Solvability of Matrix Riccati inequalities

An analysis on the sign indefinite case

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Overview

We will discuss the subject in the following manner:

- The Riccati inequality. Conditions of solvability
- Areas of application
- Necessary conditions for solvability based on Hamiltonian matrices
- Monotonic transformation of Hamiltonian matrices
- Special transformation of Hamiltonian matrices
- General Riccati inequalities case
  - `Jordan blocks of Hamiltonian Matrices with pure imaginary eigenvalues`
  - `Main result`

Problem under consideration

The matrix Riccati inequality arises in the theory of absolute stability, H_∞ control problem, linear-quadratic (LQ) control problem, and optimal estimation problem.

It has the form:

$$HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$$

- Where A, B, G, Γ are given matrices of dimensions $n \times n, n \times m, n \times n$, and $m \times m$ respectively
- G, Γ are Hermitian matrices and $\det \Gamma \neq 0$
- H is Herimitian matrix, solution to the inequality

Of Interest

We are looking for necessary and sufficient conditions for existence of stabilizing and anti-stabilizing solutions of the inequality.

That is:

- Matrices H_- (Stabilizing solution) and H_+ (Anti-stabilizing solution) both satisfies the inequality.
- Matrices $A - B\Gamma^{-1}B^*H_-$ and $-(A - B\Gamma^{-1}B^*H_+)^*$ both being Hurwitz.

[0] Hurwitz matrix is a stability matrix with $Re[\lambda_i] < 0$

Sign of the Matrix Gamma (Γ)

- In the case where Γ is sign definite, the answer to this problem is given in the famous Kalman-Yakubovich lemma.
- For $x^*\Gamma x > 0$ with $\forall x \in C^n \setminus \{0\}$ then solvability of the inequality may be reduced to solvability of an ARE.

A form of Algebraic Riccati equation (ARE):

$$HA + A^*H + G - HB\Gamma^{-1}B^*H = 0$$

  - Which is closely related to the existence and properties of maximal J-orthogonal invariant subspaces of Hamiltonian matrices

[0] Inequality: $HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$

Kalman-Yakubovich Lemma

Assume the pair (A, B) is controllable and matrix A has no pure imaginary eigenvalues.

If matrix Γ is negative definite, then the inequality may be represented as Linear Matrix Inequality (LMI):

$$\begin{pmatrix} HA + A^*H + G & HB \\ B^*H & \Gamma \end{pmatrix} < 0$$

- `This may be solved via the interior point method.`
- Solvability of this inequality is also a subject of the famous Kalman-Yakubovich lemma.

[0] Controllable: Full rank, i.e. L.I. row and column vectors

Kalman-Yakubovich Lemma (Stated)

According to this lemma, the inequality has a solution, if and only if, the following frequency domain inequality holds:

$$\pi(i\omega) < 0,$$

for all $\omega \in [-\infty, \infty]$, where

$$\pi(\lambda) = \Gamma + B^*(\lambda I + A^*)^{-1} G (A - \lambda I)^{-1} B$$

- If matrix Γ is not sign definite, then $\pi(i\omega) < 0$ is obviously no longer necessary for solvability of the inequality.

[0] Inequality: $HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$

Motivation

To be presented is:

Necessary and sufficient conditions of solvability, a generalization of Kalman Yakubovich lemma but for the case of sign-indefinite quadratic forms.

Hamiltonian Matrix R

Consider:

$$R = \begin{pmatrix} A & -B\Gamma^{-1}B^* \\ -G & -A^* \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

The set of eigenvalues of Hamiltonian matrix R is symmetric with respect to the imaginary axis.

- Indeed, Jordan blocks corresponding to λ and $-\bar{\lambda}$ of R coincide.

[0] JR is clearly Hermitian and therefore matrix R is (J-) Hamiltonian

Hamiltonian Matrix (Split Eigenvalues)

Assume matrix R has no eigenvalues on the imaginary axis.

$\exists n \times n$ matrices Λ, X_1, Ψ_1, X_2, $\Psi_2 s.t. \lambda_i$ of Λ satisfy $Re[\lambda_i] < 0$ and

$$R \begin{pmatrix} X_1 & X_2 \\ \Psi_1 & \Psi_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ \Psi_1 & \Psi_2 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda^* \end{pmatrix}.$$

Then $A - B\Gamma^{-1}B^*H$ and $-(A - B\Gamma^{-1}B^*H)$ are Hurwitz, and H is both a stabilizing solution and anti-stabilizing solution of the Riccati equation.

[0] Riccati equation: $HA + A^*H + G - HB\Gamma^{-1}B^*H = 0$

Hamiltonian Matrix (Continued)

If H is a solution of the Riccati inequality, then $\exists \Delta G > 0$ s.t.

$$HA + A^*H + G + \Delta G - HB\Gamma^{-1}B^*H = 0$$

Assume H is a stabilizing solution of equation. Then:

$$R_{new} = \begin{pmatrix} A & -B\Gamma^{-1}B^* \\ -G - \Delta G & -A^* \end{pmatrix}$$

and we have $A - B\Gamma^{-1}B^*H = X_1 \Lambda X_1^{-1}$, therefore Λ is Hurwitz.

This means that R_{new} has no pure imaginary eigenvalues.

[0] Inequality: $HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$

Special Transformation

Assume V_1, V_2 are $n \times m$ matrices, and $V = col(V_1, V_2)$.

$$R(V) = R + V(JV)^* = \begin{pmatrix} A - V_1V_2^* & -B\Gamma^{-1}B^* + V_1V_1^* \\ -G - V_2V_2^* & -A^* + V_2V_1^* \end{pmatrix}$$

For a solution H of it's corresponding equation, we have:

$$HA + A^*H + G - HB\Gamma^{-1}B^*H = -(V_2 - HV_1)(V_2 - HV_1)^* \leq 0$$

- If right hand side is strictly negative, then H solves inequality.

[0] Inequality: $HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$

Special Case

Assume R has no pure imaginary eigenvalues, then for sufficiently small positive ϵ, $R(\epsilon I)$ also has no pure imaginary eigenvalues.

$\forall \delta > 0$, $\exists \tilde{R}$ such that:

$$\tilde{R} = \begin{pmatrix} \tilde{A} & -\tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^* + \epsilon I \\ -\tilde{G} - \epsilon I & -\tilde{A}^* \end{pmatrix}$$

for $\|R(\epsilon I) - \tilde{R}\| < \delta$, where matrices X_1, X_2 are nonsingular

If H is a stabilizing solution, then

$$\begin{aligned} HA + A^*H + G - HB\Gamma^{-1}B^*H = \\ = H(A - \tilde{A}) + (A - \tilde{A})^*H + (G - \tilde{G} + \epsilon I) - \\ -(HB\Gamma^{-1}B^*H - H\tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^*H) + H\epsilon H. \end{aligned}$$

- For sufficiently small δ, right hand side of the equality is negative. Thus, H is a solution to the inequality.

Special Case (Continued)

Since matrix $\tilde{A} - \tilde{B}\tilde{\Gamma}^{-1}\tilde{B}^* H$ is Hurwitz, for sufficiently small number δ the matrix $A - B\Gamma^{-1}B^* H$ is Hurwitz, and H is a stabilizing solution of the inequality.

It turns out that the same conclusion is true for the anti-stabilizing solution of the inequality.

- Thus, if matrix R has no pure imaginary eigenvalues, then the Riccati inequality has both stabilizing and anti-stabilizing solutions.

[0] Inequality: $HA + A^*H + G - HB\Gamma^{-1}B^*H < 0$

General Case

Definition

Let P be a subspace of $\mathbf{C}^{2n}$, and T a matrix with set of columns forming a basis of P. Denote $n_+(P)$: numbers of positive eigenvalues, $n_0(P)$: numbers of zero eigenvalues, and $n_-(P)$: number of negative eigenvalues of matrix T^*iJT.

Denote by $J_1, \ldots, J_m$ all Jordan blocks of matrix R with pure imaginary eigenvalues $i\omega_1, \ldots, i\omega_m$ respectively:

$$J_j = \begin{pmatrix} i\omega_j & 1 & 0 & \ldots & 0 \\ 0 & i\omega_j & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & i\omega_j \end{pmatrix}$$

General Case (Definition 2)

Definition

We say that the block J_j contains $n_+(J_j)$ eigenvalues $i\omega_j$ of the first type, and $n_-(J_j)$ eigenvalues $i\omega_j$ of the second type.

For every subspace P_j there exist a number $\beta_j \in \{-1, 1\}$, and a matrix S_j such that the columns of S_j span P_j, $RS_j = S_j J_j$,

$$S_j^* J S_j = \epsilon_j \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & (-1)^2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{m_j} & 0 & \dots & 0 & 0 \end{pmatrix}$$

General Case (Continued)

We know that:

1. if the size n_j of J_j is even, then $n_+(J_j) - n_-(J_j) = 0$ and $\epsilon_j = (-1)^{n_j/2}\beta_j$;
2. if the size n_j of J_j is odd, then $n_+(J_j) - n_-(J_j) = \beta_j$ and $\epsilon_j = (-1)^{(n_j-1)/2} i\beta_j$.

Moreover, matrices S_j for distinct j are J-orthogonal: $S_{j_1}^* J S_{j_2} = 0$ if $j_1 \neq j_2$.

General Case (Continued)

Denote by S_+ (S_-) a matrix whose columns span the R-invariant subspace associated to eigenvalues with positive (respectively, negative) real parts.

S_+ and S_- are J-orthogonal to S_j for every $j = 1, \ldots, m$ and all columns of matrices S_-, S_+, $S_1, \ldots, S_m$ form a basis of $\mathbf{C}^{2n}$.

Fix $j \in \{1, \ldots, m\}$ and consider V such that JV is orthogonal to S_-, S_+, and all matrices S_k with $k \neq j$. Then

$$\begin{aligned}\det(R + V(JV)^* - \lambda I) &= \det(R - \lambda I)(1 + (JV)^*(R - \lambda I)^{-1}V) \\ &= \det(R - \lambda I)(1 + (JV)^* S_j (J_j - \lambda I)^{-1}(S_j^* J S_j)^{-1} S_j^* JV)\end{aligned}$$

General Case (Continued)

Denote $P_j = S_j^* J S_j / \epsilon_j$, then we have $P_j^{-1} = (-1)^{m_j - 1} P_j$, and

$$(J_j - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{i\omega_j - \lambda} & \frac{-1}{(i\omega_j - \lambda)^2} & \cdots & \frac{(-1)^{m_j - 1}}{(i\omega_j - \lambda)^{m_j}} \\ 0 & \frac{1}{i\omega_j - \lambda} & \cdots & \frac{(-1)^{m_j - 2}}{(i\omega_j - \lambda)^{m_j - 1}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{i\omega_j - \lambda} \end{pmatrix}$$

General Case (Continued)

For even $m_j = 2r_j$ we have $\epsilon_j = (-1)^{r_j}\beta_j$, and for $\lambda = i\omega$
We have $(J_j - i\omega I)^{-1}(\epsilon_j P_j)^{-1} =$

$$= \beta \begin{pmatrix} \frac{1}{(\omega_j-\omega)^{2r_j}} & \frac{i}{(\omega_j-\omega)^{2r_j-1}} & \dots & \frac{i(-1)^{r_j+1}}{\omega_j-\omega} \\ \frac{-i}{(\omega_j-\omega)^{2r_j-1}} & \frac{1}{(\omega_j-\omega)^{2r_j-2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{i(-1)^{r_j}}{\omega_j-\omega} & 0 & \dots & 0 \end{pmatrix}$$

If we choose vector V such that $S_k^*(JV) = 0$ for all $k \neq j$, and $S_j^*(JV) = (\delta, 0, \dots, 0)^*$, then

$$1 + (JV)^* S_j (J_j - \lambda I)^{-1} (S_j^* J S_j)^{-1} S_j^* JV = 1 + \delta^2 \beta_j \frac{1}{(\omega_j - \omega)^{2r_j}}$$

General Case (Continued)

Now let $m_j = 2r_j + 1$, then $\epsilon_j = (-1)^{r_j} i\beta$, and for $\lambda = i\omega$, we get:

$$(J_j - i\omega I)^{-1}(\epsilon_j P_j)^{-1} = \beta \begin{pmatrix} \frac{1}{(\omega_j-\omega)^{2r_j+1}} & \frac{i}{(\omega_j-\omega)^{2r_j}} & \cdots & \frac{(-1)^{r_j}}{\omega_j-\omega} \\ \frac{-i}{(\omega_j-\omega)^{2r_j}} & \frac{1}{(\omega_j-\omega)^{2r_j-1}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{(-1)^{r_j}}{\omega_j-\omega} & 0 & \cdots & 0 \end{pmatrix}.$$

Again, if V is such that $S_k^*(JV) = 0$ for all $k \neq j$, and $S_j^*(JV) = (\delta, 0, \ldots, 0)^*$, then

$$1 + (JV)^* S_j (J_j - \lambda I)^{-1} (S_j^* J S_j)^{-1} S_j^* JV = 1 + \delta^2 \beta_j \frac{1}{(\omega_j - \omega)^{2r_j+1}}$$

General Case (Theorem 1)

Theorem

There exists a nonnegative matrix M such that the eigenvalues of matrix $R - tMJ$ with number t increasing from zero have the following behaviour.

1. For each Jordan block J_j of matrix R of odd dimension $2r_j + 1$ with index $\beta = 1$ exactly $2r_j$ eigenvalues leave imaginary axis with increasing t from zero, and the rest eigenvalue goes up imaginary axis, and has the first type.
2. For each Jordan block J_j of matrix R of odd dimension $2r_j + 1$ with index $\beta = -1$ exactly $2r_j$ eigenvalues leave imaginary axis with increasing t from zero, and the rest eigenvalue goes down imaginary axis, and has the second type.

General Case (Theorem 1 continued)

Theorem

3. For each Jordan block J_j of matrix R of even dimension $2r_j$ with index $\beta = 1$ all eigenvalues leave imaginary axis with increasing t from zero.
4. For each Jordan block J_j of matrix R of even dimension $2r_j$ with index $\beta = -1$ exactly $2r_j - 2$ eigenvalues leave imaginary axis with increasing t from zero. One of the rest eigenvalues goes up imaginary axis and has the first type, and the other eigenvalue goes down imaginary axis and has the second type.

General Case (Theorem 2)

Denote

$$s(\omega) = \sum (m_+(\omega) - m_-(\omega)) - m_0(\omega)$$

- where $m_+(\omega)$ is the number of odd dimensional Jordan blocks matrix R with eigenvalues $i\omega_j$ such that $\omega_j < \omega$ and $\beta = 1$
- $m_-(\omega)$ is the number of odd dimensional Jordan blocks of matrix R with eigenvalues $i\omega_j$ such that $\omega_j \leq \omega$, $\beta = -1$ and
- $m_0(\omega)$ is the number of even dimensional Jordan blocks of matrix R with eigenvalues $i\omega$ such that $\beta = -1$.

General Case (Theorem 3)

Theorem

$\exists M$ with $M > 0$ s.t. $R - MJ$ has no pure imaginary eigenvalues of odd multiplicity $\iff$ for every pure imaginary eigenvalue $i\omega_j$ of R we have

$$s(\omega) \geq 0$$

Taking into account our results earlier, we get the following main result:

Theorem (Main Result)

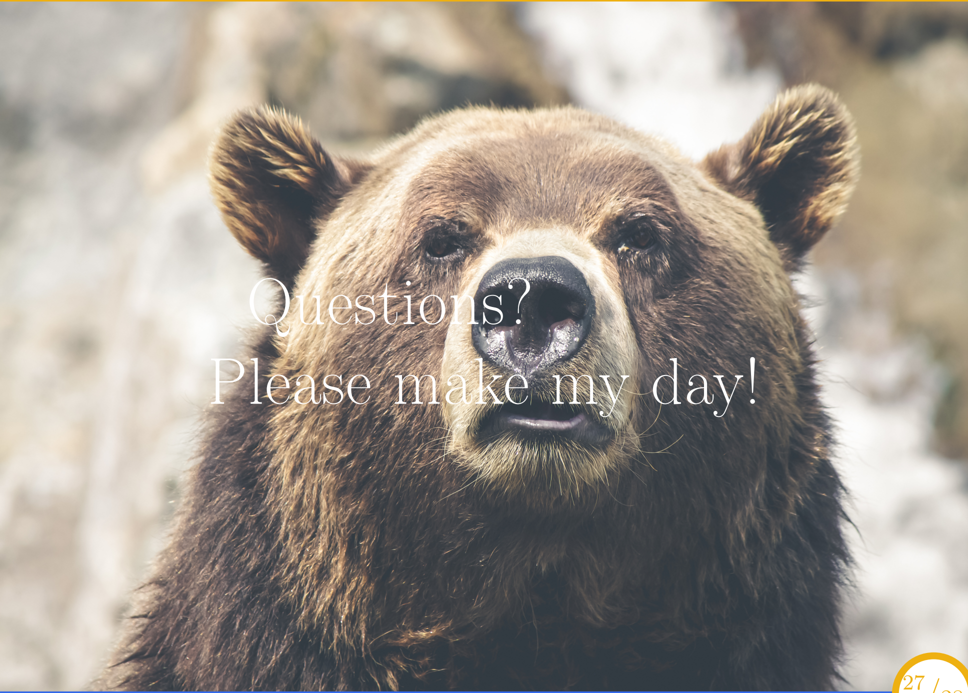
The Riccati inequality has a solution $\iff$ inequality $s(\omega) \geq 0$ holds $\forall \omega \in R$.

[0] $s(\omega) = \sum(m_+(\omega) - m_-(\omega)) - m_0(\omega)$

Conclusion

We considered conditions of solvability of the Riccati inequality

- First presented necessary conditions for sovability based on Hamiltonian matrices
- Then considered a monotonic transformation of Hamiltonian matrices
- Used a special transformation of Hamiltonian matrices
- Considered a general case
  - `Presented the main result, a new theorem`



Questions?

Please make my day!

Thank You!