time step t, the user has a positive probability $\lambda_a(t)$ to pull each arm $a \in A$, with $\sum_{a \in A} \lambda_a(t) = 1, \forall t$. In other words, the probability $\lambda_a(t)$ can be viewed as the preference rate in time step t. In this paper, we adopt the widely used multinomial logit model in the literature to model $\lambda_a(t)$ as follows:

$$\lambda_a(t) = \frac{F\big(S_a(t-1) + \theta_a\big)}{\sum_{i \in A} F\big(S_i(t-1) + \theta_i\big)}, \tag{1}$$

where the function $F : \mathbb{R} \to (0, +\infty)$ is an increasing feedback function of the accumulative reward of an arm a, and $\theta_a > 0$ denotes the initial preference bias of arm a. Several important remarks for the preference model in (1) are in order: (i) The multinomial logit model is based on the behavioral theory of utility and has been widely applied in the marketing literature to model the brand choice behavior (Guadagni and Little, 2008; Gupta, 1988). The multinomial logit model is also used in the social network literature to model preferential attachment (Barabási and Albert, 1999), where the probability that a link connects a new node j with another existing node i is linearly proportional to the degree of i. Besides, the preferential attachment phenomenon also characterizes user behavior influenced by history. Therefore, we assume in (1) that the user can access the history and will be influenced accordingly. (ii) For the feedback function $F(\cdot)$ in (1), a simple example is $F(x) = x^\alpha$ for some $\alpha > 1$, i.e., users are more influenced by the accumulative reward in the past with a larger α-value.

2) Incentive mechanism modeling: In the classic MAB model, the service provider can directly control which arm to pull (Shah et al., 2018). In our model, however, the user randomly selects an arm depending on the current preference rate, which is in turn influenced by the history and the incentive. The service provider aims at maximizing total reward in the long run, but cannot directly control which arm to pull. The service provider can only offer a certain amount of incentive on the arm that he prefers, so as to increase the users' preferences of pulling this particular arm. In this paper, we model the influence of the incentives following the so-called "coupon effects on brand choice behaviors" in the economics literature (Papatla and Krishnamurthi, 1996; Bawa and Shoemaker, 1987). Simply speaking, the relationship between coupons and choices is nonlinear, and the redemption rate increases with respect to the coupon face value but exhibits a diminishing return effect (Bawa and Shoemaker, 1987). This type of incentive effects can be modeled as follows: in each time step t, if arm $a \in A$ is preferred by the service provider, a payment b is offered to the user to increase the user's preference on pulling arm a. The preference rates under payment b are updated as follows:

$$\hat{\lambda}_i(t) = \begin{cases} \dfrac{G'(b) + F\big(S_i(t-1)+\theta_i\big)}{G'(b) + \sum_{j\in A} F\big(S_j(t-1)+\theta_j\big)}, & i = a,\\ \dfrac{F\big(S_i(t-1)+\theta_i\big)}{G'(b) + \sum_{j\in A} F\big(S_j(t-1)+\theta_j\big)}, & i \neq a, \end{cases}$$

where $G'(\cdot)$ is an increasing function independent of $S_i(t-1)$, F and θ_i. Clearly, the above preference update model still follows the multinomial logit model, where the value $G'(b)$ can be interpreted as the perceived reward to the users for pulling arm a under payment b. We assume that the service provider has the knowledge of $G'(\cdot)$, which can be learned from historical data. Also, we can see that from the above definition that, as the payment b increases asymptotically (i.e., $b \uparrow \infty$), we have $\hat{\lambda}_a(t) \uparrow 1$ and $\hat{\lambda}_i(t) \downarrow 0$, $\forall i \neq a$, i.e., arm a is preferred with probability one.

For simplicity and convenience in our subsequent theoretical analysis, in the rest of the paper, we rewrite $\hat{\lambda}_i(t)$ in the following equivalent form: we divide both the denominator and numerator by $\sum_{i\in A} F\big(S_i(t-1)+\theta_i\big)$ and let $G(b) \triangleq G'(b)/\sum_{i\in A} F\big(S_i(t-1)+\theta_i\big)$. Then, it can be readily verified that the updated preference rate can be equivalently rewritten as:

$$\hat{\lambda}_i(t) = \begin{cases} \dfrac{\lambda_i(t) + G(b)}{1+G(b)}, & i = a,\\ \dfrac{\lambda_i(t)}{1+G(b)}, & i \neq a. \end{cases}$$

It is easy to see that $G(\cdot)$ remains an increasing function of b. Also for notational convenience, we define the accumulative payment offered up to time step t as $B_t \coloneqq \sum_{i=1}^{t} b_t$, where $b_t \in \{0, b\}$, $\forall t$.

3) Regret: Let $\Gamma_T = \sum_{t=1}^{T} X(t)$ denote the accumulative rewards up to time T. In this paper, we aim to maximize $\mathbb{E}[\Gamma_T]$ by designing an incentivized policy π with low accumulative payment, hopefully, both being logarithmic with respect to the time horizon T. A policy π is an algorithm that produces a sequence of arms that are recommended at time step $t = 1, \ldots, T$. Similar to classic MAB problems, we want to measure our performance against an oracle policy, where in hindsight the service provider knows the best arm a^* with the largest mean and can always offer an *infinite* amount of payments to users, so that the updated preference rate of arm a^* is always infinitely close to one. We denote the expected accumulative reward generated under the oracle policy up to time T as $\mathbb{E}[\Gamma_T^*] = \mu_{a^*} T$.[1] The expected (pseudo) regret is defined as follow: $\mathbb{E}[R_T] = \mu_{a^*} T - \mathbb{E}[\Gamma_T]$. Our objective is to minimize the expected regret $\mathbb{E}[R_T]$, with the expected accumulative payment $\mathbb{E}[B_T]$ being logarithmic with respect to the time horizon T.

4 Policies and Upper Bounds

In this section, we present two policies that achieve $O(\log T)$ expected regret with $O(\log T)$ accumulative payment with respect to the time horizon T. The main ideas of these two policies are

[1] It is insightful to compare our oracle policy with Shah et al. (2018). The oracle policy in Shah et al. (2018) does not achieve $\mu_{a^*} T$ expected accumulative reward up to time T due to the following key modeling difference: In Shah et al. (2018), it is assumed that the service provider can only feed a *single arm* at a time to the current user. Hence, the oracle policy keeps *only* feeding the best arm to all arriving users. However, in the early time steps, a fraction of the users may not prefer the best arm due to initial biases. Hence, the system has to spend time mitigating these initial biases, resulting in an expected accumulative reward smaller than $\mu_{a^*} T$. In contrast, we assume that the service provider can feed *all arms* to each user (closely models real-world recommender systems), and the oracle policy offers an infinite amount of payment as incentives. Thus, users will always pull the best arm with probability one in each time step, which implies $\mu_{a^*} T$ expected accumulative reward up to time T.

similar. We first perform exploration among all arms by incentivizing pulling until we know the best-empirical arm is optimal, i.e., $\hat{a}^* = a^*$ with high confidence. Then, we keep incentivizing the pulling of the best-empirical arm $\hat{a}^*$ until it dominates. To this end, we formally define the notion of dominance as follows:

Definition 1 (Dominance)**.** *An arm is said to be dominant if it produces at least half of the total reward.*

The key of the policies is to guarantee the dominance of arm $\hat{a}^*$, which induces the correct preference to arm $\hat{a}^*$, while keeping the accumulative payment sub-linear.

Before presenting the policies, we want to show that if the feedback function $F(x)$ satisfies certain conditions, then with probability one, the users' self-reinforcing preferences converge to one arm without incentive after sufficiently many time steps, i.e., an arm $a \in A$ is the only arm to be sampled. We define this event as the *monopoly by arm a*, or *mono*$_a$ for short. We then define an incentivized policy as a policy that incentivizes pulling with bounded payment for each time step for a continuous duration smaller than or equal to the time horizon T. Clearly, monopoly is necessary for the existence of an incentivized policy that induces all users' preferences to a certain arm with sub-linear total payment. Correspondingly, if such a policy exists, then it implies that the system enjoys the property that the users' self-reinforcing preferences converge to a certain arm even after the service provider stops providing incentives at some stage.

Lemma 1. (Monopoly) *There exists an incentivized policy that induces users' preferences to converge in probability to an arm over time with sub-linear payment, if and only if $F(x)$ satisfies $\sum_{i=1}^{+\infty} \frac{1}{F(i)} < +\infty$.*

Proof Sketch. Our main mathematical tool is the improved exponential embedding method. In a nutshell, this method simulates the reward generating sequence by random exponentials. Define a sequence $\{\chi_j\}_{j=1}^{\infty}$ denoting the reward generating order, where each element denotes the arm index. Note that an arm index appears in $\{\chi_j\}$ only when it is pulled and generates a unit reward. We want to construct a sequence $\{\zeta_j\}$ that has the same conditional distribution as $\{\chi_j\}$ given history $\mathcal{F}_{j-1}$. For the arm $i \in A$, consider a collection of independent exponential random variables $\{r_i(n)\}$ such that $\mathbb{E}[r_i(n)] = \frac{1}{\mu_i F(n+\theta_i)}$. We construct an infinite set $B_i = \{\sum_{k=0}^{n} r_i(k)\}_{n=0}^{\infty}$, where each element $\sum_{k=0}^{n} r_i(k)$ models the time needed for arm i to get n accumulative rewards. Then we mix up and sort B_i for all $i \in A$ to form a sequence H. Our objective sequence $\{\zeta_j\}$ is the arm index sequence out of H.

Next, we prove by induction that given the previous reward history $\mathcal{F}_{j-1}$, the constructed sequence $\{\zeta_j\}$ has the same conditional distribution as $\{\chi_j\}$. Then, the proof of Lemma 1 is done if we show that if and only if any feedback function $F(x) > 0$ satisfies $\sum_i \frac{1}{F(i)} < +\infty$, then $\mathbb{P}(\exists a \in A, \textit{mono}_a) = 1$. To prove this, we define the *attraction time* N as the time step when the monopoly happens. By leveraging the constructed sequence $\{\zeta_j\}$, we establish the necessity by showing that if $\sum_i \frac{1}{F(i)} < +\infty$ then $\mathbb{P}(N < \infty) = 1$, and the sufficiency by showing that if $\sum_i \frac{1}{F(i)} = +\infty$ then $\mathbb{P}(N = \infty) > 0$. $\square$

Remark 1. The exponential embedding technique has been widely applied (Zhu, 2009; Oliveira, 2009; Davis, 1990; Athreya and Karlin, 1968). This method embeds a discrete-time process into a continuous-time process built with exponential random variables. We improve this method and adapt it to our model, which considers more random variables. The most significant feature of the exponential embedding technique is that the random times of different arms generating unit rewards are independent and can be mathematically expressed using exponential distributions, which facilitates our subsequent analysis.

Remark 2. As a special case, if $F(x) = x^\alpha$ with $\alpha > 1$ (i.e., superlinear polynomial), then there exists an incentivized policy that induces all preferences to converge over time with sub-linear total payment, since $\sum_{i=1}^{+\infty} \frac{1}{i^\alpha} < +\infty$ with $\alpha > 1$. Previous works (Drinea et al., 2002; Khanin and Khanin, 2001) considering the balls and bins model also studied the cases $\alpha < 1$ and $\alpha = 1$. In the case $\alpha < 1$, the asymptotic preference rates of arms are all deterministic, positive, and dependent on the means and biases of arms. In the case $\alpha = 1$, the system is akin to a standard Pólya urn model, and will converge to a state where all arms have random positive preference rates that depend on the means and initial biases of the arms. In the case $\alpha > 1$, the system converges almost surely to a state where only one arm has a positive probability to generate rewards, depending on the means and initial biases of arms. Thus, the system under these three conditions exhibits completely different behaviors.

4.1 At-Least-n Greedy

The At-Least-n Greedy policy consists of three phases: the exploration phase, the exploitation phase, and the self-sustaining phase. The service provider participates in the first two phases. During the exploration phase, the At-Least-n Greedy policy explores all arms until each arm generates sufficient accumulative rewards. Then, the policy exploits the arm with the best empirical mean until it dominates (as defined in Definition 1). We show that once the best-empirical arm dominates, there is an overwhelming probability that after finite time steps in the self-sustaining phase, monopoly happens on the best-empirical arm, which implies sub-linear regret. *One of our key contributions in this paper is the discovery* that the incentive can stop as soon as the dominance happens, which is *earlier than* the monopoly is established and guarantees sub-linear accumulative payment. In both policies, we define the sample mean of arm $a \in A$ in time step t as $\hat{\mu}_a(t) = \frac{S_a(t-1)}{T_a(t-1)}$. The At-Least-$n$ Greedy policy is stated as follows:

Policy 1 (At-Least-n Greedy)**.** *Given the time horizon T, incentive sensitivity function G and a payment b satisfying $G(b) > 1$. Let $n = q \log T$, where $q \geq 1$ is a tunable parameter.*

1) Exploration Phase: *Let $\tau_n = \min(t : S_a(t) \geq n, \forall a) \wedge T$ denote the random time step where any arm has accumulative reward of at least n. For $t \in [1, \tau_n]$, incentivize users with payment b to sample arm $I(t) \in \arg\min_{a \in A} S_a(t)$, with ties broken at random.*

2) Exploitation Phase: *Let $\hat{a}^* \in \arg\max_{a \in A} \hat{\mu}_a(\tau_n + 1)$, with ties broken at random. Let $\tau_s = \tau_n + \delta\tau_n$, where $\delta = \frac{G(b)+1}{G(b)-1}$. For $t \in (\tau_n, \tau_s]$, the service provider offers payment b for users to sample arm $\hat{a}^*$.*

3) Self-Sustaining Phase: *For $t \in (\tau_s, T]$, the service provider stops offering payments and lets users sample arms based on their own preferences.*

Beyond time τ_s, arm $\hat{a}^*$ is expected to dominate. Since with enough time steps after τ_s, monopoly happens with probability one and arm $\hat{a}^*$ has high probability to emerge victorious in monopoly (to be shown in the proof of Theorem 2). If the time horizon T is large enough to cover the attraction time, then arm $\hat{a}^*$ will be sampled repeatedly after monopoly happens, while the accumulative reward generated by sub-optimal empirical arms is *independent* of T after the monopoly (which contribute to the regret). Thus, the policy achieves a sub-linear expected regret stated as follows:

Theorem 2. (At-Least-n Greedy) *Given a sensitivity function $G(\cdot)$ and a fixed payment b satisfying $G(b) > 1$, if the feedback function satisfies $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ for $\alpha > 1$, the expected regret $\mathbb{E}[R_T]$ of the At-Least-n Greedy policy is upper bounded by $\mathbb{E}[R_T] \leq$*

$$\sum_{a\in A} \frac{2\max_{i\in A}\Delta_i + [G(b)-1]\Delta_a}{[G(b)-1]\mu_a} \times q\log T + O(1),$$ *with the expected accumulative payment* $\mathbb{E}[B_T]$ *upper bounded by* $\mathbb{E}[B_T] \leq \sum_{a\in A} \frac{2[G(b)+1]}{[G(b)-1]\mu_a} \times q\log T$.

Proof Sketch. By the law of total expectation, the expected regret up to time T can be decomposed as $\mathbb{E}[R_T] \leq \mathbb{E}[R_T \mid \hat{a}^* = a^*] + T \cdot \mathbb{P}(\hat{a}^* \neq a^*)$. To bound $\mathbb{E}[R_T]$, we want to upper bound both $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$ and $\mathbb{P}(\hat{a}^* \neq a^*)$. First, the probability $\mathbb{P}(\hat{a}^* = a^*)$ is upper bounded by $O(T^{-1})$ by leveraging the Chernoff-Hoeffding bound. To bound $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$, we need to bound $\mathbb{E}[\tau_n]$ and $\mathbb{E}[\tau_s]$. Consider $\mathbb{E}[\tau_n]$, we show that the number of pulling of arm a to get a unit reward is a geometric random variable with parameter larger than $\frac{\mu_a G(b)}{1+G(b)}$. Then, for each arm $a \in A$ to obtain at least n accumulative reward, the expected time needed is upper bounded by $\mathbb{E}[\tau_n] \leq n\sum_{a\in A} \frac{1+G(b)}{\mu_a G(b)}$. For $\mathbb{E}[\tau_s]$, it follows from its definition that $\mathbb{E}[\tau_s] = (1+\delta)\mathbb{E}[\tau_n]$. According to the policy, the expected accumulative payment $\mathbb{E}[B_T]$ can be upper bounded by $b\mathbb{E}[\tau_s]$.

The next challenge is to show whether the dominant arm has a large enough probability to "win" in monopoly. By straightforward probability calculations, our choice of τ_s can be shown to be sufficiently large to guarantee that the best-empirical arm dominates in expectation. Next, we show that during the self-sustaining phase, the dominant arm has an overwhelming probability to "win" in monopoly. Construct an event $D(u,n)$ to describe the following phenomenon: the fraction of accumulative reward from a weak arm increases over time, i.e., at time step τ_s, there are $u_0 n_0$ accumulative reward from the weak arm, with n_0 total reward and the fraction $u_0 < \frac{1}{2}$. Then at time step $t' \in (\tau_s, T]$, there are un accumulative reward from the weak arm with the fraction $u > u_0$. The probability of $D(u,n)$ can be bounded as $\mathbb{P}\big(\exists n > n_0,\, D(u,n)\big) \leq Ce^{-(u_0 n_0)^\gamma}$ with constant $\gamma > 0$ using the improved exponential embedding method and a Chernoff-like bound developed in Lemma 7. Thus the arms that stay on the weak side for a long time have little chance to win back.

Lastly, with the upper bound of $\mathbb{E}[\tau_n]$ and $\mathbb{E}[\tau_s]$, we obtain $\mathbb{E}[R_T \mid \hat{a}^* = a^*] \leq n\sum_{a\in A} \frac{2\max_{i\in A}\Delta_i + [G(b)-1]\Delta_a}{[G(b)-1]\times\mu_a} + \mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]$, where $\mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*]$ represents the expected regret during time $(\tau_s, T]$. The expected regret during this phase is caused by pulling any sub-optimal arms, and can be bounded by $O(1)$, which follows from the fact that "the arm that stays on the weak side for a long time has little chance to win back" and some standard expectation calculations. Thus, the term $\mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]$ can be upper bounded by $O(1)$. This completes the proof. □

4.2 UCB-List

In this section, we propose a UCB-List policy to further improve the performance of the At-Least-n Greedy policy. The UCB-List policy is similar to At-Least-n Greedy and also consists of three phases. During the exploration phase, the service provider initially puts all arms in one set, and then makes arm selection in a way similar to the standard UCB policy. Meanwhile, it removes arms that are estimated to be sub-optimal, until only one arm is left in the set, which is viewed as the best-empirical arm. Then, the service provider incentivizes users to sample this arm until it dominates. Compared to At-Least-n Greedy, UCB-List reduces the pulling times of sub-optimal arms during the exploration phase, while still balancing the trade-off between exploration and exploitation. Formally, the UCB-list policy is stated as follows:

Policy 2 (UCB-List)**.** *Given the time horizon* T*, function* $G(\cdot)$ *and a payment* b *satisfying* $G(b) > 1$*. Define the confidence interval of arm* a *in time step* t *as* $c_a(t) = p\sqrt{\frac{\ln T}{T_a(t)}}$*, where* p *is a tunable parameter.*

Initialization: *Incentivize users with payment b to sample arm $a \in A$ for which $T_a(t) = 0$, with ties broken at random until* $\min_{a\in A} T_a(t) = 1$. *Let set $U = A$.*

1) Exploration Phase: *If $|U| > 1$, remove arm a for which $\hat{\mu}_a(t) + c_a(t) \le \min_{i\neq a, i\in U} \big[\hat{\mu}_i(t) - c_i(t)\big]$ from set U if there is any. Offer payment b to incentivize users to sample arm $I(t) \in$* $\arg\max_{a\in U} \big[\hat{\mu}_a(t) + c_a(t)\big]$, *with ties broken at random. If $|U| = 1$, let arm $\hat{a}^* = \{a : a \in U\}$, mark current time as τ_1 and proceed to the Exploitation Phase.*

2) Exploitation Phase: *Let $\tau_2 = \tau_1 + \delta\big(\tau_1 - T_{\hat{a}^*}(\tau_1)\big)$, where $\delta = \frac{G(b)+1}{G(b)-1}$. For $t \in [\tau_1, \tau_2]$, offer payment b to incentivize users to sample arm $\hat{a}^*$.*

3) Self-Sustaining Phase: *For $t \in (\tau_2, T]$, the service provider stops offering payments and lets users sample arms based on their own preferences.*

We note that the parameter p plays an important role in the UCB-List: If p is too large, it takes a long time to remove arms from set U, which prolongs the exploration phase and the incentivized duration. If p is too small, then the exploration may not be long enough to ensure a logarithmic regret.

Theorem 3. (UCB-List) *Given an incentive sensitivity function $G(\cdot)$ and a fixed payment b satisfying $G(b) > 1$, if the feedback function satisfies $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ for $\alpha > 1$, then the expected regret $\mathbb{E}[R_T]$ of the UCB-List policy is upper bounded by $\mathbb{E}[R_T] \le \sum_{a\neq a^*, a\in A} (\frac{4p^2}{\Delta_a} + \frac{\gamma_p \Delta_a}{G(b)-1}) \log T + O(1) + O(T^{-3})$, with a constant γ_p dependent on the parameter p. The expected accumulative payment $\mathbb{E}[B_T]$ is upper bounded by $\mathbb{E}[B_T] \le \dfrac{2\gamma_p G(b)}{G(b)-1} \log T$.*

Proof Sketch. The expected time needed for initialization can be proved upper bounded by $O(1)$ trivially. Then by the law of total expectation, the expected regret up to time T can be decomposed as:

$$\begin{aligned}\mathbb{E}[R_T] \le & \underbrace{\mathbb{E}[R_{\tau_1}]}_{\textbf{(a)}} + \underbrace{\mathbb{E}[R_{\tau_2} - R_{\tau_1} \mid \hat{a}^* = a^*]}_{\textbf{(b)}} \\ & + \underbrace{\mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*]}_{\textbf{(c)}} + \underbrace{T \cdot \mathbb{P}(\hat{a}^* \neq a^*)}_{\textbf{(d)}}.\end{aligned}$$

In what follows, we will bound the four terms on the right-hand-side one by one.

(a) In the exploration phase, since the regret results from the pulls of sub-optimal arms, the expected regret at time step τ_1 can be written as $\mathbb{E}[R_{\tau_1}] = \sum_{a\neq a^*, a\in A} \Delta_a \mathbb{E}[T_a(\tau_1)]$, where Δ_a is defined as $\Delta_a = \mu_{a^*} - \mu_a$ for arm $a \in A$. Thus, term (a) can be bounded if we upper bound $\mathbb{E}[T_a(\tau_1)]$ for each $a \in A$. The proof is similar to that of the standard UCB policy except that we consider in addition the case where user behaviors in our model are stochastic. Thus, there exists a positive probability for users to pull arms that are not incentivized. This possibility causes regret that can be bounded based on the convergence of the feedback function. Thus, for each arm $a \in A$, we obtain $\mathbb{E}[T_a(\tau_1)] \le \frac{8 \ln T}{\Delta_a^2} + O(1)$. By using the Chernoff-Hoeffding bound, $\mathbb{E}[\tau_1]$ can be bounded by $O(\log T)$.

(b) In the exploitation phase, the expected regret $\mathbb{E}[R_{\tau_2} - R_{\tau_1} \mid \hat{a}^* = a^*]$ is upper bounded by $O(\mathbb{E}[\tau_2])$. Thus, following the definition of τ_2, we obtain the bound $O(\log T)$ for this term. Also, since the payment is stopped after the exploitation phase, the expected accumulative payment $\mathbb{E}[B_T]$ can be bounded by $b\mathbb{E}[\tau_2]$ given time τ_2. Note that the choice of τ_2 has the same analysis as in the proof of At-Least-n Greedy.

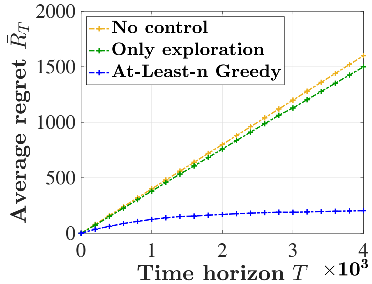
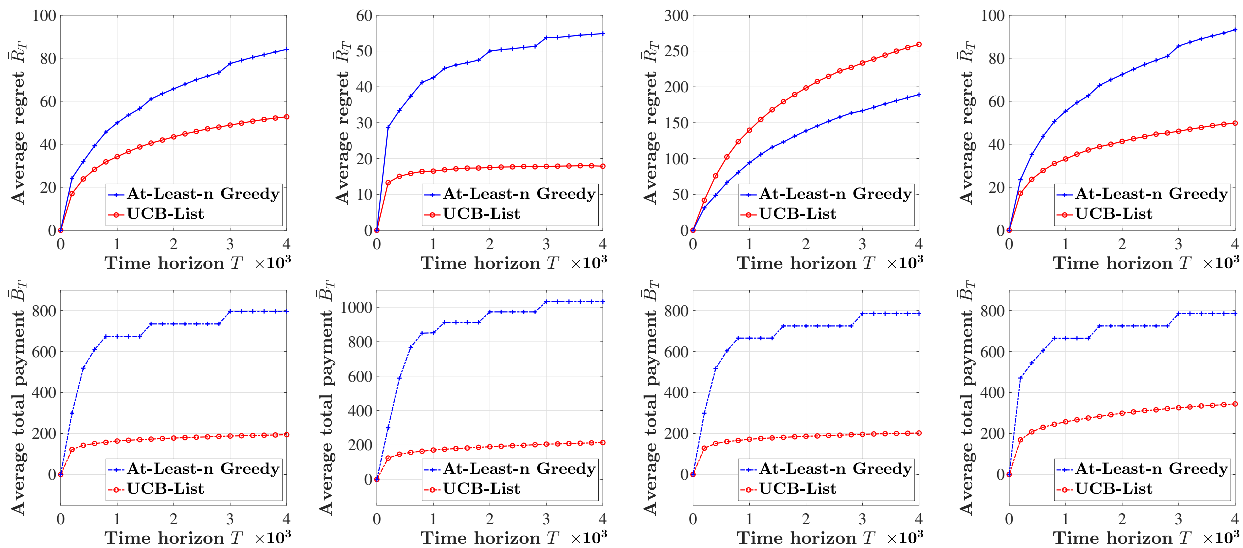


Figure 1: The performance of the At-Least-n Greedy, compared to i) the performance of the baseline with no incentive control and ii) the baseline with only incentive control during exploration phase.



(a) Benchmark result. (b) $\alpha = 1.5$. (c) $\boldsymbol{\theta} = [50, 1]$. (d) $b = 3$.

Figure 2: Performances of the At-Least-n Greedy and UCB-List policies.

(c) The third term represents the expected regret from time τ_2 to T. Similar to the proof of Theorem 2, the expected regret during this phase is resulted from the pulling of any sub-optimal arms, and can be bounded by $O(1)$ based on the result in choosing τ_2 and some standard expectation calculations.

(d) The probability $\mathbb{P}(\hat{a}^* \neq a^*)$ can be bounded by $O(T^{-4})$ using the Chernoff-Hoeffding bound.

Combining steps **(a)**–**(d)** yields the result stated in the theorem and the proof is complete. $\square$

5 Simulations

We conduct simulations to evaluate the performances of the At-Least-n Greedy and UCB-List policies.

1) Baseline Comparisons: We first compare our policies' performance with two baselines: i) no incentive control and ii) with incentive control only during exploration. We use At-Least-n Greedy as an example. Consider a two-armed model with means $\boldsymbol{\mu} = [0.2, 0.6]$ and initial biases $\boldsymbol{\theta} = [100, 1]$. We choose the feedback function as $F(x) = x^\alpha$ with $\alpha = 1.5$ and set the payment as $b = 1.5$ with an incentive sensitivity function $G(x) = x$. The results are illustrated in Figure 1, where each point is averaged over $10,000$ trials. We can observe that the average regret under no incentive control grows linearly due to the self-reinforcing preference on the suboptimal arm with large initial bias and large α. The average regret under partial incentive control is also linear since the incentive is insufficient to offset the initial bias from the suboptimal arm. The average regret of the At-Least-n Greedy follows a $\log(T)$ growth rate with $O(\log T)$ total payment.

2) At-Least-n Greedy vs. UCB-List: Next, we compare the performances of At-Least-n Greedy and UCB-List. The setup is the same as that in baseline comparisons, except that the initial bias is set as $\boldsymbol{\theta} = [10, 1]$ and $\alpha = 1.1$. For UCB-List, we set the parameter $p = 0.33$. As the means of arms get closer, it is better to choose a small p, since a smaller p prevents the exploration phase being too large. For At-Least-n Greedy, we set the parameter $q = 1.5$. Four groups of simulations are conducted and the results are shown in Figure 2. Figure 2a illustrates the performance of both average regret and average total payment. Figure 2a also serves as a benchmark result, which is compared to the other three groups of results. In each of the three Figures 2b–2d, only one parameter is changed compared to the benchmark group. This helps to observe the changes in average regret and average total payment. In Figure 2b, all settings are the same as Figure 2a except that $\alpha = 1.5$. In Figure 2c, all settings are the same as those in Figure 2a except that $\boldsymbol{\theta} = [50, 1]$. In Figure 2d, all settings are the same as Figure 2a except that $b = 3$.

We can see that both policies achieve $O(\log T)$ average regrets and $O(\log T)$ average total payments. This indicates that: i) both policies balance the trade-off between exploration and exploitation so that an order-optimal regret can be reached; ii) both policies balance the trade-off between maximizing the total reward and keeping the total payment grow at rate $O(\log T)$. In Figure 2b, the result shows that both policies achieve a smaller average regret, because the self-reinforcing preferences are easier to converge to the arm incentivized by the service provider under a larger α. Also, the At-Least-n Greedy policy incurs a higher total payment because it incentivizes the pulling of sub-optimal arms more often. In Figure 2c, both policies have larger average regrets because it takes more effort to mitigate the larger initial biases. Also, the average regret shows that UCB-List is more sensitive to the initial bias than At-Least-n Greedy, since UCB-List tends to make biased decisions during the early time steps. Thus, a good choice of the parameter p is important to achieve good performance under UCB-List. In Figure 2d, as the payment for each time step increases from 1.5 to 3, the average regret and average total payment are not affected significantly since the total incentivized duration decreases correspondingly.

6 Conclusion

We proposed and studied an incentivized bandit model with self-reinforcing preferences. Two incentivized bandit policies are proposed to achieve $O(\log T)$ expected regrets with $O(\log T)$ incentivized costs, under the condition that the feedback function satisfies $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ for $\alpha > 1$. We conjecture that the feedback can be extended to a larger class of nonlinear functions. We note that the area of incentivized MAB with self-reinforcing preferences remains under-explored. Future works include, for example, incentive costs could be time-varying in each time step, which can either be dependent on the current state, or restricted by certain conditions. The self-reinforcing preferences can also be viewed as contexts, and thus this setting can be modeled by leveraging contextual bandit with more interesting properties.

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A Proof of Lemma 1

Lemma 1. (Monopoly) *There exists an incentivized policy that induces users' preferences to converge in probability to an arm over time with sub-linear payment, if and only if $F(x)$ satisfies $\sum_{i=1}^{+\infty} \frac{1}{F(i)} < +\infty$.*

Let the sequence $\{\chi_j\}_{j=1}^{\infty}$ be the arm order that generates a unit reward without being incentivized in our model, such that χ_j indicates the arm that generates the j-th unit reward, as shown in Figure 3. Next we will construct a sequence that has the same conditional distribution as $\{\chi_j\}$.

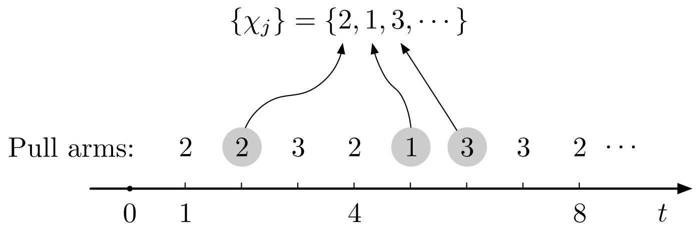

Figure 3: This figure shows an instance of sequence $\{\chi_j\}$. At time step $t = 1$, arm 2 is pulled and generates 0 reward. At time step $t = 2$, arm 2 is pulled and generates a unit reward. Thus, the first element χ_1 in $\{\chi_j\}$ is the arm index 2 that generates the first unit reward. The subsequent elements in the sequence are generated similarly.

Our main mathematical tool is the *improved exponential embedding* method. For each arm $i \in A$, we let $\{r_i(n)\}$ be a collection of independent exponential random variables such that $\mathbb{E}[r_i(n)] = \frac{1}{\mu_i F(n+\theta_i)}$. We define set $B_i := \{\sum_{k=0}^{n} r_i(k)\}_{n=0}^{\infty}$, where each element $\sum_{k=0}^{n} r_i(k)$ represents the random time needed for arm i to get n accumulative reward, and define set $G = B_1 \cup B_2 \cup \cdots \cup B_m$. Let ζ_1 be the smallest number in G and in general let ζ_j be the j-th smallest number in G. Next, we define a new random sequence $\{\zeta_j\}$, by making the j-th element of the sequence be the arm i if $\zeta_j \in B_i$. Then, we have the following lemma (to be proved later):

Lemma 4. *Given the previous reward history $\mathcal{F}_{j-1}$, the constructed sequence $\{\zeta_j\}$ is equivalent in conditional distribution to the sequence $\{\chi_j\}$.*

Next, we formally define the notion of attraction time.

Definition 2 (Attraction time)**.** *Let N denote the attraction time, such that after this time step N, monopoly happens, i.e., only one arm has positive probability to generate rewards.*

Necessity: if $\alpha > 1$ then $\mathbb{P}(N < \infty) = 1$. With the help of improved exponential embedding, the time until the accumulative reward of arm $i \in A$ approaches infinity is $\sum_{k=0}^{\infty} r_i(k)$. If the condition $\sum_i \frac{1}{F(i)} < \infty$ is satisfied, then we have

$$\mathbb{E}\big[\sum_{k=0}^{\infty} r_i(k)\big] = \frac{1}{\mu_i} \sum_{k=0}^{\infty} \frac{1}{F(k+\theta_i)} < \infty.$$

So for each arm $i \in A$, $\mathbb{P}(\sum_{k=0}^{\infty} r_i(k) < \infty) = 1$. Let $a = \arg\min_{i \in A}\{\sum_{k=0}^{\infty} r_i(k)\}$, then for each $b \neq a$, there exists a finite number K_b such that

$$\sum_{k=0}^{K_b} r_b(k) < \sum_{k=0}^{\infty} r_a(k) < \sum_{k=0}^{K_b+1} r_b(k).$$

Thus if we let $N := \max_{i\in A, i\neq a}\{\sum_{k=0}^{f_i(k)} r_i(k)\}$, then after this time N, only arm a can generate rewards.

Sufficiency: if $\mathbb{P}(N<\infty)=1$ then $\sum_i \frac{1}{F(i)} < \infty$. If we show that when $\sum_i \frac{1}{F(i)} = \infty$ we have $\mathbb{P}(N=\infty)>0$, then the proof is done. When $\sum_i \frac{1}{F(i)} = \infty$, we have

$$\mathbb{E}\big[\sum_{k=0}^{\infty} r_i(k)\big] = \frac{1}{\mu_i}\sum_{k=0}^{\infty}\frac{1}{F(k+\theta_i)} \to \infty.$$

Thus for any $i \in A$ it takes infinite time to accumulate infinite reward, which implies $\mathbb{P}(N=\infty)>0$. In fact, in this case $\mathbb{P}(N=\infty)=1$. We refer readers to Khanin and Khanin (2001) and Oliveira (2004) for further details.

A.1 Proof of Lemma 4

The proof of this lemma relies on the memoryless property of the exponential distribution as well as the following two facts:

Fact 1. *If $X_1,\cdots,X_m (m\geq 2)$ are independent exponential random variables with parameter $\lambda_1,\cdots,\lambda_m$, respectively, then $\min(X_1,\cdots,X_m)$ is also exponential with parameter $\lambda_1+\cdots+\lambda_m$.*

Fact 2. *For two independent exponential random variables $X_1 \sim exp(\lambda_1)$ and $X_2 \sim exp(\lambda_2)$, $\mathbb{P}(X_1<X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.*

Initially, in the sequence $\{\zeta_j\}$ when $j=1$ since the initial value for arm i is its bias θ_i, using the above two facts:

$$\begin{aligned}\mathbb{P}(\zeta_1=\chi_1 \mid \mathcal{F}_0) &= \mathbb{P}\bigg(r_{\chi_1}(0) < \min_{i\neq\chi_1}\{r_i(0)\}\bigg|\mathcal{F}_0\bigg)\\ &= \frac{\mu_{\chi_1}F(\theta_{\chi_1})}{\sum_{i\in A}\mu_i F(\theta_i)}.\end{aligned}$$

In our system, the first arm i that generates reward has probability $\mu_i \cdot \lambda_i(t) = \frac{\mu_i F(\theta_i)}{\sum_{j\in A}F(\theta_j)}$ to generate the first reward every time step before it does. Thus in the sequence $\{\chi_j\}$ the first element is arm i with probability $\frac{\mu_i F(\theta_i)}{\sum_{j\in A}\mu_j F(\theta_j)}$, which is equivalent to $\mathbb{P}(\zeta_1=\chi_1 \mid \mathcal{F}_0)$.
Following the technique in Davis (1990), we consider a representative case, by calculating the probability that the forth element in sequence $\{\zeta_j\}$ is arm i given $\mathcal{F}_3 = \{$the first three elements are arms $i,j,k\} = \{\zeta_1\in B_i, \zeta_2\in B_j, \zeta_3 \in B_k\}$, as shown in Figure 4. On $\mathcal{F}_3$, the distance d_k from ζ_3 to the smallest element in B_k that greater than ζ_3 is $r_k(1)$, which is an exponential random variable conditioned on $\mathcal{F}_3$. On $\mathcal{F}_3$, the distance d_j from ζ_3 to the smallest element in B_j that greater than ζ_3 is $r_j(1)+r_j(0)-r_k(0)$, noting $\mathcal{F}_3=\{r_j(0)<r_k(0)<r_j(0)+r_j(1)\}$, by the memoryless property of $r_j(1)$ we get that d_j has the distribution of $r_j(1)$ conditioned on $\mathcal{F}_3$. Similarly, we get that d_i has the exponential distribution of $r_i(1)$ conditioned on $\mathcal{F}_3$. The independence of the random variables $\{r_i(n), r_j(n), r_k(n), n\geq 0\}$ guarantees that d_i, d_j, d_k are conditionally independent given $\mathcal{F}_3$. Thus

$$\begin{aligned}\mathbb{P}(\zeta_4=i \mid \mathcal{F}_3) &= \mathbb{P}\big(d_i<\min\{d_j,d_k\} \mid \mathcal{F}_3\big)\\ &= \mathbb{P}\big(r_i(1)<\min\{r_j(1),r_k(1)\} \mid \mathcal{F}_3\big)\\ &= \frac{\mu_i F(1+\theta_i)}{\sum_{l=i,j,k}\mu_l F(1+\theta_l)}.\end{aligned}$$

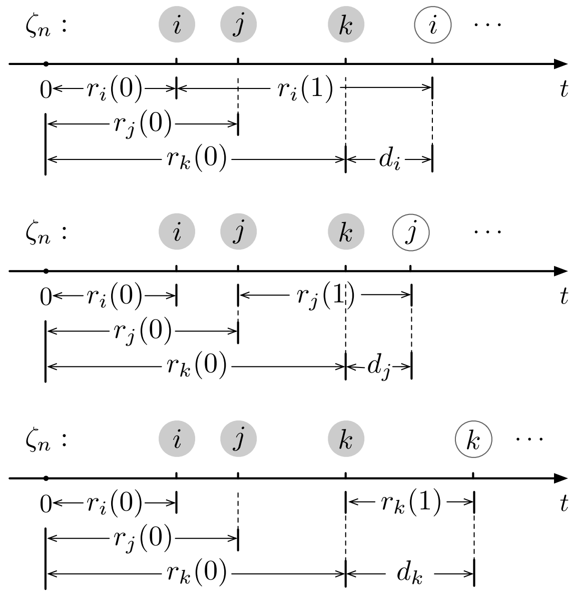

Figure 4: An example of sequences $\{\zeta_n\}$ with $\{\zeta_n\} = \{i, j, k, i, \ldots\}$, $\{\zeta_n\} = \{i, j, k, j, \ldots\}$, and $\{\zeta_n\} = \{i, j, k, k, \ldots\}$, respectively.

This is the same as the relevant conditional probability $\mathbb{P}(\chi_4 = i \mid \mathcal{F}_3)$. More generally, for any case with $m \geq 3$, on $\mathcal{F}_3$ we have:

$$\begin{aligned}\mathbb{P}(\zeta_4 = \chi_4 \mid \mathcal{F}_3) &= \mathbb{P}\big(d_{\chi_4} < \min\{d_i, d_j, d_k, \cdots\} \mid \mathcal{F}_3\big)\\ &= \mathbb{P}\big(r_{\chi_4}(0) < \min\{r_i(1), r_j(1), \cdots\} \mid \mathcal{F}_3\big)\\ &= \frac{\mu_i F(\theta_{\chi_4})}{\sum_{l=i,j,k} \mu_l F(1+\theta_l) + \sum_{\substack{a \neq i,j,k \\ a \in A}} \mu_a F(\theta_a)}.\end{aligned}$$

This is also the same as the conditional probability $\mathbb{P}(\chi_4 \mid \mathcal{F}_3)$.

B Proof of Theorem 2

Theorem 2. (At-Least-n Greedy) *Given a sensitivity function $G(\cdot)$ and a fixed payment b satisfying $G(b) > 1$, if the feedback function satisfies $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ for $\alpha > 1$, the expected regret $\mathbb{E}[R_T]$ of the At-Least-n Greedy policy is upper bounded by $\mathbb{E}[R_T] \leq \sum_{a \in A} \frac{2 \max_{i \in A} \Delta_i + [G(b)-1]\Delta_a}{[G(b)-1]\mu_a} \times q \log T + O(1)$, with the expected accumulative payment $\mathbb{E}[B_T]$ upper bounded by $\mathbb{E}[B_T] \leq \sum_{a \in A} \frac{2[G(b)+1]}{[G(b)-1]\mu_a} \times q \log T$.*

By the law of total expectation, the expected regret up to T is as follows:

$$\begin{aligned}\mathbb{E}[R_T] &= \mathbb{E}[R_T \mid \hat{a}^* = a^*]\mathbb{P}(\hat{a}^* = a^*) + \mathbb{E}[R_T \mid \hat{a}^* \neq a^*]\mathbb{P}(\hat{a}^* \neq a^*)\\ &\leq \mathbb{E}[R_T \mid \hat{a}^* = a^*] + T \cdot \mathbb{P}(\hat{a}^* \neq a^*).\end{aligned}$$

We want to bound both $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$ and $\mathbb{P}(\hat{a}^* \neq a^*)$ to get the regret bound. First we analyze the upper bound of the part $\mathbb{P}(\hat{a}^* \neq a^*)$. For each arm $a \in A$ we define $\Delta_a = \mu_{a^*} - \mu_a$. We start with the following lemma.

Lemma 5. *For each arm $a \neq a^*$, there exists a constant $\epsilon_a > 0$ independent of n such that the following hold:*

$$\mathbb{P}\left(\hat{\mu}_a(\tau_n) > \mu_a + \frac{\Delta_a}{2}\right) \leq 2e^{-2\epsilon_a n},$$

and

$$\mathbb{P}\left(\hat{\mu}_{a^*}(\tau_n) < \mu_{a^*} - \frac{\Delta_a}{2}\right) \leq 2e^{-2\epsilon_a n}.$$

Let arm $a = \arg\max_{i \in A, i \neq a^*} \hat{\mu}_i(\tau_n)$ denote the arm with largest sample mean and not equal to arm a^* at time step τ_n. We have:

$$\begin{aligned}
\mathbb{P}(\hat{a}^* \neq a^*) &\leq \mathbb{P}\left(\hat{\mu}_a(\tau_n) \geq \hat{\mu}_{a^*}(\tau_n)\right) \\
&\leq \mathbb{P}\left(\hat{\mu}_a(\tau_n) \geq \mu_a + \frac{\Delta_a}{2}\right) + \mathbb{P}\left(\hat{\mu}_{a^*}(\tau_n) \leq \mu_{a^*} - \frac{\Delta_a}{2}\right) \\
&\leq 4e^{-n\frac{\Delta_a^2}{2}}.
\end{aligned}$$

Recall that in the policy we define $n = q \log T$ for some $q > 0$. It then follows that $\mathbb{P}(\hat{a}^* \neq a^*) = O(\frac{1}{T})$.

Next, we analyze the upper bound of the part $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$. Let Γ_t denote the accumulative reward up to time step t. Then, we have:

$$\begin{aligned}
\mathbb{E}[R_T \mid \hat{a}^* = a^*] &= \mathbb{E}[\Gamma_T^*] - \mathbb{E}[\Gamma_T \mid \hat{a}^* = a^*] \\
&= \mu_{a^*} \cdot T - \mathbb{E}[\Gamma_T \mid \hat{a}^* = a^*] \\
&= \mu_{a^*} T - \big(\mathbb{E}[\Gamma_{\tau_s} \mid \hat{a}^* = a^*] + \mathbb{E}[\Gamma_T - \Gamma_{\tau_s} \mid \hat{a}^* = a^*]\big).
\end{aligned} \tag{2}$$

During the exploration phase, since each arm generates rewards at least n times, we obtain:

$$\begin{aligned}
\mathbb{E}[\Gamma_{\tau_n} \mid \tau_n] &= \mathbb{E}\left[\sum_{i \in A} \big(n + (S_i(\tau_n) - n)\big)\right] \\
&= m \cdot n + \mathbb{E}\left[\sum_{i \in A} \big(T_i(\tau_n) \cdot \mu_i - n\big)\right] \\
&= m \cdot n + \sum_{i \in A} \mu_i \left(\mathbb{E}[T_i(\tau_n)] - \frac{n}{\mu_i}\right) \\
&\geq m \cdot n + \big(\min_{j \in A} \mu_j\big) \sum_{i \in A} \left(\mathbb{E}[T_i(\tau_n)] - \frac{n}{\mu_i}\right) \\
&= m \cdot n + \left(\tau_n \cdot \min_{j \in A} \mu_j - \big(\min_{j \in A} \mu_j\big) \sum_{i \in A} \frac{n}{\mu_i}\right) \\
&= \tau_n \cdot \min_{j \in A} \mu_j + n \cdot \sum_{i \in A} \frac{\mu_i - \min_{j \in A} \mu_j}{\mu_i}.
\end{aligned} \tag{3}$$

During the exploitation phase, the service provider offers payment to users pulling arm $\hat{a}^*$, so using the bound in (3) we obtain:

$$
\begin{aligned}
&\mathbb{E}[\Gamma_{\tau_s} \mid \hat{a}^* = a^*, \tau_n, \tau_s] \\
&= \mathbb{E}[\Gamma_{\tau_n} \mid \tau_n] + \sum_{t=\tau_n+1}^{\tau_s} \mathbb{E}\Big[\frac{\lambda_{a^*}(t) + G(b)}{1+G(b)} \cdot \mu_{a^*} + \sum_{i\in A} \frac{\lambda_i(t)}{1+G(b)} \cdot \mu_i\Big] \\
&\geq \mathbb{E}[\Gamma_{\tau_n} \mid \tau_n] + \sum_{t=\tau_n+1}^{\tau_s} \mathbb{E}\Big[\frac{\lambda_{a^*}(t) + G(b)}{1+G(b)} \cdot \mu_{a^*} + \frac{(1-\lambda_{a^*}(t))}{1+G(b)} \cdot \min_{i\in A} \mu_i\Big] \\
&= \mathbb{E}[\Gamma_{\tau_n} \mid \tau_n] + \sum_{t=\tau_n+1}^{\tau_s} \mathbb{E}\Big[\frac{G(b)}{1+G(b)} \cdot \mu_{a^*} + \frac{\min_{i\in A} \mu_i}{1+G(b)} + \frac{\lambda_{a^*}(t) \max_{i\in A} \Delta_i}{1+G(b)}\Big] \\
&\geq \mathbb{E}[\Gamma_{\tau_n} \mid \tau_n] + \frac{\mu_{a^*}(\tau_s - \tau_n)G(b)}{1+G(b)} + \frac{(\tau_s-\tau_n)\min_{j\in A}\mu_j}{1+G(b)} \\
&\geq \tau_n \cdot \min_{j\in A} \mu_j + n \cdot \sum_{i\in A} \frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} + \frac{\mu_{a^*}(\tau_s - \tau_n)G(b)}{1+G(b)} + \frac{(\tau_s-\tau_n)\min_{j\in A}\mu_j}{1+G(b)} \qquad (4) \\
&= n\sum_{i\in A} \frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} + \frac{\mu_{a^*}G(b) + \min_{i\in A}\mu_i}{1+G(b)}\tau_s + \tau_n \cdot \min_{j\in A}\mu_j - \frac{\mu_{a^*}G(b) + \min_{j\in A}\mu_j}{1+G(b)}\tau_n \\
&= n\sum_{i\in A} \frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} + \frac{\mu_{a^*}G(b) + \min_{i\in A}\mu_i}{1+G(b)}\tau_s - \frac{\max_{j\in A}\Delta_j G(b)}{1+G(b)}\tau_n, \qquad (5)
\end{aligned}
$$

where (4) is obtained by replacing $\mathbb{E}[\Gamma_{\tau_n} \mid \tau_n]$ using (3). Then replacing (2) using (5) and taking expectation with respect to τ_n and τ_s, we obtain:

$$
\begin{aligned}
&\mathbb{E}[R_T \mid \hat{a}^* = a^*] \\
&\leq \mu_{a^*}T - \frac{\mu_{a^*}G(b) + \min_{i\in A}\mu_i}{1+G(b)}\mathbb{E}[\tau_s] + \frac{\max_{i\in A}\Delta_i G(b)}{1+G(b)}\mathbb{E}[\tau_n] - n\sum_{i\in A}\frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} \\
&\quad - \mathbb{E}[\Gamma_T - \Gamma_{\tau_s} \mid \hat{a}^* = a^*] \\
&= \mu_{a^*}\mathbb{E}[\tau_s] - \frac{\mu_{a^*}G(b) + \min_{i\in A}\mu_i}{1+G(b)}\mathbb{E}[\tau_s] + \frac{\max_{i\in A}\Delta_i G(b)}{1+G(b)}\mathbb{E}[\tau_n] - n\sum_{i\in A}\frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} \\
&\quad + \mu_{a^*}\big(T - \mathbb{E}[\tau_s]\big) - \mathbb{E}[\Gamma_T - \Gamma_{\tau_s} \mid \hat{a}^* = a^*] \\
&= \frac{\max_{i\in A}\Delta_i}{1+G(b)}\mathbb{E}[\tau_s] + \frac{\max_{i\in A}\Delta_i G(b)}{1+G(b)}\mathbb{E}[\tau_n] - n\sum_{i\in A}\frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} + \mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]. \qquad (6)
\end{aligned}
$$

Since $\frac{\max_{i\in A}\Delta_i}{1+G(b)}$, $\frac{\max_{i\in A}\Delta_i G(b)}{1+G(b)}$, and $n\sum_{i\in A}\frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i}$ are constants, we want to bound $\mathbb{E}[\tau_n]$, $\mathbb{E}[\tau_s]$ and $\mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]$.

First, we bound $\mathbb{E}[\tau_n]$. During the exploration phase at time step t, the service provider offers payment b to the user pulling arm i. The probability that the arm i generates reward is $\frac{\lambda_i(t)+G(b)}{1+G(b)} \cdot \mu_i > \frac{G(b)\mu_i}{1+G(b)}$. Thus, the number of attempts for arm i to generate a unit reward is a geometric random variable with parameter larger than $\frac{G(b)\mu_i}{1+G(b)}$. By the policy during the exploration phase, each arm generates at least n accumulative reward, then we obtain

$$\mathbb{E}[\tau_n] \leq n \cdot \sum_{i\in A} \frac{1+G(b)}{G(b)\mu_i} = O(n) = O(\log T).$$

Next, we consider $\mathbb{E}[\tau_s]$. Recall that in the policy we define $\tau_s = \tau_n + \delta\tau_n$, where $\delta = \frac{1+G(b)}{G(b)-1}$ is a constant. Thus $\mathbb{E}[\tau_s] = (1 + \frac{G(b)+1}{G(b)-1})\mathbb{E}[\tau_n]$. Then, the evaluation of $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$ boils down to $\mathbb{E}[\tau_n]$ and $\mathbb{E}[\tau_s]$. We obtain from (6) that

$$
\begin{aligned}
&\mathbb{E}[R_T \mid \hat{a}^* = a^*] \\
\le\ & n \sum_{i\in A} \frac{\max_{i\in A}\Delta_i}{1+G(b)} \cdot (1 + \frac{G(b)+1}{G(b)-1}) \cdot \frac{1+G(b)}{G(b)\mu_i} + n\sum_{i\in A} \frac{\max_{i\in A}\Delta_i G(b)}{1+G(b)} \cdot \frac{1+G(b)}{G(b)\mu_i} \\
&- n\sum_{i\in A} \frac{\mu_i - \min_{j\in A}\mu_j}{\mu_i} + \mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*] \\
\le\ & n\sum_{a\in A} \frac{2\max_{i\in A}\Delta_i + (G(b)-1)\Delta_a}{(G(b)-1)\mu_a} + \mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*].
\end{aligned}
$$

The expected accumulative payment $\mathbb{E}[B_T]$ can also be upper bounded by

$$
\mathbb{E}[B_T] = \mathbb{E}[\tau_s] \le \sum_{a\in A} \frac{2(G(b)+1)}{(G(b)-1)\mu_a} \cdot q\log T = O(\log T).
$$

We then want to show that this choice of τ_s is large enough to make the best empirical arm dominate. That is, the best empirical arm dominates when $S_{\hat{a}^*}(\tau_s) > \sum_{a\ne\hat{a}^*} S_a(\tau_s)$. Since $\lambda_{\hat{a}^*}(t) > 0$ and $\lambda_a(t) < 1, \forall a \ne \hat{a}^*$ for all $t \le \tau_s$, we have $\mathbb{E}[S_{\hat{a}^*}(\tau_s)] > (\tau_s - \tau_n)\frac{G(b)}{1+G(b)}$ and $\sum_{a\ne\hat{a}^*} \mathbb{E}[S_a(\tau_s)] < \tau_n + (\tau_s - \tau_n)\frac{1}{1+G(b)}$. The best empirical arm $\hat{a}^*$ is expected to dominate at time step τ_s if

$$
(\tau_s - \tau_n)\frac{G(b)}{1+G(b)} \ge \tau_n + (\tau_s - \tau_n)\frac{1}{1+G(b)},
$$

by some rearrangements we obtain $\tau_s \ge \tau_n + \frac{G(b)+1}{G(b)-1}\tau_n$.

Next, for simplicity, we consider a system with $A = \{1,2\}$, where $\mu_1 > \mu_2$ and $\theta_1, \theta_2 > 0$. The idea of the policy is that the service provider keeps offering payment b to the users pulling arm 1 to help accumulate reward from arm 1 and keep the arm in the leading side, i.e., arm 1 generates at least half of accumulative reward, until time step τ_s when arm 1 dominates and has an overwhelming chance to be the only arm that can generate rewards after monopoly happens. This phenomenon is formulated as follows: suppose at time step τ_s, $S_1(\tau_s) + S_2(\tau_s) = n_0$, and $S_2(\tau_s) = u_0 n_0$ with $0 < u_0 < \frac{1}{2}$ and $u_0 n_0 \gg \theta_1, \theta_2$. We estimate the probability of event $D(u,n)$, where at some time step $t' > \tau_s$ we have $S_1(t') + S_2(t') = n > n_0$ and $S_2(t') \ge un$ with $0 < u_0 < u < \frac{1}{2}$. We want to show that $\mathbb{P}(D(u,n))$ is very small, and with $u_0 n_0$ getting larger, $\mathbb{P}(D(u,n))$ is getting exponentially smaller. By leveraging the improved exponential embedding method, $D(u,n)$ can be expressed as:

$$
D(u,n) = \Bigg(\sum_{i=u_0 n_0}^{un-1} r_2(i) < \sum_{i=n_0-u_0 n_0}^{n-un-1} r_1(i) \Bigg).
$$

Lemma 6. *Suppose at time step τ_s there are n_0 accumulative reward with $u_0 n_0, 0 < u_0 < \frac{1}{2}$ generated by arm 2. Then there exists a constant $\gamma > 0$ such that for any $u_0 < u < \frac{1}{2}$ and all large enough n_0, it holds that:*

$$
\mathbb{P}\Bigg(\exists n > n_0, D(u,n)\Bigg) \le e^{-(u_0 n_0)^\gamma}.
$$

By the above lemma, with $u_0 n_0 = O(\tau_n) = O(\log T)$, we get $\mathbb{P}\big(D(u,n)\big) = O(e^{-(\log T)^\gamma})$. This idea can be extended to the case with arm number $m \geq 2$, by viewing the sum of accumulative reward generated from all sub-optimal arms as the accumulative reward generated from a single "super arm."

Next, we bound the last part $\mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]$. Note that the regret comes from pullings of sub-optimal arms, and the expected number of attempt for each arm to get a unit reward is $O(1)$ since $\mu_i > 0, i \in A$. Let n_0 denote the accumulative reward from all arms at time step τ_s with $u_0 n_0, 0 < u_0 < \frac{1}{2}$ rewards generated by sub-optimal arms. Then, by Lemma 6 for the unit reward generated right after τ_s, it is generated by sub-optimal arms with probability smaller than or equal to $e^{-(u_0 n_0)^\gamma}$, and when a unit reward is generated by sub-optimal arms, the next unit reward is also generated by sub-optimal arms with probability smaller than or equal to $e^{-(u_0 n_0 + 1)^\gamma}$. Thus, for any $p > 1$, we can upper bound the expected regret $\mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*]$ by

$$\begin{aligned}
\mathbb{E}[R_T - R_{\tau_s} \mid \hat{a}^* = a^*] &\leq e^{-(u_0 n_0)^\gamma} + e^{-(u_0 n_0 + 1)^\gamma} + \cdots \\
&\leq (u_0 n_0)^{-p} + (u_0 n_0 + 1)^{-p} + \cdots \qquad (7) \\
&= \sum_{k=u_0 n_0}^{\infty} \frac{1}{k^p} \\
&= O(1),
\end{aligned}$$

where (7) is obtained because e^{x^γ} grows faster than x^p for any $p > 1$. Now we get the expected regret up to time step T as $\mathbb{E}[R_T] = O(\log T)$, this completes the proof.

B.1 Proof of Lemma 5

Fact 3 (Chernoff-Hoeffding bound)**.** *Let $Z_1, \cdots, Z_n$ be independent bounded random variables with $Z_i \in [a,b]$ for all i, where $-\infty < a \leq b < \infty$. Then for all $s \geq 0$*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(Z_i - \mathbb{E}[Z_i])\right| \geq s\right) \leq \exp\left(-\frac{2ns^2}{(b-a)^2}\right).$$

Let sequences $\{X_i(t)\}$ denote the Bernoulli reward with support $\{0,1\}$ generated by arm $i \neq a^*$ at time step t. Thus, for each time step t, $X_i(t)$ is i.i.d. random variable and $\mathbb{E}[X_i(t)] = \mu_i$. At time step τ_n, by the policy each arm has at least n accumulative reward. Since $S_i(\tau_n)$ is the accumulative reward generated by arm i at time step τ_n we have $S_i(\tau_n) \geq n$. By Chernoff-Hoeffding bound, at time step τ_n for arm i we get the following:

$$\mathbb{P}\left(\hat{\mu}_i(\tau_n) > \mu_i + \frac{\Delta_i}{2}\right) \leq 2e^{-2T_i(\tau_n)(\frac{\Delta_i}{2})^2} \leq 2e^{-2S_i(\tau_n)(\frac{\Delta_i}{2})^2} \leq 2e^{-2n(\frac{\Delta_i}{2})^2}.$$

The proof for arm a^* also follows from similar arguments and thus is omitted for brevity.

B.2 Proof of Lemma 6

Suppose at some time step t there are n accumulative reward from both arms. Recall that for arm $i \in A$, $\sum_{j=n}^{\infty} r_i(j) < \infty$ and $\mathbb{E}\big[\sum_{j=n}^{\infty} r_i(j)\big] = \sum_{j=n}^{\infty} \frac{1}{\mu_i F(j+\theta_i)}$ converges. To prove Lemma 6, we want to leverage the following lemma

Lemma 7. *There exists a constant $C \in \mathbb{R}^+$ such that for all large enough n,*

$$\mathbb{P}\left(\left|\frac{\sum_{j=n}^{\infty} r_i(j)}{\mathbb{E}\big[\sum_{j=n}^{\infty} r_i(j)\big]} - 1\right| > n^{-\frac{1}{4}}\right) \leq Ce^{-n^{\frac{1}{4}}}, i \in A.$$

Given a constant t, define an event E_{n_0} where the following conditions hold simultaneously.

$$\left|\frac{\sum_{j=u_0n_0}^{\infty} r_2(j)}{\mathbb{E}\big[\sum_{j=u_0n_0}^{\infty} r_2(j)\big]} - 1\right| \le (u_0n_0)^{-\frac{1}{4}}, \tag{8}$$

$$\forall n > n_0, \left|\frac{\sum_{j=un}^{\infty} r_2(j)}{\mathbb{E}\big[\sum_{j=un}^{\infty} r_2(j)\big]} - 1\right| \le (un)^{-\frac{1}{4}}, \tag{9}$$

$$\left|\frac{\sum_{j=(1-u_0)n_0}^{\infty} r_1(j)}{\mathbb{E}\big[\sum_{j=(1-u_0)n_0}^{\infty} r_1(j)\big]} - 1\right| \le \big((1-u_0)n_0\big)^{-\frac{1}{4}}, \tag{10}$$

$$\forall n > n_0, \left|\frac{\sum_{j=(1-u)n}^{\infty} r_1(j)}{\mathbb{E}\big[\sum_{j=(1-u)n}^{\infty} r_1(j)\big]} - 1\right| \le \big((1-u)n\big)^{-\frac{1}{4}}. \tag{11}$$

By Lemma 7, we obtain the probability of event E_{n_0} as follows, there exists a constant C and $\gamma > 0$ such that

$$\mathbb{P}(E_{n_0}) \ge 1 - 2Ce^{-(u_0n_0)^{\frac{1}{4}}} - \sum_{n>n_0} 2Ce^{-(u_0n)^{\frac{1}{4}}} \ge 1 - e^{-(u_0n_0)^{\gamma}}.$$

If we show that for all large enough u_0n_0, $E_{n_0} \cap D(u,n) = 0$, then the proof is finished since it implies

$$\mathbb{P}\Big(\exists n > n_0, D(u,n)\Big) \le \mathbb{P}(E_{n_0}^c) \le e^{-(u_0n_0)^{\gamma}}.$$

We consider the definition of event $D(u,n)$. By (8)–(11), we obtain

$$\begin{aligned}\sum_{i=u_0n_0}^{un-1} r_2(i) &= \sum_{i=u_0n_0}^{\infty} r_2(i) - \sum_{i=un}^{\infty} r_2(i) \\ &\ge \big(1+o(1)\big) \sum_{i=u_0n_0}^{\infty} \frac{1}{\mu_2 F(i+\theta_2)} - \big(1+o(1)\big) \sum_{i=un}^{\infty} \frac{1}{\mu_2 F(i+\theta_2)},\end{aligned}$$

and similarly,

$$\sum_{i=n_0-u_0n_0}^{n-un-1} r_1(i) \le \big(1+o(1)\big) \sum_{i=(1-u_0)n_0}^{\infty} \frac{1}{\mu_1 F(i+\theta_1)} - \big(1+o(1)\big) \sum_{i=(1-u)n}^{\infty} \frac{1}{\mu_1 F(i+\theta_1)}.$$

Thus $E_{n_0} \cap D(u,n) \ne 0$ implies

$$\begin{aligned}&\big(1+o(1)\big) \sum_{i=u_0n_0}^{\infty} \frac{1}{\mu_2 F(i+\theta_2)} - \big(1+o(1)\big) \sum_{i=un}^{\infty} \frac{1}{\mu_2 F(i+\theta_2)} \\ &< \big(1+o(1)\big) \sum_{i=(1-u_0)n_0}^{\infty} \frac{1}{\mu_1 F(i+\theta_1)} - \big(1+o(1)\big) \sum_{i=(1-u)n}^{\infty} \frac{1}{\mu_1 F(i+\theta_1)},\end{aligned}$$

which implies

$$\sum_{i=u_0n_0}^{(1-u_0)n_0} \frac{1}{\mu_1 F(i+\theta_1)} < \big(1+o(1)\big) \sum_{i=un}^{(1-u)n} \frac{1}{\mu_1 F(i+\theta_1)}. \tag{12}$$

We want to show that (12) cannot hold as u_0n_0 goes large, which implies $E_{n_0} \cap D(u,n) = 0$. Since $F(x) = \Omega(x^\alpha)$, there exists $k > 0$ such that

$$\sum_{i=un}^{(1-u)n} \frac{1}{\mu_1 F(i+\theta_1)} \leq k\left(\frac{n_0}{n}\right)^\alpha \sum_{i=un}^{(1-u)n} \frac{1}{\mu_1 F(\frac{n_0}{n}i + \frac{n_o}{n}\theta_1)}$$
$$= k\left(\frac{n_0}{n}\right)^\alpha \sum_{i=un_0}^{(1-u)n_0} \frac{1}{\mu_1 F(i+\theta_1)},$$

and notice that $[un_0, (1-u)n_0] \subset [u_0n_0, (1-u_0)n_0]$, therefore there exists a constant $d \in (0,1)$ such that

$$\sum_{i=un}^{(1-u)n} \frac{1}{\mu_1 F(i+\theta_1)} \leq dk\left(\frac{n_0}{n}\right)^\alpha \sum_{i=u_0n_0}^{(1-u_0)n_0} \frac{1}{\mu_1 F(i+\theta_1)},$$

which conflicts with (12) since $o(1)$ goes to 0 as u_0n_0 goes to infinity, and this completes the proof.

B.3 Proof of Lemma 7

Let $R_n = \sum_{j=n}^{\infty} r_i(j)$, $h(j) = \mu_i F(j+\theta_i)$, $Z_n = \sum_{j=n}^{\infty} \frac{1}{h(j)^2}$. We first show that for any $t \in \mathbb{R}^+$, we have

$$\mathbb{P}(R_n - \mathbb{E}[R_n] > t\sqrt{Z_n}) \leq e^{-t}, \tag{13}$$

and

$$\mathbb{P}(R_n - \mathbb{E}[R_n] < -t\sqrt{Z_n}) \leq e^{-t}. \tag{14}$$

We only prove the first inequality and the proof of the second one is similar. Given a constant s, we have:

$$\begin{aligned}
\mathbb{P}(R_n - \mathbb{E}[R_n] > t\sqrt{Z_n}) &\overset{(1)}{=} \mathbb{P}\left(e^{s(R_n - \mathbb{E}[R_n])} > e^{st\sqrt{Z_n}}\right) \\
&\overset{(2)}{\leq} e^{-st\sqrt{Z_n}} \mathbb{E}\left[e^{s\sum_{j\geq n}(r_i(j) - \frac{1}{h(j)})}\right] \\
&= e^{-st\sqrt{Z_n}} \prod_{j\geq n} \mathbb{E}\left[e^{s(r_i(j) - \frac{1}{h(j)})}\right] \\
&\overset{(3)}{=} e^{-st\sqrt{Z_n}} \prod_{j\geq n} \frac{e^{-\frac{s}{h(j)}}}{1 - \frac{s}{h(j)}} \\
&= e^{-st\sqrt{Z_n}} \prod_{j\geq n} e^{\frac{-s}{h(j)}} \left[1 + \frac{s}{h(j)} + \frac{\frac{s^2}{h(j)^2}}{1 - \frac{s}{h(j)}}\right] \\
&\overset{(4)}{\leq} e^{-st\sqrt{Z_n}} \prod_{j\geq n} e^{\frac{2s^2}{h(j)^2}} \\
&\leq \exp(2s^2 Z_n - st\sqrt{Z_n}),
\end{aligned} \tag{15}$$

where (1) follows from multiplying both sides by a variable s and exponentiate both sides, (2) follows from Markov's inequality, (3) is because given random variable $X \sim Exp(\lambda)$, $\mathbb{E}[e^{aX}] = \frac{1}{1-\frac{a}{\lambda}}, a < \lambda$, and (4) follows from $e^x \geq 1 + x$. We set $s = \frac{1}{\sqrt{Z_n}}$, which is accessible since there exists n such that $\frac{1}{\sqrt{Z_n}} \leq \frac{h(n)}{2}$. Thus by (15), we obtain $\mathbb{P}(R_n - \mathbb{E}[R_n] > t\sqrt{Z_n}) \leq e^{-t}$. Next, we leverage Lemma 1 in Oliveira (2009) as our Lemma 8 to determine the relation between $\mathbb{E}[R_n]$ and $\sqrt{Z_n}$.

Lemma 8 (Oliveira (2004), Lemma 1)**.** *Define a feedback function $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ where $\alpha > 1$. Define the quantity*

$$S_r(n) = \sum_{j=n}^{\infty} \frac{1}{F(n)^r}, r \in \mathbb{R}^+, n \in \mathbb{N}.$$

Then for all $r \geq 1$, $S_r(n)$ converges and as $n \to +\infty$

$$S_r(n) \sim \frac{n}{(r\alpha - 1)F(n)^r}.$$

By leveraging Lemma 8, we obtain $\sqrt{S_2(n)} = O(n^{-\frac{1}{2}} S_1(n))$. Note that $S_1(n) = \mu_i \mathbb{E}[R_n]$ and $S_2(n) = \mu_i^2 Z_n$. Therefore, we obtain the relation between $\mathbb{E}[R_n]$ and $\sqrt{Z_n}$ as $\sqrt{Z_n} = O(n^{-\frac{1}{2}} \mathbb{E} R_n)$. Then we replace t by $n^{\frac{1}{4}}$ in both (13) and (14), and we get the inequality in Lemma 7.

C Proof of Theorem 3

Theorem 3. (UCB-List) *Given an incentive sensitivity function $G(\cdot)$ and a fixed payment b satisfying $G(b) > 1$, if the feedback function satisfies $F(x) = \Omega(x^\alpha)$ and $F(x) = O(x^\alpha \ln x)$ for $\alpha > 1$, then the expected regret $\mathbb{E}[R_T]$ of the UCB-List policy is upper bounded by $\mathbb{E}[R_T] \leq \sum_{a \neq a^*, a \in A} (\frac{4p^2}{\Delta_a} + \frac{\gamma_p \Delta_a}{G(b) - 1}) \log T + O(1) + O(T^{-3})$, with a constant γ_p dependent on the parameter p. The expected accumulative payment $\mathbb{E}[B_T]$ is upper bounded by $\mathbb{E}[B_T] \leq \frac{2\gamma_p G(b)}{G(b) - 1} \log T$.*

We start in a similar way as the proof of Theorem 2. By the law of total expectation, the expected regret up to T is as follows:

$$\begin{aligned}
\mathbb{E}[R_T] &= \mathbb{E}[R_T \mid \hat{a}^* = a^*]\mathbb{P}(\hat{a}^* = a^*) + \mathbb{E}[R_T \mid \hat{a}^* \neq a^*]\mathbb{P}(\hat{a}^* \neq a^*) \\
&\leq \mathbb{E}[R_T \mid \hat{a}^* = a^*] + T \cdot \mathbb{P}(\hat{a}^* \neq a^*).
\end{aligned}$$

We want to upper bound both $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$ and $\mathbb{P}(\hat{a}^* \neq a^*)$ to get the regret bound. We first consider $\mathbb{E}[R_T \mid \hat{a}^* = a^*]$. After decomposing, we have:

$$\begin{aligned}
\mathbb{E}[R_T \mid \hat{a}^* = a^*] &= \mathbb{E}[R_{\tau_2} \mid \hat{a}^* = a^*] + \mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*] \\
&= \mathbb{E}[R_{\tau_1}] + \mathbb{E}[R_{\tau_2} - R_{\tau_1} \mid \hat{a}^* = a^*] + \mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*]. \qquad (16)
\end{aligned}$$

Note that after initialization, i.e., let t_0 be the time step when initialization is finished, each arm a has $T_a(t_0) \geq 1$ since the number of attempts for each arm a to get a unit reward is a Geometric random variable with parameter larger than $\frac{G(b)\mu_a}{1+G(b)}$, which is independent with time. During the exploration phase, since the regret happens because of pullings of sup-optimal arms, the expected regret after t time steps can be written as

$$\sum_{a \neq a^*, a \in A} \Delta_a \mathbb{E}[T_a(t)].$$

Thus we can bound the expected regret during the exploration phase $\mathbb{E}[R_{\tau_1}]$ by bounding each $\mathbb{E}[T_a(\tau_1)]$ for $a \neq a^*$, $a \in A$. Let l be a random positive integer. Then for each arm $a \neq a^*, a \in A$, we have:

$$\begin{aligned}
\mathbb{E}[T_a(\tau_1)] &\overset{(1)}{=} O(1) + \sum_{t=t_0+1}^{\tau_1} \mathbb{P}\big(I(t) = a\big) \\
&\leq O(1) + l + \sum_{t=t_0+1}^{\tau_1} \mathbb{P}\big(I(t) = a, T_a(t-1) \geq l\big) \\
&\overset{(2)}{\leq} O(1) + l + \sum_{t=t_0+1}^{\tau_1} \Bigg\{\mathbb{P}\bigg(\hat{\mu}_{a^*}(t) + c_{a^*}(t) \leq \hat{\mu}_a(t) + c_a(t), T_a(t-1) \geq l\bigg) \\
&\quad \cdot \mathbb{E}\bigg[\hat{\lambda}_a(t)\bigg|\hat{\mu}_{a^*}(t) + c_{a^*}(t) \leq \hat{\mu}_a(t) + c_a(t)\bigg] \\
&\quad + \mathbb{E}\bigg[\hat{\lambda}_a(t)\bigg|\hat{\mu}_{a^*}(t) + c_{a^*}(t) > \hat{\mu}_a(t) + c_a(t)\bigg]\Bigg\} \\
&\leq O(1) + l + \sum_{t=t_0+1}^{\tau_1} \Bigg\{\mathbb{P}\bigg(\min_{0<s<t} \hat{\mu}_{a^*}(s) + p\sqrt{\frac{\ln T}{s}} \leq \max_{l \leq s_a < t} \hat{\mu}_a(s_a) + p\sqrt{\frac{\ln T}{s_a}}\bigg) \\
&\quad \cdot \mathbb{E}\bigg[\frac{\lambda_a(t) + G(b)}{1 + G(b)}\bigg] + \mathbb{E}\bigg[\frac{\lambda_a(t)}{1 + G(b)}\bigg]\Bigg\} \\
&\leq O(1) + l \\
&\quad + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{s_a=l}^{t-1} \mathbb{P}\bigg(\hat{\mu}_{a^*}(s) + p\sqrt{\frac{\ln T}{s}} \leq \hat{\mu}_a(s_a) + p\sqrt{\frac{\ln T}{s_a}}\bigg) + \sum_{t=1}^{\infty} \mathbb{E}\bigg[\frac{\lambda_a(t)}{1 + G(b)}\bigg],
\end{aligned}$$

where in (1) the $O(1)$ is the time period for initialization independent of t, and in (2) we consider two cases for $I(t) = a$: the user pulls arm a with and without incentive. Similar to the proof of upper bound of UCB1 (Auer et al., 2002), the part $\hat{\mu}_{a^*}(s) + p\sqrt{\frac{\ln T}{s}} \leq \hat{\mu}_a(s_a) + p\sqrt{\frac{\ln T}{s_a}}$ implies that at least one of the following inequalities holds

$$\hat{\mu}_{a^*}(s) \leq \mu_{a^*} - p\sqrt{\frac{\ln T}{s}}, \tag{17}$$

$$\hat{\mu}_a(s_a) \geq \mu_a + p\sqrt{\frac{\ln T}{s_a}}, \tag{18}$$

$$\mu_{a^*} < \mu_a + 2p\sqrt{\frac{\ln T}{s_a}}. \tag{19}$$

Eqs.(17) and (18) can be bounded by the Chernoff-Hoeffding bound as follows

$$\mathbb{P}\bigg(\hat{\mu}_{a^*}(s) \leq \mu_{a^*} - p\sqrt{\frac{\ln T}{s}}\bigg) \leq T^{-2p^2},$$

$$\mathbb{P}\bigg(\hat{\mu}_a(s_a) \geq \mu_a + p\sqrt{\frac{\ln T}{s_a}}\bigg) \leq T^{-2p^2}.$$

Let $l = \frac{4p^2 \ln T}{{\Delta_a}^2}$, then Eq.(19) is false since

$$\mu_{a^*} - \mu_a - 2p\sqrt{\frac{\ln T}{s_a}} \geq \mu_{a^*} - \mu_a - \Delta_a = 0.$$

Thus

$$\begin{aligned}\mathbb{E}[T_a(\tau_1)] &\le O(1) + l + \sum_{t=1}^{\infty}\sum_{s=1}^{t-1}\sum_{s_a=\frac{8\ln T}{\Delta_a{}^2}}^{t-1} 2T^{-2p^2} + \sum_{t=1}^{\infty}\mathbb{E}\Big[\frac{\lambda_a(t)}{1+G(b)}\Big]\\ &\le O(1) + l + \sum_{t=1}^{\infty}\sum_{s=1}^{t}\sum_{s_a=1}^{t} 2t^{-2p^2} + \sum_{t=1}^{\infty}\mathbb{E}\Big[\frac{\lambda_a(t)}{1+G(b)}\Big]\\ &= O(1) + \frac{8\ln T}{\Delta_a{}^2} + \sum_{t=1}^{\infty} 2t^{-p^2} + \sum_{t=1}^{\infty}\mathbb{E}\Big[\frac{\lambda_a(t)}{1+G(b)}\Big].\end{aligned}$$

Thus, we can bound $\mathbb{E}[T_a(\tau_1)]$ if we upper bound $\sum_{t=1}^{\infty}\mathbb{E}\big[\frac{\lambda_a(t)}{1+G(b)}\big]$. Note that this term has the condition that the payment is offered to some arm but not arm a. Let $C_{i,t} \coloneqq S_i(t-2)+\theta_i$, then at time step $t>2$ we have

$$\begin{aligned}\frac{\lambda_a(t-1)}{\lambda_a(t)} &= \lambda_a(t-1)\cdot\frac{\sum_{i\in A} F\big(S_i(t-2)+\sum_{j\in A}\mu_j\frac{\lambda_a(t-2)+G(b)}{1+G(b)}+\theta_i\big)}{F\big(1+\mu_a\lambda_a(t-2)\big)}\\ &\overset{(1)}{\ge} \lambda_a(t-1)\cdot\frac{\sum_{i\in A} F\big(S_i(t-2)+\frac{G(b)\sum_{j\in A}\mu_j}{1+G(b)}+\theta_i\big)}{F\big(1+\mu_a\lambda_a(t-2)\big)}\\ &\overset{(2)}{=} \frac{1}{\sum_{i\in A}F\big(S_i(t-2)+\theta_i\big)}\cdot\frac{\sum_{i\in A} F\big(S_i(t-2)+\frac{G(b)\sum_{j\in A}\mu_j}{1+G(b)}+\theta_i\big)}{F\big(1+\frac{\mu_a}{\sum_{i\in A}F\left(S_i(t-2)+\theta_i\right)}\big)}\\ &= \frac{1}{\sum_{i\in A}F(C_{i,t})}\cdot\frac{\sum_{i\in A}F(C_{i,t}+\frac{G(b)\sum_{j\in A}\mu_j}{1+G(b)})}{F(1+\frac{\mu_a}{\sum_{i\in A}F(C_{i,t})})},\end{aligned} \tag{20}$$

where (1) follows from $\lambda_a(t-2)\ge 0$, and (2) follows from the definition of $\lambda_a(t-1)$. When $F(x)=x^{\alpha}$ with $\alpha=1$,

$$(20) = \frac{\sum_{i\in A}C_{i,t}+\frac{mG(b)\sum_{j\in A}\mu_j}{1+G(b)}}{\sum_{i\in A}C_{i,t}+\mu_a} > 1,$$

since $m\ge 2$ and $G(b)>1$. As F increases faster and α goes larger, $\sum_{i\in A}F(C_{i,t}+\frac{G(b)\sum_{j\in A}\mu_j}{1+G(b)})$ increases faster than $\sum_{i\in A}F(C_{i,t})F(1+\frac{\mu_a}{\sum_{i\in A}F(C_{i,t})})$. Thus, for each time step $t\le\tau_1$ there exists a constant $C_t>1$ depending only on $\theta_i,\mu_i,i\in A$ and $m,G(b),t$ such that $C_t\lambda_a(t)\le\lambda_a(t-1)$. Thus, we obtain

$$\begin{aligned}\sum_{t=1}^{\infty}\mathbb{E}\Big[\frac{\lambda_a(t)}{1+G(b)}\Big] &= \frac{1}{1+G(b)}\sum_{t=1}^{\infty}\mathbb{E}[\lambda_a(t)]\\ &\le \frac{1}{1+G(b)}\Big(\frac{F(\theta_a)}{\sum_{i\in A}F(\theta_i)}+\frac{F(\theta_a)}{C_2\sum_{i\in A}F(\theta_i)}+\frac{F(\theta_a)}{C_2C_3\sum_{i\in A}F(\theta_i)}+\cdots\Big)\\ &= \frac{1}{1+G(b)}\frac{F(\theta_a)}{\sum_{i\in A}F(\theta_i)}\Big(1+\frac{1}{C_2}+\frac{1}{C_2C_3}+\frac{1}{C_2C_3C_4}+\cdots\Big),\end{aligned}$$

which converges to a constant. Since $\tau_1\le T$ we get $\mathbb{E}[T_a(\tau_1)]\le\frac{4p^2\ln T}{\Delta_a^2}+O(1)=O(\log T)$.

We next prove that given a parameter p, there exists a constant γ_p such that $\gamma_p\log T$ is sufficient to upper bound $\mathbb{E}[\tau_1]$. Let $U(t)$ denote the set of arms that can get payment at time step t. The definition of time τ_1 implies that as time $t \le \tau_1$ goes large, $\mathbb{P}(\exists a \neq a^*, a \in U(t))$ decreases toward zero. Thus, if we show that given a parameter p, there exists a constant γ_p such that when $\tau_1 = \gamma_p \log T$ we have $\mathbb{P}(\exists a \neq a^*, a \in U(\tau_1)) = O(\frac{1}{\log T})$, then the time τ_1 can be upper bounded by $O(\log T)$. Given any arm $a \neq a^*$, we have:

$$
\begin{aligned}
\mathbb{P}\big(a \in U(\tau_1)\big) &\le \mathbb{P}\Big(a \in U(\tau_1), a^* \in U(\tau_1)\Big) + \mathbb{P}\big(a^* \notin U(\tau_1)\big) \\
&\overset{(1)}{\le} \mathbb{P}\Big(\exists t \le \tau_1 : \hat{\mu}_a(t) + c_a(t) \ge \hat{\mu}_{a^*}(t) - c_{a^*}(t)\Big) + \mathbb{P}\big(a^* \notin U(\tau_1)\big) \\
&\overset{(2)}{\le} \mathbb{P}\Big(\exists t \le \tau_1 : \hat{\mu}_a(t) + c_a(t) \ge \mu_a + \frac{\Delta_a}{2}\Big) \\
&\quad + \mathbb{P}\Big(\exists t \le \tau_1 : \hat{\mu}_{a^*}(t) - c_{a^*}(t) \le \mu_{a^*} - \frac{\Delta_a}{2}\Big) + \mathbb{P}\big(a^* \notin U(\tau_1)\big),
\end{aligned}
$$

where (1) follows from the implication that if in some time step t, arms a and a^* are both in the set, then at least the upper confidence bound of arm a is larger than the lower confidence bound of arm a^*, and (2) follows from the fact that if the upper confidence bound of arm a is larger than the lower confidence bound of arm a^*, then at least one of the two front events hold. Let $\tau_1 = \gamma_p \log T$. By the Chernoff-Hoeffding bound, we obtain that there exists a constant $C_\gamma > 0$ such that

$$
\begin{aligned}
\mathbb{P}\big(a \in U(\tau_1)\big) &\le e^{-2T_a(\tau_1)\left(\frac{\Delta_a}{2} - c_a(\tau_1)\right)^2} + e^{-2T_{a^*}(\tau_1)\left(\frac{\Delta_a}{2} - c_{a^*}(\tau_1)\right)^2} + \mathbb{P}\big(a^* \notin U(\tau_1)\big) \\
&\overset{(1)}{=} (\ln T)^{-C_\gamma} + \mathbb{P}\big(a^* \notin U(\tau_1)\big),
\end{aligned}
$$

where (1) follows from the fact that $T_a(\tau_1) = O(\log T)$ for $a \neq a^*$ and $T_{a^*}(\tau_1) \le \tau_1 = \gamma_p \log T$. Then, we analyze the upper bound of the part $\mathbb{P}\big(a^* \notin U(\tau_1)\big)$, which is equivalent to $\mathbb{P}(\hat{a}^* \neq a^*)$. According to the policy, we have:

$$
\begin{aligned}
\mathbb{P}\big(a^* \notin U(\tau_1)\big) &\overset{(1)}{\le} \sum_{a \neq a^*, a \in A} \mathbb{P}\Big(\exists t \le T : \hat{\mu}_{a^*}(t) + c_{a^*}(t) < \hat{\mu}_a(t) - c_a(t)\Big) \\
&\le \mathbb{P}\Big(\exists t \le T : \hat{\mu}_{a^*}(t) + c_{a^*}(t) \le \mu_{a^*}\Big) + \sum_{\substack{a \neq a^* \\ a \in A}} \mathbb{P}\Big(\exists t \le T : \hat{\mu}_a(t) - c_a(t) \ge \mu_a\Big), \quad (21)
\end{aligned}
$$

where (1) follows from the fact that if a^* is removed from the set at time τ_1, then there exists a time step when the upper confidence bound of a^* is smaller than any lower confidence bounds of sub-optimal arms. By the Chernoff-Hoeffding bound, we obtain that there exists a constant C such that (21) $\le Ce^{-2\ln T} = O(T^{-2})$. Thus $\tau_1 = O(\log T)$ and the expected accumulative payment $\mathbb{E}[B_T]$ is upper bounded by

$$
\begin{aligned}
\mathbb{E}[B_T] &= b \cdot \mathbb{E}[\tau_2] \\
&= b\tau_1 + b \cdot \frac{G(b)+1}{G(b)-1}\big(\tau_1 - T_{\hat{a}^*}(\tau_1)\big) \\
&\le \frac{2\gamma_p G(b)}{G(b)-1} \log T.
\end{aligned}
$$

Next, we consider the part $\mathbb{E}[R_{\tau_2} - R_{\tau_1} \mid \hat{a}^* = a^*]$ in (16), (see Page 19). Recall that we defined τ_2 in the policy as $\tau_2 = \tau_1 + \frac{G(b)+1}{G(b)-1}\big(\tau_1 - T_{\hat{a}^*}(\tau_1)\big)$. During the exploitation phase, the regret comes from the pullings of any sampled sub-optimal arms. Thus, we obtain:

$$\begin{aligned}
\mathbb{E}[R_{\tau_2} - R_{\tau_1} \mid \hat{a}^* = a^*] &= \sum_{a \neq \hat{a}^*, a \in A} \Delta_a \Big(\mathbb{E}[T_a(\tau_2)] - \mathbb{E}[T_a(\tau_1)] \Big) \\
&\overset{(1)}{\leq} \sum_{a \neq \hat{a}^*, a \in A} \frac{(\tau_2 - \tau_1)\Delta_a}{G(b) + 1} \\
&= \sum_{a \neq \hat{a}^*, a \in A} \frac{\gamma_p \Delta_a}{G(b) - 1} \log T,
\end{aligned}$$

where (1) follows from the fact that during the exploitation phase, all the incentives are on the best empirical arm $\hat{a}^*$, and for $a \neq \hat{a}^*$, $\lambda_a(t) \leq 1$ for any t. The choice of τ_2 is sufficient to make the sampled best arm dominate at time step τ_2 and have overwhelming probability to stay in leading side in monopoly after τ_2. The proof is the same as that in the proof of Theorem 2. The expected regret of the last part $\mathbb{E}[R_T - R_{\tau_2} \mid \hat{a}^* = a^*] = O(1)$ and the proof is also the same as that in the proof of Theorem 2.

Now we get the expected regret up to time step T as $\mathbb{E}[R_T] = O(\log T)$ with expected accumulative payment $\mathbb{E}[B_T] = O(\log T)$, which completes the proof.