Computational Solution to the Transient 2D Heat Equation

Kevin Juan (kj89)

November 16th 2018

The Physical System

This project aims to solve the 2D transient heat equation using a finite difference method. The partial differential equation to be solved is shown below:

$$\rho C_p \frac{\partial T}{\partial t} = k\nabla^2 T + \dot{q} \tag{1}$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\dot{q}}{\rho C_p} \tag{2}$$

$$\alpha = \frac{k}{\rho C_p} \tag{3}$$

The constant α is denoted as the thermal diffusivity, which is dependent on the material of interest. In both equations, the term $\dot{q}$ can be a heat source, a heat sink, or 0 to simplify the system. The heat equation is a parabolic PDE, which is not only applicable to heat transfer, but also areas like mass transfer and financial mathematics where it is used heavily in probability theory.

The project aims to examine how the temperature profile varies with time for different types of systems. To simplify the calculations, a flat plate geometry in Cartesian coordinates with constant temperature boundaries will bound the equation and system of interest. This type of geometry is common in electric stove-tops, CPU heat spreaders, and fins found in CPU/GPU heat sinks on coolers. The project will start by examining the heat equation without the source/sink term found in Equations 1 and 2 for aluminum, PVC, and pyrolytic graphite due to their vastly different thermal diffusivities. If time permitting, the same systems will be examined with the added heat source/sink term. It would be interesting to examine the differences in results between a constant heat source/sink and a time-dependent heat source/sink that may be oscillatory or monotonically increasing or decreasing, permitting the problem does not become too computationally or time intensive.

Numerical Methods

This project aims to tackle a time-dependent partial differential equation in two dimensions. The Crank-Nicolson method tested in class has shown to have excellent stability and accuracy. Where Crank-Nicolson begins to fall short is its ability to perform the same task in higher dimensions. The method worked well for 1D systems because the tri-diagonal matrix that needed to be solved was relatively easy for the problem size. In higher dimensions, the tri-diagonal matrix becomes a band-diagonal matrix, which can become prohibitively expensive computationally as more equations are needed to solve the system.

An alternative method is the alternating direction implicit method (ADI). This method is commonly implemented for multidimensional unsteady problems like the 2D transient heat equation. As outlined in Press et al[1], Özışık et al[2], Pletcher et al[3], and Majumdar et al[4], discretization is done such that multiple systems are solved in one time step rather than one.

Consider the time step from n to $n+1$. Discretization would be done such that from n to $n+\frac{1}{2}$ the simple implicit method is used for one direction, x, while the simple explicit method is used for the other direction y. For the other half of the time step, the implicit method is used for y and the explicit method is used for x. These alternations are done for the entire simulation. We thus get the two finite difference equations shown below for Equation 2.

$$\frac{T_{i,j}^{n+\frac{1}{2}} - T_{i,j}^{n}}{\frac{\Delta t}{2}} = \alpha\left(\frac{T_{i-1,j}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i+1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n} - 2T_{i,j}^{n} + T_{i,j+1}^{n}}{(\Delta y)^2}\right) + \frac{1}{\rho C_p}\dot{q}_{i,j}^{n+\frac{1}{2}} \tag{4}$$

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \alpha\left(\frac{T_{i-1,j}^{n+\frac{1}{2}} - 2T_{i,j}^{n+\frac{1}{2}} + T_{i+1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{(\Delta y)^2}\right) + \frac{1}{\rho C_p}\dot{q}_{i,j}^{n+1} \tag{5}$$

It is vastly more convenient to rearrange the equations as such:

$$-r_x T_{i-1,j}^{n+\frac{1}{2}} + (1+2r_x)T_{i,j}^{n+\frac{1}{2}} - r_x T_{i+1,j}^{n+\frac{1}{2}} = r_y T_{i,j-1}^{n} + (1-2r_y)T_{i,j}^{n} + r_y T_{i,j+1}^{n} + \frac{\Delta t}{2\rho C_p}\dot{q}_{i,j}^{n+\frac{1}{2}} \tag{6}$$

$$-r_y T_{i,j-1}^{n+1} + (1+2r_y)T_{i,j}^{n+1} - r_y T_{i,j+1}^{n+1} = r_x T_{i-1,j}^{n+\frac{1}{2}} + (1-2r_x)T_{i,j}^{n+\frac{1}{2}} + r_x T_{i+1,j}^{n+\frac{1}{2}} + \frac{\Delta t}{2\rho C_p}\dot{q}_{i,j}^{n+1} \tag{7}$$

We note that $r_x = \frac{\alpha \Delta t}{2(\Delta x)^2}$ and $r_y = \frac{\alpha \Delta t}{2(\Delta y)^2}$. Like the Crank-Nicolson method, both of these finite difference equations can be solved using a tri-diagonal matrix equation rather than a more expensive band-diagonal method.

Expected Results

The goal of this project is to obtain various 2D temperature profiles at different times for the different materials and scenarios. It would also be interesting to create an animation that shows the temperature profile change with time.

References

[1] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, editors. *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press, Cambridge, UK ; New York, 3rd ed edition, 2007. OCLC: ocn123285342.

[2] M. Necati Özışık, Helcio R. B. Orlande, Marcelo Jose Colaco, and Renato Machado Cotta. *Finite difference methods in heat transfer*. Taylor & Francis, CRC Press, Boca Raton, second edition edition, 2017.

[3] Richard H Pletcher, Dale A Anderson, and John C Tannehill. *Computational fluid mechanics and heat transfer*. 2016. OCLC: 1027194464.

[4] Pradip Majumdar. *Computational methods for heat and mass transfer*. 2011. OCLC: 897338030.