Indirect Proof Method

In section 7.6 we were introduced to the indirect proof method. This method is very powerful and very widely used in advanced logic and mathematics. The basic idea is that if one assumption leads to a contradiction, then that assumption's negation must be true. Why is that? Well, if something leads to a contradiction it is probably bad in some way. The idea is that if a contradiction is derivable from a particular assumption, then the assumption leads to / entails something that is always false. But, we want assumptions that if true, would necessitate the truth of the conclusion (i.e. we want valid arguments). Since an assumption that leads to something that can never be true cannot be used to make an argument valid, it must not be rejected. However, the negation of this faulty assumption would, then, have to acceptable since because the negation of an assumption that leads to a contradiction, cannot also lead to a contradiction (give the way negation works). Note the following way of thinking about the matter:

> Let *F* stand for a contradiction, and *T* stand for a tautology:
> If P ⊃ F then, by transposition, ~F ⊃ ~P. Since F's negation is a tautology, then ~F ⊃ ~P can be re-written as T ⊃ ~P. Hence we are allowed to infer ~P when P leads to a contradiction because ~P validly follows from a tautology.

The way we will structure proofs involving the indirect method mirrors that of the conditional method.

When we want to begin an indirect proof, we draw an indented vertical line. For the first line we cite AIP (assumed for indirect proof) and then begin our subproof. Then, once we have derived a contradiction, or created one from the previous lines and our 18 rules, then we can discharge so long as the contradiction is on the last line of the proof. Then, we end the indented vertical line and slide back to the indentation of the main proof, citing the lines of the indirect proof, and stating IP (indirect proof).

1. (*A* ∨ *B*) ⊃ (*C* • *D*)	
2. *C* ⊃ ~*D*	/ ~*A*
3. *A*	AIP
4. *A* ∨ *B*	3, Add
5. *C* • *D*	1, 4, MP
6. *C*	5, Simp
7. ~*D*	2, 6, MP
8. *D* • *C*	5, Com
9. *D*	8, Simp
10. *D* • ~*D*	7, 9, Conj
11. ~*A*	3–10, IP

> The indirect proof sequence (lines 3–10) begins by assuming the negation of the conclusion. Since the conclusion is a negated statement, it shortens the proof to assume A instead of $\sim\sim A$. This assumption, which is tagged "AIP" (assumption for indirect proof), leads to a contradiction in line 10. Since any assumption that leads to a contradiction is false, the indirect sequence is discharged (line 11) by asserting the negation of the assumption made in line 3. This line is then tagged with the designation "IP" (indirect proof) together with the numerals indicating the scope of the indirect sequence from which it is obtained.

Just like conditional proof, indirect proofs can also be used as parts of the main proof. We can get parts that we need for the main proof by doing an indirect proof along the way.

Note that it would be very difficult to derive ~F without the indirect method. Much like conditional proof, indirect proofs allow us to get what we want with less work while still retaining validity.	

1. $E \supset [(F \vee G) \supset (H \cdot J)]$	
2. $E \cdot \sim(J \vee K)$	/ $\sim(F \vee K)$
3. E	2, Simp
4. $(F \vee G) \supset (H \cdot J)$	1, 3, MP
5. $\sim(J \vee K) \cdot E$	2, Com
6. $\sim(J \vee K)$	5, Simp
7. $\sim J \cdot \sim K$	6, DM
| 8. F	AIP
| 9. $F \vee G$	8, Add
| 10. $H \cdot J$	4, 9, MP
| 11. $J \cdot H$	10, Com
| 12. J	11, Simp
| 13. $\sim J$	7, Simp
| 14. $J \cdot \sim J$	12, 13, Conj
15. $\sim F$	8–14, IP
16. $\sim K \cdot \sim J$	7, Com
17. $\sim K$	16, Simp
18. $\sim F \cdot \sim K$	15, 17, Conj
19. $\sim(F \vee K)$	18, DM

The more challenging part of the indirect method is isolating what to assume for contradiction and when. While there is no hard and fast rule which will always apply, there are a variety of techniques you can utilize, some of which are the following:

- Assess whether your conclusion is itself a tautology. If it is, then assume it's negation. The negation of any tautology is itself a contradiction! Discharge the negation of the negation, and, by double negation, you arrive at your conclusion. Boy, that was quick.
- If your conclusion looks substantially different from your premises, just assume the negation of the conclusion and see if you can get a contradiction with this new assumption your premises.
- Try and apply as many of the rules as possible. Once you have reached a brick wall, see if the negation of anything you currently have would assist you in getting closer to your conclusion. If it would, then AIP it!