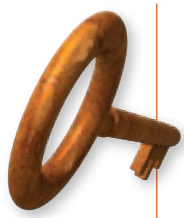
7 Natural Deduction in Propositional Logic



7.1 Rules of Implication I

PREVIEW • *Suppose you are going on a hike. The departure point and the ending point have been predetermined, as have the supplies you will take. You have been given a trail map with several alternate routes. Such a hike is similar to solving the problems in this section of the book. The departure point and supplies are like premises, the ending point is like the conclusion, and the alternate routes are like the intermediate steps linking the premises to the conclusion. In most cases the choice of routes allows for creativity.*

Natural deduction is a method for deriving the conclusion of valid arguments expressed in the symbolism of propositional logic. The method consists in using sets of **rules of inference** (valid argument forms) to derive either a conclusion directly, or a series of intermediate conclusions that links the premises of an argument with the stated conclusion. Natural deduction gets its name from the fact that it resembles the ordinary step-by-step reasoning process people use in daily life. It also resembles the method used in geometry to derive theorems relating to lines and figures; but whereas each step in a geometrical proof depends on some mathematical principle, each step in a logical proof depends on a rule of inference.

Natural deduction is similar in some respects to the truth table method for testing arguments. Both methods can be used to prove a valid argument valid; but natural deduction is largely useless for invalid arguments, so we still need truth tables. However, the method of natural deduction is more illuminating than truth tables are. Natural deduction shows how a conclusion "comes out" of the premises, whereas truth tables show nothing of the sort. Also, while truth tables are relatively automatic and mechanical, natural deduction requires insight and creativity. As a result, most students find that natural deduction is challenging and fun, while truth tables can be tedious, especially for long arguments.

The first eight rules of inference are called ***rules of implication*** because they consist of simple, valid argument forms whose premises *imply* their conclusions. The first four rules of implication should be familiar from Section 6.6. They are listed together with an illustration of their use as follows.

1. *Modus ponens* (MP)

$p \supset q$	If Su Lin is a panda, then Su Lin is cute.
p	Su Lin is a panda.
q	Su Lin is cute.

2. *Modus tollens* (MT)

$p \supset q$	If Koko is a koala, then Koko is cuddly.
$\sim q$	Koko is not cuddly.
$\sim p$	Koko is not a koala.

3. Pure hypothetical syllogism (HS)

$p \supset q$	If Leo is a lion, then Leo roars.
$q \supset r$	If Leo roars, then Leo is fierce.
$p \supset r$	If Leo is a lion, then Leo is fierce.

4. Disjunctive syllogism (DS)

$p \vee q$	Scooter is either a mouse or a rat.
$\sim p$	Scooter is not a mouse.
q	Scooter is a rat.

Modus ponens says that given a conditional statement and its antecedent on lines by themselves, we can assert its consequent on a line by itself. ***Modus tollens*** says that given a conditional statement and the negation of its consequent on lines by themselves, we can assert the negation of its antecedent on a line by itself. **Pure hypothetical syllogism** ("hypothetical syllogism" for short) says that given two conditional statements on lines by themselves such that the consequent of one is identical with the antecedent of the other, we can assert on a line by itself a conditional statement whose antecedent is the antecedent of the first conditional and whose consequent is the consequent of the second conditional. Note in the rule that the two premises hook together like links of a chain.

Disjunctive syllogism says that given a disjunctive statement and the negation of the left-hand disjunct on lines by themselves, we can assert the right-hand disjunct on a line by itself. Because of the way this rule is written, only the right-hand disjunct can be asserted as the conclusion. However, once we are supplied with the commutativity rule (see Section 7.3), we will be able to switch the order of the disjuncts in the first line, and this will allow us to assert what was originally the left-hand disjunct as the conclusion. Disjunctive syllogism is otherwise called the method of elimination. The short premise eliminates one of the alternatives in the disjunctive premise, leaving the other as the conclusion.

These four rules will be sufficient to derive the conclusion of many simple arguments in propositional logic. Further, once we are supplied with all eighteen rules together with conditional proof, the resulting system will be sufficient to derive the conclusion of any valid argument in propositional logic. Conversely, since each rule is a valid argument form unto itself, any conclusion derived from their correct use results in a valid argument.

Applying the rules of inference rests on the ability to visualize more- or less-complex arrangements of simple propositions as substitution instances of the rules. For a fairly simple substitution instance of *modus ponens*, consider the following:

1. $\sim A \supset B$	$p \supset q$
2. $\sim A$	p
3. B	q

When $\sim A$ and B are mentally substituted, respectively, in place of the p and q of the *modus ponens* rule, then you should be able to see that the argument on the left is an instance of the rule. The fact that A is preceded by a tilde is irrelevant.

Here is a more complex example:

1. $(A \bullet B) \supset (C \vee D)$	$p \supset q$
2. $A \bullet B$	p
3. $C \vee D$	q

In this case, if you mentally substitute $A \bullet B$ and $C \vee D$, respectively, in place of p and q in the rule, you can see that the argument on the left is an instance of *modus ponens*. This example illustrates the fact that any pair of compound statements can be uniformly substituted in place of p and q to produce a substitution instance of the rule.

Finally, the order of the premises never makes a difference:

1. A	p
2. $A \supset (B \supset C)$	$p \supset q$
3. $B \supset C$	q

In this case, if you mentally substitute A and $B \supset C$ in place of p and q, you can see, once again, that the argument on the left is an instance of *modus ponens*. The fact that the order of the premises is reversed makes no difference.

These arguments are all instances of ***modus ponens*** (MP):

$\sim F \supset (G \equiv H)$	$(A \vee B) \supset \sim(C \bullet D)$	$K \bullet L$
$\sim F$	$A \vee B$	$(K \bullet L) \supset [(R \supset S) \bullet (T \supset U)]$
$G \equiv H$	$\sim(C \bullet D)$	$(R \supset S) \bullet (T \supset U)$

Now let us use the rules of inference to construct a proof. Such a proof consists of a sequence of propositions, each of which is either a premise or is derived from preceding propositions by application of a rule of inference and the last of which is the conclusion of the original argument. Let us begin with the following example:

> If the Astros switch leagues,then the Braves will not win the pennant.
> If the Cubs retain their manager, then the Braves will win the pennant.
> The Astros will switch leagues. Therefore, the Cubs will not retain their manager.

The first step is to symbolize the argument, numbering the premises and writing the intended conclusion to the right of the last premise, separated by a slash mark:

1. $A \supset \sim B$
2. $C \supset B$
3. A $\quad\quad$ / $\sim C$

The next step is to derive the conclusion through a series of inferences. For this step, always begin by trying to "find" the conclusion in the premises. The conclusion to be derived is $\sim C$, and we see that C appears in the antecedent of line 2. We could derive $\sim C$ from line 2 by *modus tollens* if we had $\sim B$, so now we look for $\sim B$. Turning our attention to line 1, we see that we could derive $\sim B$ by *modus ponens* if we had A, and we do have A on line 3. Thus, we have now thought through the entire proof, and we can begin to write it out. First, we derive $\sim B$ by *modus ponens* from lines 1 and 3:

1. $A \supset \sim B$
2. $C \supset B$
3. A $\quad\quad$ / $\sim C$
4. $\sim B$ $\quad\quad$ 1, 3, MP

The justification for line 4 is written to the right, directly beneath the slash mark. If you have trouble understanding how line 4 was derived, imagine substituting A and $\sim B$ in place of p and q in the *modus ponens* rule. Then you can see that lines 1, 3, and 4 constitute a substitution instance of that rule.

The final step is to derive $\sim C$ from lines 2 and 4 by *modus tollens*:

1. $A \supset \sim B$
2. $C \supset B$
3. A $\quad\quad$ / $\sim C$
4. $\sim B$ $\quad\quad$ 1, 3, MP
5. $\sim C$ $\quad\quad$ 2, 4, MT

The proof is now complete. The justification for line 5 is written directly beneath the justification for line 4.

These arguments are all instances of ***modus tollens*** (MT):

$$\frac{\begin{array}{l}(D \vee F) \supset K \\ \sim K\end{array}}{\sim(D \vee F)} \qquad \frac{\begin{array}{l}\sim G \supset \sim(M \vee N) \\ \sim\sim(M \vee N)\end{array}}{\sim\sim G} \qquad \frac{\begin{array}{l}\sim T \\ \left[(H \vee K) \bullet (L \vee N)\right] \supset T\end{array}}{\sim\left[(H \vee K) \bullet (L \vee N)\right]}$$

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The next example is already translated into symbols:

1. $A \supset B$
2. $\sim A \supset (C \vee D)$
3. $\sim B$
4. $\sim C$ / D

Once again, to derive the conclusion, always begin by trying to "find" it in the premises. The intended conclusion is D, and after inspecting the premises we find D in line 2. If we had the consequent of that line, $C \vee D$, on a line by itself, we could derive D by disjunctive syllogism if we had $\sim C$. And we do have $\sim C$ on line 4. Also, we could derive $C \vee D$ by *modus ponens* if we had $\sim A$, so now we look for $\sim A$. Turning our attention to line 1, we see that we could derive $\sim A$ by *modus tollens* if we had $\sim B$, and we do have $\sim B$ on line 3. Thus, we have now thought through the entire proof, and we can write it out:

1.	$A \supset B$	
2.	$\sim A \supset (C \vee D)$	
3.	$\sim B$	
4.	$\sim C$	/ D
5.	$\sim A$	1, 3, MT
6.	$C \vee D$	2, 5, MP
7.	D	4, 6, DS

As usual, the justification for each line is written directly beneath the slash mark preceding the intended conclusion. If you have trouble understanding line 6, imagine substituting $\sim A$ and $C \vee D$ in place of p and q in the *modus ponens* rule. Then you can see that lines 2, 5, and 6 constitute a substitution instance of that rule.

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These arguments are all instances of **pure hypothetical syllogism** (HS):

$A \supset (D \bullet F)$	$\sim M \supset (R \supset S)$	$(L \supset N) \supset [(S \vee T) \bullet K]$
$(D \bullet F) \supset \sim H$	$(C \vee K) \supset \sim M$	$(C \equiv F) \supset (L \supset N)$
$A \supset \sim H$	$(C \vee K) \supset (R \supset S)$	$(C \equiv F) \supset [(S \vee T) \bullet K]$

Here is another example.

1. $F \supset G$
2. $F \vee H$
3. $\sim G$
4. $H \supset (G \supset I)$ / $F \supset I$

The intended conclusion is $F \supset I$. When we attempt to find it in the premises, we see no such statement. This tells us that we should adopt "Plan B" and look at the main operator of the conclusion, a strategy that often gives us a clue as to how the conclusion can be built up. In this case the main operator is a horseshoe, so we ask: What rule gives us a horseshoe—that is, a conditional statement? The answer is: pure hypothetical syllogism.

Having tentatively settled on pure hypothetical syllogism for deriving the conclusion, we now look for two conditional statements, which, when combined, will yield $F \supset I$. In line 1 we find $F \supset G$ and in line 4 we find $G \supset I$, but before we can apply pure hypothetical syllogism, we must obtain $G \supset I$ on a line by itself. Examining line 4, we see that $G \supset I$ could be derived by *modus ponens*, if we had H on a line by itself, and examining line 2, we see that H could be derived by disjunctive syllogism if we had $\sim F$ on a line by itself. Turning to line 1, we see that $\sim F$ could be derived by *modus tollens* if we had ~G on a line by itself, and we do have $\sim G$ on line 3. Thus, we have now thought through the entire proof, and we can write it out:

1.	$F \supset G$	
2.	$F \vee H$	
3.	$\sim G$	
4.	$H \supset (G \supset I)$	/ $F \supset I$
5.	$\sim F$	1, 3, MT
6.	H	2, 5, DS
7.	$G \supset I$	4, 6, MP
8.	$F \supset I$	1, 7, HS

In addition to adopting Plan B, which involves looking at the conclusion's main operator for a clue about deriving the conclusion, this proof teaches another important lesson about every proof. Every line in a proof can be used multiple times. In line 5, we used line 1 in conjunction with the *modus tollens* rule, and in line 8 we used that same line again in conjunction with pure hypothetical syllogism. Any line can be used as many times as we need it.

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These arguments are all instances of **disjunctive syllogism** (DS):

$U \vee \sim(W \bullet X)$	$\sim(E \vee F)$	$\sim B \vee [(H \supset M) \bullet (S \supset T)]$
$\sim U$	$(E \vee F) \vee (N \supset K)$	$\sim\sim B$
$\sim(W \bullet X)$	$N \supset K$	$(H \supset M) \bullet (S \supset T)$

The next example is more complex:

1.	$\sim(A \bullet B) \vee [\sim(E \bullet F) \supset (C \supset D)]$	
2.	$\sim\sim(A \bullet B)$	
3.	$\sim(E \bullet F)$	
4.	$D \supset G$	/ $C \supset G$

Again, when we attempt to find the intended conclusion in the premises, we see no such statement. But then we notice that the main operator of the conclusion is a horseshoe, and we find $C \supset D$ on line 1 and $D \supset G$ on line 4. We could derive the conclusion by pure hypothetical syllogism if we could obtain $C \supset D$ on a line by itself. Examining line 1, we see that we could derive $C \supset D$ by *modus ponens* if we could obtain both $\sim(E \bullet F) \supset (C \supset D)$ and $\sim(E \bullet F)$ on lines by themselves, and we see that $\sim(E \bullet F)$ appears on line 3. Also, examining line 1, we see that we could derive $\sim(E \bullet F) \supset (C \supset D)$ by disjunctive syllogism if we had $\sim\sim(A \bullet B)$ on a line by itself, and we do have it on line 2. Thus, we can now write out the proof:

1. $\sim(A \bullet B) \vee [\sim(E \bullet F) \supset (C \supset D)]$
2. $\sim\sim(A \bullet B)$
3. $\sim(E \bullet F)$
4. $D \supset G$ / $C \supset G$
5. $\sim(E \bullet F) \supset (C \supset D)$ 1, 2, DS
6. $C \supset D$ 3, 5, MP
7. $C \supset G$ 4, 6, HS

If you have trouble seeing how lines 5 and 6 are derived, for line 5 imagine substituting $\sim(A \bullet B)$ and $\sim(E \bullet F) \supset (C \supset D)$, in place of the p and q of the disjunctive syllogism rule. Then you can see that lines 1, 2, and 5 constitute a substitution instance of that rule. For line 6, imagine substituting $\sim(E \bullet F)$ and $(C \supset D)$ in place of p and q in the *modus ponens* rule. Then you can see that lines 5, 3, and 6 constitute a substitution instance of that rule.

In applying the four rules of inference introduced in this section, we have noted that various expressions first had to be obtained on lines by themselves. If this procedure is not followed, the resulting proof will likely be invalid. For an example of an invalid application of *modus ponens*, consider the following:

1. $A \supset (B \supset C)$
2. B
3. C 1, 2, MP (invalid)

This inference is invalid because $B \supset C$ must first be obtained on a line by itself. In deriving the conclusion of an argument we always assume the premises are true. But if we assume line 1 of this proof is true, this does not entail that $B \supset C$ is true. What line 1 says is that *if* A is true, then $B \supset C$ is true. Thus, $B \supset C$ cannot be treated as a premise. We do not know if it is true or false.

Here are some additional examples of invalid inferences:

1. $A \vee B$
2. A
3. $\sim B$ 1, 2, HS (invalid—line 2 must negate A in line 1, not assert it)

1. $A \supset B$
2. B
3. A 1, 2, MP (invalid—line 2 must assert the antecedent of line 1, not the consequent)

1. $A \supset B$
2. $A \supset C$
3. $B \supset C$ 1, 2, HS (invalid—the consequent of one conditional must be identical with the antecedent of the other)

1. $A \supset B$
2. $\sim A$
3. $\sim B$ 1, 2, MT (invalid—line 2 must negate the consequent of line 1, not the antecedent)

1. $(A \supset B) \supset C$	
2. $\sim B$	
3. $\sim A$	1, 2, MT (invalid—$A \supset B$ must first be obtained on a line by itself)

1. $A \supset (B \supset C)$	
2. $C \supset D$	
3. $B \supset D$	1, 2, HS (invalid—$B \supset C$ must first be obtained on a line by itself)

1. $(A \vee B) \supset C$	
2. $\sim A$	
3. B	1, 2, DS (invalid—$A \vee B$ must first be obtained on a line by itself)

We conclude this section with some strategies for applying the first four rules of inference.

Strategy 1: Always begin by attempting to "find" the conclusion in the premises. If the conclusion is not present in its entirety in the premises, look at the main operator of the conclusion. This will provide a clue as to how the conclusion should be derived.

Strategy 2: If the conclusion contains a letter that appears in the consequent of a conditional statement in the premises, consider obtaining that letter via *modus ponens:*

1. $A \supset B$	
2. $C \vee A$	
3. A	/ B
4. B	1, 3, MP

Strategy 3: If the conclusion contains a negated letter and that letter appears in the antecedent of a conditional statement in the premises, consider obtaining the negated letter via *modus tollens:*

1. $C \supset B$	
2. $A \supset B$	
3. $\sim B$	/ $\sim A$
4. $\sim A$	2, 3, MT

Strategy 4: If the conclusion is a conditional statement, consider obtaining it via pure hypothetical syllogism:

1. $B \supset C$	
2. $C \supset A$	
3. $A \supset B$	/ $A \supset C$
4. $A \supset C$	1, 3, HS

Strategy 5: If the conclusion contains a letter that appears in a disjunctive statement in the premises, consider obtaining that letter via disjunctive syllogism:

1. $A \supset B$	
2. $A \vee C$	
3. $\sim A$	/ C
4. C	2, 3, DS

Of course, these strategies apply to deriving any line prior to the conclusion, just as they apply to deriving the conclusion.

EXERCISE 7.1

I. For each of the following lists of premises, derive the conclusion and supply the justification for it. There is only one possible answer for each problem.

★(1) 1. $G \supset F$
2. $\sim F$
3. ________ ____

(2) 1. S
2. $S \supset M$
3. ________ ____

(3) 1. $R \supset D$
2. $E \supset R$
3. ________ ____

★(4) 1. $B \vee C$
2. $\sim B$
3. ________ ____

(5) 1. N
2. $N \vee F$
3. $N \supset K$
4. ________ ____

(6) 1. $\sim J \vee P$
2. $\sim J$
3. $S \supset J$
4. ________ ____

★(7) 1. $H \supset D$
2. $F \supset T$
3. $F \supset H$
4. ________ ____

(8) 1. $S \supset W$
2. $\sim S$
3. $S \vee N$
4. ________ ____

(9) 1. $F \supset \sim A$
2. $N \supset A$
3. $\sim F$
4. $\sim A$
5. ________ ____

★(10) 1. $H \supset A$
2. A
3. $A \vee M$
4. $G \supset H$
5. ________ ____

(11) 1. $W \vee B$
2. W
3. $B \supset T$
4. $W \supset A$
5. ________ ____

(12) 1. $K \supset \sim R$
2. $\sim R$
3. $R \vee S$
4. $R \supset T$
5. ________ ____

★(13) 1. $\sim C \supset \sim F$
2. $L \supset F$
3. $\sim \sim F$
4. $F \vee \sim L$
5. ________ ____

(14) 1. $N \supset \sim E$
2. $\sim \sim S$
3. $\sim E \vee \sim S$
4. $\sim S \vee N$
5. ________ ____

(15) 1. $\sim R \supset \sim T$
2. $\sim T \vee B$
3. $C \supset \sim R$
4. $\sim C$
5. ________ ____

★(16) 1. $\sim K$
2. $\sim K \supset \sim P$
3. $\sim K \vee G$
4. $G \supset Q$
5. ________ ____

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(17) 1. $F \vee (A \supset C)$
2. $A \vee (C \supset F)$
3. A
4. $\sim F$
5. ________ ____

(18) 1. $(R \supset M) \supset D$
2. $M \supset C$
3. $D \supset (M \vee E)$
4. $\sim M$
5. ________ ____

★(19) 1. $(S \vee C) \supset L$
2. $\sim S$
3. $\sim L$
4. $S \supset (K \supset L)$
5. ________ ____

(20) 1. $(A \vee W) \supset (N \supset Q)$
2. $Q \supset G$
3. $\sim A$
4. $(Q \supset G) \supset (A \vee N)$
5. ________ ____

II. The following symbolized arguments are missing a premise. Write the premise needed to derive the conclusion (last line), and supply the justification for the conclusion. Try to construct the simplest premise needed to derive the conclusion.

★(1) 1. $B \vee K$
2. ________
3. K ____

(2) 1. $N \supset S$
2. ________
3. S ____

(3) 1. $K \supset T$
2. ________
3. $\sim K$ ____

★(4) 1. $C \supset H$
2. ________
3. $R \supset H$ ____

(5) 1. $F \supset N$
2. $N \supset T$
3. ________
4. $\sim F$ ____

(6) 1. $W \vee T$
2. $A \supset W$
3. ________
4. $A \supset T$ ____

★(7) 1. $M \supset B$
2. $Q \supset M$
3. ________
4. M ____

(8) 1. $C \vee L$
2. $L \supset T$
3. ________
4. L ____

(9) 1. $E \supset N$
2. $T \vee \sim E$
3. $S \supset E$
4. ________
5. E ____

★(10) 1. $H \supset A$
2. $S \supset H$
3. $\sim M \vee H$
4. ________
5. $\sim H$ ____

(11) 1. $T \supset N$
2. $G \supset T$
3. $H \vee T$
4. ________
5. $F \supset T$ ____

(12) 1. $G \supset C$
2. $M \vee G$
3. $T \vee \sim G$
4. ________
5. G ____

★(13) 1. $\sim S \supset \sim B$
2. $R \vee \sim B$
3. $\sim B \supset \sim S$
4. ________
5. $\sim\sim B$ ____

(14) 1. $\sim R \supset D$
2. $\sim J \supset \sim R$
3. $N \vee \sim R$
4. ________
5. $\sim F \supset \sim R$ ____

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(15) 1. $\sim S \vee \sim P$
2. $\sim K \supset P$
3. $\sim P \supset F$
4. ______
5. $\sim P$ ____

★(16) 1. $J \supset E$
2. $B \vee \sim J$
3. $\sim Z \supset J$
4. ______
5. J ____

(17) 1. $(H \supset C) \supset A$
2. $N \supset (F \supset K)$
3. $(E \bullet R) \supset K$
4. ______
5. $H \supset K$ ____

(18) 1. $(S \supset M) \supset G$
2. $S \supset (M \bullet G)$
3. $G \supset (R \supset \sim S)$
4. ______
5. $\sim S$ ____

★(19) 1. $(W \vee \sim F) \supset H$
2. $(H \vee G) \supset \sim F$
3. $T \supset (F \supset G)$
4. ______
5. $\sim F$ ____

(20) 1. $(H \bullet A) \vee T$
2. $\sim S \supset (P \supset T)$
3. $(N \vee T) \supset P$
4. ______
5. T ____

III. Use the first four rules of inference to derive the conclusions of the following symbolized arguments.

★(1) 1. $\sim C \supset (A \supset C)$
2. $\sim C$ / $\sim A$

(2) 1. $F \vee (D \supset T)$
2. $\sim F$
3. D / T

(3) 1. $(K \bullet B) \vee (L \supset E)$
2. $\sim(K \bullet B)$
3. $\sim E$ / $\sim L$

★(4) 1. $P \supset (G \supset T)$
2. $Q \supset (T \supset E)$
3. P
4. Q / $G \supset E$

(5) 1. $\sim W \supset [\sim W \supset (X \supset W)]$
2. $\sim W$ / $\sim X$

(6) 1. $J \supset (K \supset L)$
2. $L \vee J$
3. $\sim L$ / $\sim K$

★(7) 1. $\sim S \supset D$
2. $\sim S \vee (\sim D \supset K)$
3. $\sim D$ / K

(8) 1. $A \supset (E \supset \sim F)$
2. $H \vee (\sim F \supset M)$
3. A
4. $\sim H$ / $E \supset M$

(9) 1. $\sim G \supset (G \vee \sim A)$
2. $\sim A \supset (C \supset A)$
3. $\sim G$ / $\sim C$

★(10) 1. $N \supset (J \supset P)$
2. $(J \supset P) \supset (N \supset J)$
3. N / P

(11) 1. $G \supset [\sim O \supset (G \supset D)]$
2. $O \vee G$
3. $\sim O$ / D

(12) 1. $\sim M \vee (B \vee \sim T)$
2. $B \supset W$
3. $\sim\sim M$
4. $\sim W$ / $\sim T$

★(13) 1. $R \supset (G \vee \sim A)$
2. $(G \vee \sim A) \supset \sim S$
3. $G \supset S$
4. R / $\sim A$

(14) 1. $(L \equiv N) \supset C$
2. $(L \equiv N) \vee (P \supset \sim E)$
3. $\sim E \supset C$
4. $\sim C$ / $\sim P$

(15) 1. $\sim J \supset [\sim A \supset (D \supset A)]$
2. $J \vee \sim A$
3. $\sim J$ / $\sim D$

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★(16) 1. $(B \supset \sim M) \supset (T \supset \sim S)$
2. $B \supset K$
3. $K \supset \sim M$
4. $\sim S \supset N$ / $T \supset N$

(17) 1. $H \vee (Q \vee F)$
2. $R \vee (Q \supset R)$
3. $R \vee \sim H$
4. $\sim R$ / F

(18) 1. $\sim A \supset (B \supset \sim C)$
2. $\sim D \supset (\sim C \supset A)$
3. $D \vee \sim A$
4. $\sim D$ / $\sim B$

★(19) 1. $\sim G \supset [G \vee (S \supset G)]$
2. $(S \vee L) \supset \sim G$
3. $S \vee L$ / L

(20) 1. $H \supset [\sim E \supset (C \supset \sim D)]$
2. $\sim D \supset E$
3. $E \vee H$
4. $\sim E$ / $\sim C$

(21) 1. $\sim B \supset [(A \supset K) \supset (B \vee \sim K)]$
2. $\sim J \supset K$
3. $A \supset \sim J$
4. $\sim B$ / $\sim A$

★(22) 1. $(C \supset M) \supset (N \supset P)$
2. $(C \supset N) \supset (N \supset M)$
3. $(C \supset P) \supset \sim M$
4. $C \supset N$ / $\sim C$

(23) 1. $(R \supset F) \supset [(R \supset \sim G) \supset (S \supset Q)]$
2. $(Q \supset F) \supset (R \supset Q)$
3. $\sim G \supset F$
4. $Q \supset \sim G$ / $S \supset F$

(24) 1. $\sim A \supset [A \vee (T \supset R)]$
2. $\sim R \supset [R \vee (A \supset R)]$
3. $(T \vee D) \supset \sim R$
4. $T \vee D$ / D

★(25) 1. $\sim N \supset [(B \supset D) \supset (N \vee \sim E)]$
2. $(B \supset E) \supset \sim N$
3. $B \supset D$
4. $D \supset E$ / $\sim D$

7

IV. Translate the following arguments into symbolic form and use the first four rules of inference to derive the conclusion of each. The letters to be used for the simple statements are given in parentheses after each exercise. Use these letters in the order in which they are listed.

★**1.** If the average child watches more than five hours of television per day, then either his power of imagination is improved or he becomes conditioned to expect constant excitement. The average child's power of imagination is not improved by watching television. Also, the average child does watch more than five hours of television per day. Therefore, the average child is conditioned to expect constant excitement. (*W*, *P*, *C*)

2. If a ninth planet exists, then its orbit is perpendicular to that of the other planets. Either a ninth planet is responsible for the death of the dinosaurs, or its orbit is not perpendicular to that of the other planets. A ninth planet is not responsible for the death of the dinosaurs. Therefore, a ninth planet does not exist. (*E*, *O*, *R*)

3. If quotas are imposed on textile imports only if jobs are not lost, then the domestic textile industry will modernize only if the domestic textile industry is not destroyed. If quotas are imposed on textile imports, the domestic textile industry will modernize. The domestic textile industry will modernize only if jobs are not lost. Therefore, if quotas are imposed on textile imports, the domestic textile industry will not be destroyed. (*Q*, *J*, *M*, *D*)

★4. If teachers are allowed to conduct random drug searches on students only if teachers are acting in loco parentis, then if teachers are acting in loco parentis, then students have no Fourth Amendment protections. Either students have no Fourth Amendment protections or if teachers are allowed to conduct random drug searches on students, then teachers are acting in loco parentis. It is not the case that students have no Fourth Amendment protections. Therefore, teachers are not allowed to conduct random drug searches on students. (*R*, *L*, *F*)

5. Either funding for nuclear fusion will be cut or if sufficiently high temperatures are achieved in the laboratory, nuclear fusion will become a reality. Either the supply of hydrogen fuel is limited, or if nuclear fusion becomes a reality, the world's energy problems will be solved. Funding for nuclear fusion will not be cut. Furthermore, the supply of hydrogen fuel is not limited. Therefore, if sufficiently high temperatures are achieved in the laboratory, the world's energy problems will be solved. (*C*, *H*, *R*, *S*, *E*)

6. Either the continents are not subject to drift or if Antarctica was always located in the polar region, then it contains no fossils of plants from a temperate climate. If the continents are not subject to drift, then Antarctica contains no fossils of plants from a temperate climate. But it is not the case that Antarctica contains no fossils of plants from a temperate climate. Therefore, Antarctica was not always located in the polar region. (*D*, *L*, *F*)

★7. If terrorists take more hostages, then terrorist demands will be met if and only if the media give full coverage to terrorist acts. Either the media will voluntarily limit the flow of information or if the media will recognize they are being exploited by terrorists, they will voluntarily limit the flow of information. Either the media will recognize they are being exploited by terrorists or terrorists will take more hostages. The media will not voluntarily limit the flow of information. Therefore, terrorist demands will be met if and only if the media give full coverage to terrorist acts. (*H*, *D*, *A*, *V*, *R*)

8. Either we take recycling seriously or we will be buried in garbage. If we incinerate our garbage only if our health is jeopardized, then we do not take recycling seriously. If our landfills are becoming exhausted, then if we incinerate our garbage, then toxic ash will be produced. If toxic ash is produced, then our health is jeopardized. Our landfills are becoming exhausted. Therefore, we will be buried in garbage. (*R*, *B*, *I*, *H*, *L*, *T*)

9. If the drug interdiction program is strengthened only if cocaine becomes more readily available, then either the number of addicts is decreasing or the war on drugs is failing. If the drug interdiction program is strengthened, then smugglers will shift to more easily concealable drugs. If smugglers shift to more easily concealable drugs, then cocaine will become more readily available. Furthermore, the number of addicts is not decreasing. Therefore, the war on drugs is failing. (*D*, *C*, *N*, *W*, *S*)

★10. If the death penalty is not cruel and unusual punishment, then either it is cruel and unusual punishment or if society is justified in using it, then it will deter

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other criminals. If the death penalty is cruel and unusual punishment, then it is both cruel and unusual and its use degrades society as a whole. It is not the case that both the death penalty is cruel and unusual and its use degrades society as a whole. Furthermore, the death penalty will not deter other criminals. Therefore, society is not justified in using the death penalty. (*C*, *J*, *D*, *U*)

7.2 Rules of Implication II

PREVIEW • *While on your hike, you encounter a stream, about 20 feet wide, that you must cross. The stream has several rocks protruding above the surface for 100 feet or so. To get across you must select a series of stepping-stones, placed fairly close together, starting near one bank and ending near the other. Crossing the stream illustrates the kind of strategic thinking you must use in constructing proofs. Continued practice with these proofs is guaranteed to improve your capacity for strategic thinking in many areas of life.*

Four additional rules of implication are listed here. Constructive dilemma should be familiar from Chapter 6. The other three are new. They are listed together with an illustration of their use as follows.*

5. Constructive dilemma (CD)

$(p \supset q) \bullet (r \supset s)$	If Oscar is a dog, then you'll have fleas, and if Oscar is a cat, then you'll have fur balls.
$p \lor r$	Oscar is either a dog or a cat.
$q \lor s$	You'll have either fleas or fur balls.

6. Simplification (Simp)

$p \bullet q$	Eliza has long legs and runs fast.
p	Eliza has long legs.

7. Conjunction (Conj)

p	Roxy has big eyes.
q	Roxy has a tail.
$p \bullet q$	Roxy has big eyes and a tail.

8. Addition (Add)

p	Theo has spots.
$p \lor q$	Theo has either spots or stripes.

Some textbooks include a rule called absorption *by which the statement form $p \supset (q \bullet p)$ is deduced from $p \supset q$. This rule is necessary only if conditional proof is not presented. This textbook opts in favor of conditional proof, to be introduced shortly.*

Like the previous four rules, these are fairly easy to understand, but if there is any doubt about them their validity may be proven by means of a truth table.

Constructive dilemma can be understood as involving two *modus ponens* steps. The first premise states that if we have p then we have q, and if we have r then we have s. But since, by the second premise, we do have either p or r, it follows by *modus ponens* that we have either q or s. Constructive dilemma is the only form of dilemma that will be included as a rule of inference. By the rule of transposition, which will be presented in Section 7.4, any argument that is a substitution instance of the destructive dilemma form can be easily converted into a substitution instance of constructive dilemma. Destructive dilemma, therefore, is not needed as a rule of inference.

These arguments are both instances of **constructive dilemma** (CD):

$$\begin{array}{l} {\sim}M \vee N \\ ({\sim}M \supset S) \bullet (N \supset {\sim}T) \\ \hline S \vee {\sim}T \end{array} \qquad \begin{array}{l} [(K \supset T) \supset (A \bullet B)] \bullet [(H \supset P) \supset (A \bullet C)] \\ (K \supset T) \vee (H \supset P) \\ \hline (A \bullet B) \vee (A \bullet C) \end{array}$$

Simplification states that if two propositions are given as true on a single line, then each of them is true separately. According to the strict interpretation of the simplification rule, only the left-hand conjunct may be stated in the conclusion. Once the commutativity rule for conjunction has been presented, however (see Section 7.3), we will be justified in replacing a statement such as $H \bullet K$ with $K \bullet H$. Once we do this, the K will appear on the left, and the appropriate conclusion is K.

These arguments are all instances of **simplification** (Simp):

$$\begin{array}{l} {\sim}F \bullet (U \equiv E) \\ \hline {\sim}F \end{array} \qquad \begin{array}{l} (M \vee T) \bullet (S \supset R) \\ \hline M \vee T \end{array} \qquad \begin{array}{l} [(X \supset Z) \bullet M] \bullet (G \supset H) \\ \hline (X \supset Z) \bullet M \end{array}$$

Conjunction states that two propositions—for example, H and K—asserted separately on different lines may be conjoined on a single line. The two propositions may be conjoined in whatever order we choose (either $H \bullet K$ or $K \bullet H$) without appeal to the commutativity rule for conjunction.

These arguments are all instances of **conjunction** (Conj):

$$\begin{array}{l} {\sim}E \\ {\sim}G \\ \hline {\sim}E \bullet {\sim}G \end{array} \qquad \begin{array}{l} C \supset M \\ D \supset N \\ \hline (C \supset M) \bullet (D \supset N) \end{array} \qquad \begin{array}{l} R \supset (H \bullet T) \\ K \supset (H \bullet O) \\ \hline [R \supset (H \bullet T)] \bullet [K \supset (H \bullet O)] \end{array}$$

Addition states that whenever a proposition is asserted on a line by itself it may be joined disjunctively with any proposition we choose. In other words, if G is asserted to be true by itself, it follows that $G \vee H$ is true. This may appear somewhat puzzling at first, but once one realizes that $G \vee H$ is a much weaker statement than G by itself, the puzzlement should disappear. The new proposition must, of course, always be joined disjunctively (not conjunctively) to the given proposition. If G is stated on a line by itself, we are *not* justified in writing $G \bullet H$ as a consequence of addition.

These arguments are all instances of **addition** (Add):

$$\frac{S}{S \vee \sim T} \qquad \frac{(C \bullet D)}{(C \bullet D) \vee (K \bullet \sim P)} \qquad \frac{W \equiv Z}{(W \equiv Z) \vee [A \supset (M \supset O)]}$$

The use of these four rules may now be illustrated. Consider the following argument:

1. $A \supset B$	
2. $(B \vee C) \supset (D \bullet E)$	
3. A	/ D

As usual, we begin by looking for the conclusion in the premises. D appears in the consequent of the second premise, which we can derive via simplification if we first obtain $B \vee C$. This expression as such does not appear in the premises, but from lines 1 and 3 we see that we can derive B by itself via *modus ponens*. Having obtained B, we can derive $B \vee C$ via addition. The proof has now been thought through and can be written out as follows:

1. $A \supset B$	
2. $(B \vee C) \supset (D \bullet E)$	
3. A	/ D
4. B	1, 3, MP
5. $B \vee C$	4, Add
6. $D \bullet E$	2, 5, MP
7. D	6, Simp

Another example:

1. $K \supset L$	
2. $(M \supset N) \bullet S$	
3. $N \supset T$	
4. $K \vee M$	/ $L \vee T$

Seeing that $L \vee T$ does not appear as such in the premises, we look for the separate components. Finding L and T as the consequents of two distinct conditional statements causes us to think that the conclusion can be derived via constructive dilemma. If a constructive dilemma can be set up, it will need a disjunctive statement as its second premise, and such a statement appears on line 4. Furthermore, each of the components of this statement, K and M, appears as the antecedent of a conditional statement, exactly as they both should for a dilemma. The only statement that is missing now is $M \supset T$. Inspecting line 2 we see that we can obtain $M \supset N$ via simplification, and putting this together with line 3 gives us $M \supset T$ via hypothetical syllogism. The completed proof may now be written out:

1. $K \supset L$	
2. $(M \supset N) \bullet S$	
3. $N \supset T$	
4. $K \vee M$	/ $L \vee T$
5. $M \supset N$	2, Simp
6. $M \supset T$	3, 5, HS
7. $(K \supset L) \bullet (M \supset T)$	1, 6, Conj
8. $L \vee T$	4, 7, CD

Another example:

1. $\sim M \bullet N$	
2. $P \supset M$	
3. $Q \bullet R$	
4. $(\sim P \bullet Q) \supset S$	/ $S \vee T$

When we look for $S \vee T$ in the premises we find S in the consequent of line 4 but no T at all. This signals an important principle: Whenever the conclusion of an argument contains a letter not found in the premises, addition must be used to introduce the missing letter. Addition is the *only* rule of inference that can introduce new letters. To introduce T by addition, however, we must first obtain S on a line by itself. S can be derived from line 4 via *modus ponens* if we first obtain $\sim P \bullet Q$. This, in turn, can be derived via conjunction, but first $\sim P$ and Q must be obtained individually on separate lines. Q can be derived from line 3 via simplification and $\sim P$ from line 2 via *modus tollens*, but the latter step requires that we first obtain $\sim M$ on a line by itself. Since this can be derived from line 1 via simplification, the proof is now complete. It may be written out as follows:

1. $\sim M \bullet N$	
2. $P \supset M$	
3. $Q \bullet R$	
4. $(\sim P \bullet Q) \supset S$	/ $S \vee T$
5. $\sim M$	1, Simp
6. $\sim P$	2, 5, MT
7. Q	3, Simp
8. $\sim P \bullet Q$	6, 7, Conj
9. S	4, 8, MP
10. $S \vee T$	9, Add

Addition is used together with disjunctive syllogism to derive the conclusion of arguments having inconsistent premises. As we saw in Chapter 6, such arguments are always valid. The procedure is illustrated as follows:

1. S	
2. $\sim S$	/ T
3. $S \vee T$	1, Add
4. T	2, 3, DS

With arguments of this sort the conclusion is always introduced via addition and then separated via disjunctive syllogism. Since addition can be used to introduce any letter or arrangement of letters we choose, it should be clear from this example that inconsistent premises validly entail any conclusion whatever.

To complete this presentation of the eight rules of implication, let us consider some of the typical ways in which they are *misapplied*. Examples are as follows:

1. $P \vee (S \cdot T)$
2. S — 1, Simp (invalid—$S \cdot T$ must first be obtained on a line by itself)

1. K
2. $K \cdot L$ — 1, Add (invalid—the conclusion must be a disjunctive statement)

1. $M \vee N$
2. M — 1, Simp (invalid—simplification is possible only with a conjunctive premise; line 1 is a disjunction)

1. $G \supset H$
2. $G \supset (H \vee J)$ — 1, Add (improper—J must be added to the whole line, not just to the consequent of line 1)

1. $L \supset M$
2. $L \supset N$
3. $M \cdot N$ — 1, 2, Conj (invalid—M and N must first be obtained on lines by themselves)

1. $\sim(P \cdot Q)$
2. $\sim P$ — 1, Simp (invalid—parentheses must be removed first)

1. $\sim(P \vee Q)$
2. $\sim P$
3. Q — 1, 2, DS (invalid—parentheses must be removed first)

The use of addition in the $G \supset H$ example is called "improper" because the letter that is added is not added to the whole line. It turns out, however, that even though the addition rule is not correctly applied here, the inference is still valid. Hence, this inference is not called "invalid," as the others are. As for the last two examples, a rule will be presented in the next section (De Morgan's rule) that will allow us to remove parentheses preceded by negation signs. But even after the parentheses have been removed from these examples, the inferences remain invalid.

Like the previous section, this one ends with a few strategies for applying the last four rules of implication:

Strategy 6: If the conclusion contains a letter that appears in a conjunctive statement in the premises, consider obtaining that letter via simplification:

1. $A \supset B$
2. $C \cdot B$
3. $C \supset A$ — / C
4. C — 2, Simp

Strategy 7: If the conclusion is a conjunctive statement, consider obtaining it via conjunction by first obtaining the individual conjuncts:

1. $A \supset C$
2. B
3. $\sim C$ / $B \bullet \sim C$
4. $B \bullet \sim C$ 2, 3, Conj

Strategy 8: If the conclusion is a disjunctive statement, consider obtaining it via constructive dilemma or addition:

1. $(A \supset B) \bullet (C \supset D)$
2. $B \supset C$
3. $A \vee C$ / $B \vee D$
4. $B \vee D$ 1, 3, CD

1. $A \vee C$
2. B
3. $C \supset D$ / $B \vee D$
4. $B \vee D$ 2, Add

Strategy 9: If the conclusion contains a letter not found in the premises, addition *must* be used to introduce that letter.

Strategy 10: Conjunction can be used to set up constructive dilemma:

1. $A \supset B$
2. $C \supset D$
3. $A \vee C$ / $B \vee D$
4. $(A \supset B) \bullet (C \supset D)$ 1, 2, Conj
5. $B \vee D$ 3, 4, CD

7

EXERCISE **7.2**

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification. In problems 4 and 8 you can add any statement you choose.

★(1) 1. $S \vee H$
2. $B \bullet E$
3. $R \supset G$
4. ______ ______, Simp

(2) 1. $(N \supset T) \bullet (F \supset Q)$
2. $(N \supset R) \vee (F \supset M)$
3. $N \vee F$
4. ________________ ______, CD

(3) 1. D
2. W
3. ______ ______, Conj

★(4) 1. H
2. ______ ______, Add

(5) 1. $R \bullet (N \vee K)$
2. $(G \bullet T) \vee S$
3. $(Q \bullet C) \supset (J \bullet L)$
4. ________________ ______, Simp

(6) 1. $\sim R \vee P$
2. $(P \supset \sim D) \bullet (\sim R \supset S)$
3. $(\sim R \supset A) \bullet (P \supset \sim N)$
4. ______________ _____, CD

★(7) 1. $(Q \vee K) \bullet \sim B$
2. $(M \bullet R) \supset D$
3. $(W \bullet S) \vee (G \bullet F)$
4. ______________ _____, Simp

(8) 1. $E \bullet G$
2. _______ _____, Add

(9) 1. $\sim B$
2. $F \vee N$
3. _____ _____, Conj

★(10) 1. $S \vee \sim C$
2. $(S \supset \sim L) \bullet (\sim C \supset M)$
3. $(\sim N \supset S) \bullet (F \supset \sim C)$
4. ________________ _____, CD

II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1) 1. $G \supset N$
2. $G \bullet K$
3. _______ _____
4. $G \vee T$ _____

(2) 1. $\sim A$
2. $A \vee E$
3. _______ _____
4. $\sim A \bullet E$ _____

(3) 1. $B \supset N$
2. $B \vee K$
3. $K \supset R$
4. ______ _____
5. $N \vee R$ _____

★(4) 1. T
2. $T \supset G$
3. $(T \vee U) \supset H$
4. _____________ _____
5. H _____

(5) 1. $S \supset E$
2. $E \vee (S \bullet P)$
3. $\sim E$
4. _________ _____
5. S _____

(6) 1. N
2. $N \supset F$
3. $(N \supset A) \bullet (F \supset C)$
4. ________________ _____
5. $A \vee C$ _____

★(7) 1. J
2. $\sim L$
3. $F \supset L$
4. ______ _____
5. $\sim F \bullet J$ _____

(8) 1. $(E \supset B) \bullet (Q \supset N)$
2. $K \supset E$
3. $B \supset K$
4. _________________ _____
5. $E \supset K$ _____

(9) 1. $G \vee N$
2. $\sim G$
3. $\sim G \supset (H \bullet R)$
4. ____________ _____
5. H _____

★(10) 1. M
2. $(M \bullet E) \supset D$
3. E
4. ____________ _____
5. D _____

III. Use the first eight rules of inference to derive the conclusions of the following symbolized arguments:

★(1) 1. $\sim M \supset Q$
2. $R \supset \sim T$
3. $\sim M \vee R$ / $Q \vee \sim T$

(2) 1. $N \supset (D \bullet W)$
2. $D \supset K$
3. N / $N \bullet K$

(3) 1. $E \supset (A \bullet C)$
2. $A \supset (F \bullet E)$
3. E / F

★(4) 1. $(H \vee \sim B) \supset R$
2. $(H \vee \sim M) \supset P$
3. H / $R \bullet P$

(5) 1. $G \supset (S \bullet T)$
2. $(S \vee T) \supset J$
3. G / J

(6) 1. $(L \vee T) \supset (B \bullet G)$
2. $L \bullet (K \equiv R)$ / $L \bullet B$

★(7) 1. $(\sim F \vee X) \supset (P \vee T)$
2. $F \supset P$
3. $\sim P$ / T

(8) 1. $(N \supset B) \bullet (O \supset C)$
2. $Q \supset (N \vee O)$
3. Q / $B \vee C$

(9) 1. $(U \vee W) \supset (T \supset R)$
2. $U \bullet H$
3. $\sim R \bullet \sim J$ / $U \bullet \sim T$

★(10) 1. $(D \vee E) \supset (G \bullet H)$
2. $G \supset \sim D$
3. $D \bullet F$ / M

(11) 1. $(B \vee F) \supset (A \supset G)$
2. $(B \vee E) \supset (G \supset K)$
3. $B \bullet \sim H$ / $A \supset K$

(12) 1. $(P \supset R) \supset (M \supset P)$
2. $(P \vee M) \supset (P \supset R)$
3. $P \vee M$ / $R \vee P$

★(13) 1. $(C \supset N) \bullet E$
2. $D \vee (N \supset D)$
3. $\sim D$ / $\sim C \vee P$

(14) 1. $F \supset (\sim T \bullet A)$
2. $(\sim T \vee G) \supset (H \supset T)$
3. $F \bullet O$ / $\sim H \bullet \sim T$

(15) 1. $(\sim S \vee B) \supset (S \vee K)$
2. $(K \vee \sim D) \supset (H \supset S)$
3. $\sim S \bullet W$ / $\sim H$

★(16) 1. $(C \vee \sim G) \supset (\sim P \bullet L)$
2. $(\sim P \bullet C) \supset (C \supset D)$
3. $C \bullet \sim R$ / $D \vee R$

(17) 1. $[A \vee (K \bullet J)] \supset (\sim E \bullet \sim F)$
2. $M \supset [A \bullet (P \vee R)]$
3. $M \bullet U$ / $\sim E \bullet A$

(18) 1. $\sim H \supset (\sim T \supset R)$
2. $H \vee (E \supset F)$
3. $\sim T \vee E$
4. $\sim H \bullet D$ / $R \vee F$

★(19) 1. $(U \bullet \sim\sim P) \supset Q$
2. $\sim O \supset U$
3. $\sim P \supset O$
4. $\sim O \bullet T$ / Q

(20) 1. $(M \vee N) \supset (F \supset G)$
2. $D \supset \sim C$
3. $\sim C \supset B$
4. $M \bullet H$
5. $D \vee F$ / $B \vee G$

(21) 1. $(F \bullet M) \supset (S \vee T)$
2. $(\sim S \vee A) \supset F$
3. $(\sim S \vee B) \supset M$
4. $\sim S \bullet G$ / T

★(22) 1. $(\sim K \bullet \sim N) \supset$
$[(\sim P \supset K) \bullet (\sim R \supset G)]$
2. $K \supset N$
3. $\sim N \bullet B$
4. $\sim P \vee \sim R$ / G

(23) 1. $(\sim A \vee D) \supset (B \supset F)$
2. $(B \vee C) \supset (A \supset E)$
3. $A \vee B$
4. $\sim A$ / $E \vee F$

(24) 1. $(J \supset K) \cdot (\sim O \supset \sim P)$
2. $(L \supset J) \cdot (\sim M \supset \sim O)$
3. $\sim K \supset (L \vee \sim M)$
4. $\sim K \cdot G$ / $\sim P$

★(25) 1. $(\sim M \cdot \sim N) \supset [(\sim M \vee H) \supset (K \cdot L)]$
2. $\sim M \cdot (C \supset D)$
3. $\sim N \cdot (F \equiv G)$ / $K \cdot \sim N$

(26) 1. $(P \vee S) \supset (E \supset F)$
2. $(P \vee T) \supset (G \supset H)$
3. $(P \vee U) \supset (E \vee G)$
4. P / $F \vee H$

(27) 1. $(S \supset Q) \cdot (Q \supset \sim S)$
2. $S \vee Q$
3. $\sim Q$ / $P \cdot R$

★(28) 1. $(D \supset B) \cdot (C \supset D)$
2. $(B \supset D) \cdot (E \supset C)$
3. $B \vee E$ / $D \vee B$

(29) 1. $(R \supset H) \cdot (S \supset I)$
2. $(\sim H \cdot \sim L) \supset (R \vee S)$
3. $\sim H \cdot (K \supset T)$
4. $H \vee \sim L$ / $I \vee M$

(30) 1. $(W \cdot X) \supset (Q \vee R)$
2. $(S \vee F) \supset (Q \vee W)$
3. $(S \vee G) \supset (\sim Q \supset X)$
4. $Q \vee S$
5. $\sim Q \cdot H$ / R

IV. Translate the following arguments into symbolic form and use the first eight rules of inference to derive the conclusion of each. Use the letters in the order in which they are listed.

★1. If topaz is harder than quartz, then it will scratch quartz and also feldspar. Topaz is harder than quartz and it is also harder than calcite. Therefore, either topaz will scratch quartz or it will scratch corundum. (*T*, *Q*, *F*, *C*, *O*)

2. If clear-cutting continues in primary forests and the Endangered Species Act is not repealed, then either the Endangered Species Act will be repealed or thousands of animal species will become extinct. Clear-cutting continues in primary forests. The Endangered Species Act will not be repealed. Therefore, thousands of animal species will become extinct. (*C*, *E*, *T*)

3. If either executive salaries are out of control or exorbitant bonuses are paid, then either shareholders will be cheated or ordinary workers will be paid less. Executive salaries are out of control and the rich are getting richer. If shareholders are cheated, then future investors will stay away; also, if ordinary workers are paid less, then consumer spending will decline. If either future investors stay away or consumer spending declines, then the economy will suffer. Therefore, the economy will suffer. (*S*, *B*, *C*, *P*, *R*, *F*, *D*, *E*)

★4. Either animals are mere mechanisms or they feel pain. If either animals feel pain or they have souls, then they have a right not to be subjected to needless pain and humans have a duty not to inflict needless pain on them. It is not the case that animals are mere mechanisms. Therefore, animals have a right not to be subjected to needless pain. (*M*, *P*, *S*, *R*, *D*)

5. If half the nation suffers from depression, then if either the insurance companies have their way or the psychiatrists have their way, then everyone will be taking antidepressant drugs. If either half the nation suffers from depression

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or sufferers want a real cure, then it is not the case that everyone will be taking antidepressant drugs. Half the nation suffers from depression. Therefore, it is not the case that either the insurance companies or the psychiatrists will have their way. (*H*, *I*, *P*, *E*, *W*)

6. If either parents get involved in their children's education or the school year is lengthened, then if the children learn phonics, their reading will improve and if they are introduced to abstract concepts earlier, their math will improve. If either parents get involved in their children's education or nebulous subjects are dropped from the curriculum, then either the children will learn phonics or they will be introduced to abstract concepts earlier. Parents will get involved in their children's education, and writing lessons will be integrated with other subjects. Therefore, either the children's reading or their math will improve. (*P*, *S*, *L*, *R*, *I*, *M*, *N*, *W*)

★7. If either manufacturers will not concentrate on producing a superior product or they will not market their product abroad, then if they will not concentrate on producing a superior product, then the trade deficit will worsen. Either manufacturers will concentrate on producing a superior product or the trade deficit will not worsen. Manufacturers will not concentrate on producing a superior product. Therefore, today's business managers lack imagination. (*C*, *M*, *T*, *B*)

8. If either medical fees or malpractice awards escape restrictions, then health care costs will soar and millions of poor will go uninsured. If the lawyers get their way, then malpractice awards will escape restrictions. If the doctors get their way, then medical fees will escape restrictions. Either the doctors or the lawyers will get their way, and insurance companies will resist reform. Therefore, health care costs will soar. (*F*, *A*, *H*, *P*, *L*, *D*, *I*)

9. If we are less than certain the human fetus is a person, then we must give it the benefit of the doubt. If we are certain the human fetus is a person, then we must accord it the right to live. If either we must give the fetus the benefit of the doubt or accord it the right to live, then we are not less than certain the fetus is human and it is not merely a part of the mother's body. Either we are less than certain the human fetus is a person or we are certain about it. If we are certain the human fetus is a person, then abortion is immoral. Therefore, abortion is immoral. (*L*, *G*, *C*, *A*, *M*, *I*)

★10. If the assassination of terrorist leaders violates civilized values and also is not effective in the long run, then if it prevents terrorist atrocities, then it is effective in the long run. If the assassination of terrorist leaders violates civilized values, then it is not effective in the long run. The assassination of terrorist leaders violates civilized values and is also illegal. If the assassination of terrorist leaders is not effective in the long run, then either it prevents terrorist atrocities or it justifies acts of revenge by terrorists. Therefore, the assassination of terrorist leaders justifies acts of revenge by terrorists and also is not effective in the long run. (*V*, *E*, *P*, *I*, *J*)

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7.3 Rules of Replacement I

PREVIEW • *After crossing the stream, you encounter a rocky cliff about 25 feet high that you must scale. You can now practice some of your rock-climbing skills. As you climb, you must find a series of crevices in the rock for your hands, and little ledges for your feet. The crevices and ledges must not be too far apart, and they must lead continuously upward. This involves a higher level of strategic thinking than crossing the stream required. The following section of the book will give you great practice for this kind of thinking.*

Unlike the rules of implication, which are basic argument forms, the ten **rules of replacement** are expressed in terms of pairs of logically equivalent statement forms, either of which can replace the other in a proof sequence. To express these rules, a new symbol, called a **double colon** (::), is used to designate logical equivalence. This symbol is a *metalogical* symbol in that it makes an assertion not about things but about symbolized statements: It asserts that the expressions on either side of it have the same truth value regardless of the truth values of their components. Underlying the use of the rules of replacement is an **axiom of replacement**, which asserts that within the context of a proof, logically equivalent expressions may replace each other. The first five rules of replacement are as follows:

9. De Morgan's rule (DM):
   $\sim(p \bullet q) :: (\sim p \vee \sim q)$
   $\sim(p \vee q) :: (\sim p \bullet \sim q)$

10. Commutativity (Com):
   $(p \vee q) :: (q \vee p)$
   $(p \bullet q) :: (q \bullet p)$

11. Associativity (Assoc):
   $[p \vee (q \vee r)] :: [(p \vee q) \vee r]$
   $[p \bullet (q \bullet r)] :: [(p \bullet q) \bullet r]$

12. Distribution (Dist):
   $[p \bullet (q \vee r)] :: [(p \bullet q) \vee (p \bullet r)]$
   $[p \vee (q \bullet r)] :: [(p \vee q) \bullet (p \vee r)]$

13. Double negation (DN):
   $p :: \sim\sim p$

De Morgan's rule (named after the nineteenth-century logician Augustus De Morgan) was discussed in Section 6.1 in connection with translation. There it was pointed out that "Not both *p* and *q*" is logically equivalent to "Not *p* or not *q*," and that "Not either *p* or *q*" is logically equivalent to "Not *p* and not *q*." When applying De Morgan's rule, one should keep in mind that it holds only for conjunctive and disjunctive statements (not for conditionals or biconditionals). The rule may be summarized as follows: When moving a tilde inside or outside a set of parentheses, a dot switches with a wedge and vice versa.

Commutativity asserts that the truth value of a conjunction or disjunction is unaffected by the order in which the components are listed. In other words, the component statements may be commuted, or switched for one another, without affecting the truth value. The validity of this rule should be immediately apparent. You may recall from arithmetic that the commutativity rule also applies to addition and multiplication and asserts, for example, that $3 + 5$ equals $5 + 3$, and that 2×3 equals 3×2. However, it does *not* apply to division; $2 \div 4$ does not equal $4 \div 2$. A similar lesson applies in logic: The commutativity rule applies only to conjunction and disjunction; it does *not* apply to implication.

Associativity states that the truth value of a conjunctive or disjunctive statement is unaffected by the placement of parentheses when the same operator is used throughout. In other words, the way in which the component propositions are grouped, or associated with one another, can be changed without affecting the truth value. The validity of this rule is quite easy to see, but if there is any doubt about it, it may be readily checked by means of a truth table. You may recall that the associativity rule also applies to addition and multiplication and asserts, for example, that $3 + (5 + 7)$ equals $(3 + 5) + 7$, and that $2 \times (3 \times 4)$ equals $(2 \times 3) \times 4$. But it does *not* apply to division: $(8 \div 4) \div 2$ does not equal $8 \div (4 \div 2)$. Analogously, in logic, the associativity rule applies only to conjunctive and disjunctive statements; it does *not* apply to conditional statements. Also note, when applying this rule, that the order of the letters remains unchanged; only the placement of the parentheses changes.

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Distribution, like De Morgan's rule, applies only to conjunctive and disjunctive statements. When a proposition is conjoined to a disjunctive statement in parentheses or disjoined to a conjunctive statement in parentheses, the rule allows us to put that proposition together with each of the components inside the parentheses, and also to go in the reverse direction. In the first form of the rule, a statement is distributed through a disjunction, and in the second form, through a conjunction. While the rule may not be immediately obvious, it is easy to remember: The operator that is at first outside the parentheses goes inside, and the operator that is at first inside the parentheses goes outside. Note also how distribution differs from commutativity and associativity. The latter two rules apply only when the *same* operator (either a dot or a wedge) is used throughout a statement. Distribution applies when a dot and a wedge appear *together* in a statement.

Double negation is fairly obvious and needs little explanation. The rule states simply that pairs of tildes immediately adjacent to one another may be either deleted or introduced without affecting the truth value of the statement.

There is an important difference between the rules of implication, treated in the first two sections of this chapter, and the rules of replacement. The rules of implication derive their name from the fact that each is a simple argument form in which the premises imply the conclusion. To be applicable in natural deduction, certain lines in a proof must be interpreted as substitution instances of the argument form in question. Stated another way, the rules of implication are applicable only to *whole lines* in a proof. For example, step 3 in the following proof is not a legitimate application of *modus ponens*, because the first premise in the *modus ponens* rule is applied to only a *part* of line 1.

1. $A \supset (B \supset C)$
2. B
3. C — 1, 2, MP (invalid)

The rules of replacement, on the other hand, are not rules of implication but rules of logical equivalence. Since, by the axiom of replacement, logically equivalent statement forms can always replace one another in a proof sequence, the rules of replacement can be applied either to a whole line or to any part of a line. Step 2 in the following proof is a quite legitimate application of De Morgan's rule, even though the rule is applied only to the consequent of line 1:

1. $S \supset \sim(T \bullet U)$
2. $S \supset (\sim T \vee \sim U)$ — 1, DM (valid)

Another way of viewing this distinction is that the rules of implication are "one-way" rules, whereas the rules of replacement are "two-way" rules. The rules of implication allow us to proceed only from the premise lines of a rule to the conclusion line, but the rules of replacement allow us to replace either side of an equivalence expression with the other side.

Application of the first five rules of replacement may now be illustrated. Consider the following argument:

1. $A \supset \sim(B \bullet C)$
2. $A \bullet C$ — / $\sim B$

Examining the premises, we find B in the consequent of line 1. This leads us to suspect that the conclusion can be derived via *modus ponens*. If this is correct, the tilde would then have to be taken inside the parentheses via De Morgan's rule and the resulting $\sim C$ eliminated by disjunctive syllogism. The following completed proof indicates that this strategy yields the anticipated result:

1. $A \supset \sim(B \bullet C)$	
2. $A \bullet C$	/ $\sim B$
3. A	2, Simp
4. $\sim(B \bullet C)$	1, 3, MP
5. $\sim B \vee \sim C$	4, DM
6. $C \bullet A$	2, Com
7. C	6, Simp
8. $\sim\sim C$	7, DN
9. $\sim C \vee \sim B$	5, Com
10. $\sim B$	8, 9, DS

The rationale for line 6 is to get C on the left side so that it can be separated via simplification. Similarly, the rationale for line 9 is to get $\sim C$ on the left side so that it can be eliminated via disjunctive syllogism. Line 8 is required because, strictly speaking, the negation of $\sim C$ is $\sim\sim C$—not simply C. Thus, C must be replaced with $\sim\sim C$ to set up the disjunctive syllogism. If your instructor permits it, you can combine commutativity and double negation with other inferences on a single line, as the following

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shortened proof illustrates. However, we will avoid this practice throughout the remainder of the book.

1.	$A \supset \sim(B \cdot C)$	
2.	$A \cdot C$	/ $\sim B$
3.	A	2, Simp
4.	$\sim(B \cdot C)$	1, 3, MP
5.	$\sim B \vee \sim C$	4, DM
6.	C	2, Com, Simp
7.	$\sim B$	5, 6, Com, DN, DS

Another example:

1.	$D \cdot (E \vee F)$	
2.	$\sim D \vee \sim F$	/ $D \cdot E$

The conclusion requires that we get *D* and *E* together. Inspection of the first premise suggests distribution as the first step in achieving this. The completed proof is as follows:

1.	$D \cdot (E \vee F)$	
2.	$\sim D \vee \sim F$	/ $D \cdot E$
3.	$(D \cdot E) \vee (D \cdot F)$	1, Dist
4.	$(D \cdot F) \vee (D \cdot E)$	3, Com
5.	$\sim(D \cdot F)$	2, DM
6.	$D \cdot E$	4, 5, DS

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Some proofs require that we use distribution in the reverse manner. Consider this argument:

1.	$(G \cdot H) \vee (G \cdot J)$	
2.	$(G \vee K) \supset L$	/ L

The conclusion can be obtained from line 2 via *modus ponens* if we first obtain $G \vee K$ on a line by itself. Since *K* does not occur in the first premise at all, it must be introduced by addition. Doing this requires in turn that we obtain *G* on a line by itself. Distribution applied to line 1 provides the solution:

1.	$(G \cdot H) \vee (G \cdot J)$	
2.	$(G \vee K) \supset L$	/ L
3.	$G \cdot (H \vee J)$	1, Dist
4.	G	3, Simp
5.	$G \vee K$	4, Add
6.	L	2, 5, MP

Application of the associativity rule is illustrated in the next proof:

1.	$M \vee (N \vee O)$	
2.	$\sim O$	/ $M \vee N$
3.	$(M \vee N) \vee O$	1, Assoc
4.	$O \vee (M \vee N)$	3, Com
5.	$M \vee N$	2, 4, DS

Before O can be eliminated via disjunctive syllogism from line 1, it must be moved over to the left side. Associativity and commutativity together accomplish this objective.

In some arguments the attempt to "find" the conclusion in the premises is not immediately successful. When confronted with such an argument, one should often begin by "deconstructing" the conclusion using the rules of replacement. In other words, one should first apply the rules of replacement to the conclusion to see how it is put together. After this is done, how the premises entail the conclusion may be evident. This procedure is justified by the fact that the rules of replacement are two-way rules. As a result, after the conclusion is deconstructed, it can be derived by using the same rules in reverse order. Here is an example of such an argument:

1. $K \supset (F \vee B)$	
2. $G \bullet K$	/ $(F \bullet G) \vee (B \bullet G)$

If immediate inspection does not reveal how the conclusion should be derived, we may begin by applying the rules of replacement to the conclusion. The form of the conclusion suggests the distribution rule, but first we must use commutativity to move the G's to the left-hand side. The deconstruction proceeds as follows:

$(F \bullet G) \vee (B \bullet G)$	
$(G \bullet F) \vee (B \bullet G)$	Com
$(G \bullet F) \vee (G \bullet B)$	Com
$G \bullet (F \vee B)$	Dist

Now we see that if we can obtain G on a line by itself, and $F \vee B$ on a line by itself, we can combine them on a single line via the conjunction rule. We can then derive the conclusion via distribution and commutativity. Inspection of the premises reveals that G can be derived from line 2 of the premises by simplification, and $F \vee B$ can be derived from line 1 by *modus ponens*. The completed proof is as follows:

1. $K \supset (F \vee B)$	
2. $G \bullet K$	/ $(F \bullet G) \vee (B \bullet G)$
3. G	2, Simp
4. $K \bullet G$	2, Com
5. K	4, Simp
6. $F \vee B$	1, 5, MP
7. $G \bullet (F \vee B)$	3, 6, Conj
8. $(G \bullet F) \vee (G \bullet B)$	7, Dist
9. $(F \bullet G) \vee (G \bullet B)$	8, Com
10. $(F \bullet G) \vee (B \bullet G)$	9, Com

Here are some strategies for applying the first five rules of replacement. Most of them show how these rules may be used together with other rules.

Strategy 11: Conjunction can be used to set up De Morgan's rule:

1. $\sim A$	
2. $\sim B$	
3. $\sim A \bullet \sim B$	1, 2, Conj
4. $\sim (A \vee B)$	3, DM

Willard Van Orman Quine 1908–2000


AP Photo/Julia Malakie

Prior to his death in the year 2000, Willard Van Orman Quine was widely considered to be, as Stuart Hampshire put it, "the most distinguished and influential of living philosophers." At that time, over 2,000 scholarly articles had been written about his work.

Quine was born in Akron, Ohio, in 1908 to a father who founded a heavy equipment company and a mother who taught elementary school. He earned his bachelor's degree in mathematics from Oberlin College, where he graduated *summa cum laude* in 1930. He then entered Harvard University, where he switched to philosophy so he could study under Alfred North Whitehead. He earned his PhD in a record two years. Except for four years during World War II, when he served in the Navy decoding messages from German submarines, Quine remained affiliated with Harvard for the remainder of his life.

Quine wrote twenty-two books, the first five of which dealt with mathematical logic. One of the goals of the earlier books was to show how the foundations of mathematics could be laid out in less than a fourth of the space taken by Whitehead and Russell's *Principia Mathematica.* One of his most famous publications was "Two Dogmas of Empiricism," which shook the pillars of analytic philosophy by undermining the sacrosanct distinction between analytic and synthetic statements. As a result of this work, even the truths of logic and mathematics became subject to the dictates of empirical experience.

As a boy, Quine had a fascination with collecting stamps and drawing maps, which, as an adult, he translated into a zest for world travel. He visited 118 countries, became fluent in six different languages, delivered lectures all over the world, and was awarded the first Schock Prize (Stockholm, 1993) and the Kyoto Prize (Tokyo, 1996). He was married twice, raised two children from each marriage, loved Dixieland jazz, and played the banjo, mandolin, and piano. He was singularly unpretentious, had an unfailing curiosity about a vast range of topics, and delighted in teaching freshman logic as well as advanced courses in philosophy. He died in Boston at the age of ninety-two.

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Strategy 12: Constructive dilemma can be used to set up De Morgan's rule:

1.	$(A \supset \sim B) \bullet (C \supset \sim D)$	
2.	$A \vee C$	
3.	$\sim B \vee \sim D$	1, 2, CD
4.	$\sim(B \bullet D)$	3, DM

Strategy 13: Addition can be used to set up De Morgan's rule:

1.	$\sim A$	
2.	$\sim A \vee \sim B$	1, Add
3.	$\sim(A \bullet B)$	2, DM

Strategy 14: Distribution can be used in two ways to set up disjunctive syllogism:

1. $(A \vee B) \bullet (A \vee C)$
2. $\sim A$
3. $A \vee (B \bullet C)$ 1, Dist
4. $B \bullet C$ 2, 3, DS

1. $A \bullet (B \vee C)$
2. $\sim(A \bullet B)$
3. $(A \bullet B) \vee (A \bullet C)$ 1, Dist
4. $A \bullet C$ 2, 3, DS

Strategy 15: Distribution can be used in two ways to set up simplification:

1. $A \vee (B \bullet C)$
2. $(A \vee B) \bullet (A \vee C)$ 1, Dist
3. $A \vee B$ 2, Simp

1. $(A \bullet B) \vee (A \bullet C)$
2. $A \bullet (B \vee C)$ 1, Dist
3. A 2, Simp

Strategy 16: If inspection of the premises does not reveal how the conclusion should be derived, consider using the rules of replacement to deconstruct the conclusion. (See the final example in this section.)

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EXERCISE 7.3

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification. For double negation, avoid the occurrence of triple tildes. Exercise 6 has two possible answers.

★(1) 1. $\sim(E \supset H)$
2. $\sim(N \vee G)$
3. $\sim A \vee D$
4. ________ _____, DM

(2) 1. $G \supset (N \supset K)$
2. $R \vee (D \supset F)$
3. $S \bullet (T \vee U)$
4. ____________ _____, Dist

(3) 1. $M \vee (G \vee T)$
2. $P \bullet (S \supset N)$
3. $D \bullet (R \vee K)$
4. ____________ _____, Assoc

★(4) 1. $B \supset W$
2. $G \equiv F$
3. $S \bullet A$
4. _______ _____, Com

(5) 1. $\sim\sim R \vee T$
2. $\sim N \vee \sim B$
3. $\sim A \supset \sim H$
4. __________ _____, DN

(6) 1. $(F \vee N) \supset (K \bullet D)$
2. $(H \bullet Z) \vee (H \bullet W)$
3. $(P \supset H) \vee (P \supset N)$
4. ____________________ _____, Dist

★(7) 1. $\sim(G \bullet \sim Q)$
2. $\sim(K \equiv \sim B)$
3. $\sim T \supset \sim F$
4. ___________ _____, DM

(8) 1. $G \supset (\sim L \supset T)$
2. $L \equiv (\sim R \supset \sim C)$
3. $J \supset (S \vee \sim N)$
4. ______________ _____, Com

(9) 1. $S \supset (M \supset D)$
2. $(K \bullet G) \vee B$
3. $(E \bullet H) \bullet Q$
4. _____________ _____, Assoc

★(10) 1. $\sim R \vee \sim P$
2. $\sim F \supset \sim W$
3. $G \bullet \sim A$
4. __________ _____, DM

(11) 1. $\sim B \vee E$
2. $\sim E \bullet \sim A$
3. $\sim C \supset \sim R$
4. __________ _____, DN

(12) 1. $\sim G \bullet (S \supset A)$
2. $\sim S \supset (B \bullet K)$
3. $\sim Q \vee (T \bullet R)$
4. _____________ _____, Dist

★(13) 1. $F \supset (\sim S \vee M)$
2. $H \supset (\sim L \bullet \sim D)$
3. $N \supset (\sim G \supset \sim C)$
4. _____________ _____, DM

(14) 1. $F \supset (P \supset \sim E)$
2. $C \vee (S \bullet \sim B)$
3. $M \bullet (R \bullet \sim T)$
4. ______________ _____, Assoc

(15) 1. $(D \vee \sim K) \bullet (D \vee \sim W)$
2. $(S \vee \sim Z) \vee (P \vee \sim T)$
3. $(Q \supset \sim N) \bullet (Q \supset \sim F)$
4. ______________ ____, Dist

II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1) 1. $K \vee C$
2. $\sim C$
3. ______ ____
4. K ____

(2) 1. $G \supset (R \vee N)$
2. $\sim R \bullet \sim N$
3. __________ ____
4. $\sim G$ ____

(3) 1. $H \bullet T$
2. ______ ____
3. T ____

★(4) 1. $(L \bullet S) \bullet F$
2. ________ ____
3. L ____

(5) 1. $\sim B \vee K$
2. ________ ____
3. $\sim(B \bullet \sim K)$ ____

(6) 1. $C \supset \sim A$
2. A
3. ______ ____
4. $\sim C$ ____

★(7) 1. $(D \bullet M) \vee (D \bullet N)$
2. ______________ ____
3. D ____

(8) 1. $(U \vee T) \supset R$
2. $T \vee U$
3. __________ ____
4. R ____

(9) 1. $\sim L \vee M$
2. L
3. ______ ____
4. M ____

★(10) 1. $D \vee (N \bullet H)$
2. __________ ____
3. $D \vee N$ ____

(11) 1. $(K \vee E) \bullet (K \vee G)$
2. $\sim K$
3. ______________ ____
4. $E \bullet G$ ____

(12) 1. $(N \supset T) \bullet (F \supset Q)$
2. $F \vee N$
3. ______________ ____
4. $T \vee Q$ ____

★(13) 1. $(M \vee G) \vee T$
2. $\sim M$
3. __________ ____
4. $G \vee T$ ____

(14) 1. $(\sim A \supset T) \bullet (\sim S \supset K)$
2. $\sim(A \bullet S)$
3. ______________ ____
4. $T \vee K$ ____

(15) 1. $\sim R$
2. _______ ____
3. $\sim(R \bullet T)$ ____

III. Use the first thirteen rules of inference to derive the conclusions of the following symbolized arguments:

★(1) 1. $(\sim M \supset P) \bullet (\sim N \supset Q)$
2. $\sim(M \bullet N)$ / $P \vee Q$

(2) 1. $\sim S$ / $\sim(F \bullet S)$

(3) 1. $J \vee (K \bullet L)$
2. $\sim K$ / J

★(4) 1. $\sim(N \bullet T)$
2. T / $\sim N$

(5) 1. $H \supset \sim A$
2. A / $\sim(H \vee \sim A)$

(6) 1. $R \supset \sim B$
2. $D \vee R$
3. B / D

★(7) 1. $T \supset (B \vee E)$
2. $\sim E \bullet T$ / B

(8) 1. $(O \vee M) \supset S$
2. $\sim S$ / $\sim M$

(9) 1. $Q \vee (L \vee C)$
2. $\sim C$ / $L \vee Q$

★(10) 1. $(K \bullet H) \vee (K \bullet L)$
2. $\sim L$ / H

(11) 1. $\sim(\sim E \bullet \sim N) \supset T$
2. $G \supset (N \vee E)$ / $G \supset T$

(12) 1. $H \bullet (C \bullet T)$
2. $\sim(\sim F \bullet T)$ / F

★(13) 1. $(E \bullet I) \vee (M \bullet U)$
2. $\sim E$ / $\sim(E \vee \sim M)$

(14) 1. $\sim(J \vee K)$
2. $B \supset K$
3. $S \supset B$ / $\sim S \bullet \sim J$

(15) 1. $(G \bullet H) \vee (M \bullet G)$
2. $G \supset (T \bullet A)$ / A

★(16) 1. $(Q \bullet N) \vee (N \bullet T)$
2. $(Q \vee C) \supset \sim N$ / T

(17) 1. $\sim(U \vee R)$
2. $(\sim R \vee N) \supset (P \bullet H)$
3. $Q \supset \sim H$ / $\sim Q$

(18) 1. $\sim(F \bullet A)$
2. $\sim(L \vee \sim A)$
3. $D \supset (F \vee L)$ / $\sim D$

★(19) 1. $[(I \vee M) \vee G] \supset \sim G$
2. $M \vee G$ / M

(20) 1. $E \supset \sim B$
2. $U \supset \sim C$
3. $\sim(\sim E \bullet \sim U)$ / $\sim(B \bullet C)$

(21) 1. $\sim(K \vee F)$
2. $\sim F \supset (K \vee C)$
3. $(G \vee C) \supset \sim H$ / $\sim(K \vee H)$

★(22) 1. $S \vee (I \bullet \sim J)$
2. $S \supset \sim R$
3. $\sim J \supset \sim Q$ / $\sim(R \bullet Q)$

(23) 1. $(J \vee F) \vee M$
2. $(J \vee M) \supset \sim P$
3. $\sim F$ / $\sim(F \vee P)$

(24) 1. $(K \bullet P) \vee (K \bullet Q)$
2. $P \supset \sim K$ / $Q \vee T$

★(25) 1. $E \vee \sim(D \vee C)$
2. $(E \vee \sim D) \supset C$ / E

(26) 1. $A \bullet (F \bullet L)$
2. $A \supset (U \vee W)$
3. $F \supset (U \vee X)$ / $U \vee (W \bullet X)$

(27) 1. $(T \bullet R) \supset P$
2. $(\sim P \bullet R) \bullet G$
3. $(\sim T \vee N) \supset H$ / H

★(28) 1. $P \vee (I \bullet L)$
2. $(P \vee I) \supset \sim(L \vee C)$
3. $(P \bullet \sim C) \supset (E \bullet F)$ / $F \vee D$

(29) 1. $B \vee (S \bullet N)$
2. $B \supset \sim S$
3. $S \supset \sim N$ / $B \vee W$

(30) 1. $(\sim M \vee E) \supset (S \supset U)$
2. $(\sim Q \vee E) \supset (U \supset H)$
3. $\sim(M \vee Q)$ / $S \supset H$

★(31) 1. $(\sim R \vee D) \supset \sim(F \bullet G)$
2. $(F \bullet R) \supset S$
3. $F \bullet \sim S$ / $\sim(S \vee G)$

(32) 1. $\sim Q \supset (C \bullet B)$
2. $\sim T \supset (B \bullet H)$
3. $\sim(Q \bullet T)$ / B

(33) 1. $\sim(A \bullet G)$
2. $\sim(A \bullet E)$
3. $G \vee E$ / $\sim(A \bullet F)$

★(34) 1. $(M \bullet N) \vee (O \bullet P)$
2. $(N \vee O) \supset \sim P$ / N

(35) 1. $(T \bullet K) \vee (C \bullet E)$
2. $K \supset \sim E$
3. $E \supset \sim C$ / $T \bullet K$

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IV. Translate the following arguments into symbolic form and then use the first thirteen rules of inference to derive the conclusion of each. Use the translation letters in the order in which they are listed.

★**1.** Either health-care costs are skyrocketing and they are attributable to greedy doctors, or health-care costs are skyrocketing and they are attributable to greedy hospitals. If health-care costs are skyrocketing, then both the government should intercede and health care may have to be rationed. Therefore, health-care costs are skyrocketing and health care may have to be rationed. (*S*, *D*, *H*, *I*, *R*)

2. Either the ancient Etruscans were experienced city planners and they invented the art of writing or they were highly skilled engineers and they invented the art of writing. If the ancient Etruscans were bloodthirsty numskulls (as scholars once thought), they did not invent the art of writing. Therefore, the ancient Etruscans were not bloodthirsty numskulls (as scholars once thought). (*C*, *I*, *H*, *B*)

3. It is not the case that either the Earth's molten core is stationary or that it contains no iron. If it is not the case that both the Earth's molten core is stationary and has a regular topography, then either the Earth's core contains no iron or the direction of the Earth's magnetic field is subject to change. Therefore, the direction of the Earth's magnetic field is subject to change. (*S*, *C*, *R*, *D*)

★**4.** Either mosquito genes can be cloned or mosquitoes will become resistant to all insecticides and the incidence of encephalitis will increase. If either mosquito genes can be cloned or the incidence of encephalitis increases, then mosquitoes will not become resistant to all insecticides. Therefore, either mosquito genes can be cloned or mosquitoes will multiply out of control. (*G*, *R*, *E*, *M*)

5. Protein engineering will prove to be as successful as genetic engineering, and new enzymes will be developed for producing food and breaking down industrial wastes. If protein engineering proves to be as successful as genetic engineering and new enzymes are developed for breaking down industrial wastes, then it is not the case that new enzymes will be developed for producing food but not medicines. Therefore, protein engineering will prove to be as successful as genetic engineering and new enzymes will be developed for producing medicines. (*E*, *P*, *B*, *M*)

6. If workers have a fundamental right to a job, then unemployment will be virtually nonexistent but job redundancy will become a problem. If workers have no fundamental right to a job, then production efficiency will be maximized but job security will be jeopardized. Workers either have or do not have a fundamental right to a job. Therefore, either unemployment will be virtually nonexistent or production efficiency will be maximized. (*F*, *U*, *R*, *P*, *S*)

★**7.** If China is to reduce its huge trade surplus, then it must either convince its citizens to spend more or it must move its manufacturing facilities to other countries. It is not the case that China will either increase its imports or convince its citizens to spend more. Furthermore, it is not the case that China will

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either allow foreign construction companies to compete on an equal footing or move its manufacturing facilities to other countries. Therefore, China will not reduce its huge trade surplus. (*R*, *C*, *M*, *I*, *A*)

8. If women are by nature either passive or uncompetitive, then it is not the case that there are lawyers who are women. If men are by nature either insensitive or without the ability to nurture, then it is not the case that there are kindergarten teachers who are men. There are lawyers who are women and kindergarten teachers who are men. Therefore, it is not the case that either women by nature are uncompetitive or men by nature are without the ability to nurture. (*P*, *U*, *L*, *I*, *W*, *K*)

9. It is not the case that either the sun's interior rotates faster than its surface or Einstein's general theory of relativity is wrong. If the sun's interior does not rotate faster than its surface and eccentricities in the orbit of Mercury can be explained by solar gravitation, then Einstein's general theory of relativity is wrong. Therefore, eccentricities in the orbit of Mercury cannot be explained by solar gravitation. (*S*, *E*, *M*)

★10. Either school dropout programs are not as effective as they could be, or they provide basic thinking skills and psychological counseling to their students. Either school dropout programs are not as effective as they could be, or they adequately prepare their students for getting a job and working effectively with others. Either school dropout programs do not provide psychological counseling to their students or they do not provide adequate preparation for working effectively with others. Therefore, school dropout programs are not as effective as they could be. (*E*, *B*, *P*, *G*, *W*)

V. The following dialogue contains eight arguments. Translate each into symbolic form and then use the first thirteen rules of inference to derive the conclusion of each.

With This Ring

"Hi. I didn't expect to see you here," says Ken as he catches sight of Gina on the steps of the church. "You must be friends with the bride."

"I am," she says, "and are you a friend of the groom?"

"A friend of a friend of the groom," he replies. "So I don't know too many people here."

"Well, I'll be happy to keep you company until the ceremony starts," Gina says. "And it looks like things are running late, so we'll have a few minutes."

She looks around and sighs. "Every time I attend a wedding," she says, "I feel sad for a lesbian couple I know who would give just about anything to get married. Unfortunately this state doesn't allow same-sex marriage."

"Well, I don't think that's unfortunate," says Ken. "If marriage is sacred, then we shouldn't tamper with it; and if that's the case, then we shouldn't allow same-sex marriage. And I do think marriage is sacred, so we shouldn't allow same-sex marriage."

Gina looks shocked. Ken continues. "Look," he says, "the Bible condemns homosexuality. If either Leviticus or Romans is true, then homosexuality is an abomination and it must be avoided. And if it must be avoided or it's contrary to nature, then if it's a sin, then

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it must not be allowed. Now, we know that Romans is true, and if the Bible condemns homosexuality, then it's a sin. Thus, same-sex marriage must not be allowed."

"Obviously you're injecting religion into the issue," Gina responds, as she waves to a friend in the gathering crowd. "But the First Amendment to the Constitution says that the state must not act either to establish a religion or to interfere with religious practices. If the state bars same-sex marriage for your reasons, it acts to establish a religion. So, it must not bar same-sex marriage for your reasons."

Ken opens his mouth to object, but this time it's Gina who's on a roll. "Also," she continues, "our country is based on the principle of equality. If straight couples can get married, and obviously they can, then either same-sex couples can get married or same-sex couples are not equal to straight couples. And if same-sex couples can get married, then our law has to change and other state laws have to change as well. Now, if our country is based on the principle of equality, then same-sex couples are equal to straight couples. Thus, our law has to change."

"Look," says Ken, "Marriage has always been between a man and a woman. And if that is so and tradition is worth preserving, then if we allow same-sex marriage, then the very concept of marriage will change and gender roles will switch. Now tradition is worth preserving and gender roles shouldn't switch. Thus, we cannot allow same-sex marriage."

"Ha!" says Gina. "I can see why you don't want gender roles to switch. You can't see yourself in the kitchen preparing meals and washing dishes."

"Well, I don't really relish the idea," Ken admits. "I think God wants us to keep things the way they are. But here's another reason. One of the chief purposes of marriage is raising children, and if that is so, then it's important that the children grow up well adjusted. But if the children are to be well adjusted, then they must have both a male role model and a female role model. If the parents are both men, then the children will have no female role model; if they're both women, then the children will have no male role model. Clearly if the marriage is a same-sex marriage, then the parents are either both men or they are both women. Therefore, the marriage must not be a same-sex marriage."

"Your reasoning is a bit shortsighted," Gina says. "In a same-sex marriage with children, the parents are either both men or they are both women. This much I grant you. But if they are both men, then surely they have close female friends, and if that is so, then the marriage has both male and female role models. If the parents are both women, then surely they have close male friends, and if that is so, then the marriage has both female and male role models. If a marriage has both male and female role models, then the children will be well adjusted. Therefore, in a same-sex marriage with children, the children will be well adjusted. What do you think of that?"

"Well," says Ken, as he scratches his head, "I wonder if those surrogate role models would be as effective as male and female parents. But in either event there is the option of civil unions. Why won't that satisfy you?"

"Civil unions fall short of marriage in too many ways," Gina replies. "They're valid only in the state in which they are performed, and they don't allow the partners either to file a joint federal tax return or to receive Social Security survivor benefits. If they're valid only in the state in which they're performed, then if the couple moves to a different state, then if one partner is hospitalized, the other partner will have no visitation rights. If the partners can't file a joint federal tax return, then they must file as single taxpayers. And if that is so, then they might pay much more in taxes. If the partners don't receive Social Security survivor benefits, then if a partner receiving Social Security benefits dies, then the other will not receive anything as a survivor. Suppose that two partners in a civil union move to a different state, and that one partner, who receives Social Security benefits, is hospitalized and eventually dies. The conclusion is that the other partner will not have either visitation rights while the hospitalized partner is alive or survivor benefits after that partner dies, and the partners might pay much more in taxes. Does that seem like a fair substitute for marriage?"

"Maybe not," Ken says, as he stretches to see above the crowd. "But I still think there's something unnatural about same-sex marriage. Anyway, here are the bride and groom. Let's go inside."

"Good, let's go," says Gina.

7.4 Rules of Replacement II

PREVIEW • *Your hike comes to an end on the bank of a river with several Class 3 rapids. Fortunately, before starting out on the hike you had arranged to have your kayak dropped off at this location. Kayaking down the river will pose an even greater—but more exhilarating—challenge than climbing the cliff. Selecting the best route through the rapids requires strategic vision, and negotiating the rocks without capsizing demands skill. You must be constantly aware of where you are and where you want to go. This section of the book supports further practice with this kind of thinking.*

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The remaining five rules of replacement are as follows:

14. Transposition (Trans):
    $(p \supset q) :: (\sim q \supset \sim p)$
15. Material implication (Impl):
    $(p \supset q) :: (\sim p \vee q)$
16. Material equivalence (Equiv):
    $(p \equiv q) :: [(p \supset q) \cdot (q \supset p)]$
    $(p \equiv q) :: [(p \cdot q) \vee (\sim p \cdot \sim q)]$
17. Exportation (Exp):
    $[(p \cdot q) \supset r] :: [p \supset (q \supset r)]$
18. Tautology (Taut):
    $p :: (p \vee p)$
    $p :: (p \cdot p)$

Transposition asserts that the antecedent and consequent of a conditional statement may switch places if and only if tildes are inserted before both or tildes are removed from both. The rule is fairly easy to understand and is easily proved by a truth table.

Material implication is less obvious than transposition, but it can be illustrated by substituting actual statements in place of the letters. For example, the statement "If you bother me, then I'll punch you in the nose" ($B \supset P$) is logically equivalent to "Either you stop bothering me or I'll punch you in the nose" ($\sim B \vee P$). The rule states that a horseshoe may be replaced by a wedge if the left-hand component is negated, and the reverse replacement is allowed if a tilde is deleted from the left-hand component.

Material equivalence has two formulations. The first is the same as the definition of material equivalence given in Section 6.1. The second formulation is easy to remember through recalling the two ways in which $p \equiv q$ may be true. Either p and q are both true or p and q are both false. This, of course, is the meaning of $[(p \bullet q) \vee (\sim p \bullet \sim q)]$.

Exportation is also fairly easy to understand. It asserts that the statement "If we have both p and q, then we have r" is logically equivalent to "If we have p, then if we have q, then we have r." As an illustration of this rule, the statement "If Bob and Sue told the truth, then Jim is guilty" is logically equivalent to "If Bob told the truth, then if Sue told the truth, then Jim is guilty."

Tautology, the last rule introduced in this section, is obvious. Its effect is to eliminate redundancy in disjunctions and conjunctions.

The following proofs illustrate the use of these five rules.

1. $\sim A$	/ $A \supset B$

In this argument the conclusion contains a letter not found in the premise. Obviously, addition must be used to introduce the B. The material implication rule completes the proof:

1. $\sim A$	/ $A \supset B$
2. $\sim A \vee B$	1, Add
3. $A \supset B$	2, Impl

Here is another example:

1. $F \supset G$	
2. $F \vee G$	/ G

To derive the conclusion of this argument, some method must be found to link the two premises together and eliminate the F. Hypothetical syllogism provides the solution, but first the second premise must be converted into a conditional. Here is the proof:

1. $F \supset G$	
2. $F \vee G$	/ G
3. $\sim\sim F \vee G$	2, DN
4. $\sim F \supset G$	3, Impl
5. $\sim F \supset \sim\sim G$	4, DN
6. $\sim G \supset F$	5, Trans
7. $\sim G \supset G$	1, 6, HS
8. $\sim\sim G \vee G$	7, Impl
9. $G \vee G$	8, DN
10. G	9, Taut

Another example:

1. $J \supset (K \supset L)$	/ $K \supset (J \supset L)$

The conclusion can be obtained by simply rearranging the components of the single premise. Exportation provides the simplest method:

1. $J \supset (K \supset L)$	/ $K \supset (J \supset L)$
2. $(J \bullet K) \supset L$	1, Exp
3. $(K \bullet J) \supset L$	2, Com
4. $K \supset (J \supset L)$	3, Exp

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Another example:

1. $M \supset N$
2. $M \supset O$ $\quad$ / $M \supset (N \bullet O)$

As with the F and G example, some method must be found to link the two premises together. In this case, however, hypothetical syllogism will not work. The solution lies in setting up a distribution step:

1.	$M \supset N$	
2.	$M \supset O$	/ $M \supset (N \bullet O)$
3.	$\sim M \vee N$	1, Impl
4.	$\sim M \vee O$	2, Impl
5.	$(\sim M \vee N) \bullet (\sim M \vee O)$	3, 4, Conj
6.	$\sim M \vee (N \bullet O)$	5, Dist
7.	$M \supset (N \bullet O)$	6, Impl

Another example:

1. $P \supset Q$
2. $R \supset (S \bullet T)$
3. $\sim R \supset \sim Q$
4. $S \supset (T \supset P)$ $\quad$ / $P \equiv R$

The conclusion is a biconditional, and there are only two ways that a biconditional can be obtained from such premises—namely, via the two formulations of the material equivalence rule. The fact that the premises are all conditional statements suggests the first formulation of this rule. Accordingly, we must try to obtain $P \supset R$ and $R \supset P$. Again, the fact that the premises are themselves conditionals suggests hypothetical syllogism to accomplish this. Premises 1 and 3 can be used to set up one hypothetical syllogism; premises 2 and 4 provide the other. Here is the proof:

1.	$P \supset Q$	
2.	$R \supset (S \bullet T)$	
3.	$\sim R \supset \sim Q$	
4.	$S \supset (T \supset P)$	/ $P \equiv R$
5.	$Q \supset R$	3, Trans
6.	$P \supset R$	1, 5, HS
7.	$(S \bullet T) \supset P$	4, Exp
8.	$R \supset P$	2, 7, HS
9.	$(P \supset R) \bullet (R \supset P)$	6, 8, Conj
10.	$P \equiv R$	9, Equiv

As we saw in Section 7.3, if it is not readily apparent how the conclusion should be derived, we can use the rules of replacement to deconstruct the conclusion. This will usually provide insight on how best to proceed. Again, this technique is justified because the rules of replacement are two-way rules. As a result, they can be applied in reverse order in the completed proof. Here is an example:

1. $\sim S \supset K$
2. $S \supset (R \vee M)$ $\quad$ / $\sim R \supset (\sim M \supset K)$

In deconstructing the conclusion, the form of the conclusion suggests exportation, and the result of this step suggests De Morgan's rule. For further insight, we apply transposition to the latter step. Each step follows from the one preceding it:

$\sim R \supset (\sim M \supset K)$	
$(\sim R \bullet \sim M) \supset K$	Exp
$\sim(R \vee M) \supset K$	DM
$\sim K \supset \sim\sim(R \vee M)$	Trans
$\sim K \supset (R \vee M)$	DN

Now, examining the premises in light of the deconstruction suggests that we begin by setting up a hypothetical syllogism. This will give us the last step in the deconstruction. We can then obtain the conclusion by repeating the deconstruction steps in reverse order. The completed proof is as follows:

1. $\sim S \supset K$	
2. $S \supset (R \vee M)$	/ $\sim R \supset (\sim M \supset K)$
3. $\sim K \supset \sim\sim S$	1, Trans
4. $\sim K \supset S$	3, DN
5. $\sim K \supset (R \vee M)$	2, 4, HS
6. $\sim(R \vee M) \supset \sim\sim K$	5, Trans
7. $\sim(R \vee M) \supset K$	6, DN
8. $(\sim R \bullet \sim M) \supset K$	7, DM
9. $\sim R \supset (\sim M \supset K)$	8, Exp

Here is another example:

1. $K \supset M$	
2. $L \supset M$	/ $(K \vee L) \supset M$

In deconstructing the conclusion, the form of the premises suggests that we use some procedure that will combine M separately with K and L. This, in turn, suggests distribution; but before we can use distribution, we must eliminate the horseshoe via material implication. The deconstruction is as follows:

$(K \vee L) \supset M$	
$\sim(K \vee L) \vee M$	Impl
$(\sim K \bullet \sim L) \vee M$	DM
$M \vee (\sim K \bullet \sim L)$	Com
$(M \vee \sim K) \bullet (M \vee \sim L)$	Dist
$(\sim K \vee M) \bullet (M \vee \sim L)$	Com
$(\sim K \vee M) \bullet (\sim L \vee M)$	Com
$(K \supset M) \bullet (\sim L \vee M)$	Impl
$(K \supset M) \bullet (L \supset M)$	Impl

Now, examining the premises in light of the last line of the deconstruction suggests that we begin by joining the premises together via the conjunction rule. The conclusion can

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then be obtained by reversing the steps of the deconstruction:

1.	$K \supset M$	
2.	$L \supset M$	/ $(K \vee L) \supset M$
3.	$(K \supset M) \bullet (L \supset M)$	1, 2, Conj
4.	$(\sim K \vee M) \bullet (L \supset M)$	3, Impl
5.	$(\sim K \vee M) \bullet (\sim L \vee M)$	4, Impl
6.	$(M \vee \sim K) \bullet (\sim L \vee M)$	5, Com
7.	$(M \vee \sim K) \bullet (M \vee \sim L)$	6, Com
8.	$M \vee (\sim K \bullet \sim L)$	7, Dist
9.	$(\sim K \bullet \sim L) \vee M$	8, Com
10.	$\sim(K \vee L) \vee M$	9, DM
11.	$(K \vee L) \supset M$	10, Impl

Note that whenever we use this strategy of working backward from the conclusion, the rules of replacement are the *only* rules we may use. We may not use the rules of implication, because these rules are one-way rules.

This section ends with some strategies that show how the last five rules of replacement can be used together with various other rules.

Strategy 17: Material implication can be used to set up hypothetical syllogism:

1.	$\sim A \vee B$	
2.	$\sim B \vee C$	
3.	$A \supset B$	1, Impl
4.	$B \supset C$	2, Impl
5.	$A \supset C$	3, 4, HS

Strategy 18: Exportation can be used to set up *modus ponens:*

1.	$(A \bullet B) \supset C$	
2.	A	
3.	$A \supset (B \supset C)$	1, Exp
4.	$B \supset C$	2, 3, MP

Strategy 19: Exportation can be used to set up *modus tollens:*

1.	$A \supset (B \supset C)$	
2.	$\sim C$	
3.	$(A \bullet B) \supset C$	1, Exp
4.	$\sim(A \bullet B)$	2, 3, MT

Strategy 20: Addition can be used to set up material implication:

1.	A	
2.	$A \vee \sim B$	1, Add
3.	$\sim B \vee A$	2, Com
4.	$B \supset A$	3, Impl

Strategy 21: Transposition can be used to set up hypothetical syllogism:

1.	$A \supset B$	
2.	$\sim C \supset \sim B$	
3.	$B \supset C$	2, Trans
4.	$A \supset C$	1, 3, HS

Strategy 22: Transposition can be used to set up constructive dilemma:

1. $(A \supset B) \cdot (C \supset D)$
2. $\sim B \vee \sim D$
3. $(\sim B \supset \sim A) \cdot (C \supset D)$ 1, Trans
4. $(\sim B \supset \sim A) \cdot (\sim D \supset \sim C)$ 3, Trans
5. $\sim A \vee \sim C$ 2, 4, CD

Strategy 23: Constructive dilemma can be used to set up tautology:

1. $(A \supset C) \cdot (B \supset C)$
2. $A \vee B$
3. $C \vee C$ 1, 2, CD
4. C 3, Taut

Strategy 24: Material implication can be used to set up tautology:

1. $A \supset \sim A$
2. $\sim A \vee \sim A$ 1, Impl
3. $\sim A$ 2, Taut

Strategy 25: Material implication can be used to set up distribution:

1. $A \supset (B \cdot C)$
2. $\sim A \vee (B \cdot C)$ 1, Impl
3. $(\sim A \vee B) \cdot (\sim A \vee C)$ 2, Dist

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EXERCISE 7.4

I. For each of the following lists of premises, derive the indicated conclusion and complete the justification. For tautology, derive a conclusion that is simpler than the premise.

★(1) 1. $H \vee F$
2. $N \vee \sim S$
3. $\sim G \vee Q$
4. _______ _____, Impl

(2) 1. $R \supset (S \supset N)$
2. $T \supset (U \vee M)$
3. $K \cdot (L \supset W)$
4. ____________ _____, Exp

(3) 1. $G \equiv R$
2. $H \supset P$
3. $\sim F \vee T$
4. _______ _____, Trans

★(4) 1. $(B \supset N) \cdot (N \supset B)$
2. $(R \vee F) \cdot (F \vee R)$
3. $(K \supset C) \vee (C \supset K)$
4. __________________ _____, Equiv

(5) 1. $E \lor \sim E$
2. $A \lor A$
3. $G \bullet \sim G$
4. ________ _____, Taut

(6) 1. $S \lor \sim M$
2. $\sim N \bullet \sim T$
3. $\sim L \supset Q$
4. ________ _____, Trans

★(7) 1. $\sim C \supset \sim F$
2. $D \lor \sim P$
3. $\sim R \bullet Q$
4. ________ _____, Impl

(8) 1. $E \supset (R \bullet Q)$
2. $(G \bullet N) \supset Z$
3. $(S \supset M) \supset P$
4. ________ _____, Exp

(9) 1. $(D \bullet H) \lor (\sim D \bullet \sim H)$
2. $(F \supset J) \bullet (\sim F \supset \sim J)$
3. $(N \lor T) \bullet (\sim N \lor \sim T)$
4. ________ _____, Equiv

★(10) 1. $L \supset (A \supset A)$
2. $K \supset (R \lor \sim R)$
3. $S \supset (G \bullet G)$
4. ________ _____, Taut

(11) 1. $K \bullet (S \lor B)$
2. $\sim F \supset \sim J$
3. $\sim E \lor \sim M$
4. ________ _____, Trans

(12) 1. $H \supset (K \bullet J)$
2. $(N \lor E) \supset B$
3. $C \supset (H \supset A)$
4. ________ _____, Exp

★(13) 1. $(A \supset \sim C) \bullet (C \supset \sim A)$
2. $(W \supset \sim T) \bullet (\sim T \supset W)$
3. $(M \supset \sim E) \bullet (\sim M \supset E)$
4. ________ _____, Equiv

(14) 1. $(\sim K \lor M) \equiv S$
2. $T \lor (F \bullet G)$
3. $R \equiv (N \bullet \sim H)$
4. ________ _____, Impl

(15) 1. $(S \lor S) \supset D$
2. $K \supset (T \bullet \sim T)$
3. $(Q \supset Q) \supset M$
4. ________ _____, Taut

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II. In the following symbolized arguments, derive the line needed to obtain the conclusion (last line), and supply the justification for both lines.

★(1)
1. $\sim J \vee M$
2. $M \supset B$
3. ______ ____
4. $J \supset B$ ____

(2)
1. $(J \bullet F) \supset N$
2. J
3. ______ ____
4. $F \supset N$ ____

(3)
1. $C \supset A$
2. $A \supset C$
3. ______ ____
4. $C \equiv A$ ____

★(4)
1. $(G \supset K) \bullet (T \supset K)$
2. $G \vee T$
3. ______ ____
4. K ____

(5)
1. $(G \supset B) \bullet (\sim C \supset \sim H)$
2. $G \vee H$
3. ______ ____
4. $B \vee C$ ____

(6)
1. $J \supset (M \supset Q)$
2. $J \bullet M$
3. ______ ____
4. Q ____

★(7)
1. $H \supset (\sim C \vee R)$
2. ______ ____
3. $(H \bullet C) \supset R$ ____

(8)
1. $\sim G \supset \sim T$
2. $G \supset N$
3. ______ ____
4. $T \supset N$ ____

(9)
1. $K \supset (A \supset F)$
2. $\sim F$
3. ______ ____
4. $\sim(K \bullet A)$ ____

★(10)
1. $H \supset \sim H$
2. ______ ____
3. $\sim H$ ____

(11)
1. $\sim S$
2. ______ ____
3. $S \supset K$ ____

(12)
1. $M \supset (M \supset D)$
2. ______ ____
3. $M \supset D$ ____

★(13)
1. $(N \supset A) \bullet (\sim N \supset \sim A)$
2. ______ ____
3. $N \equiv A$ ____

(14)
1. $E \bullet R$
2. ______ ____
3. $E \equiv R$ ____

(15)
1. $Q \supset (\sim W \supset \sim G)$
2. ______ ____
3. $(Q \bullet G) \supset W$ ____

III. Use the eighteen rules of inference to derive the conclusions of the following symbolized arguments.

★(1)
1. $(S \bullet K) \supset R$
2. K / $S \supset R$

(2)
1. $T \supset (F \vee F)$
2. $\sim(F \bullet F)$ / $\sim T$

(3)
1. $G \supset E$
2. $H \supset \sim E$ / $G \supset \sim H$

★(4)
1. $S \equiv Q$
2. $\sim S$ / $\sim Q$

(5)
1. $\sim N \vee P$
2. $(N \supset P) \supset T$ / T

(6) 1. $F \supset B$
2. $B \supset (B \supset J)$ / $F \supset J$

★(7) 1. $(B \supset M) \bullet (D \supset M)$
2. $B \vee D$ / M

(8) 1. $Q \supset (F \supset A)$
2. $R \supset (A \supset F)$
3. $Q \bullet R$ / $F \equiv A$

(9) 1. $T \supset (\sim T \vee G)$
2. $\sim G$ / $\sim T$

★(10) 1. $(B \supset G) \bullet (F \supset N)$
2. $\sim(G \bullet N)$ / $\sim(B \bullet F)$

(11) 1. $(J \bullet R) \supset H$
2. $(R \supset H) \supset M$
3. $\sim(P \vee \sim J)$ / $M \bullet \sim P$

(12) 1. T / $S \supset T$

★(13) 1. $K \supset (B \supset \sim M)$
2. $D \supset (K \bullet M)$ / $D \supset \sim B$

(14) 1. $(O \supset C) \bullet (\sim S \supset \sim D)$
2. $(E \supset D) \bullet (\sim E \supset \sim C)$ / $O \supset S$

(15) 1. $\sim(U \bullet W) \supset X$
2. $U \supset \sim U$ / $\sim(U \vee \sim X)$

★(16) 1. $T \supset R$
2. $T \supset \sim R$ / $\sim T$

(17) 1. $S \vee \sim N$
2. $\sim S \vee Q$ / $N \supset Q$

(18) 1. $M \supset (U \supset H)$
2. $(H \vee \sim U) \supset F$ / $M \supset F$

★(19) 1. $\sim R \vee P$
2. $R \vee \sim P$ / $R \equiv P$

(20) 1. $\sim H \supset B$
2. $\sim H \supset D$
3. $\sim(B \bullet D)$ / H

(21) 1. $J \supset (G \supset L)$ / $G \supset (J \supset L)$

★(22) 1. $S \supset (L \bullet M)$
2. $M \supset (L \supset R)$ / $S \supset R$

(23) 1. $F \supset (A \bullet K)$
2. $G \supset (\sim A \bullet \sim K)$
3. $F \vee G$ / $A \equiv K$

(24) 1. $(I \supset E) \supset C$
2. $C \supset {\sim}C$ / I

★(25) 1. $T \supset G$
2. $S \supset G$ / $(T \vee S) \supset G$

(26) 1. $H \supset U$ / $H \supset (U \vee T)$

(27) 1. $Q \supset (W \bullet D)$ / $Q \supset W$

★(28) 1. $P \supset ({\sim}E \supset B)$
2. ${\sim}(B \vee E)$ / ${\sim}P$

(29) 1. $(G \supset J) \supset (H \supset Q)$
2. $J \bullet {\sim}Q$ / ${\sim}H$

(30) 1. $I \vee (N \bullet F)$
2. $I \supset F$ / F

★(31) 1. $K \equiv R$
2. $K \supset (R \supset P)$
3. ${\sim}P$ / ${\sim}R$

(32) 1. $C \supset ({\sim}L \supset Q)$
2. $L \supset {\sim}C$
3. ${\sim}Q$ / ${\sim}C$

(33) 1. $(E \supset A) \bullet (F \supset A)$
2. $E \vee G$
3. $F \vee {\sim}G$ / A

★(34) 1. $(F \bullet H) \supset N$
2. $F \vee S$
3. H / $N \vee S$

(35) 1. $T \supset (H \bullet J)$
2. $(H \vee N) \supset T$ / $T \equiv H$

(36) 1. $T \supset {\sim}(A \supset N)$
2. $T \vee N$ / $T \equiv {\sim}N$

★(37) 1. $(D \supset E) \supset (E \supset D)$
2. $(D \equiv E) \supset {\sim}(G \bullet {\sim}H)$
3. $E \bullet G$ / $G \bullet H$

(38) 1. $(O \supset R) \supset S$
2. $(P \supset R) \supset {\sim}S$ / ${\sim}R$

(39) 1. $(L \vee P) \supset U$
2. $(M \supset U) \supset I$
3. P / I

★(40) 1. $A \equiv W$
2. ${\sim}A \vee {\sim}W$
3. $R \supset A$ / ${\sim}(W \vee R)$

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(41) 1. $(S \vee T) \supset (S \supset \sim T)$
2. $(S \supset \sim T) \supset (T \supset K)$
3. $S \vee T$ / $S \vee K$

(42) 1. $G \equiv M$
2. $G \vee M$
3. $G \supset (M \supset T)$ / T

★(43) 1. $O \supset (Q \bullet N)$
2. $(N \vee E) \supset S$ / $O \supset S$

(44) 1. $H \equiv I$
2. $H \supset (I \supset F)$
3. $\sim(H \vee I) \supset F$ / F

★(45) 1. $P \supset A$
2. $Q \supset B$ / $(P \vee Q) \supset (A \vee B)$

IV. Translate the following arguments into symbolic form and then use the eighteen rules of inference to derive the conclusion of each. Use the translation letters in the order in which they are listed.

★1. If sports-shoe manufacturers decline to use kangaroo hides in their products, then Australian hunters will cease killing millions of kangaroos yearly. It is not the case that both Australian hunters will cease killing millions of kangaroos yearly and the kangaroo will not be saved from extinction. Therefore, if sports-shoe manufacturers decline to use kangaroo hides in their products, then the kangaroo will be saved from extinction. (D, C, S)

2. If there is a direct correlation between what a nation spends for health care and the health of its citizens, then America has the lowest incidence of disease and the lowest mortality rates of any nation on Earth. But America does not have the lowest mortality rates of any nation on Earth. Therefore, there is not a direct correlation between what a nation spends for health care and the health of its citizens. (C, D, M)

3. It is not the case that strict controls exist on either the manufacture or the sale of handguns. Therefore, if strict controls exist on the sale of handguns, then the use of handguns in the commission of crimes has decreased. (M, S, U)

★4. If birth-control devices are made available in high school clinics, then the incidence of teenage pregnancy will decrease. Therefore, if both birth-control information and birth-control devices are made available in high school clinics, then the incidence of teenage pregnancy will decrease. (D, P, I)

5. If Congress enacts a law that either establishes a religion or prohibits the free exercise of religion, then that law is unconstitutional. Therefore, if Congress enacts a law that establishes a religion, then that law is unconstitutional. (E, P, U)

6. If cigarette smokers are warned of the hazards of smoking and they continue to smoke, then they cannot sue tobacco companies for any resulting lung cancer or emphysema. Cigarette smokers are warned of the hazards of smoking.

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Therefore, if cigarette smokers continue to smoke, they cannot sue tobacco companies for any resulting lung cancer or emphysema. (W, C, S)

★7. If grade-school children are assigned daily homework, then their achievement level will increase dramatically. But if grade-school children are assigned daily homework, then their love for learning may be dampened. Therefore, if grade-school children are assigned daily homework, then their achievement level will increase dramatically but their love for learning may be dampened. (G, A, L)

8. If a superconducting particle collider is built, then the data yielded will benefit scientists of all nations and it deserves international funding. Either a superconducting particle collider will be built, or the ultimate nature of matter will remain hidden and the data yielded will benefit scientists of all nations. Therefore, the data yielded by a superconducting particle collider will benefit scientists of all nations. (S, D, I, U)

9. If parents are told that their unborn child has Tay–Sachs disease, then if they go ahead with the birth, then they are responsible for their child's pain and suffering. Therefore, if parents are not responsible for their child's pain and suffering, then if they go ahead with the birth, then they were not told that their unborn child had Tay–Sachs disease. (T, G, R)

★10. Vitamin E is an antioxidant and a useless food supplement if and only if it does not reduce heart disease. It is not the case either that vitamin E does not reduce heart disease or is not an antioxidant. Therefore, vitamin E is not a useless food supplement. (A, U, R)

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V. The following dialogue contains ten arguments. Translate each into symbolic form and then use the eighteen rules of inference to derive the conclusion of each.

Is This the End?

Brian and Molly are at the memorial service of a mutual friend who had died suddenly the week before. "I'm still shocked to think that Karl is gone," Molly says.

"I know you were quite close to him," Brian says. "But do you think in some sense Karl could still be with us? I mean do you think there could be such a thing as postmortem persistence of consciousness—life after death, as most people say?"

"I wish there were," Molly replies, "and that's what makes death so tragic. As I see it, the mind is totally dependent on the brain, and if that's so, when the brain dies, the mind dies. If the mind dies, then consciousness dies, too. Thus, if the brain dies, then consciousness dies—which means there's no life after death."

"But what makes you think that the mind is totally dependent on the brain?" Brian asks.

"Our day-to-day experience provides lots of evidence," Molly replies. "If you drink alcohol, your mind is affected. If you smoke marijuana, your mind is affected. If your mind is affected by these things, then you have firsthand experience that the mind is dependent on the brain. Thus, if you either smoke marijuana or drink alcohol, then you have firsthand experience that the mind is dependent on the brain."

"So the mind is affected by the brain. Anyone with ordinary sensation knows that," Brian retorts. "If your eye receives a visual stimulus, then that stimulus is sent to the brain and your mind is affected. If your ear receives an auditory stimulus, then your mind is affected. Thus, if either your eye or your ear receives a stimulus, then your mind is affected. But that doesn't prove that the mind is *necessarily* dependent on the brain. And there are lots of reasons for saying that it isn't."

"What reasons are those?" Molly asks.

"Well, we learned about Plato in Introduction to Philosophy," Brian replies. "And Plato held that the mind can conceive ideal objects such as perfect justice and perfect triangularity. Now, if either of these concepts came through the senses, then perfect ideals exist in nature. But no perfect ideals exist in nature. And if the concept of triangularity did not come through the senses, then the mind produced it independently of the brain. But if that is the case or the concept of triangularity is innate, then the mind is not necessarily dependent on the brain. I'll leave the conclusion up to you"

"Very interesting," Molly replies, "but I question whether the mind is really capable of conceiving ideal objects such as perfect justice and perfect triangularity. For me, these things are just words. But there are other reasons for thinking that the mind is *necessarily* dependent on the brain. Consider the visual cortex. The visual cortex is part of the brain. If the visual cortex isn't stimulated, there is no visual sensation. But if visual sensation occurs only if the visual cortex is stimulated, and if the visual cortex is part of the brain, then visual sensation is dependent on the brain. And if that is true and visual sensation is a function of the mind, then the mind is necessarily dependent on the brain. Therefore, if visual sensation is a function of the mind, then the mind is necessarily dependent on the brain."

"Furthermore," Molly continues, "there are many cases where strokes have caused loss of memory, and also loss of speech. But if remembering is a mental function, then if the mind is not necessarily dependent on the brain, then strokes do not cause loss of memory. Therefore, if remembering is a mental function, then the mind is necessarily dependent on the brain."

"It may indeed be the case," Brian replies, "that memory—or at least certain kinds of memory—are dependent on the brain. And the same may be true of sensation. But that doesn't prove that consciousness as such is brain dependent. It seems to me that consciousness as such is a nonmaterial process, and that it can occur only in a nonmaterial entity, such as a soul. And if those two claims are true and the soul is immortal, then consciousness survives the death of the body. Thus, if the soul is immortal, then consciousness survives the death of the body."

"If memory goes with the brain," Molly replies, "then I wonder if the consciousness you speak of is in any way *your* consciousness. But setting that aside, are there any reasons for thinking that the soul is immortal?"

"I think there are," Brian replies. "If the soul is nonmaterial, then it has no parts, and if it has no parts, then it can't come 'a-part'—in other words it can't disintegrate. And if it can't disintegrate, then if nothing can destroy it, then it is immortal. But the soul can be destroyed only if God destroys it, and God does not destroy souls. Therefore, if the soul is nonmaterial, then it is immortal. I think Leibniz invented that argument."

"Fine," Molly says. "But what makes you think that you have a nonmaterial soul in the first place?"

"Well," Brian replies, "according to Descartes, I am essentially either a mind or a body. But if I can doubt that I have a body, then I am not essentially a body. And I can doubt that I have a body. For example I can imagine that I am in *The Matrix,* and that all of my sensations are illusions. If I am essentially a mind, then if the essence of mind is to be nonextended, then I am a nonextended substance. But the essence of mind, being different from the essence of body, is to be nonextended. And if I am a nonextended substance, then I am (or have) a nonmaterial soul. Therefore, I am (or have) a nonmaterial soul."

"Your argument is so abstruse that I don't find it very persuasive," says Molly, as she scratches her head. "I think the evidence is overwhelming that humans are the product of biological evolution, and if that is true and humans have souls, then there is a point in the course of evolution where humans either received or developed a soul. But there is no evidence that humans ever received a soul. Also, there is no evidence that humans ever developed a soul. Therefore, humans do not have souls."

"Wow, that sounds pretty far out," Brian replies. "Well, it looks like the service is ready to start, so we'll have to hang this up. But maybe we can continue it at a later date."

"Maybe we can," Molly replies.

7.5 Conditional Proof

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PREVIEW • *Returning to our hiking metaphor, suppose, while in the middle of your hike, you come to a fork in the trail. You consult your trail map but find no reference to it. Since the left fork seems to lead off in the direction you want to go, you formulate the assumption that this fork is a shortcut that connects with the main trail. You take the left fork and find that you were correct. Then you alter your trail map to show the existence of the fork and the alternate route. Making this alteration resembles conditional proof.*

Conditional proof is a method for deriving a conditional statement (either the conclusion or some intermediate line) that offers the usual advantage of being both shorter and simpler to use than the direct method. Moreover, some arguments have conclusions that cannot be derived by the direct method, so some form of conditional proof must be used on them. Conditional proof may thus be seen as completing the rules of inference. The method consists of assuming the antecedent of the required conditional statement on one line, deriving the consequent on a subsequent line, and then "discharging" this sequence of lines in a conditional statement that exactly replicates the one that was to be obtained.

Any argument whose conclusion is a conditional statement is an immediate candidate for conditional proof. Consider the following example:

1. $A \supset (B \bullet C)$
2. $(B \vee D) \supset E$ / $A \supset E$