1 Magnus expansion for $A_t \in \mathbb{R}^{2,2}$, $B = 0$ deterministic

We will concern ourselves with the following SDE:

$$dX_t = A_t X_t dW_t$$

with

$$A_t = \begin{bmatrix} f^{11}(t) & f^{12}(t) \\ 0 & f^{22}(t) \end{bmatrix}$$

Here we chose:

$$f^{11}(t) \equiv 2, f^{12}(t) = t, f^{22}(t) \equiv -1,$$

1.1 Parameters

Parameter	value
t_0	0
T	1
`N_fine`	10001
N	101
`M_fine`	1000
M	1000
d	2

1.2 Computational Times

Method	Log	Matrix Exp	Total
exact	0	0	0.748717
euler	0	0	4.78936
m1	0.0423386	0.792471	0.83481
m2	0.0432706	0.558621	0.601892
m3	0.0789595	0.608386	0.687345

1.3 Errors

(i) Errors for $X(1, 1, :, :)$:

(a) Reference method: exact

Error	euler	m1	m2	m3
(abs error) L2	0.10599	15.5912	0	0
(rel error) min	0	0	0	0
(rel error) max	0.0235695	6.38906	0	0
(rel error) mean	0.0154253	2.22658	0	0

(ii) Errors for $X(1, 2, :, :)$:

(a) Reference method: exact

Error	euler	m1	m2	m3
(abs error) L2	0.0195706	2.1297	0.503118	0.206861
(rel error) min	0.00912873	0.423194	0.07581	0.0504957
(rel error) max	0.0570297	2.59084	0.50578	0.212776
(rel error) mean	0.0296236	1.13136	0.259956	0.0998967

(iii) Errors for $X(2, 2, :, :)$:

(a) Reference method: exact

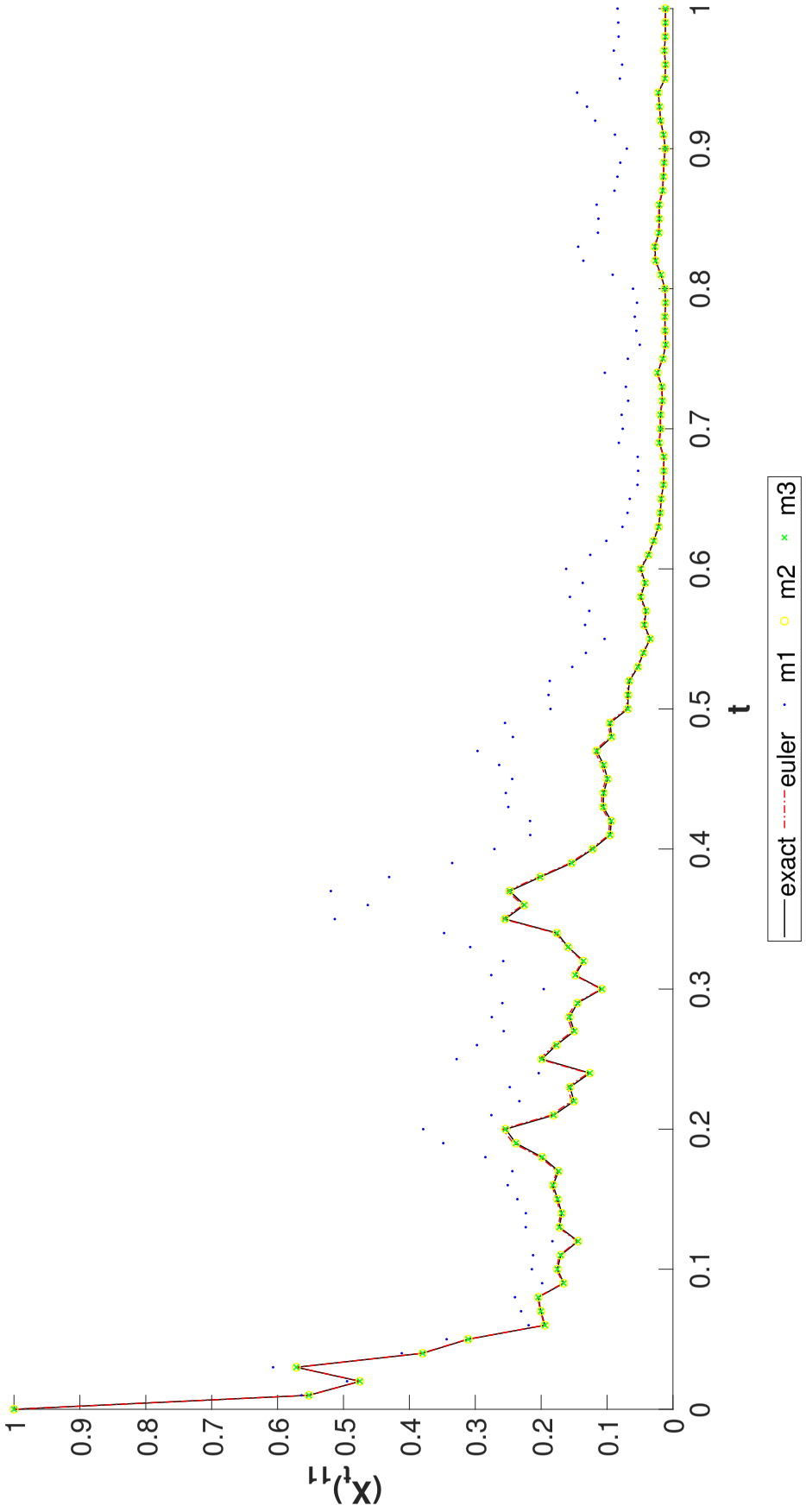
Error	euler	m1	m2	m3
(abs error) L2	0.00625928	0.477018	0	0
(rel error) min	0	0	0	0
(rel error) max	0.00579287	0.648721	0	0
(rel error) mean	0.00380367	0.300689	0	0

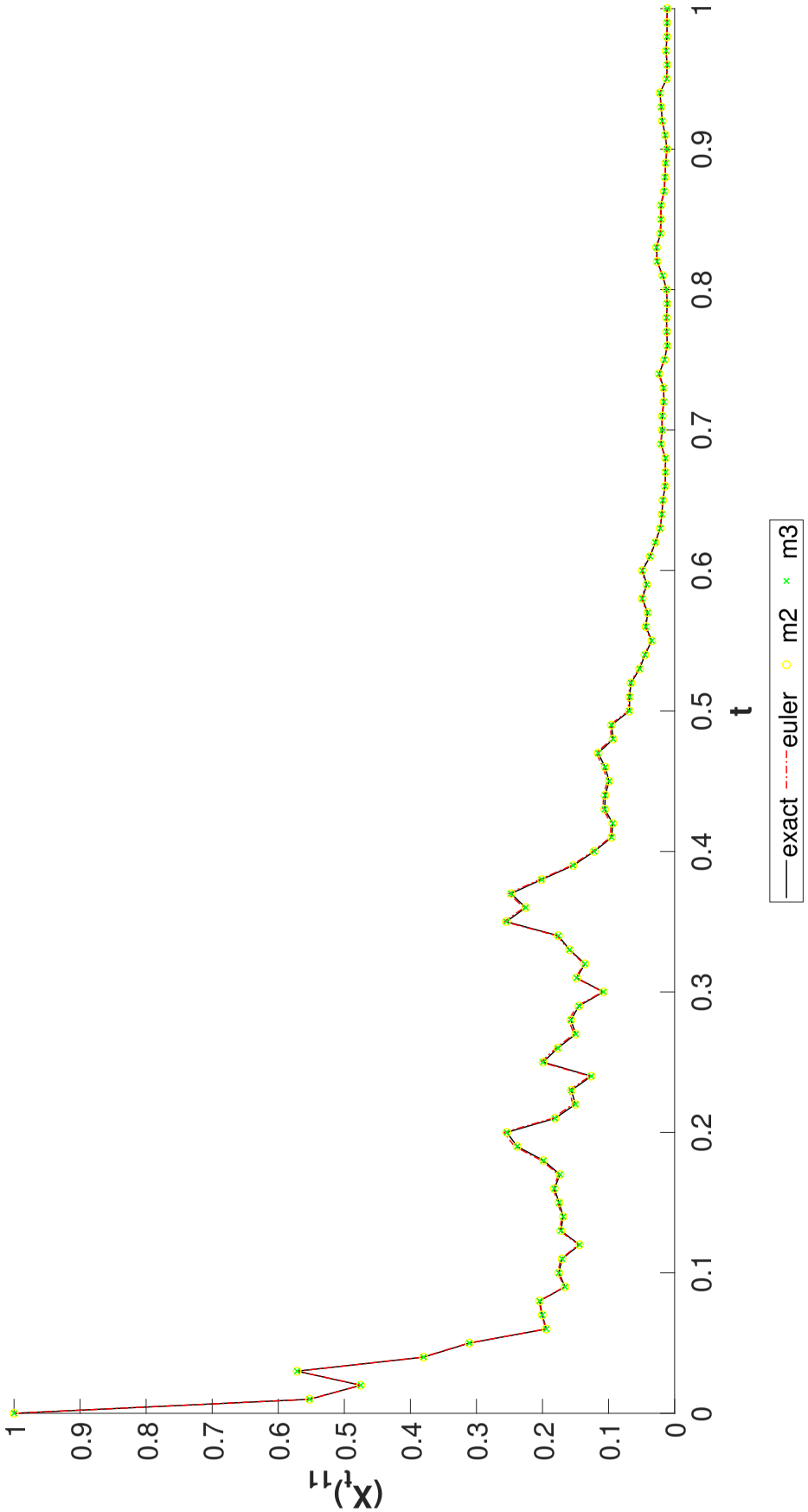
(iv) Total Errors:

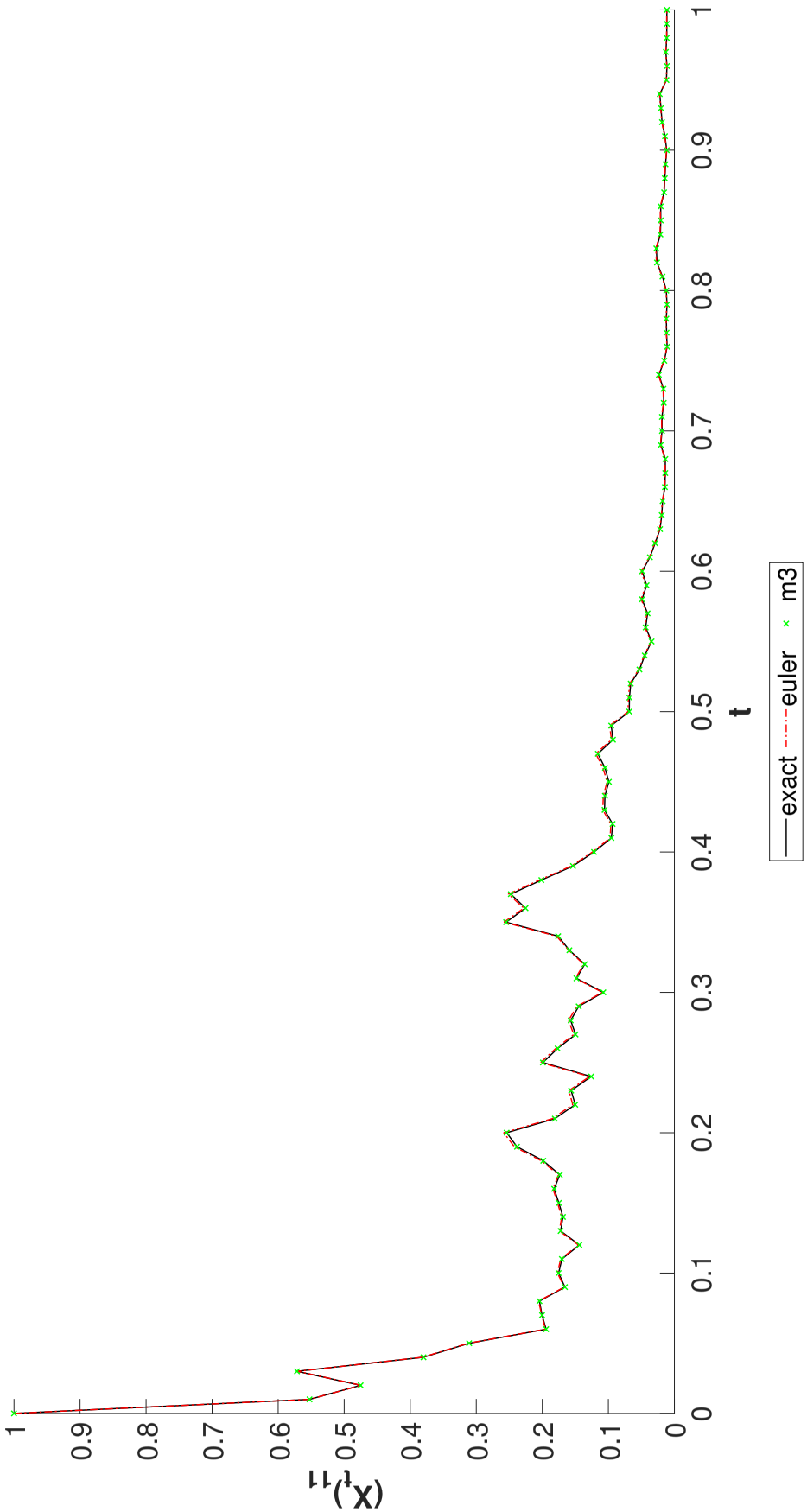
(a) Reference method: exact

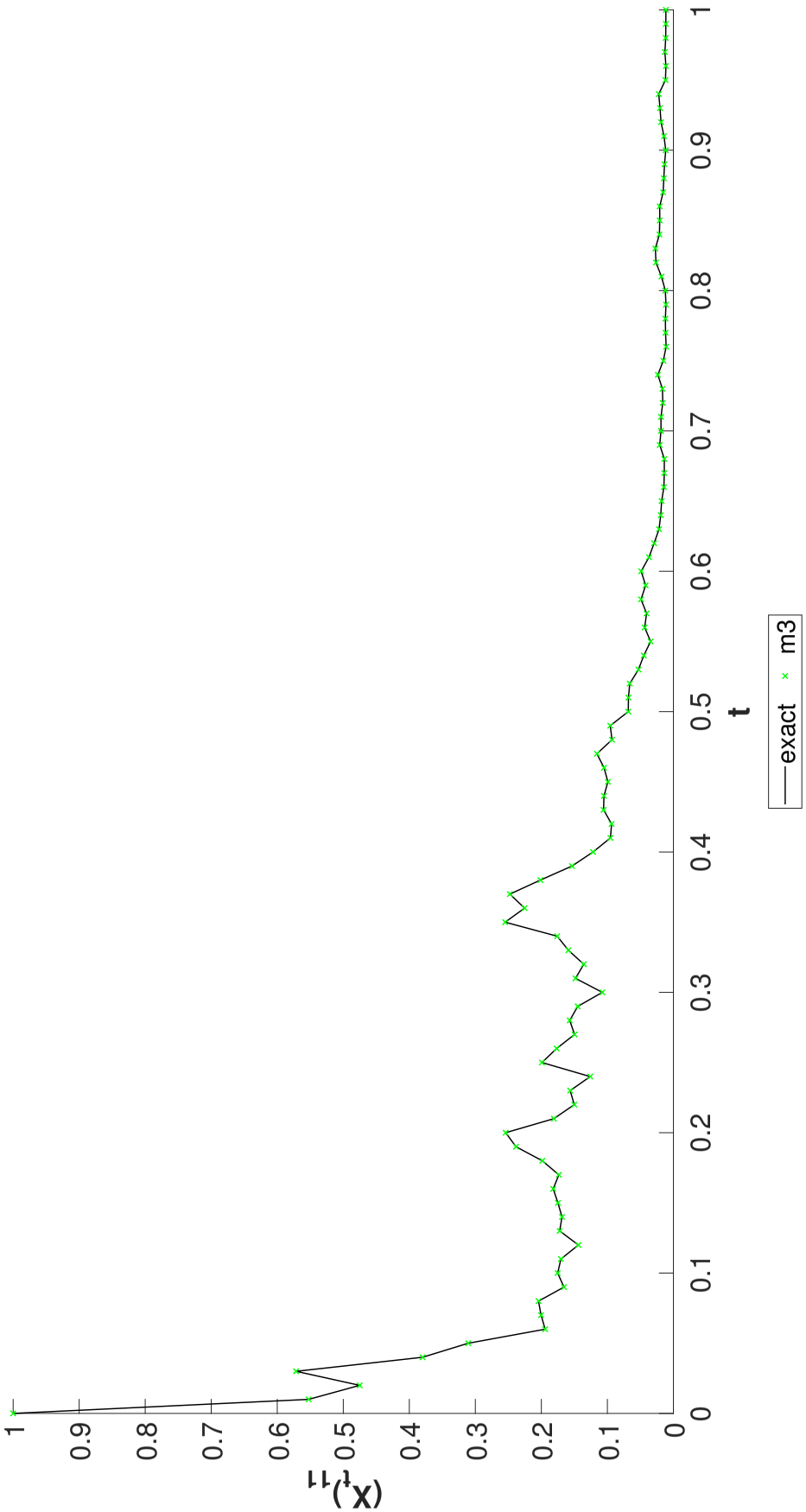
Method	$\mathbb{E}[Err_1]$
euler	1.12 %
m1	109 %
m2	3.17 %
m3	1.5 %

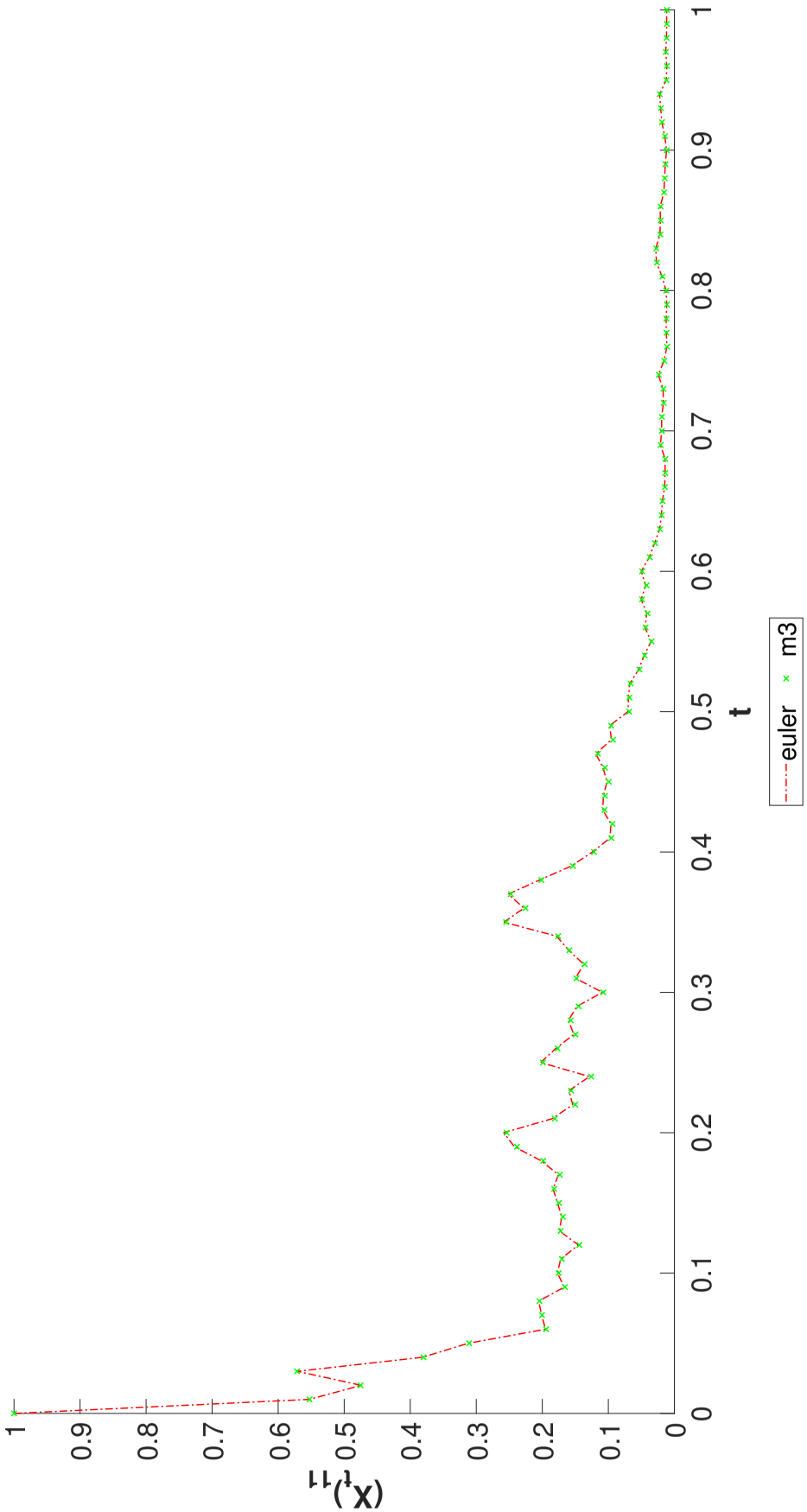
1.4 Plots


exact
euler
m1
m2
m3
t
$(X_t)_{11}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

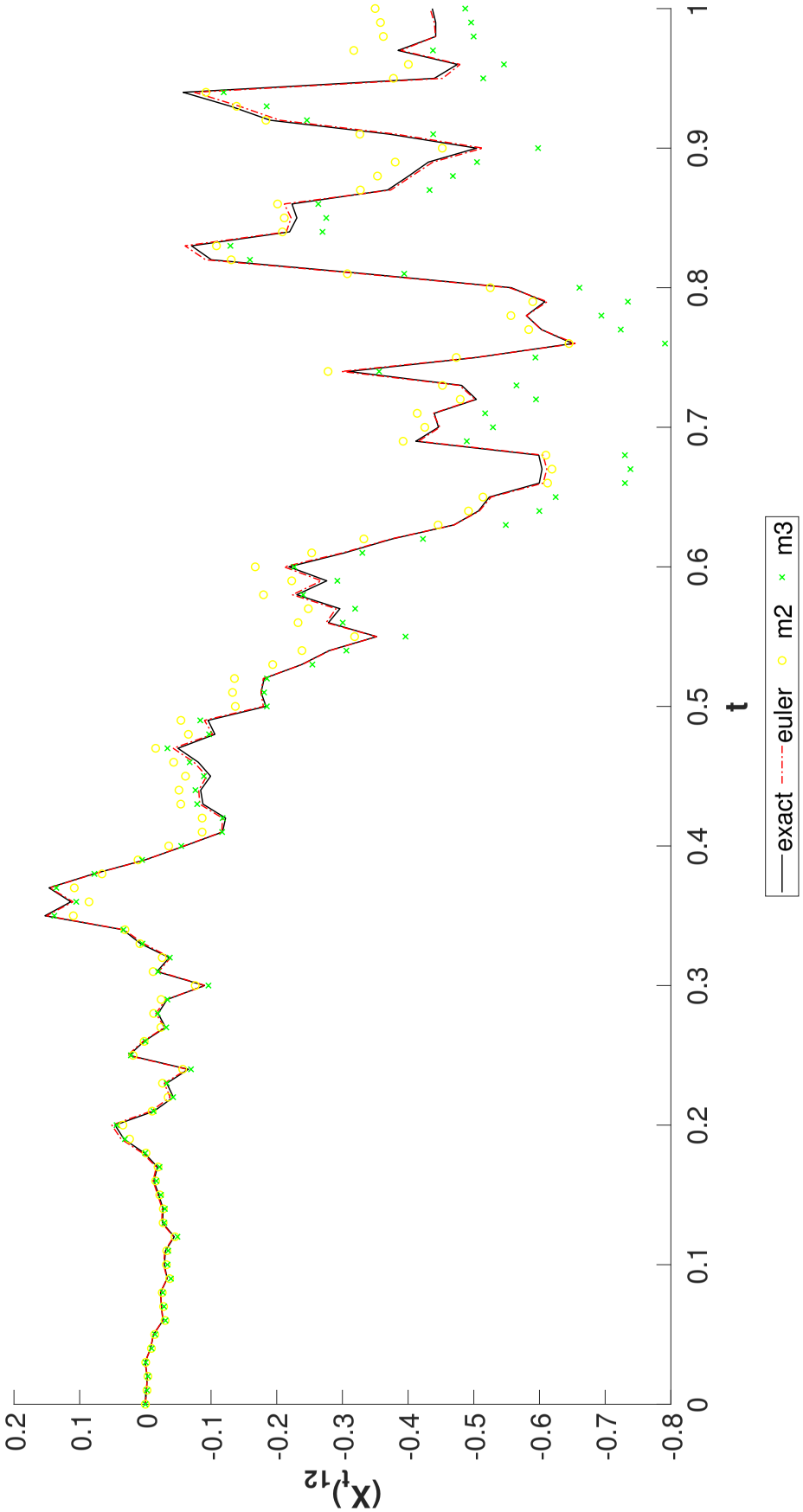

(X_t)_11
t
exact
euler
m2
m3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

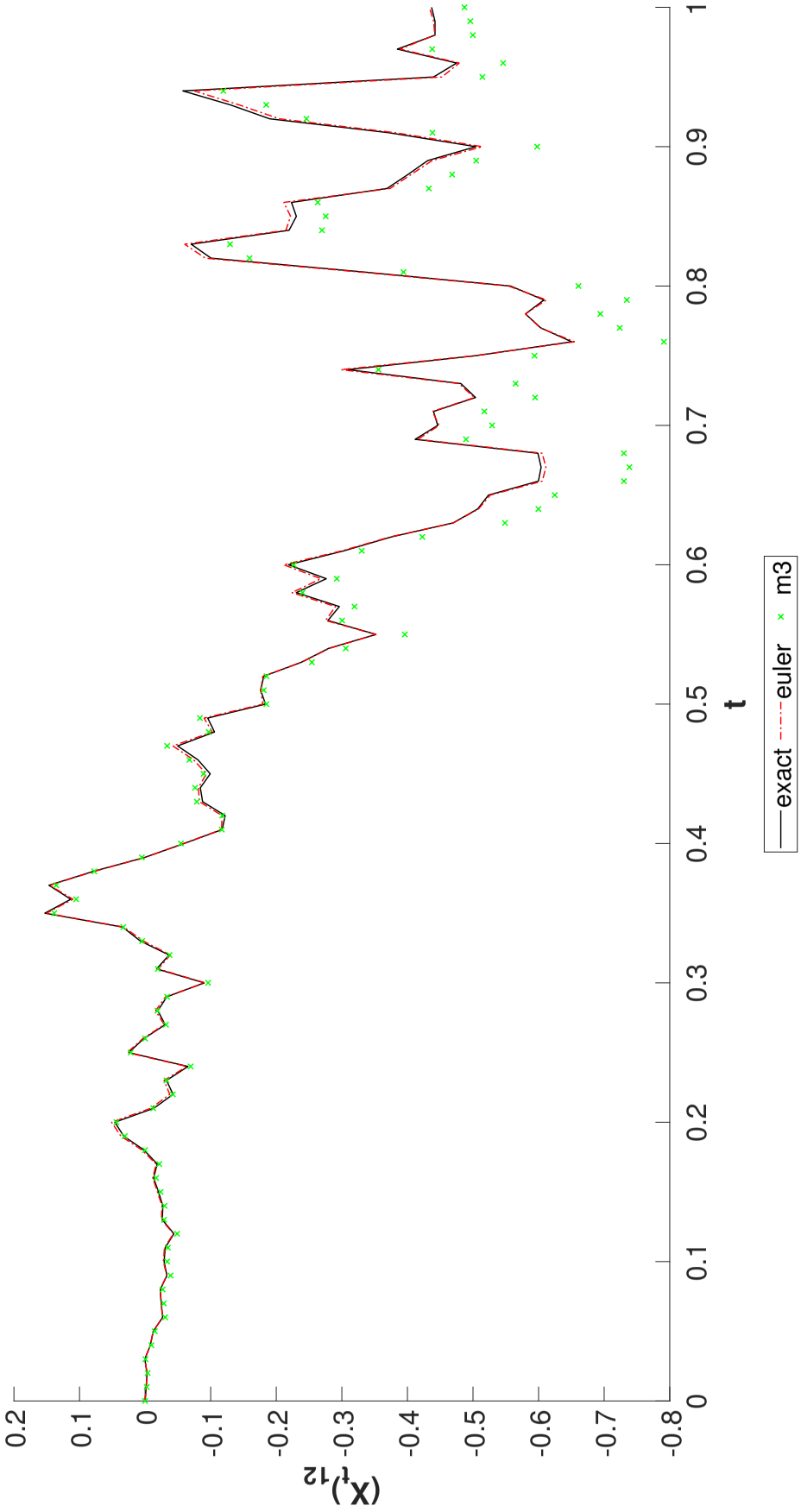

(x_t)_11
t
exact
euler
m3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

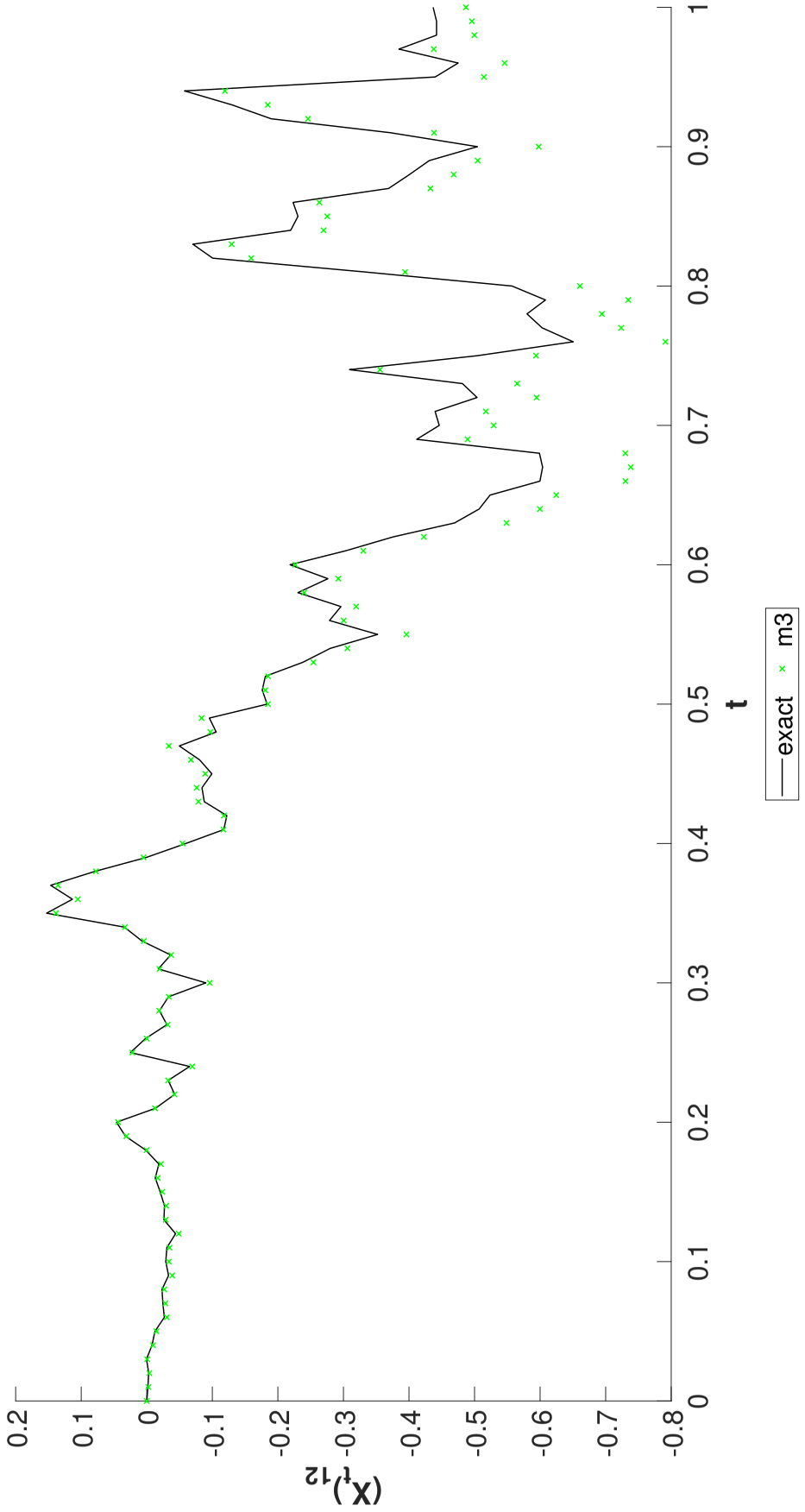

(X_t)_11
t
exact
m3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

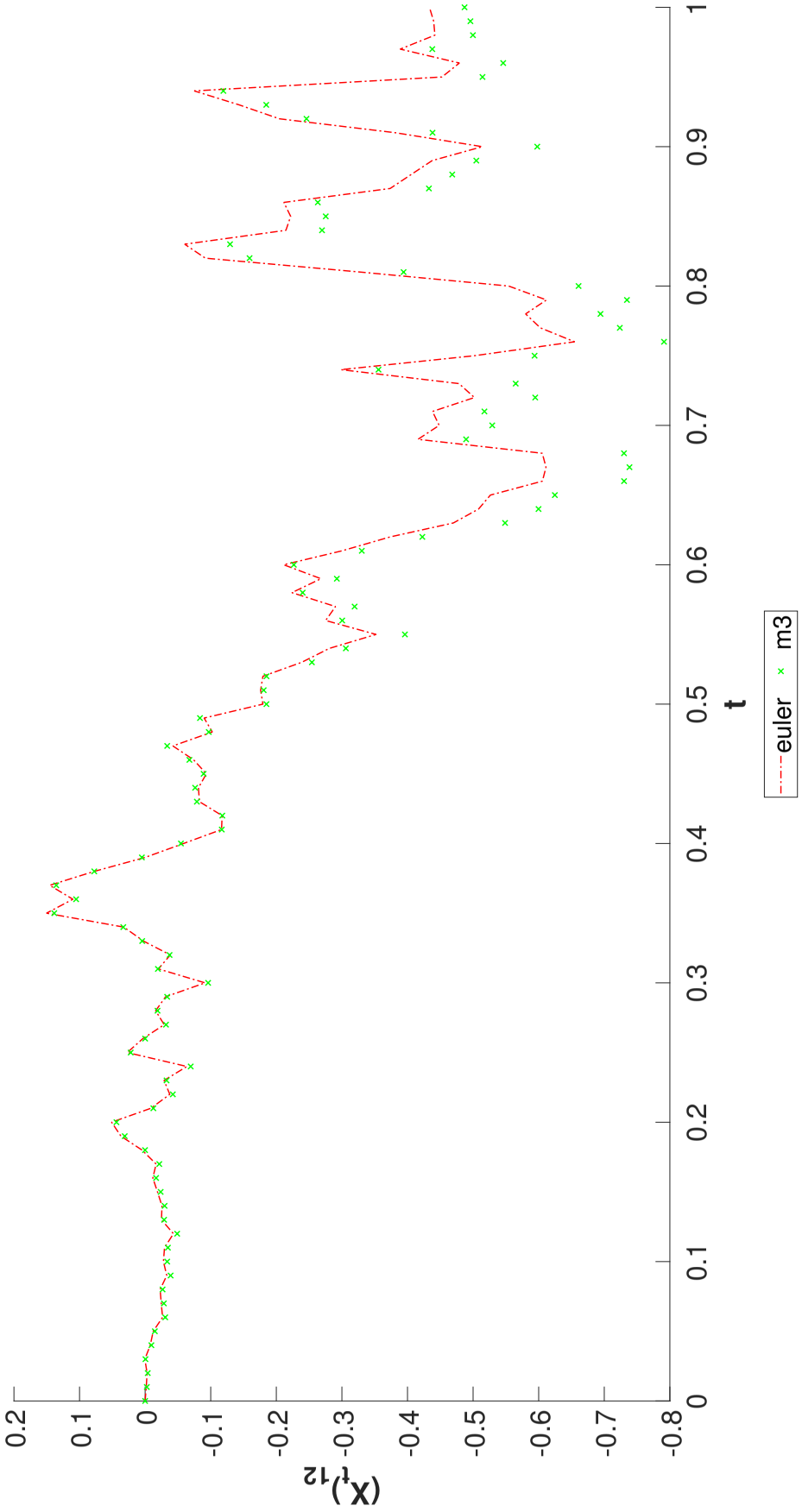

(X_t)_11
t
euler
m3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

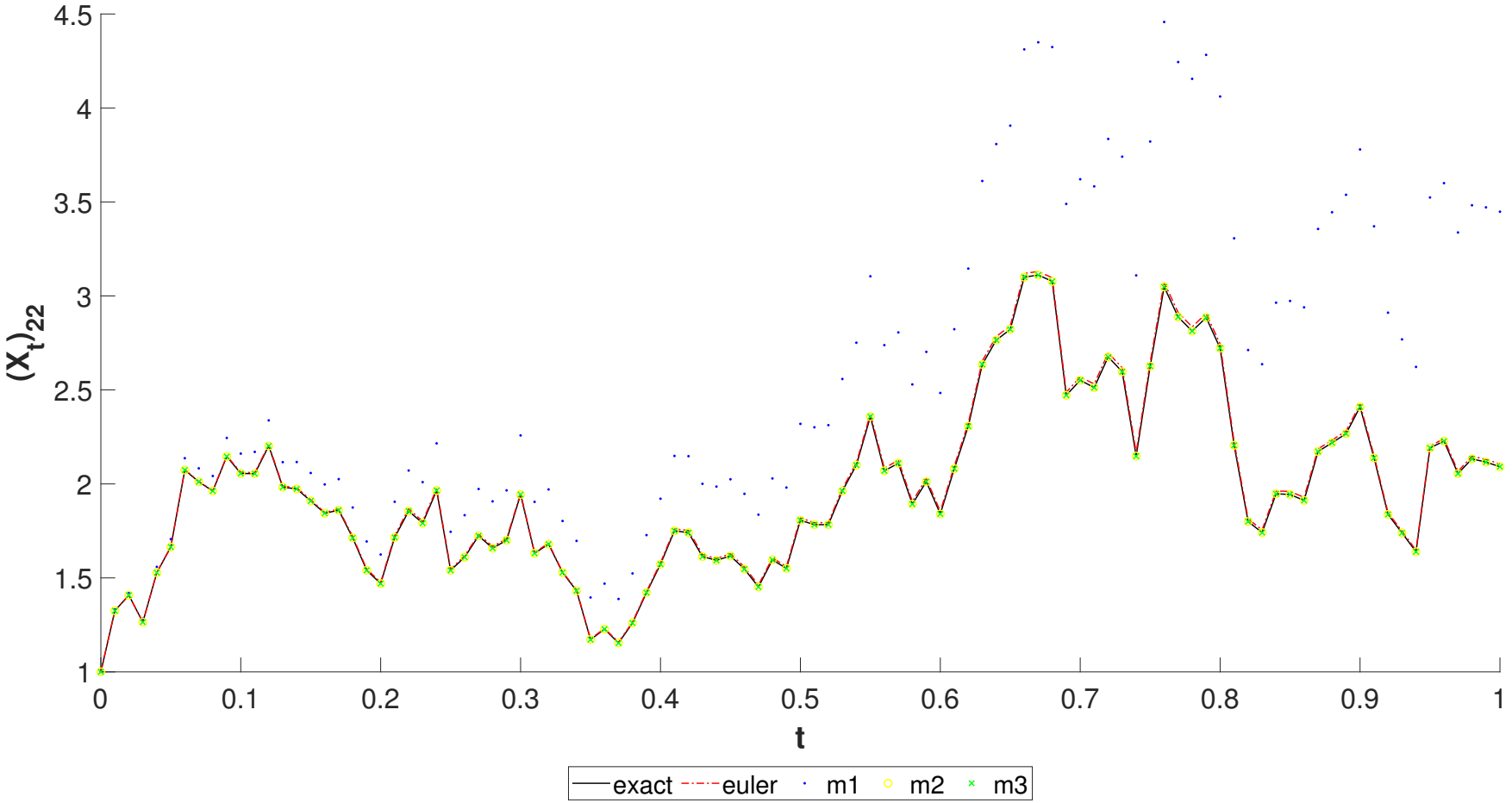

0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
$(X_t)_{12}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
euler
m1
m2
m3

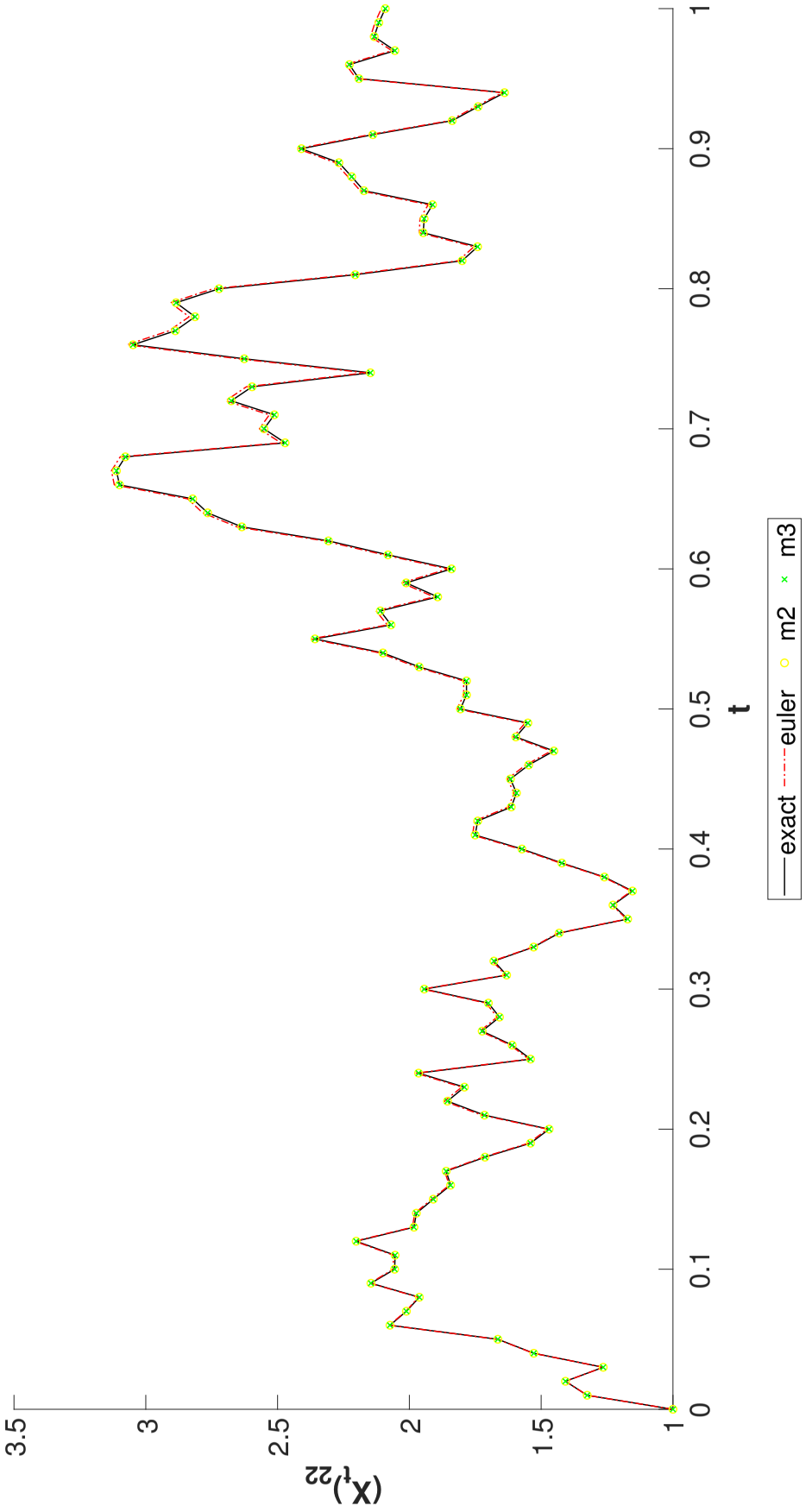

0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
(X_t)_12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
euler
m2
m3

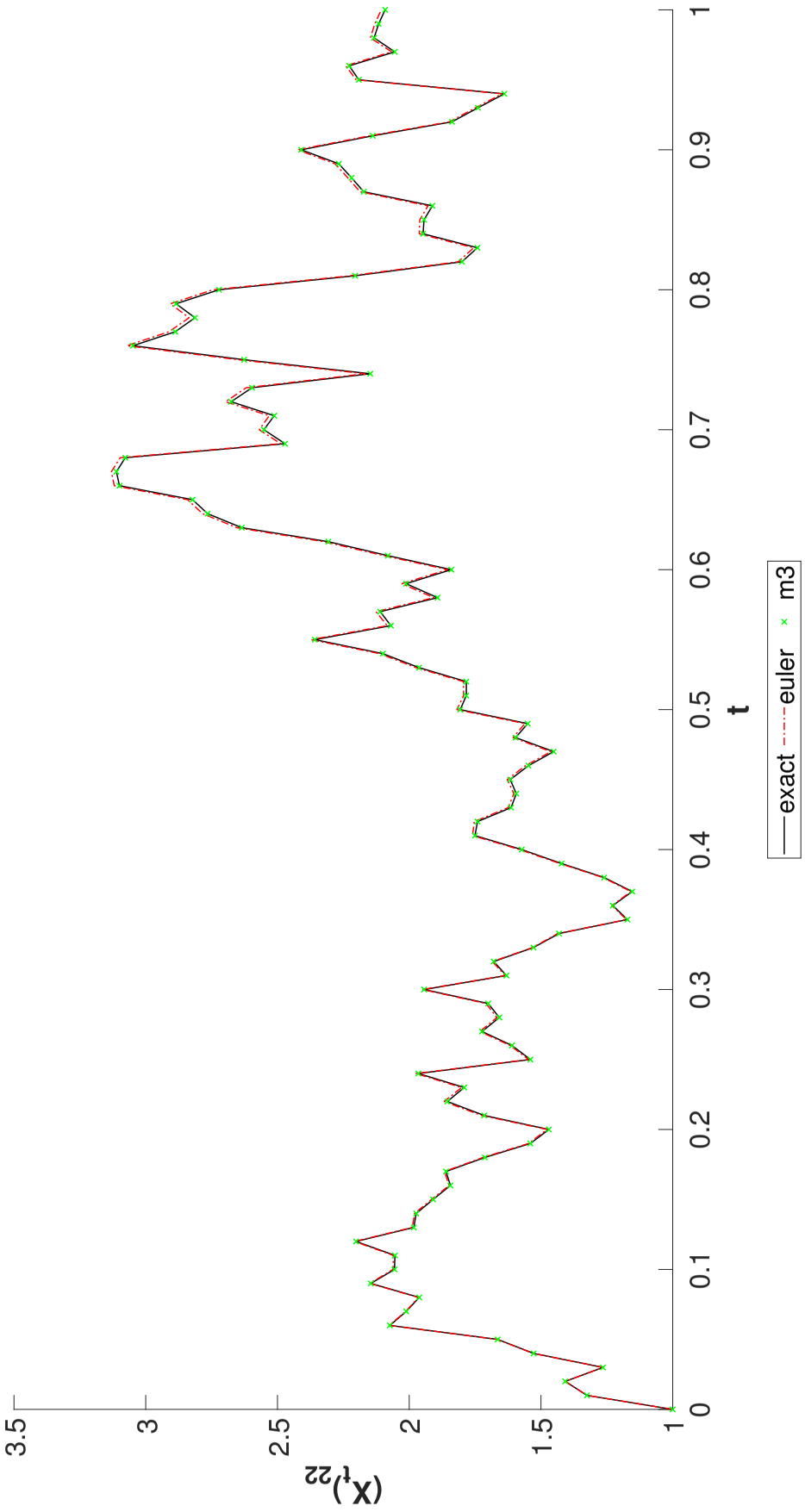

0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
(X_t)_12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
euler
m3

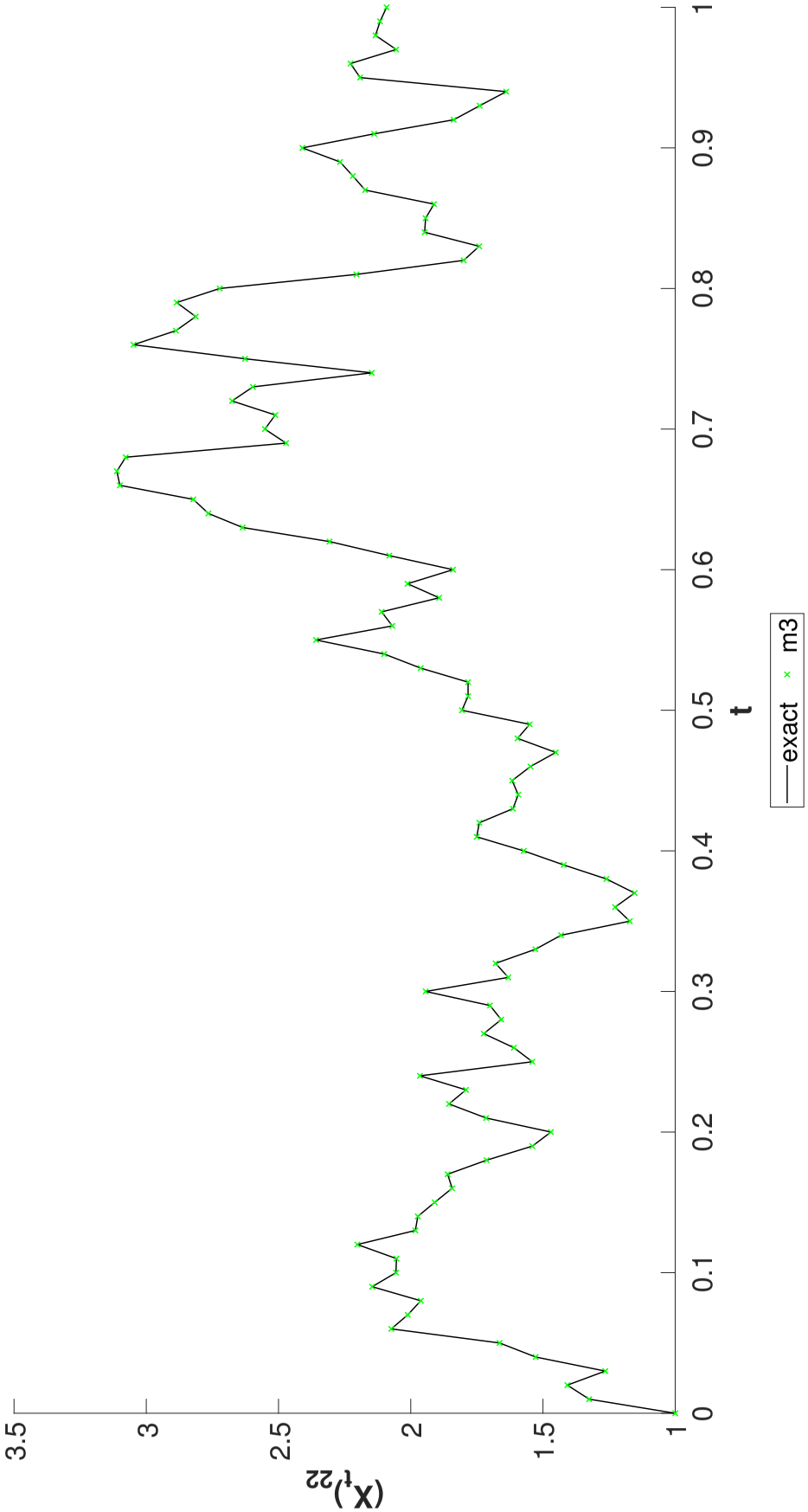

0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
$(X_t)_{12}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
m3

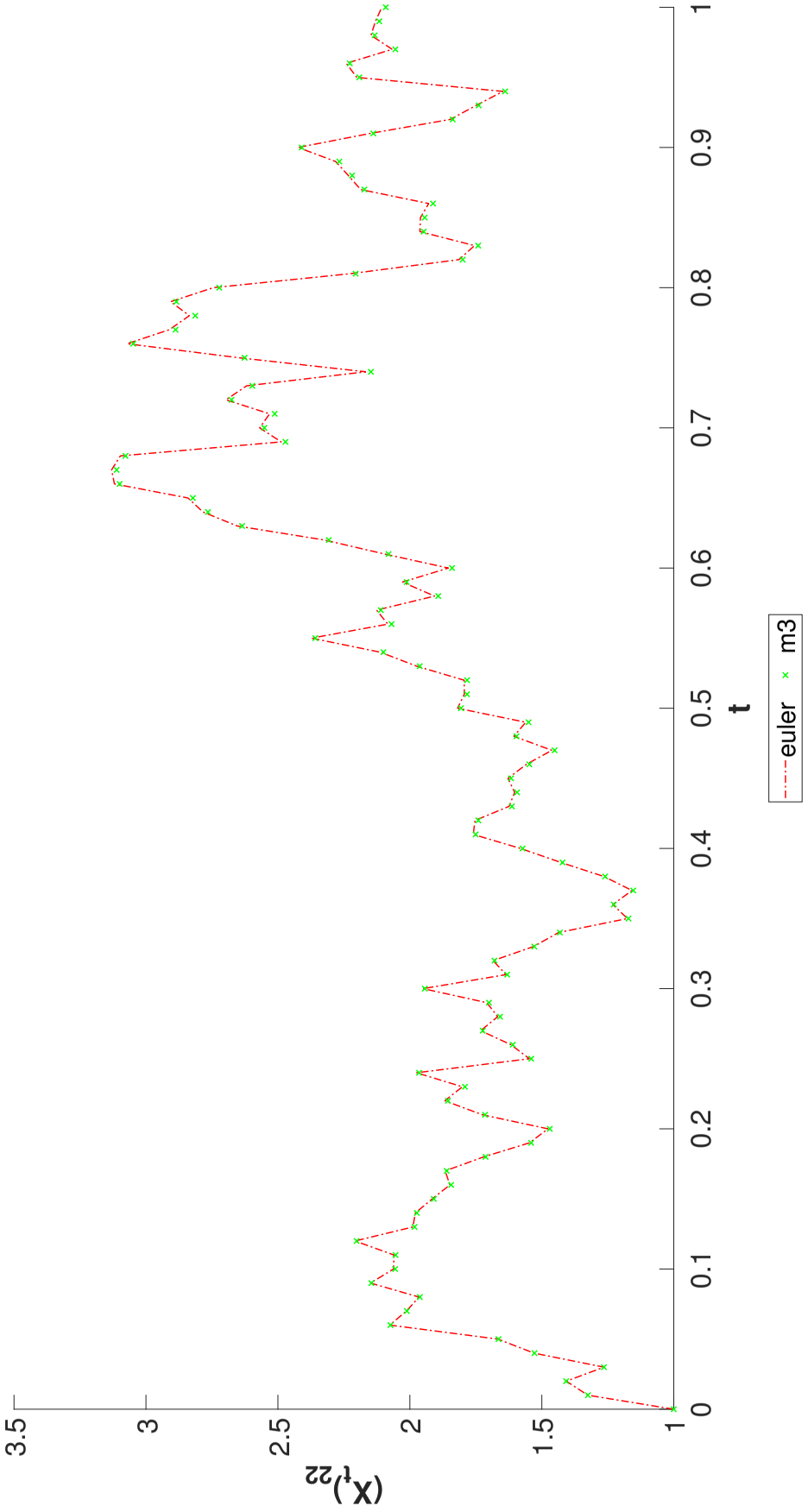

0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
(X_t)_12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
euler
m3


4.5
4
3.5
3
2.5
2
1.5
1
$(X_t)_{22}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
euler
m1
m2
m3

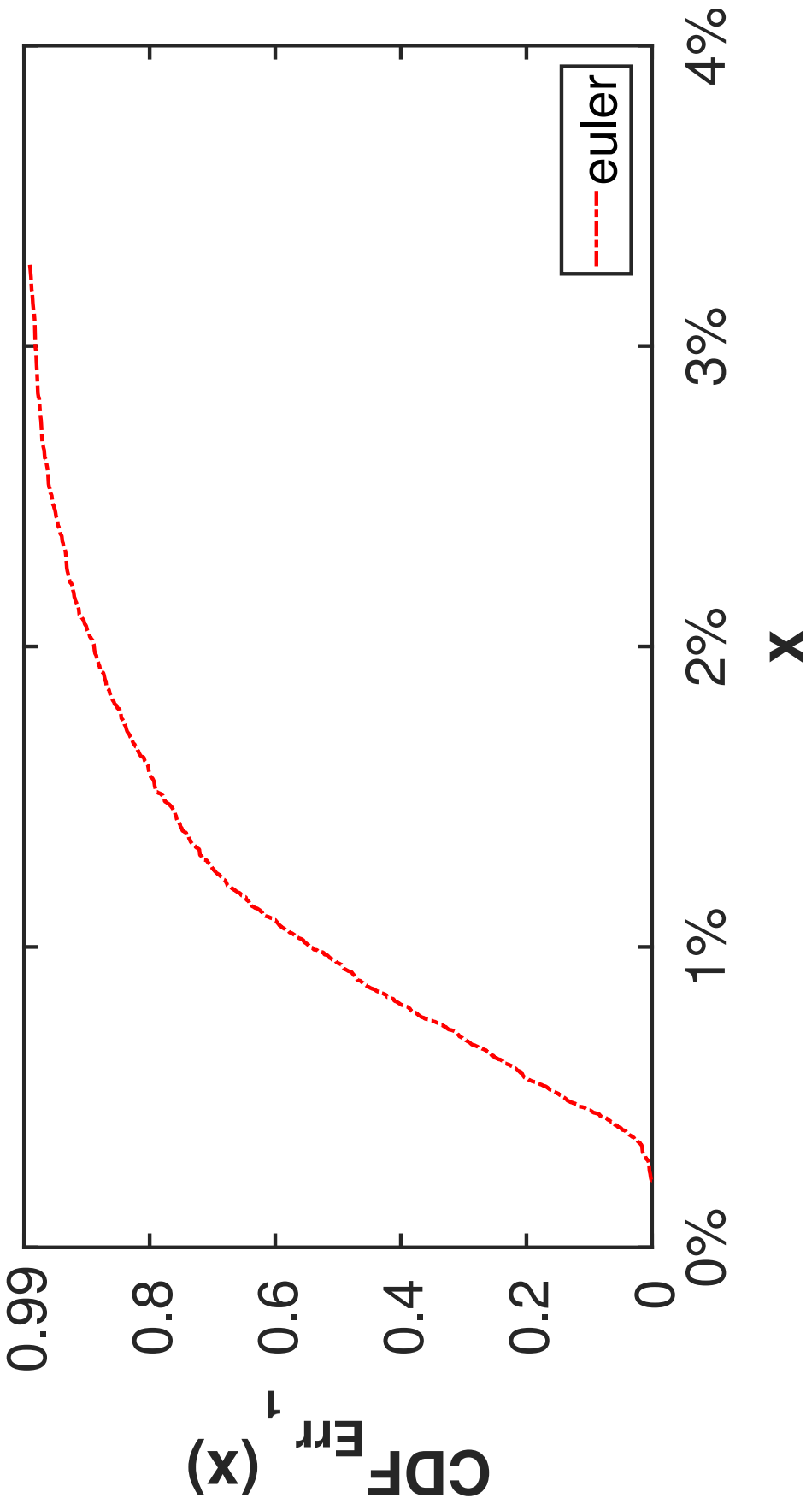

3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
(X_t)_22
exact
euler
m2
m3

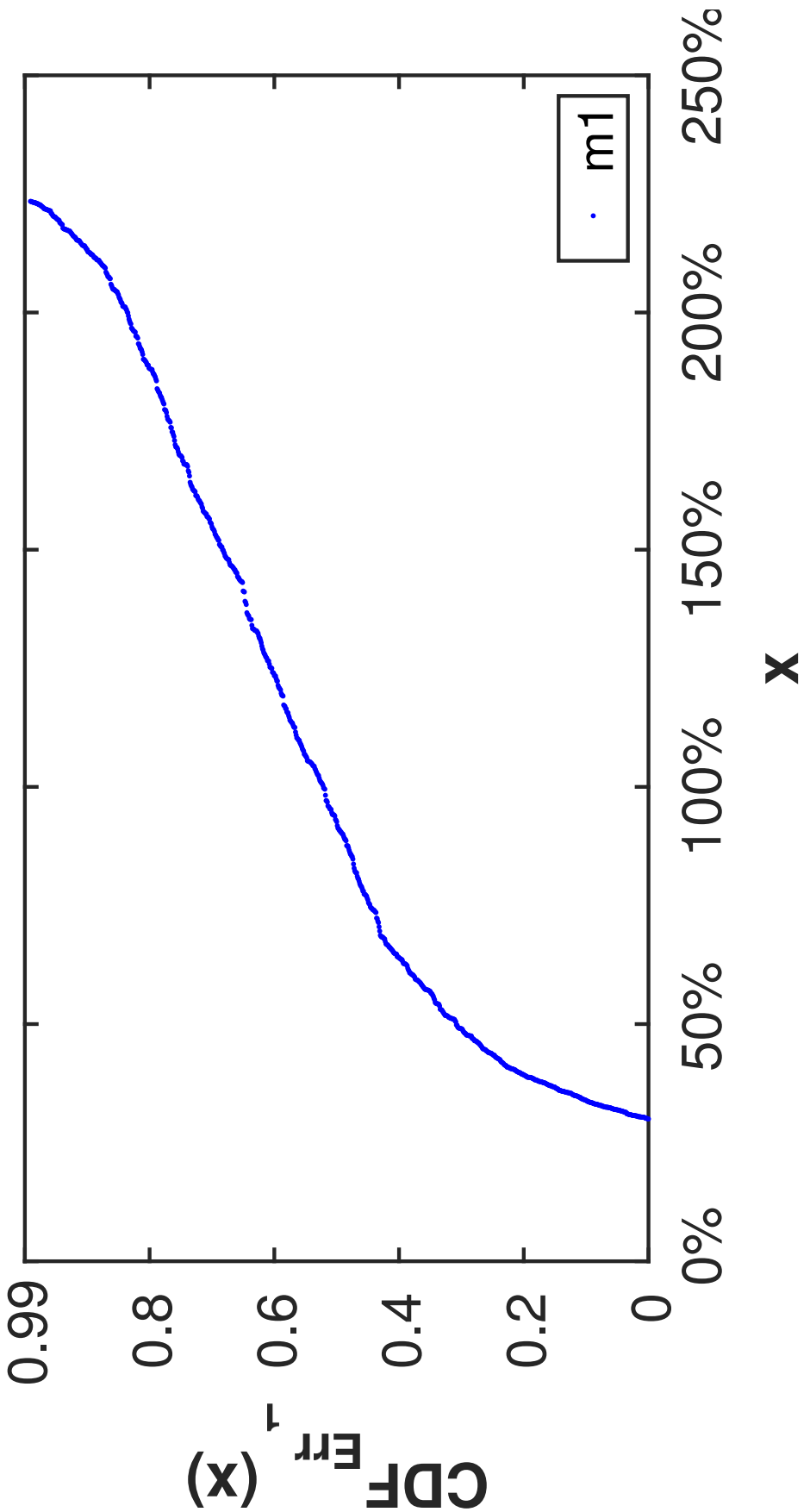

(X_t)_22
t
exact
euler
m3
3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

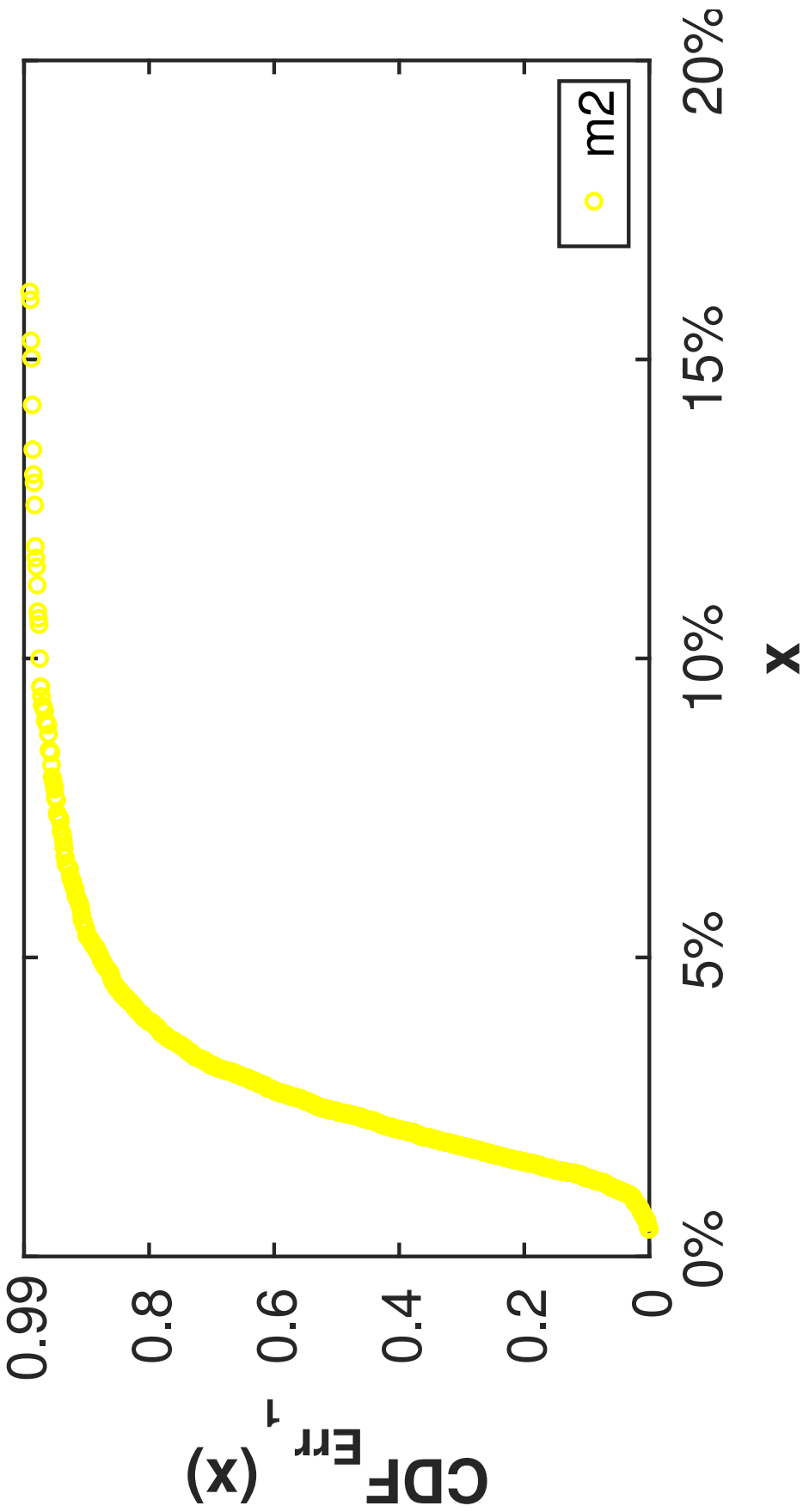

(X_t)_22
3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exact
m3

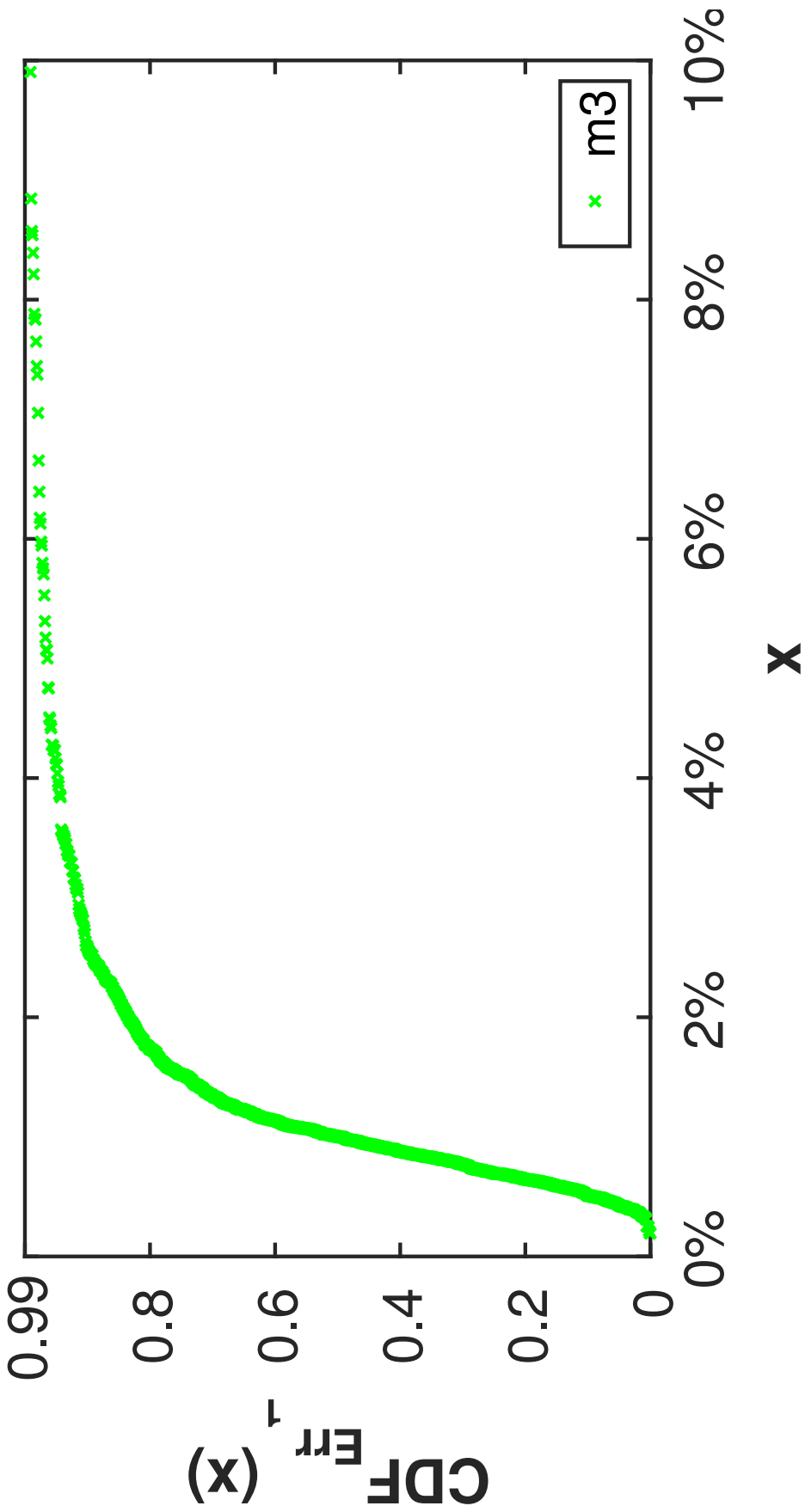

3.5
3
2.5
2
1.5
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
(X_t)_22
euler
m3

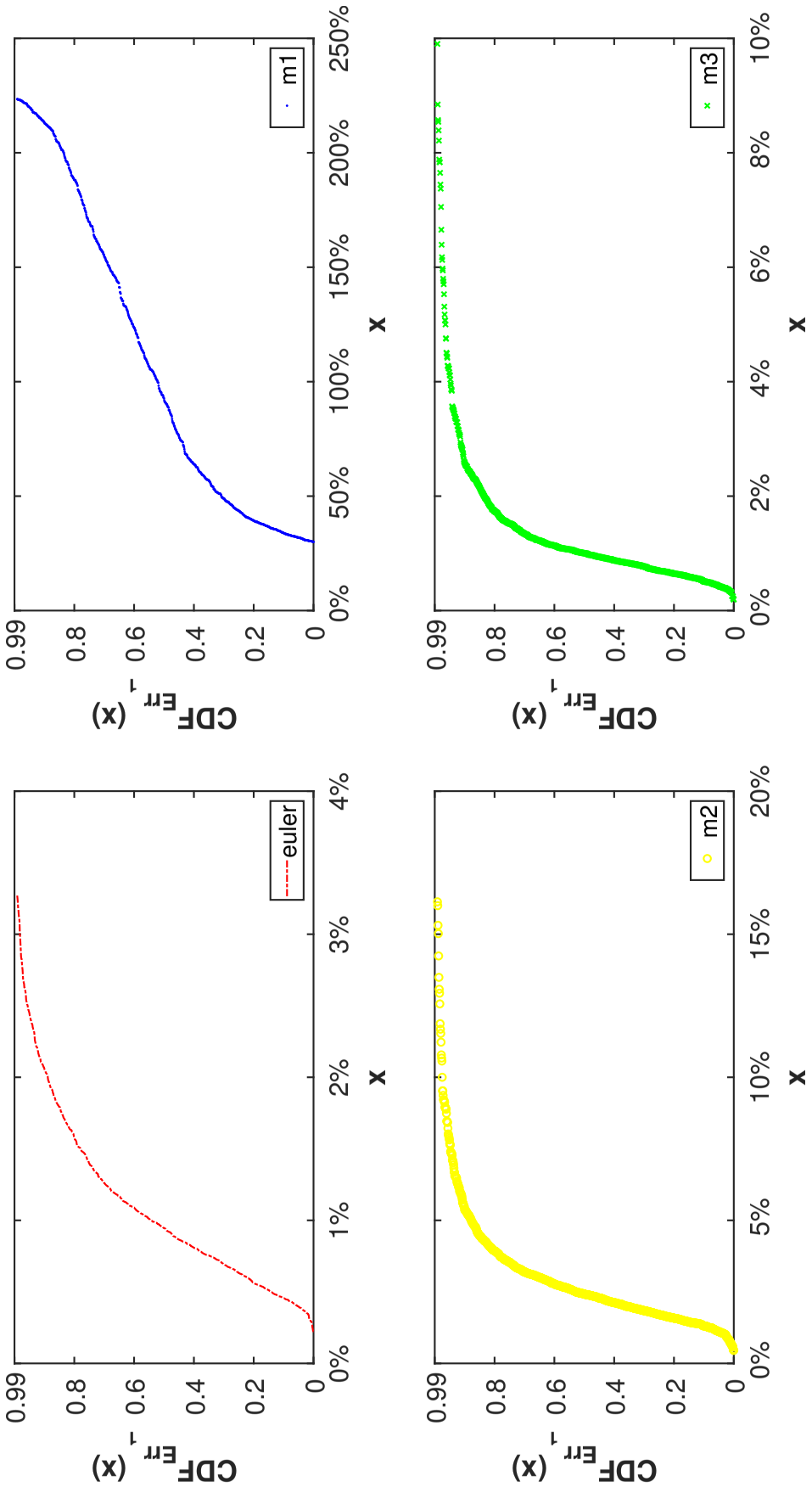
1.5 Error Plots


euler
0.99
0.8
0.6
0.4
0.2
0
0%
1%
2%
3%
4%
x
CDF_Err_1(x)


m1
0.99
0.8
0.6
0.4
0.2
0
0%
50%
100%
150%
200%
250%
x
CDF_err_1(x)


m2
0.99
0.8
0.6
0.4
0.2
0
0%
5%
10%
15%
20%
x
CDF_err_1(x)


m3
0.99
0.8
0.6
0.4
0.2
0
0%
2%
4%
6%
8%
10%
x
CDF_err_1(x)


m1
euler
m3
m2
$CDF_{Err_1}(x)$
x
0
0.2
0.4
0.6
0.8
0.99
0%
50%
100%
150%
200%
250%
1%
2%
3%
4%
5%
10%
15%
20%
6%
8%
10%