"Well," Brian replies, "according to Descartes, I am essentially either a mind or a body. But if I can doubt that I have a body, then I am not essentially a body. And I can doubt that I have a body. For example I can imagine that I am in *The Matrix,* and that all of my sensations are illusions. If I am essentially a mind, then if the essence of mind is to be nonextended, then I am a nonextended substance. But the essence of mind, being different from the essence of body, is to be nonextended. And if I am a nonextended substance, then I am (or have) a nonmaterial soul. Therefore, I am (or have) a nonmaterial soul."

"Your argument is so abstruse that I don't find it very persuasive," says Molly, as she scratches her head. "I think the evidence is overwhelming that humans are the product of biological evolution, and if that is true and humans have souls, then there is a point in the course of evolution where humans either received or developed a soul. But there is no evidence that humans ever received a soul. Also, there is no evidence that humans ever developed a soul. Therefore, humans do not have souls."

"Wow, that sounds pretty far out," Brian replies. "Well, it looks like the service is ready to start, so we'll have to hang this up. But maybe we can continue it at a later date."

"Maybe we can," Molly replies.

aplia™
Visit Aplia for section-specific problem sets.

7.5 Conditional Proof

7

PREVIEW • *Returning to our hiking metaphor, suppose, while in the middle of your hike, you come to a fork in the trail. You consult your trail map but find no reference to it. Since the left fork seems to lead off in the direction you want to go, you formulate the assumption that this fork is a shortcut that connects with the main trail. You take the left fork and find that you were correct. Then you alter your trail map to show the existence of the fork and the alternate route. Making this alteration resembles conditional proof.*

Conditional proof is a method for deriving a conditional statement (either the conclusion or some intermediate line) that offers the usual advantage of being both shorter and simpler to use than the direct method. Moreover, some arguments have conclusions that cannot be derived by the direct method, so some form of conditional proof must be used on them. Conditional proof may thus be seen as completing the rules of inference. The method consists of assuming the antecedent of the required conditional statement on one line, deriving the consequent on a subsequent line, and then "discharging" this sequence of lines in a conditional statement that exactly replicates the one that was to be obtained.

Any argument whose conclusion is a conditional statement is an immediate candidate for conditional proof. Consider the following example:

1. $A \supset (B \bullet C)$
2. $(B \vee D) \supset E$ $\quad$ / $A \supset E$

Using the direct method to derive the conclusion of this argument would require a proof having at least twelve lines, and the precise strategy to be followed in constructing it might not be immediately obvious. Nevertheless, we need only give cursory inspection to the argument to see that the conclusion does indeed follow from the premises. The conclusion states that if we have A, we then have E. Let us suppose, for a moment, that we do have A. We could then derive $B \bullet C$ from the first premise via *modus ponens*. Simplifying this expression we could derive B, and from this we could get $B \vee D$ via addition. E would then follow from the second premise via *modus ponens*. In other words, if we assume that we have A, we can get E. But this is exactly what the conclusion says. Thus, we have just proved that the conclusion follows from the premises.

The method of conditional proof consists of incorporating this simple thought process into the body of a proof sequence. A conditional proof for this argument requires only eight lines and is substantially simpler than a direct proof:

1. $A \supset (B \bullet C)$	
2. $(B \vee D) \supset E$	/ $A \supset E$
│ 3. A	ACP
│ 4. $B \bullet C$	1, 3, MP
│ 5. B	4, Simp
│ 6. $B \vee D$	5, Add
│ 7. E	2, 6, MP
8. $A \supset E$	3–7, CP

Lines 3 through 7 are indented to indicate their hypothetical character: They all depend on the assumption introduced in line 3 via ACP (assumption for conditional proof). These lines, which constitute the conditional proof sequence, tell us that if we assume A (line 3), we can derive E (line 7). In line 8 the conditional sequence is discharged in the conditional statement $A \supset E$, which simply reiterates the result of the conditional sequence. Since line 8 is not hypothetical, it is written adjacent to the original margin, under lines 1 and 2. A vertical line is added to the conditional sequence to emphasize the indentation.

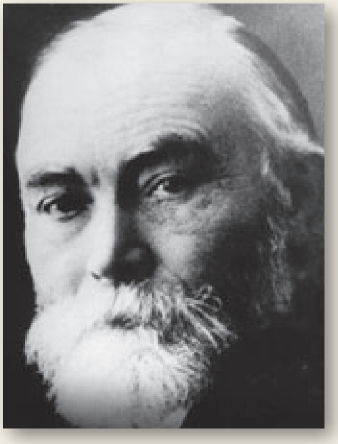
The first step in constructing a conditional proof is to decide what should be assumed on the first line of the conditional sequence. While any statement whatsoever *can* be assumed on this line, only the right statement will lead to the desired result. The clue is always provided by the conditional statement to be obtained in the end. The antecedent of this statement is what must be assumed. For example, if the statement to be obtained is $(K \bullet L) \supset M$, then $K \bullet L$ should be assumed on the first line. This line is always indented and tagged with the designation "ACP." Once the initial assumption has been made, the second step is to derive the consequent of the desired conditional statement at the end of the conditional sequence. To do this, we simply apply the ordinary rules of inference to any previous line in the proof (including the assumed line), writing the result directly below the assumed line. The third and final step is to discharge the conditional sequence in a conditional statement. The antecedent of this conditional statement is whatever appears on the first line of the conditional sequence,

Gottlob Frege 1848–1925

The German mathematician, logician, and philosopher Gottlob Frege (pronounced fray-ga) was born in Wismar, a small town in northern Germany on the Baltic Sea. His parents taught at a private girls' school, which his father had helped to found. Frege attended the local gymnasium, where he studied mathematics, and then the University of Jena, where he studied mathematics, philosophy, and chemistry. After two years he transferred to the University of Göttingen, earning a doctor's degree in mathematics at age twenty-four. He then returned to the University of Jena, where he taught until retiring in 1917. While there he married Margaret Liesburg, who bore him at least two children. The children died young, but years later the couple adopted a son, Alfred.

Frege spent his entire life analyzing the concept of number, developing theories of logic and language, and attempting to reduce arithmetic to logic. In 1879 he published the *Begriffsschrift* ("Concept-Script"), a work written in the tradition of Leibniz that develops a purely formal symbolic language to express any proposition in any area of human discourse. Five years later he published *Die Grundlagen der Arithmetik* ("The Foundations of Arithmetic"), a less technical work containing few symbols that outlined his goal of reducing arithmetic to logic. Then, nine years later he published Volume 1 of *Grundgesetze der Arithmetik* ("Basic Laws of Arithmetic"), which attempted to accomplish the first phase of this reduction.

None of these works were well received, for several reasons: They were ahead of their time, the symbolic notation of the technical works struck readers as bizarre, and they were written in German, whereas most of the new work in logic was being done by English speakers. In fact, the last of these works was so badly reviewed that Frege was forced to publish Volume 2 at his own expense. To make matters worse, in 1902, while Volume 2 was in proof, Frege received a letter from Bertrand Russell that left him "thunderstruck." He was later to remark that it had destroyed his entire life's work.



Pictorial Press Ltd/Alamy

Basic Law 5 of the *Grundgesetze* provides for the creation of classes of things merely by describing the properties of their members. So Russell invited Frege to create the class of all classes that are not members of themselves, and he then asked whether this very class is a member of itself. If it is a member of itself, then it is one of those classes that are not members of themselves; but if is not a member of itself, then, again, it is one of those classes, and it is a member of itself. The derivation of this contradiction (which has come to be called Russell's paradox) meant that the axioms of the *Grundgesetze* were fatally inconsistent. Frege attempted a last-minute modification of his system, but the change proved unworkable.

Despite this setback, Frege is universally recognized today as one of the most important logicians and philosophers of all time. Single-handedly he developed quantification theory and predicate logic, and his analysis of the concept of number led to a general theory of meaning that introduced the important distinction between *Sinn* ("sense") and *Beduetung* ("reference"). Also, his work on concept clarification initiated the current movement known as analytic philosophy.

and the consequent is whatever appears on the last line. For example, if $A \vee B$ is on the first line and $C \bullet D$ is on the last, the sequence is discharged by $(A \vee B) \supset (C \bullet D)$. This discharging line is always written adjacent to the original margin and is tagged with the designation "CP" (conditional proof) together with the numerals indicating the first through the last lines of the sequence.

Conditional proof can also be used to derive a line other than the conclusion of an argument. The following proof, which illustrates this fact, incorporates two conditional sequences one after the other within the scope of a single direct proof:

1. $G \supset (H \bullet I)$	
2. $J \supset (K \bullet L)$	
3. $G \vee J$	/ $H \vee K$
	4. G	ACP
	5. $H \bullet I$	1, 4, MP
	6. H	5, Simp
7. $G \supset H$	4–6, CP
	8. J	ACP
	9. $K \bullet L$	2, 8, MP
	10. K	9, Simp
11. $J \supset K$	8–10, CP
12. $(G \supset H) \bullet (J \supset K)$	7, 11, Conj
13. $H \vee K$	3, 12, CD

The first conditional proof sequence gives us $G \supset H$, and the second $J \supset K$. These two lines are then conjoined and used together with line 3 to set up a constructive dilemma, from which the conclusion is derived.

This proof sequence provides a convenient opportunity to introduce an important rule governing conditional proof. The rule states that after a conditional proof sequence has been discharged, no line in the sequence may be used as a justification for a subsequent line in the proof. If, for example, line 5 in the proof just given were used as a justification for line 9 or line 12, this rule would be violated, and the corresponding inference would be invalid. Once the conditional sequence is discharged, it is sealed off from the remaining part of the proof. The logic behind this rule is easy to understand. The lines in a conditional sequence are hypothetical in that they depend on the assumption stated in the first line. Because no mere assumption can provide any genuine support for anything, neither can any line that depends on such an assumption. When a conditional sequence is discharged, the assumption on which it rests is expressed as the antecedent of a conditional statement. This conditional statement *can* be used to support subsequent lines because it makes no claim that its antecedent is true. The conditional statement merely asserts that *if* its antecedent is true, then its consequent is true, and this, of course, is what has been established by the conditional sequence from which it is obtained.

Just as a conditional sequence can be used within the scope of a direct proof to derive a desired statement, one conditional sequence can be used within the scope of another to derive a desired statement. The following proof provides an example:

1. $L \supset [M \supset (N \vee O)]$		
2. $M \supset {\sim}N$		/ $L \supset ({\sim}M \vee O)$
	3. L	ACP
	4. $M \supset (N \vee O)$	1, 3, MP
	5. M	ACP
	6. $N \vee O$	4, 5, MP
	7. ${\sim}N$	2, 5, MP
	8. O	6, 7, DS
	9. $M \supset O$	5–8, CP
	10. ${\sim}M \vee O$	9, Impl
11. $L \supset ({\sim}M \vee O)$		3–10, CP

The rule introduced in connection with the previous example applies unchanged to examples of this sort. No line in the sequence 5–8 could be used to support any line subsequent to line 9, and no line in the sequence 3–10 could be used to support any line subsequent to line 11. Lines 3 or 4 could, of course, be used to support any line in the sequence 5–8.

One final reminder regarding conditional proof is that every conditional proof must be discharged. It is absolutely improper to end a proof on an indented line. If this rule is ignored, any conclusion one chooses can be derived from any set of premises. The following invalid proof illustrates this mistake:

1. P		/ $Q \supset R$
	2. ${\sim}Q$	ACP
	3. ${\sim}Q \vee R$	2, Add
	4. $Q \supset R$	2, Impl

7

EXERCISE 7.5

I. Use conditional proof and the eighteen rules of inference to derive the conclusions of the following symbolized arguments. Having done so, attempt to derive the conclusions without using conditional proof.

★(1) 1. $N \supset O$
2. $N \supset P$ / $N \supset (O \bullet P)$

(2) 1. $F \supset E$
2. $(F \bullet E) \supset R$ / $F \supset R$

(3) 1. $G \supset T$
2. $(T \vee S) \supset K$ / $G \supset K$

★(4) 1. $(G \vee H) \supset (S \bullet T)$
2. $(T \vee U) \supset (C \bullet D)$ / $G \supset C$

(5) 1. $A \supset {\sim}(A \vee E)$ / $A \supset F$