

LECTURE 5: DUALITY AND SENSITIVITY ANALYSIS

1. Dual linear program
2. Duality theory
3. Sensitivity analysis
4. Dual simplex method

Introduction to dual linear program

- Given a constraint matrix \mathbf{A} , right hand side vector \mathbf{b} , and cost vector \mathbf{c} , we have a corresponding linear programming problem:

$$\begin{array}{ll} \text{Min} & \mathbf{c}^T \mathbf{x} \\ (\text{LP}) \quad \text{s. t.} & \mathbf{Ax} = \mathbf{b} \quad \text{Primal Problem} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

- Questions:
 - Can we use the **same data** set of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ to construct another **linear programming problem**?
 - If so, how is this new linear program **related** to the original primal problem?

Guessing

- \mathbf{A} is an m by n matrix, \mathbf{b} is an m -vector, \mathbf{c} is an n -vector.
- Can the roles of \mathbf{b} and \mathbf{c} can be switched?
 - If so, we'll have m variables and n constraints.
Correspondingly,
 - (1) the transpose of matrix \mathbf{A} should be considered to accommodate rows for columns, and vise versa?
 - (2) nonnegative variables for free variables?
 - (3) equality constraints for inequality constraints?
 - (4) minimizing objective for maximizing objective?

Dual linear program

- Consider the following linear program

$$\text{Max } \mathbf{b}^T \mathbf{w}$$

$$(D) \quad \text{s. t. } \mathbf{A}^T \mathbf{w} \leq \mathbf{c} \quad \text{Dual Problem}$$

$$\mathbf{w} \in R^m$$

- Remember the original linear program

$$\text{Min } \mathbf{c}^T \mathbf{x}$$

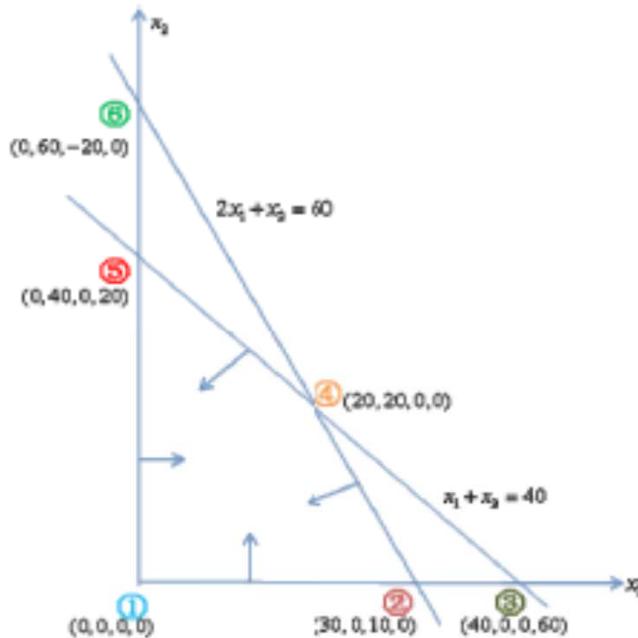
$$(LP) \quad \text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{Primal Problem}$$

$$\mathbf{x} \geq 0$$

Example

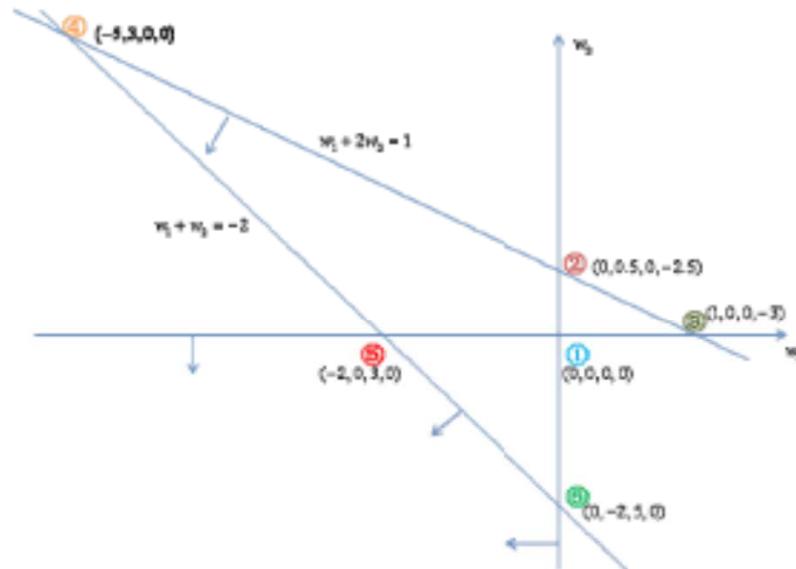
Minimize $x_1 - 2x_2$

$$(P) \text{ subject to } \begin{aligned} x_1 + x_2 + x_3 &= 40 \\ 2x_1 + x_2 + x_4 &= 60 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$



Maximize $40w_1 + 60w_2$

$$(D) \text{ subject to } \begin{aligned} w_1 + 2w_2 + w_3 &= 1 \\ w_1 + w_2 + w_4 &= -2 \\ w_1 \leq 0, w_2 \leq 0 \\ w_3 \geq 0, w_4 \geq 0 \end{aligned}$$



Dual of LP in other form

- Symmetric pair

$$\begin{aligned} (\text{P}) \quad & \min \quad \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} (\text{D}) \quad & \max \quad \mathbf{b}^T \mathbf{w} \\ \text{s. t.} \quad & \mathbf{A}^T \mathbf{w} \leq \mathbf{c} \\ & \mathbf{w} \geq 0 \end{aligned}$$

$$\begin{aligned} & \min \quad \mathbf{c}^T \mathbf{x} + \mathbf{0}^T \mathbf{s} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} - I\mathbf{s} = \mathbf{b} \\ & \mathbf{x}, \mathbf{s} \geq 0 \end{aligned}$$

$$\begin{aligned} & \max \quad \mathbf{b}^T \mathbf{w} \\ \text{s. t.} \quad & \begin{bmatrix} \mathbf{A}^T \\ -I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix} \end{aligned}$$

Karmarkar's form LP

$$\min \quad \mathbf{c}^T \mathbf{x}$$

$$\text{s. t. } a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

$$x_1 + x_2 + \cdots + x_n = 1$$

$$x_1, x_2, \dots, x_n \geq 0$$

$$\max \quad w_{m+1}$$

$$\text{s. t. } a_{11}w_1 + \cdots + a_{m1}w_m + w_{m+1} \leq c_1$$

⋮

$$(DKLP) \quad a_{12}w_1 + \cdots + a_{m2}w_m + w_{m+1} \leq c_2$$

⋮

$$a_{1n}w_1 + \cdots + a_{mn}w_m + w_{m+1} \leq c_n$$

Dual (KLP) is always feasible. For example:

$$w_1 = w_2 = \cdots = w_m = 0$$

$$w_{m+1} = \min\{c_1, c_2, \dots, c_n\} \text{ then}$$

$\mathbf{w} = (w_1, \dots, w_m, w_{m+1})^T$ is feasible to
(DKLP).

How are they related ?

We denote the original LP by (P) as the **primal problem** and the new LP by (D) as **dual problem**.

- (P) and (D) are defined by the **same data set (A , b , c)**.
- Problem (D) is a linear program with m variables and n constraints. The right-hand-side vector and the cost vector **change roles** in (P) and (D).
- What else?

Many interesting questions

- **Feasibility:**
 - Can problems (P) and (D) be both feasible?
 - One is feasible, while the other is infeasible?
 - Both are infeasible?
- **Basic solutions:**
 - Is there any relation between the basic solutions of (P) and that of (D)? bfs ? optimal solutions?
- **Optimality:**
 - Can problems (P) and (D) both have a unique optimal solution?
 - Both have infinitely many?
 - One unique, the other infinitely many?

Examples

- Both (P) and (D) are infeasible:

$$(P) \quad \begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s.t.} & -x_1 + x_2 \geq 1 \\ & x_1 - x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

$$(D) \quad \begin{array}{ll} \max & w_1 + 2w_2 \\ \text{s.t.} & -w_1 + w_2 \leq 1 \\ & w_1 - w_2 \leq -2 \\ & w_1, w_2 \geq 0 \end{array}$$

- Both (P) and (D) have infinitely many optimal solutions:

$$(P) \quad \begin{array}{ll} \min & x_1 - x_2 \\ \text{s.t.} & -x_1 + x_2 \geq 1 \\ & x_1 - x_2 \geq -1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$(D) \quad \begin{array}{ll} \max & w_1 - w_2 \\ \text{s.t.} & -w_1 + w_2 \leq 1 \\ & w_1 - w_2 \leq -1 \\ & w_1, w_2 \geq 0 \end{array}$$

The primal optimal solution set is $\{(x_1, x_2) | -x_1 + x_2 = 1, x_1, x_2 \geq 0\}$. The dual optimal solution set is $\{(w_1, w_2) | -w_1 + w_2 = 1, w_1, w_2 \geq 0\}$. Both of them have infinitely many optimal solutions.

Dual relationship

- If we take (D) as the primal problem, what'll happen?
- Convert (D) into its standard form by
 1. taking $\mathbf{w} = \mathbf{u} - \mathbf{v}$
 2. adding slacks s
 3. minimizing objective

$$(P) \quad \begin{aligned} & \text{-Min } [-\mathbf{b}^T \mid \mathbf{b}^T \mid 0] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} \\ & \text{s. t. } [\mathbf{A}^T \mid -\mathbf{A}^T \mid \mathbf{I}] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} = \mathbf{c} \\ & \mathbf{u}, \mathbf{v}, \mathbf{s} \geq 0 \end{aligned}$$

New dual

$$(D) \quad \begin{aligned} & \text{-Max } \mathbf{c}^T \mathbf{w} \\ & \text{s. t. } \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} -\mathbf{b} \\ \mathbf{b} \\ 0 \end{bmatrix} \\ & \mathbf{w} \text{ unrestricted } \in R^n \end{aligned}$$

Dual of the dual = primal

- Rearrange the new dual

$$\begin{aligned}
 & -\text{Max} && \mathbf{c}^T \mathbf{w} \\
 (\text{D}) \quad \text{s. t.} & & \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} -\mathbf{b} \\ \mathbf{b} \\ 0 \end{bmatrix} \\
 & & \mathbf{w} \text{ unrestricted } \in R^n
 \end{aligned}$$

- We have

$$\begin{aligned}
 & -\text{Max} && \mathbf{c}^T \mathbf{w} \\
 \text{s. t.} & & \mathbf{A}\mathbf{w} \leq -\mathbf{b} \\
 & & -\mathbf{A}\mathbf{w} \leq \mathbf{b} \\
 & & \mathbf{w} \leq 0
 \end{aligned}$$

$$\text{Min} \quad -\mathbf{c}^T \mathbf{w}$$

$$\begin{aligned}
 \text{s. t.} & & -\mathbf{A}\mathbf{w} = \mathbf{b} \\
 & & \mathbf{w} \leq 0
 \end{aligned}$$

For $\mathbf{x} := -\mathbf{w}$, we have

$$\begin{aligned}
 & \text{Min} && \mathbf{c}^T \mathbf{x} \\
 \text{s. t.} & & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & & \mathbf{x} \geq 0
 \end{aligned}$$

Lemma 4.1 Dual of the Dual=Primal.

Example

$$\text{Minimize } x_1 - 2x_2$$

$$(P) \text{ subject to } x_1 + x_2 + x_3 = 40$$

$$2x_1 + x_2 + x_4 = 60$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\text{Maximize } 40w_1 + 60w_2$$

$$(D) \text{ subject to } w_1 + 2w_2 + w_3 = 1$$

$$w_1 + w_2 + w_4 = -2$$

$$w_1 \leq 0, w_2 \leq 0$$

$$w_3 \geq 0, w_4 \geq 0$$

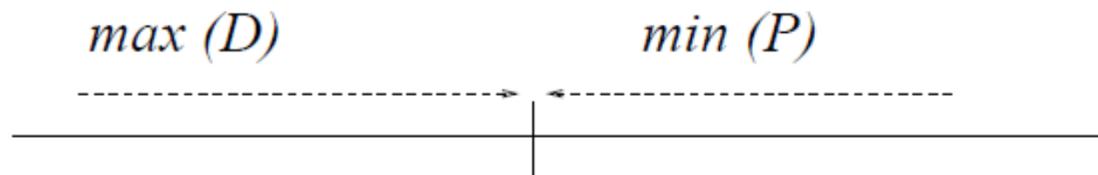
Weak duality theorem

If \mathbf{x} is a primal feasible solution to (P) and \mathbf{w} is a dual feasible solution to (D), then

$$\begin{aligned}\mathbf{c}^T \mathbf{x} &= \mathbf{x}^T \mathbf{c} \\ &\geq \mathbf{x}^T \mathbf{A}^T \mathbf{w} \\ &= \mathbf{b}^T \mathbf{w}\end{aligned}$$

(Weak Duality Theorem):

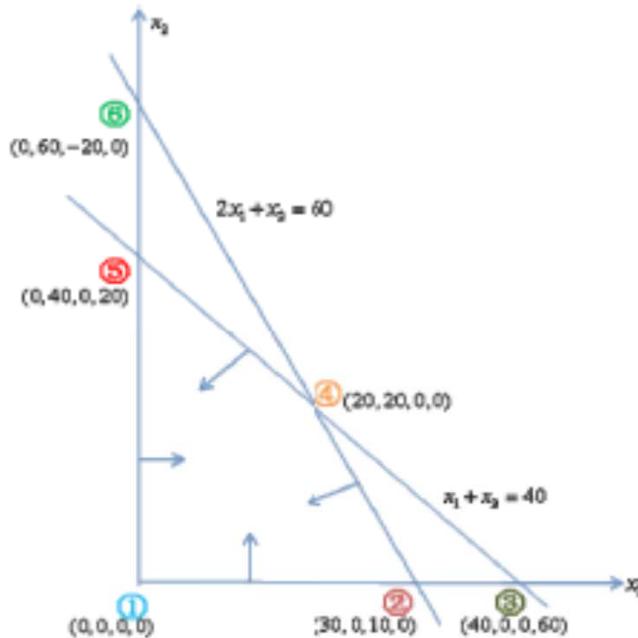
If \mathbf{x} is a **primal feasible** solution to (P) and \mathbf{w} is a **dual feasible** solution to (D), then $\mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{w}$.



Example

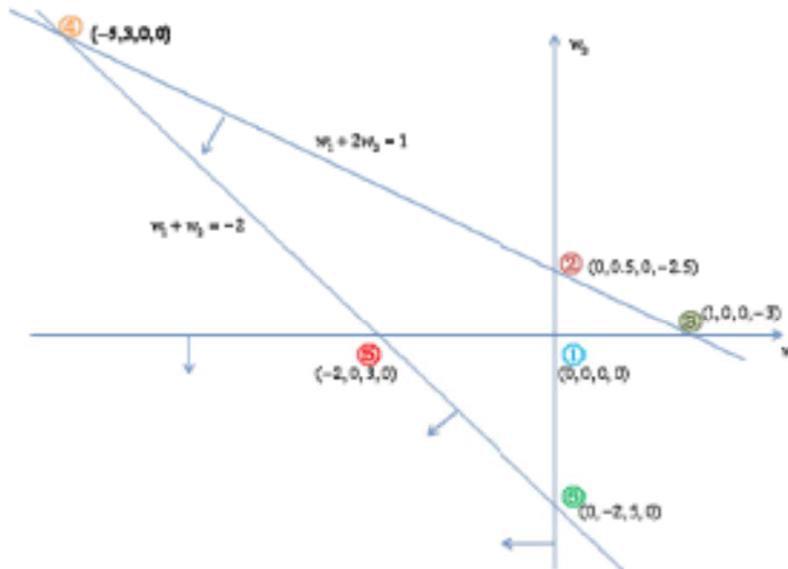
Minimize $x_1 - 2x_2$

$$(P) \text{ subject to } \begin{aligned} x_1 + x_2 + x_3 &= 40 \\ 2x_1 + x_2 + x_4 &= 60 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$



Maximize $40w_1 + 60w_2$

$$(D) \text{ subject to } \begin{aligned} w_1 + 2w_2 + w_3 &= 1 \\ w_1 + w_2 + w_4 &= -2 \\ w_1 \leq 0, w_2 \leq 0 \\ w_3 \geq 0, w_4 \geq 0 \end{aligned}$$



Corollaries

1. If \mathbf{x} is primal feasible, \mathbf{w} is dual feasible, and

$$\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{w},$$

then \mathbf{x} is **primal optimal**, and \mathbf{w} is **dual optimal**.

2. If the primal is **unbounded below**, then the dual is **infeasible**.

(Is the converse statement true? -- watch for infeasibility)

3. If the dual is **unbounded above**, then the primal is **infeasible**.

(Is the converse statement true?)

Strong duality theorem

- Questions:
 - Can the results of weak duality be **stronger**?
 - **Is there any gap** between the primal optimal value and the dual optimal value?
- Strong Duality Theorem:
 - (a) If either the primal or the dual has a finite optimum, then so does the other and $\min \mathbf{c}^T \mathbf{x} = \max \mathbf{b}^T \mathbf{w}$ (No duality gap!)
 - (b) If either problem has an unbounded objective, then the other has no feasible solution.

Proof of strong duality theorem

- Proof:

Note that the dual of the dual is the primal and the fact that
“If \mathbf{x} is primal feasible, \mathbf{w} is dual feasible and

$$\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{w},$$

then \mathbf{x} is primal optimal and \mathbf{w} is dual optimal.”

We only need to show that

“*if the primal has a finite optimal bfs \mathbf{x} , then there exists a dual feasible solution \mathbf{w} such that*

$$\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{w}.$$

Proof - continue

- Applying the simplex method at the optimal bsf \mathbf{x} with basis B , we define $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$.

Then

$$\begin{aligned}\mathbf{c} - \mathbf{A}^T \mathbf{w} &= \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix} - \begin{bmatrix} \mathbf{B}^T \\ \mathbf{N}^T \end{bmatrix} \mathbf{w} \\ &= \begin{bmatrix} \mathbf{c}_B - \mathbf{B}^T (\mathbf{B}^T)^{-1} \mathbf{c}_B \\ \mathbf{c}_N - \mathbf{N}^T (\mathbf{B}^T)^{-1} \mathbf{c}_B \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ r_N \end{bmatrix} \geq 0\end{aligned}$$

Thus \mathbf{w} is dual feasible
and

$$\begin{aligned}\mathbf{c}^T \mathbf{x} &= \mathbf{c}_B^T \mathbf{x}_B \\ &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} \\ &= \mathbf{w}^T \mathbf{b} \\ &= \mathbf{b}^T \mathbf{w}.\end{aligned}$$

- (b) It is a direct consequence of the Weak Duality Theorem.

Implications

- (1) The simplex multiplier \mathbf{w} corresponding to a primal optimal solution \mathbf{x} is a dual optimal solution.
- (2) At each iteration of the simplex method, the simplex multiplier \mathbf{w} always satisfies that $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{w}$.

However, \mathbf{w} is not dual feasible unless $r_N \geq 0$.

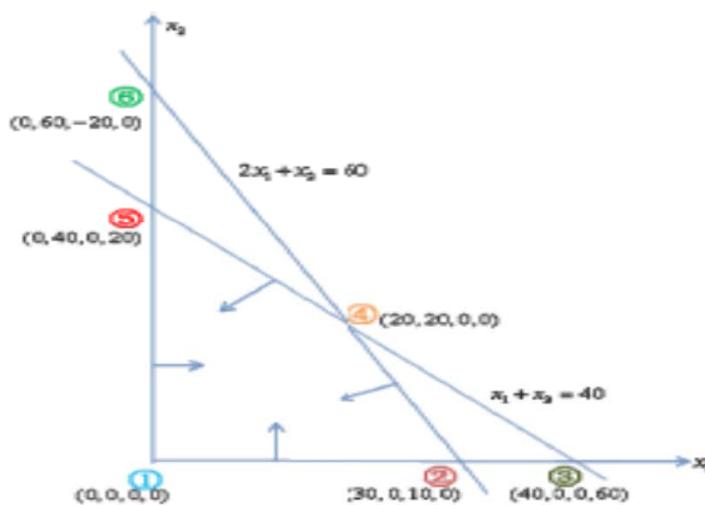
- (3) Revised Simplex Method

Keep primal feasibility
and $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{w}$ (no duality gap)
but seeks for dual feasibility.

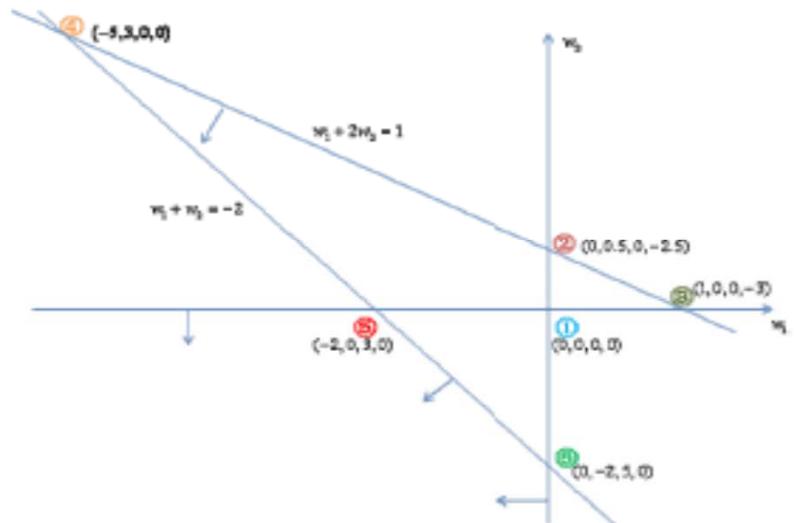
Corresponding basic solutions

- At a primal basic solution \mathbf{x} with basis B , we defined a dual basic solution $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$

$$\begin{aligned}
 & \text{Minimize } x_1 - 2x_2 \\
 (P) \quad & \text{subject to } x_1 + x_2 + x_3 = 40 \\
 & 2x_1 + x_2 + x_4 = 60 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$



$$\begin{aligned}
 & \text{Maximize } 40w_1 + 60w_2 \\
 (D) \quad & \text{subject to } w_1 + 2w_2 + w_3 = 1 \\
 & w_1 + w_2 + w_4 = -2 \\
 & w_1 \leq 0, w_2 \leq 0 \\
 & w_3 \geq 0, w_4 \geq 0
 \end{aligned}$$



Further implications of strong duality theorem

- **Theorem of Alternatives**

“Existence of solutions of systems of equalities and inequalities”

- Famous Farkas Lemma (another form)

The system

$$(I) \quad Ax = b, \quad x \geq 0$$

has no solution if and only if the system

$$(II) \quad A^T w \leq 0, \quad b^T w > 0$$

has solution.

Two systems

$$(I) \quad Ax = b, \quad x \geq 0$$

$$(II) \quad A^T w \leq 0, \quad b^T w > 0$$

Either (I) or (II) has a solution but NOT both.

Proof of Farkas Lemma

Consider LP and its dual

$$\begin{array}{ll} (\text{P}) \quad \text{Min} & 0^T \mathbf{x} \\ \text{s. t.} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

$$\begin{array}{ll} (\text{D}) \quad \text{Max} & \mathbf{b}^T \mathbf{w} \\ \text{s. t.} & \mathbf{A}^T \mathbf{w} \leq 0 \end{array}$$

- Since $w = 0$ is dual feasible, we know
(P) is infeasible if and only if (D) is unbounded above.
Note that
 - (P) is infeasible if and only if (I) has no solution.
 - (D) is unbounded above if and only if (II) has a solution.Hence,
(I) has no solution if and only if (II) has a solution.

Complementary slackness

- Consider the symmetric pair LP

$$(P) \quad \min \quad \mathbf{c}^T \mathbf{x}$$

$$\text{s. t. } \mathbf{A}\mathbf{x} \geq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

$$(D) \quad \max \quad \mathbf{b}^T \mathbf{w}$$

$$\text{s. t. } \mathbf{A}^T \mathbf{w} \leq \mathbf{c}$$

$$\mathbf{w} \geq 0$$

Let \mathbf{x} be primal feasible, \mathbf{w} be dual feasible.

Define

$$\mathbf{s} = \mathbf{A}\mathbf{x} - \mathbf{b} \geq 0$$

$$\mathbf{r} = \mathbf{c} - \mathbf{A}^T \mathbf{w} \geq 0$$

$\mathbf{s} \in R^m$: primal slackness

$\mathbf{r} \in R^n$: dual slackness

Observations

1. If $r^T x = 0$
and $w^T s = 0$ } complementary slackness

then

$$\underline{(c^T - w^T A)x = 0}$$

and

$$\underline{w^T(Ax - b) = 0}$$

Hence

$$c^T x = w^T Ax = w^T b = b^T w$$

Thus x is primal optimal and w is dual optimal.

2. On the contrary side, for a feasible pair (x, w) ,

$$c^T x \geq w^T Ax \geq w^T b$$

If x is primal optimal and w is dual optimal,
then $c^T x = w^T Ax = w^T b$

Hence

$$(c^T - w^T A)x = 0$$

and

$$w^T(Ax - b) = 0$$

Complementary slackness theorem

- Theorem:

Let (P) and (D) be a “symmetric pair”,

\mathbf{x} is primal feasible, \mathbf{w} is dual feasible.

Then \mathbf{x} , \mathbf{w} are optimal solution pair if and only if

$$\begin{cases} r_j = 0 \text{ or } x_j = 0 & \forall j = 1, 2, \dots, n. \\ s_i = 0 \text{ or } w_i = 0 & \forall i = 1, 2, \dots, m. \end{cases}$$

Example

- Consider a relaxed knapsack problem:

$$\text{Max} \quad 3x_1 + 4x_2 + 9x_3 + 2x_4 + 5x_5$$

s.t.

$$4x_1 + 7x_2 + 10x_3 + 3x_4 + 7x_5 \leq 20$$

$$x_j \geq 0, \quad \text{for all } j$$

- Its dual becomes

min	$20y$	$4y^* > 3$	$x^*1 = 0$	
s.t.	$4y \geq 3$	$7y^* > 4$	$x^*2 = 0$	
	$7y \geq 4$			
	$10y \geq 9$	$10y^* = 9$	$x^*3 = ?$	$x^*3 = 2 \quad z^* = 18$
	$3y \geq 2$	$3y^* > 2$	$x^*4 = 0$	
	$7y \geq 5$			
	$y \geq 0$	$7y^* > 5$	$x^*5 = 0$	

dual optimal solution is $y^* = 0.9 \quad 20 - (4x_1 + x_2 + 10x_3 + 3x_4 + 7x_5) = 0$

Complementary slackness for standard form LP

$$\begin{aligned} (\text{P}) \quad & \min \quad \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} (\text{D}) \quad & \max \quad \mathbf{b}^T \mathbf{w} \\ \text{s. t.} \quad & \mathbf{A}^T \mathbf{w} \leq \mathbf{c} \end{aligned}$$

the condition $\mathbf{w}^T \mathbf{s} = \mathbf{0}$ is always true.

The complementary slackness condition reduces to $\mathbf{r}^T \mathbf{x} = \mathbf{0}$.

Kuhn-Tucker condition

- Theorem:

\mathbf{x} is optimal for the problem

$$\begin{aligned}(P) \quad & \min \quad \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0\end{aligned}$$

if and only if there exist \mathbf{w} and \mathbf{r} such that

- (1) $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ (Primal feasibility)
- (2) $\mathbf{A}^T \mathbf{w} + \mathbf{r} = \mathbf{c}, \mathbf{r} \geq 0$ (Dual feasibility)
- (3) $\mathbf{r}^T \mathbf{x} = 0$ (Complementary Slackness)

Implication

- Solving a linear programming problem is equivalent to solving a system of linear inequalities.

Economic interpretation of duality

- Is there any special meaning of the **dual variables**?
- What is a **dual problem** trying to do?
- What's the role of the **complementary slackness** in decision making?

Dual variables

- Consider a nondegenerate linear program:

$$(P) \quad \min \quad \mathbf{c}^T \mathbf{x} \quad (\text{minimize total cost})$$

$$\text{s. t. } \mathbf{Ax} = \mathbf{b} \quad (\text{satisfy demands})$$

$$\mathbf{x} \geq 0 \quad (\text{different services})$$

Assume that \mathbf{x}^* nondegenerate optimal bfs

$$\mathbf{x}^* = \begin{pmatrix} \mathbf{x}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$$

$$z^* = \mathbf{c}^T \mathbf{x}^* = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}$$

Since $\mathbf{x}_B^* = \mathbf{B}^{-1}\mathbf{b} > 0$,

we have $\mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) > 0$, when $\Delta\mathbf{b}$ is small enough!

Then

$$\bar{\mathbf{x}}^* = \begin{pmatrix} \bar{\mathbf{x}}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \\ 0 \end{pmatrix}$$

is an optimal bfs to

$$(\bar{P}) \quad \min \quad \mathbf{c}^T \mathbf{x} \\ \text{s. t. } \mathbf{Ax} = \mathbf{b} + \Delta\mathbf{b} \\ \mathbf{x} \geq 0$$

with $\bar{z}^* = \mathbf{c}_B^T \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b})$.
(Why? No change in r_q !)

Dual variable for shadow price

- Moreover, we have

$$\begin{aligned}\Delta z &= \bar{z}^* - z^* \\ &= \mathbf{c}_B^T \mathbf{B}^{-1} (\mathbf{b} + \Delta \mathbf{b}) - c_B^T B^{-1} \mathbf{b} \\ &= \frac{\mathbf{c}_B^T \mathbf{B}^{-1}}{\mathbf{w}^T} \Delta \mathbf{b}\end{aligned}$$

(simplex multiplier for (P) at optimum!)

Hence w_i is the “marginal price” of the i th demand.

- Note: **dual variable** w_i indicates the minimum unit price that one has to charge for additional demand i . It is also called **shadow price** or **equilibrium price**.

Dual LP problem

Consider the following production scenario:

- **n products** to be produced

x_j = amount of product j , $j = 1, 2, \dots, n$

- **m resources** in hand

b_i = amount of resource i , $i = 1, 2, \dots, m$

- market **selling price** for each product is known

c_1, c_2, \dots, c_n

- **technology matrix** is given by

[a_{ij}] : each product j consumes a_{ij} units of
resource i , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

A manufacturer's view

- Maximize total sales (hence profit)

$$\sum_{j=1}^n c_j x_j$$

- Resource limitation

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i, \quad i = 1, 2, \dots, m$$

- Production Requirement

$$x_j \geq 0, \quad j = 1, 2, \dots, n$$

$$(P) \quad \max \quad \mathbf{c}^T \mathbf{x}$$

$$\text{s. t. } \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

Getting Resources

- m resources to purchase from a supplier

w_i = unit price to purchase resource i ,
 $i = 1, 2, \dots, m$

- Free information market:

Supplier knows your selling price c_j for product x_j and he/she wants to get the most out of you. i.e.

$$a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m \geq c_j, \quad j = 1, 2, \dots, n$$

Therefore, we have the dual

$$\min \quad \mathbf{b}^T \mathbf{w} \quad (\text{minimize total spending})$$

$$\text{s. t. } \mathbf{A}^T \mathbf{w} \geq \mathbf{c} \quad (\text{price accepted by the supplier})$$

$$\mathbf{w} \geq 0$$

Complementary slackness condition

- Observation:

1. w_i^* is the maximum marginal price the manufacturer is willing to pay the supplier for resource i .
2. When resource i is not fully utilized, i.e.,

$$a_{i1}x_1^* + a_{i2}x_2^* + \dots + a_{in}x_n^* < b_i$$

the complementary slackness condition implies $w_i^* = 0$.

This means the manufacturer is not willing to pay a penny for buying any additional amount !

3. When the supplier asks too much, i.e.,

$$a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_n > c_j$$

then $x_j = 0$. This means the manufacturer is not going to produce any product j !

Sensitivity analysis

- Sensitivity is a post-optimality analysis of a linear program

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

in which, some components of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ may change after obtaining an optimal solution \mathbf{x}^* with an optimal basis \mathbf{B}^* and an optimal objective value z^* .

Questions of interests:

Will \mathbf{x}^* remain optimal ?

\mathbf{B}^* remain optimal ?

or How will they change accordingly?

Fundamental concepts

- No matter how the data (A , b , c) change, we need to make sure that

1. Feasibility (**c** is not involved):

bfs \mathbf{x} is feasible if and only $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$

2. Optimality (**b** is not involved):

\mathbf{x} is optimal $\iff r_q \geq 0 \quad \forall q \in \mathbf{N}.$

$$r_q = c_q - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_q$$

Change in the cost vector

- Scenario: \mathbf{x}^* optimal solution, $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$

Given $\mathbf{c}' = \begin{bmatrix} \mathbf{c}'_B \\ \mathbf{c}'_N \end{bmatrix} \in R^n$ be a perturbation,

Consider

$$\mathbf{c} \rightarrow \mathbf{c} + \alpha \mathbf{c}' = \begin{bmatrix} \mathbf{c}_B + \alpha \mathbf{c}'_B \\ \mathbf{c}_N + \alpha \mathbf{c}'_N \end{bmatrix} \stackrel{\Delta}{=} \bar{\mathbf{c}}, \quad (\text{P}') \quad \text{s. t.} \quad \min z(\alpha) = \underline{(\mathbf{c} + \alpha \mathbf{c}')^T \mathbf{x}} = \bar{\mathbf{c}}^T \mathbf{x}$$

$$\alpha \in R$$

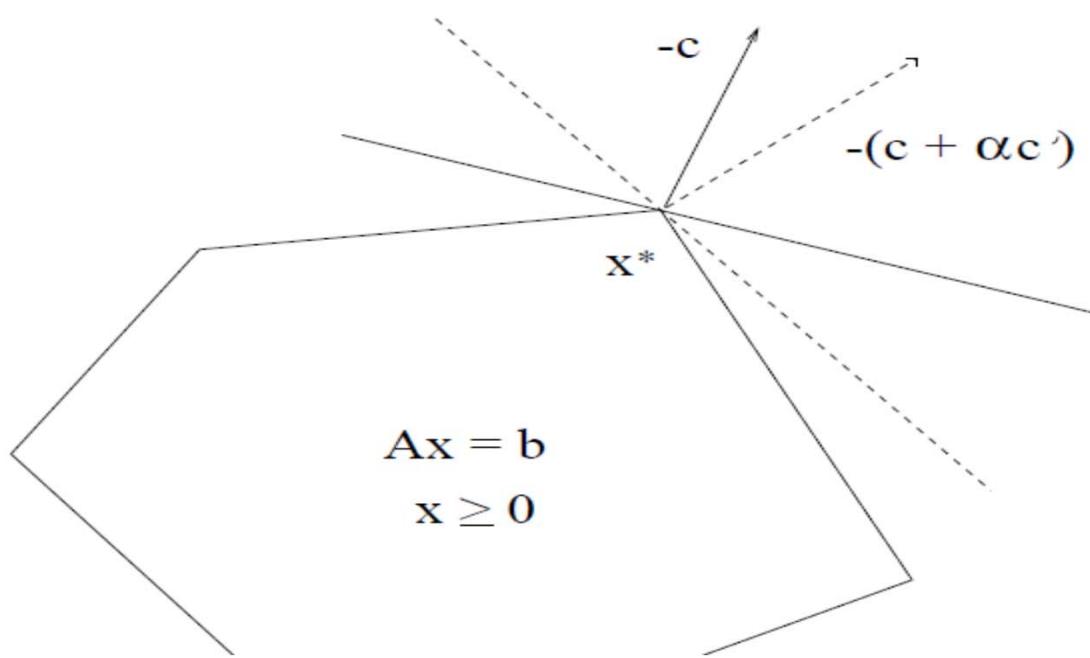
$$\mathbf{x} \geq 0$$

- Question:

Within which range of $[\underline{\alpha}, \bar{\alpha}]$, the current optimal solution \mathbf{x}^* remains to be optimal?

Note: when $\mathbf{c}' = (0, 0, 1, 0, \dots, 0)$, we have the regular sensitivity analysis on each cost coefficient.

Geometric view



- Fact:

As α changes,
 $\mathbf{x}^*(\alpha)$ changes,
 $\mathbf{B}(\alpha)$ changes,
 $z^*(\alpha)$ changes

Analysis

- Compare

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A} \mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & z(\alpha) = \underline{(\mathbf{c} + \alpha \mathbf{c}')^T \mathbf{x}} = \bar{\mathbf{c}}^T \mathbf{x} \\ (\text{P}') \quad & \text{s. t.} \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

(1) (P) and (P') have the same feasible domain,
hence \mathbf{x}^* is feasible to (P') for any α .

(2) \mathbf{x}^* remains optimal to (P') if $\bar{r}_N^T = \bar{\mathbf{c}}_N^T - \bar{\mathbf{c}}_B^T \mathbf{B}^{-1} \mathbf{N} \geq 0$
i.e., $(\mathbf{c}_N + \alpha \mathbf{c}'_N)^T - (\mathbf{c}_B + \alpha \mathbf{c}'_B)^T \mathbf{B}^{-1} \mathbf{N} \geq 0$

$$\frac{(\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}) + \alpha (\mathbf{c}'_N^T - \mathbf{c}'_B^T \mathbf{B}^{-1} \mathbf{N})}{r_N^T} \geq 0$$

$$\alpha r_N'^T \geq -r_N^T$$

Analysis - continue

(3) Case 1. For $r'_q > 0, q \in N$

$\alpha \geq \frac{-r_q}{r'_q}$ is required, thus

$$\underline{\alpha} = \max\left\{\frac{-r_q}{r'_q} \mid r'_q > 0, q \in N\right\}$$

otherwise

$$\underline{\alpha} = -\infty, \text{ if } r'_q \leq 0, \forall q \in N$$

Case 2. For $r'_q < 0, q \in N$

$\alpha \leq \frac{-r_q}{r'_q}$ is required, thus

$$\bar{\alpha} = \min\left\{\frac{-r_q}{r'_q} \mid r'_q < 0, q \in N\right\}$$

otherwise

$$\bar{\alpha} = \infty, \text{ if } r'_q \geq 0, \forall q \in N$$

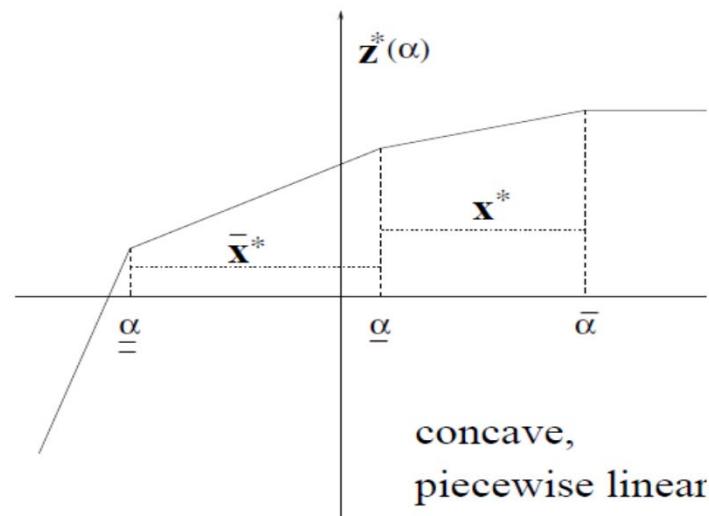
Analysis - continue

(5) For $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, \mathbf{x}^* remains optimal.

$$\begin{aligned} z^*(\alpha) &= (\mathbf{c}_B^T + \alpha \mathbf{c}'_B^T) \mathbf{B}^{-1} \mathbf{b} \\ &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} + \alpha \mathbf{c}'_B^T \mathbf{B}^{-1} \mathbf{b} \\ &\quad k : \text{constant} \\ &= z^* + k \alpha \end{aligned}$$

Thus $z^*(\alpha)$ is linear in α .

(6) .



(7) Take

$$\mathbf{c}' = e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow j\text{th place}$$

then $[c_j - \underline{\alpha}, c_j + \bar{\alpha}]$ gives stable range of the j th cost coefficient, or how sensitive the cost component is.

Change in the r-h-s vector

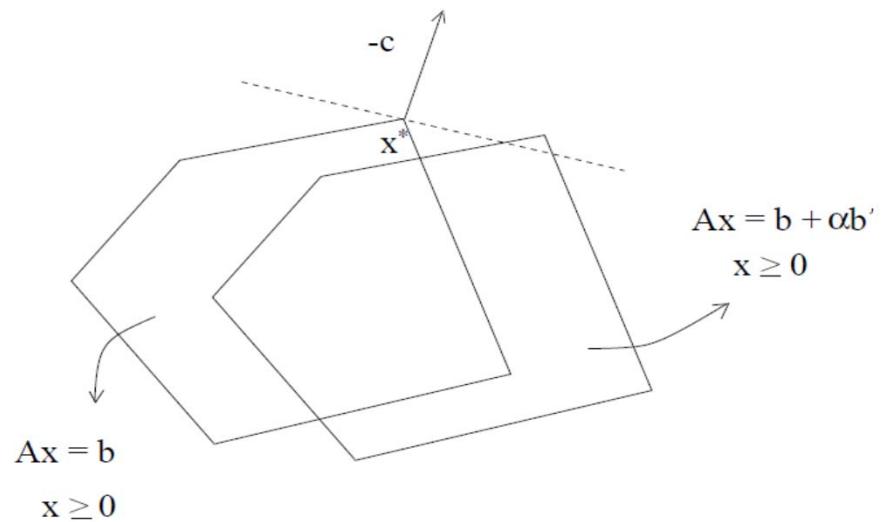
- Scenario:

Let $\mathbf{b}' \in R^m$ be a perturbation.

$$\min \mathbf{c}^T \mathbf{x}$$

$$(P') \quad \text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha \mathbf{b}'$$

$$\mathbf{x} \geq 0$$



- Fact: \mathbf{x}^* may become infeasible !
- Question:

Within which range $[\underline{\alpha}, \bar{\alpha}]$,
will \mathbf{B} remain as an optimal basis?

Analysis

$$\min \quad \mathbf{c}^T \mathbf{x}$$

$$(P') \quad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha\mathbf{b}' \quad \begin{array}{l} (\text{i}) \quad r_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} \geq 0, \text{ and} \\ (\text{ii}) \quad x(\alpha) = \left[\frac{\mathbf{B}^{-1}(\mathbf{b} + \alpha\mathbf{b}')}{\mathbf{0}} \right] \geq 0 \end{array}$$

$$\mathbf{x} \geq 0 \quad \begin{array}{l} (\text{2}) \quad (\text{i}) \text{ always holds, since} \\ \mathbf{c}, \mathbf{B}, \mathbf{N} \text{ no change!} \end{array}$$

(3) We need $\mathbf{B}^{-1}(\mathbf{b} + \alpha\mathbf{b}') \geq 0$ but (ii) is not always true.

$$i.e., \underline{\mathbf{B}^{-1}\mathbf{b}} + \alpha\underline{\mathbf{B}^{-1}\mathbf{b}'} \geq 0$$

$$\bar{\mathbf{b}} \quad \bar{\mathbf{b}'} \quad i.e., \alpha\bar{\mathbf{b}'} \geq -\bar{\mathbf{b}}$$

(i) For $\bar{\mathbf{b}}'_p > 0$, $p \in \mathbf{B}$, we need

$$\alpha \geq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p}$$

(ii) For $\bar{\mathbf{b}}'_p < 0$, $p \in \mathbf{B}$, we need

$$\alpha \leq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p}$$

$$\text{Thus } \underline{\alpha} = \begin{cases} \max\left\{\frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p} \mid \bar{\mathbf{b}}'_p > 0, p \in \mathbf{B}\right\}, \\ -\infty \end{cases}$$

$$\text{Thus } \bar{\alpha} = \begin{cases} \min\left\{\frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}}'_p} \mid \bar{\mathbf{b}}'_p < 0, p \in \mathbf{B}\right\}, \\ +\infty \end{cases}$$

Analysis

(4) When $\alpha \in [\underline{\alpha}, \bar{\alpha}]$,

$$\mathbf{x}^*(\alpha) = \begin{pmatrix} \mathbf{B}^{-1}(\mathbf{b} + \alpha \mathbf{b}') \\ \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} + \alpha \mathbf{B}^{-1}\mathbf{b}' \\ \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} + \alpha \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b}' \\ \mathbf{0} \end{pmatrix}$$

$$= \mathbf{x}^* + \alpha \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b}' \\ \mathbf{0} \end{pmatrix}$$

linear in α !

(5)

$$\begin{aligned} z^*(\alpha) &= \mathbf{c}_B^T \mathbf{x}^*(\alpha)_B \\ &= \mathbf{c}_B^T (\mathbf{x}_B^* + \alpha \mathbf{B}^{-1} \mathbf{b}') \\ &= \mathbf{c}_B^T \mathbf{x}_B^* + \alpha \frac{\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b}'}{k} \end{aligned}$$

$$= z^* + k \alpha$$

again, linear in α !

Changes in the constraint matrix

- Since both feasibility and optimality are involved, a general analysis is difficult.
- We only consider simple cases such as adding a new variable, removing a variable, adding a new constraint.

Adding a new variable

- Why? A new product, service or activity is introduced.

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} + c_{n+1} x_{n+1} \\ (P') \quad \text{s. t.} \quad & \mathbf{Ax} + A_{n+1} x_{n+1} = \mathbf{b} \\ & \mathbf{x}, x_{n+1} \geq 0 \end{aligned}$$

- Analysis:

- (1) $\begin{bmatrix} \mathbf{x}^* \\ 0 \end{bmatrix}$ is a bfs of (P') with $[\mathbf{B} \mid \mathbf{N}, A_{n+1}]$. (3) If $r_{n+1} < 0$, then x_{n+1} enters the basis and continue the revised simplex method to find an optimal solution of (P') .
- (2) $\begin{bmatrix} \mathbf{x}^* \\ 0 \end{bmatrix}$ is an optimal solution of (P') if $r_{n+1} = c_{n+1} - \mathbf{c}_B^T \mathbf{B}^{-1} A_{n+1} \geq 0$.

Removing a variable

- Why? An activity is no longer available.
 - (a) if $x_k^* = 0$, then \mathbf{x}^* remains optimal by deleting x_k^* .
 - (b) if $x_k^* > 0$, then \mathbf{x}_k has to leave the basis.
Can this be done ? Consider

$$\begin{array}{ll}\min & x_k \\ \text{(Phase I)} & \text{s. t. } \mathbf{Ax} = \mathbf{b}\end{array}$$

$$\mathbf{x} \geq 0$$

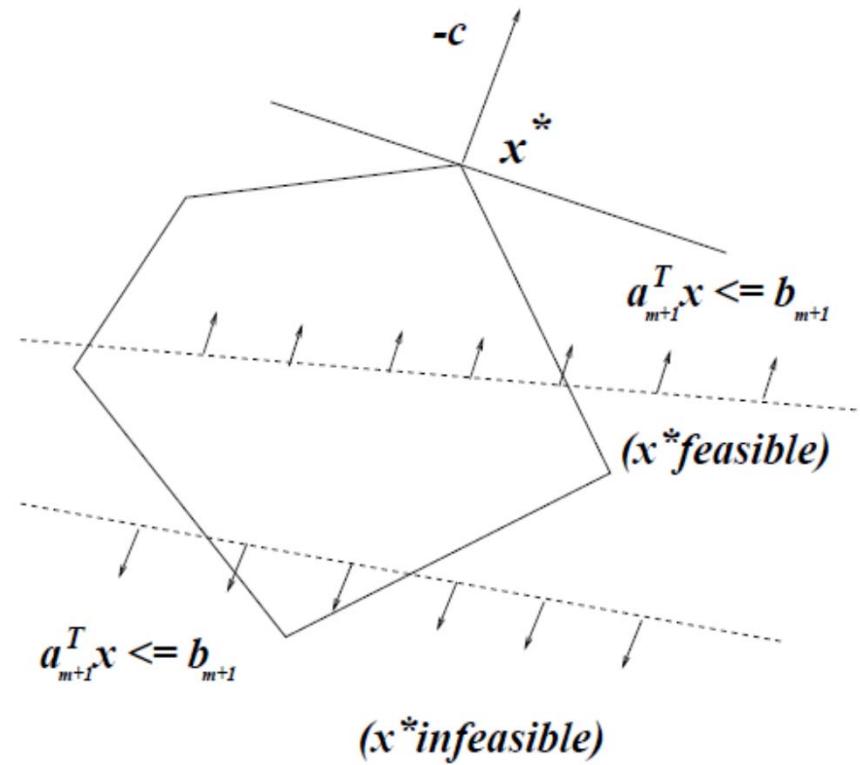
- (1) \mathbf{x}^* is a current bfs to start the revised simplex solution.
- (2) If $z_{PhI}^* > 0$, then removing x_k will cause infeasibility.
If $z_{PhI}^* = 0$, then we can start from there to solve the new problem.

Adding a new constraint

- Why? A new restriction is enforced.

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ (P') \quad \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{a}_{m+1}^T \mathbf{x} \leq b_{m+1} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\mathbf{a}_{m+1}^T = (a_{m+1,1}, a_{m+1,2}, \dots, a_{m+1,n})$$



Analysis

- (1) If $\mathbf{a}_{m+1}^T \mathbf{x}^* \leq b_{m+1}$ then \mathbf{x}^* remains optimal!
- (2) If not, \mathbf{x}^* is not feasible and we have to find a new basis of dimensionality $m + 1$.

(3) Consider

$$\min \quad \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N + 0x_{n+1}$$

$$(P') \text{ s. t. } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$

$$(\mathbf{a}_{m+1})_B^T \mathbf{x}_B + (\mathbf{a}_{m+1})_N^T \mathbf{x}_N + x_{n+1} = b_{m+1}$$

$$\mathbf{x}_B, \mathbf{x}_N, x_{n+1} \geq 0$$

$$\bar{\mathbf{B}} := \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ (\mathbf{a}_{m+1})_B^T & 1 \end{pmatrix}$$

then $\bar{\mathbf{B}}$ is a nonsingular $(m + 1) \times (m + 1)$ matrix, and

$$\bar{\mathbf{B}}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix}$$

i.e. $\bar{\mathbf{B}}$ is a basis for (P') !

Analysis

(4) The reduced cost

$$\begin{aligned} r'_q &= c_q - \left[\begin{array}{c} \mathbf{c}_B \\ 0 \end{array} \right]^T \bar{\mathbf{B}}^{-1} \left[\begin{array}{c} A_q \\ a_{m+1,q} \end{array} \right] \\ &= c_q - \left[\begin{array}{c|c} \mathbf{c}_B^T \mathbf{B}^{-1} & 0 \end{array} \right] \left[\begin{array}{c} A_q \\ a_{m+1,q} \end{array} \right] \\ &= c_q - \mathbf{c}_B^T \mathbf{B}^{-1} A_q \\ &= r_q, \quad \forall q \in \mathbb{N} \end{aligned}$$

Since \mathbf{B} is an optimal basis to (P) , we know

$$r'_q = r_q \geq 0, \quad \forall q \in \mathbb{N}$$

i.e., $\bar{\mathbf{B}}$ provides a dual feasible solution

$$\mathbf{w}^T = \mathbf{c}_B^T \bar{\mathbf{B}}^{-1} \text{ for } (P').$$

(5) Define

$$\begin{aligned} \bar{\mathbf{x}}_{\bar{B}} &= \bar{\mathbf{B}}^{-1} \left(\frac{\mathbf{b}}{b_{m+1}} \right) \\ \bar{\mathbf{x}}_N &= \mathbf{0} \end{aligned}$$

then $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_{\bar{B}} | \bar{\mathbf{x}}_N)$ is an optimal solution of (P')
if $\bar{\mathbf{x}}_{\bar{B}} \geq 0$.

(6) If

$$\bar{\mathbf{x}}_{\bar{B}} = \bar{\mathbf{B}}^{-1} \left(\frac{\mathbf{b}}{b_{m+1}} \right) \not\geq 0$$

then we can apply the dual simplex method
with

$$\begin{aligned} \mathbf{w}^T &= \mathbf{c}_{\bar{B}}^T \bar{\mathbf{B}}^{-1} \\ &= (\mathbf{c}_B^T, \ 0) \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix} \\ &= (\mathbf{c}_B^T \mathbf{B}^{-1} | 0) \end{aligned}$$

to solve (P') .

Dual simplex method

- What's the dual simplex method?
 - It is a **simplex based** algorithm that works on the **dual problem** directly. In other words, it hops from one vertex to another vertex along some edge directions in the dual space.
- It keeps **dual feasibility** and **complementary slackness**, but seeks **primal feasibility**.

Motivation

- Applying the (revised) simplex method to solve the dual problem:

$$(D) \quad \max \quad b^T w \quad m \text{ variables}$$

$$\text{s. t. } A^T w \leq c \quad n \text{ constraints}$$

$$(D') \quad (-) \min \quad -b^T u + b^T v + 0^T s \quad 2m + n \text{ variables}$$

(i) Dimensionality becomes larger.

$$\text{s. t. } A^T u - A^T v + s = c \quad n \text{ constraints}$$

(ii) Keep (*dual*) feasibility

Maintain complementary slackness

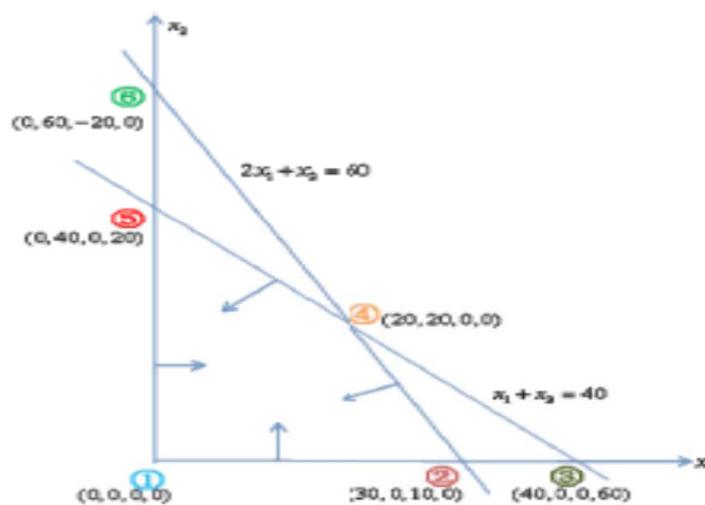
$$u, v, s \geq 0$$

Seek (*primal*) feasibility

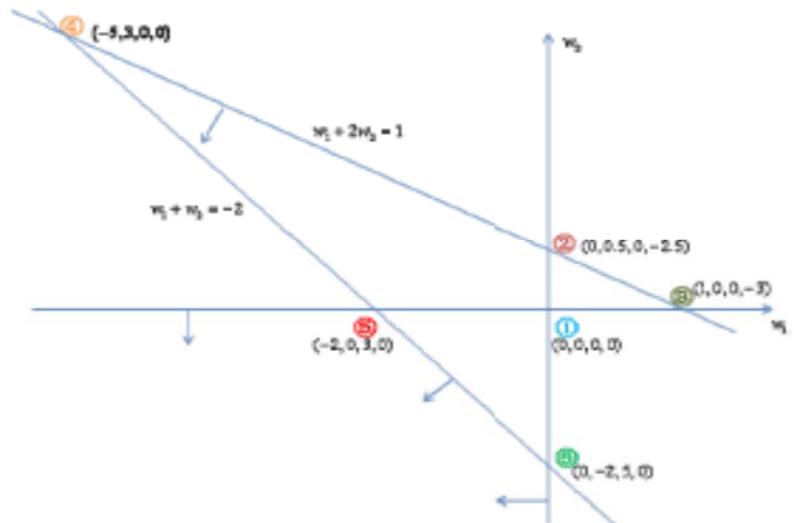
Corresponding basic solutions

- At a **primal basic solution** \mathbf{x} with basis B , we defined a dual basic solution $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$

$$\begin{aligned}
 & \text{Minimize } x_1 - 2x_2 \\
 (P) \quad & \text{subject to } x_1 + x_2 + x_3 = 40 \\
 & 2x_1 + x_2 + x_4 = 60 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$



$$\begin{aligned}
 & \text{Maximize } 40w_1 + 60w_2 \\
 (D) \quad & \text{subject to } w_1 + 2w_2 + w_3 = 1 \\
 & w_1 + w_2 + w_4 = -2 \\
 & w_1 \leq 0, w_2 \leq 0 \\
 & w_3 \geq 0, w_4 \geq 0
 \end{aligned}$$



Basic ideas of dual simplex method

- Starting with a dual basic feasible solution

(1) Start with a basis

$$A = [B \mid N]$$

such that

$w^T := c_B^T B^{-1}$ is dual feasible, i.e.,

$$A^T w \leq c$$

(2) Further define

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

then

$$\begin{aligned} r^T x &= (c^T - w^T A)x \\ &= c^T x - w^T Ax \\ &= c_B^T B^{-1} b - c_B^T B^{-1} b \\ &= 0 \end{aligned}$$

i.e., complementary slackness condition holds.

Basic ideas of dual simplex method

- Checking optimality

(3) Since

$$\mathbf{A}\mathbf{x} = [\mathbf{B} \mid \mathbf{N}] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \mathbf{B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{b}.$$

If $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$, then

we have primal feasibility

and $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$ is primal optimal,
 $\mathbf{w}^T := \mathbf{c}_B^T \mathbf{B}^{-1}$ is dual optimal.

Basic ideas of dual simplex method

- Pivoting move

If there exists $p \in \mathbf{B}$ such that $x_p < 0$

then A_p may leave the basis

and $x_p \leftarrow 0$ becomes nbv.

And we have to pivot-in a nbv x_q for $q \in \mathbf{N}$.

Related issues

- Question 1: Which x_q is entering the basis?
- Analysis:
 - (1) We should keep dual feasibility and complementary slackness.
 - (2) Complementary slackness condition always holds according to the way we define \mathbf{w} and \mathbf{x} .
 - (3) The real problem is to keep dual feasibility while x_p leaves and x_q enters.

Related issues

- Question 2:
Where is the dual feasibility information?
- Guess: must be from the **fundamental matrix** and vector **c**.

$$\begin{aligned}\mathbf{M}^{-1} &= \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\ \mathbf{c}^T \mathbf{M}^{-1} &= (\mathbf{c}_B^T \mid \mathbf{c}_N^T) \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\ &= (\mathbf{c}_B^T \mathbf{B}^{-1} \mid \mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}) \\ &= (\mathbf{w}^T \mid \mathbf{r}_N^T)\end{aligned}$$

$\nearrow \qquad \nwarrow$

dual variable reduced cost

$$r_N \geq 0 \iff \text{dual feasibility}$$

Answer:

$x_p \longrightarrow \text{nonbasic}$

$x_q \longleftarrow \text{basic}$

$\mathbf{M} \longleftarrow \text{update}$

We check

$\mathbf{c}^T \mathbf{M}^{-1} = (\mathbf{w}^T \mid \mathbf{r}_N^T)$
for dual feasibility!

Related issues

- Question 3: When $x_p \rightarrow$ nonbasic
 $x_q \leftarrow$ basic
How will \mathbf{M}^{-1} change?
- Answer: Sherman-Morrison-Woodbury formula.

Sherman, J.; Morrison, W. J. (1949). "Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix (abstract)". Annals of Mathematical Statistics 20: 621.
doi:10.1214/aoms/1177729959

Sherman-Morrison-Woodbury Formula

- Lemma: M : nonsingular $n \times n$ matrix.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in R^n$$

If $w := 1 + v^T M^{-1} u \neq 0$

then

$(M + uv^T)$ is nonsingular and

$$(M + uv^T)^{-1} = M^{-1} - (1/w)M^{-1}uv^TM^{-1}.$$

Proof: Check that

$$(M + uv^T)[M^{-1} - (1/w)M^{-1}uv^TM^{-1}] = I.$$

Proof from wikipedia.org/wiki

We verify the properties of the inverse. A matrix Y (in this case the right-hand side of the Sherman–Morrison formula) is the inverse of a matrix X (in this case $A + uv^T$) if and only if $XY = YX = I$.

We first verify that the right hand side (Y) satisfies $XY = I$. Note that $v^T A^{-1} u$ is a scalar, and so **($1 + v^T A^{-1} u$) can be factored out.**

$$\begin{aligned} XY &= (A + uv^T) \left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \right) \\ &= AA^{-1} + uv^TA^{-1} - \frac{AA^{-1}uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{(1+v^TA^{-1}u)uv^TA^{-1}}{1+v^TA^{-1}u} \\ &= I + uv^TA^{-1} - uv^TA^{-1} \\ &= I. \end{aligned}$$

In the same way, it is verified that

$$YX = \left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \right) (A + uv^T) = I.$$

Observations

- (1) It takes $o(n^3)$ elementary operations to invert the matrix $(M + uv^T)$ directly.
- (2) The formula only takes $o(n^2)$ elementary operations.
- (3) Sometimes, we call it *rank-one* updating method.
- (4) For our case

$$\left\{ \begin{array}{l} x_p \longrightarrow \text{out} \\ x_q \longleftarrow \text{in} \end{array} \right\} M \text{ becomes } \bar{M}$$

$$\bar{M} = M + e_q(e_p - e_q)^T.$$

Example

$$A = \left[\begin{array}{cc|ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \end{array} \right]$$

The q th row (e_q^T) of M is replaced by e_p^T in \bar{M} ,

$$M + e_q(e_p - e_q)^T =$$

$$M = \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \longrightarrow e_5^T$$

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$x_1 \rightarrow \text{out}, p = 1, x_5 \rightarrow \text{in}, q = 5$

$$\bar{M} = \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right] \longrightarrow e_1^T = \bar{M}.$$

$$= \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

Analysis

Let $u = e_q$, $v = (e_p - e_q)$

Since $(\mathbf{w}, r_N^T) = c^T M^{-1}$, we have

$$\bar{M}^{-1} = M^{-1} - \frac{M^{-1} + e_q(e_p - e_q)^T M^{-1}}{1 + (e_p - e_q)^T M^{-1} e_q}$$

$$(\bar{\mathbf{w}}, \bar{r}_N^T) = c^T \bar{M}^{-1}$$

Notice that

$$\begin{aligned} e_q^T M^{-1} &= q\text{th row of } M^{-1} \\ &= e_q^T \end{aligned}$$

Hence

$$\begin{aligned} \bar{M}^{-1} &= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{1 + e_p^T M^{-1} e_q - e_q^T e_q} \\ &= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q} \end{aligned}$$

We define

$$u^T = e_p^T \mathbf{B}^{-1}$$

$$y_j = u^T A_j$$

$$\gamma = \frac{r_q}{y_q}$$

then

$$\bar{\mathbf{w}} = \mathbf{w} + \gamma u$$

$$\bar{r}_j = r_j - \gamma y_j, \quad \forall j \in \mathbf{N} - \{q\}$$

$$\bar{r}_p = -\gamma$$

Observations

1. $u^T = e_p^T \mathbf{B}^{-1} \implies u^T$ is the p th row of \mathbf{B}^{-1} .
2. $y_q = u^T A_q = -d_p^q$ where $\mathbf{d}^q = -\mathbf{B}^{-1} A_q$.
3. To maintain dual feasibility, we need

$$\bar{r}_p = -\gamma = -\frac{r_q}{y_q} \geq 0$$

and

$$\bar{r}_j = r_j - \gamma y_j \geq 0, \quad \forall j \in \mathbf{N} - \{q\}.$$

Case 1:

If $\exists j \in \mathbf{N}$ such that $y_j < 0$

then $\frac{-r_j}{y_j} \geq -\gamma$ is required.

Therefore q is chosen by the min-ratio test,
i.e.,

$$\frac{-r_q}{y_q} = \min\left\{\frac{-r_j}{y_j} \mid y_j < 0, j \in \mathbf{N}\right\}$$

Case 2:

If $y_j \geq 0, \forall j \in \mathbf{N}$

then $y_j = u^T A_j = e_p^T \mathbf{B}^{-1} A_j, j \in \mathbf{N}$.

and $e_p^T \mathbf{B}^{-1} \mathbf{A} = e_p^T \mathbf{B}^{-1} [\mathbf{B} \mid \mathbf{N}] \geq 0$.

Thus $e_p^T \mathbf{B}^{-1} \mathbf{A} \mathbf{x} \geq 0, \forall \mathbf{x} \geq 0, \mathbf{x}$ feasible.

||

$$e_p^T \mathbf{B}^{-1} \mathbf{b} = e_p^T \mathbf{x}_B = x_p. \rightarrow \leftarrow x_p < 0$$

Therefore there is no primal feasible solution!

Dual simplex method

Step 1 (starting with a feasible basic solution):

In the primal problem, given,

$$\mathbf{B} = [\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \mathbf{A}_{j_3}, \dots, \mathbf{A}_{j_m}]$$

$$\tilde{\mathbf{B}} = [j_1, j_2, j_3, \dots, j_m]$$

A dual basic feasible solution \mathbf{w} can be obtained by solving

$$\mathbf{B}^T \mathbf{w} = \mathbf{c}_B$$

Compute the reduced cost \mathbf{r} with

$$r_j = c_j - \mathbf{w}^T \mathbf{A}_j, \quad \forall j \notin \tilde{\mathbf{B}}$$

Step 2 (checking for optimality):

Compute \mathbf{x}_B by solving

$$\mathbf{Bx}_B = \mathbf{b}$$

If $\mathbf{x}_B \geq \mathbf{0}$, then STOP. The current solution

$$\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

is optimal.

Otherwise go to Step 3.

Step 3 (leaving the basis):

Choose a basic variable $x_{j_p} < 0$ with index $j_p \in \tilde{\mathbf{B}}$.

Step 4 (checking for infeasibility):

Compute \mathbf{u} by solving

$$\mathbf{B}^T \mathbf{u} = \mathbf{e}_p$$

Also compute

$$y_j = \mathbf{u}^T \mathbf{A}_j, \quad \forall j \notin \tilde{\mathbf{B}}$$

If $y_j \geq 0, \forall j \notin \tilde{\mathbf{B}}$; then STOP. The primal problem is infeasible.

Otherwise go to Step 5.

Dual simplex method

Step 5 (entering the basis):

Choose a nonbasic variable x_q by the minimum ratio test

$$\frac{-r_q}{y_q} = \min \left\{ \frac{-r_j}{y_j} \mid y_j < 0, j \notin \tilde{\mathbf{B}} \right\}.$$

Set

$$\frac{-r_q}{y_q} = -\gamma$$

Step 6 (updating the reduced costs):

$$r_j \leftarrow r_j - \gamma y_j \quad \forall j \notin \tilde{\mathbf{B}}, \quad j \neq q$$

$$r_{j_p} \leftarrow -\gamma$$

Step 7 (updating the current solution and basis):
Compute \mathbf{d} by solving

$$\mathbf{B}\mathbf{d} = -\mathbf{A}_q$$

Set

$$x_q \leftarrow \alpha = \frac{x_{j_p}}{y_q} = \left(\frac{-x_{j_p}}{d_p} \right)$$

$$x_{j_i} \leftarrow x_{j_i} + \alpha d_{j_i}, \quad \forall j_i \in \tilde{\mathbf{B}}, \quad i \neq p$$

$$\mathbf{B} \leftarrow \mathbf{B} + [A_q - A_{j_p}] \mathbf{e}_p^T$$

$$\tilde{\mathbf{B}} \leftarrow \tilde{\mathbf{B}} \cup \{q\} \setminus \{j_p\}$$

Go to Step 2.

Example

Minimize $-2x_1 - x_2$

$$\begin{array}{lllll} \text{s.t.} & x_1 + x_2 + x_3 & = & 2 \\ & x_1 + x_4 & = & 1 \\ & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Step 1 (starting): Choose $\mathbf{B} = \{1, 4\}$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Then the dual solution

$$\mathbf{w} = \mathbf{c}_B^T \mathbf{B}^{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Computing r_j , $\forall j \notin \tilde{\mathbf{B}}$, we have $r_2 = 1, r_3 = 2$, which implies that \mathbf{w} is dual feasible.

Step 2 (checking for optimality):

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

the corresponding primal vector is infeasible.

Step 3 (leaving the basis):

$x_4 < 0$ (the second element in $\tilde{\mathbf{B}}$), x_4 leaves the basis and let $p = 2$.

Step 4 (checking infeasibility):

$$\mathbf{u}^T = \mathbf{e}_2^T \mathbf{B}^{-1} = [-1 \quad 1]$$

and

$$y_2 = \mathbf{u}^T \mathbf{A}_2 = -1, \quad y_3 = \mathbf{u}^T \mathbf{A}_3 = -1$$

Step 5 (entering the basis):

Take the minimum ratio test

$$-\frac{r_2}{y_2} = \min \left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1 = -\gamma$$

Therefore x_2 is entering the basis and $p = 2$.

Example - continue

Step 6 (updating the reduced cost):

$$r_4 = -\gamma = 1 \text{ and } r_3 = 2 - \gamma y_3 = 1$$

(note that r_2 has been changed from 1 to 0 as x_2 enters the basis.)

Step 7 (updating current solution and basis):

Solving for \mathbf{d} in $\mathbf{Bd} = -\mathbf{A}_2$, we obtain

$$\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Also

$$x_2 = \alpha = \frac{x_4}{y_2} = 1$$

$$x_1 = 2 - 1 \times 1 = 1$$

Thus the new primal vector has $x_1 = x_2 = 1$ (and nonbasic variables $x_3 = x_4 = 0$).

Since it's nonnegative, we know it's a optimal solution to the original linear program.

The corresponding optimal basis \mathbf{B} becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

How to start the dual simplex method?

$$(P) \quad \begin{aligned} & \min && \mathbf{c}^T \mathbf{x} \\ & \text{s. t.} && \mathbf{A}\mathbf{x} = \mathbf{b} \\ & && \mathbf{x} \geq 0 \end{aligned}$$

$$(D) \quad \begin{aligned} & \max && \mathbf{b}^T \mathbf{w} \\ & \text{s. t.} && \mathbf{A}^T \mathbf{w} \leq \mathbf{c} \end{aligned}$$

$\mathbf{A} = [\mathbf{B} \mid \mathbf{N}]$, $\mathbf{B}_{m \times m}$ nonsingular matrix.

$$\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$$

$\mathbf{A}^T \mathbf{w} \leq \mathbf{c}$? If not, consider:

$$(P') \quad \begin{aligned} & \min && \mathbf{c}^T \mathbf{x} \\ & \text{s. t.} && \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{e} \\ & && \mathbf{x} \geq 0 \end{aligned}$$

$$(D') \quad \begin{aligned} & \max && \mathbf{e}^T \mathbf{B}^T \mathbf{w} \\ & \text{s. t.} && \mathbf{A}^T \mathbf{w} \leq \mathbf{c} \end{aligned}$$

$$\mathbf{e}^T = (1, 1, \dots, 1)$$

Observations

- (1) $\mathbf{x} = \left(\begin{smallmatrix} e \\ 0 \end{smallmatrix}\right)$ is a primal bfs of (P') .
- (2) (D) and (D') have the same feasible domain.
- (3) Apply the revised simplex method to (P') ,
either it stops at an optimal solution, or find
 (P') is unbounded.
- (4) If it stops at an optimal solution, then
 $\mathbf{w}^{*T} = \mathbf{c}_{B^*}^T (\mathbf{B}^*)^{-1}$ is feasible to (D') .
Hence \mathbf{w}^* is feasible to (D) .
- (5) If (P') is unbounded, then we find a feasible
direction \mathbf{d} , such that $\mathbf{A}\mathbf{d} = 0$, $\mathbf{d} \geq 0$ and
 $\mathbf{c}^T \mathbf{d} < 0$.
Hence (P) is also unbounded
and (D) must be infeasible!

Remarks

- (1) Solving a standard form LP by the dual simplex method is **mathematically equivalent** to solving its dual LP by the revised (primal) simplex method.
- (2) The work of solving an LP by the dual simplex method is **about the same** as of by the revised (primal) simplex method.
- (3) The dual simplex method is useful for the **sensitivity analysis**.

Complexity of the simplex method

- Total # of elementary operations
= (# of elementary operations at each iteration) \times (# of iterations).
- # of elementary operations at each iteration of the revised simplex method $O(mn)$.
- From practical experience, the simplex method takes about (αm) iterations where $e^\alpha < \log_2(2 + n/m)$. Hence it is of $O(m^2n)$.
- From the worst-case analysis, Klee and Minty [1972] showed a class of examples (in the d -dimensional space) which $2^d - 1$ iterations for the simplex method.

Worst case performance of the simplex method

Klee-Minty Example:

- Victor Klee, George J. Minty, “How good is the simplex algorithm?” in (O. Shisha edited) Inequalities, Vol. III (1972), pp. 159-175.

$$(2 \text{ dim}) \quad \min -x_2$$

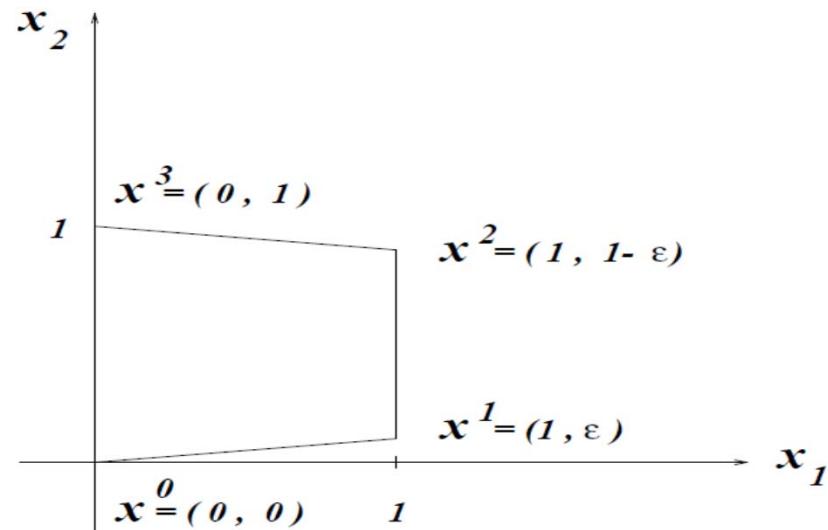
$$\text{s. t. } x_1 \geq 0$$

$$x_1 \leq 1$$

$$x_2 \geq \epsilon x_1 \quad \left(0 < \epsilon < \frac{1}{2}\right)$$

$$x_2 \leq 1 - \epsilon x_1$$

$$x_1, x_2 \geq 0$$



$\mathbf{x}^0 \rightarrow \mathbf{x}^1 \rightarrow \mathbf{x}^2 \rightarrow \mathbf{x}^3$ (optimal)

$2^2 - 1 = 3$ iterations

Klee-Minty Example

$$(3 \text{ dim}) \quad \min -x_3$$

$$\text{s. t. } x_1 \geq 0$$

$$x_1 \leq 1$$

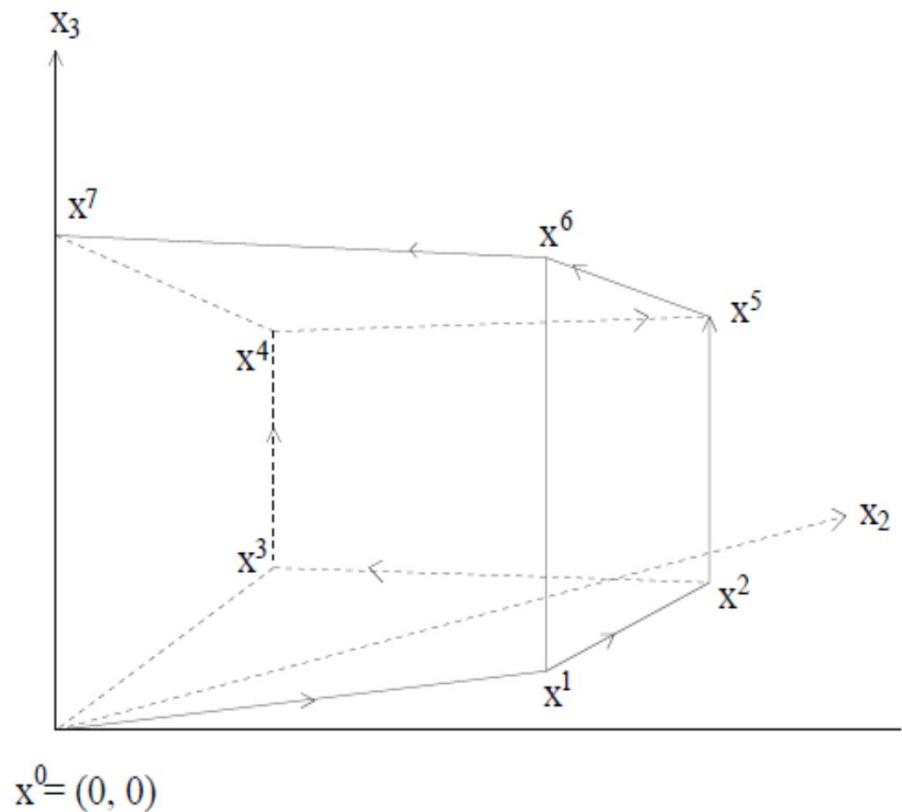
$$x_2 \geq \epsilon x_1$$

$$x_2 \leq 1 - \epsilon x_1$$

$$x_3 \geq \epsilon x_2$$

$$x_3 \leq 1 - \epsilon x_2$$

$$x_1, x_2, x_3 \geq 0$$



$$2^3 - 1 = 7 \text{ iterations}$$

Klee-Minty Example

$$(d \text{ dim}) \quad \min -x_d$$

$$\text{s. t. } x_1 \geq 0$$

$$x_1 \leq 1$$

$$x_2 \geq \epsilon x_1$$

$$x_2 \leq 1 - \epsilon x_1$$

:

$$x_d \geq \epsilon x_{d-1}$$

$$x_d \leq 1 - \epsilon x_{d-1}$$

$$x_i \geq 0$$

$2^d - 1$ iterations

Hence, in theory, the simplex method is not a polynomial-time algorithm. It is an *exponential time* algorithm!

