<u>Modeling the Motion of a Spring-System in a Seismograph Using Euler's Method</u>

Description of the real-world problem

The world has been experiencing a large number of earthquakes lately, with one study estimating 20,000 earthquakes every year. The need to predict and study earthquakes have become more important over years. This simple numerical method model presented herein can help us to generate these seismographs to help us understand the strength, velocity, and amplitude of the earthquakes and prepare for future events [1].

The mathematical formulation of the problem

Most physics students are familiar with the fact that the force in a spring is the product of a spring constant and the displacement of that spring from its natural position (where no force is applied). However, this model is extremely simplified and there are other factors that affect the movement of a spring. Similar to the fact that friction on a surface can always be reduced but never completely eliminated, springs always have a factor that opposes spring movement--whether the spring is contracting or expanding. This factor is called damping [2]. Let's assume that the mass stays at rest when the ground moves. Then, the true displacement of the ground - z(t) - relative to the true inertial frame can be used to obtain the displacement of the seismometer frame - x(t) - relative to the assumed fixed mass. Hence, the following ODE can be obtained to model the system (Equation 1):

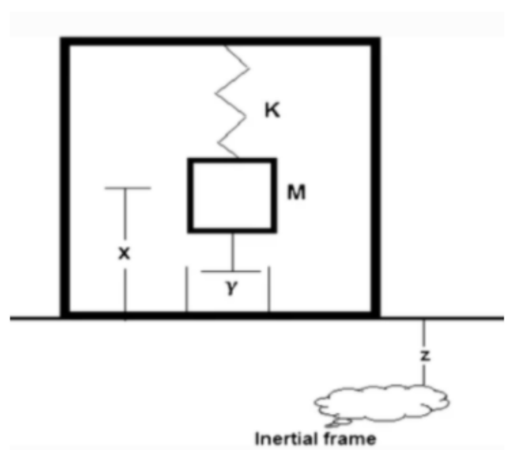

Figure 1: Set-up used to obtain the ODE to model the system

$$\frac{dx^2}{d^2t}(M) + \frac{dx}{dt}(\gamma) + x(k) = \frac{dz^2}{d^2t}(-M)$$

Equation 1

where m [kg] is the mass , γ $[N/\frac{m}{s}]$ is the velocity damping constant, k [N/m] is the spring constant and $\frac{dz^2}{d^2t}$ is the acceleration of the ground relatively to the inertial frame. Figure on the right (Figure 1) further explains the set up [4].

Description of the problem inputs and outputs

Using the simplified version of the ODE to model the system (Equation 1), the inputs of the problem include the mass of the load attached to the spring (M), the spring constant (k), the velocity damping constant (γ) and a function z(t) that would model the displacement of the ground relative to the inertial frame. This last variable will be used to simulate an earthquake - and we are going to assume it can be described as a sine wave with random noise (distortion). It is going to be differentiated twice to obtain the acceleration of the ground - that will be used to model the behaviour of the system. The objective of the project is to obtain a seismogram - a plot of the position of the seismometer frame displacement (x) with respect to time relative to the fixed position of the load attached to the spring and the plot of input signal simulating the earthquake that is going to be used for comparison and further analysis.

Description of the numerical techniques that will be applied to solve the problem

For this problem, the numerical technique to be used will be the Forward Euler's Method and it will be used to analyze the differential equation that will be derived from the specific problem inputs and design. Since Euler's method is a numerical method designed for obtaining the approximate solution of first order differential equations, we firstly need to reduce our second order ODE (Equation 2) to a system of two first order differential equations [3]. To achieve that let's $u = x' = \frac{dx}{dt}$ and hence $u' = x" = \frac{dx^2}{d^2t}$. Therefore the second order ODE (Equation 1) can be rewritten as:

$$u'(M) + u(��) + x(k) = \frac{dz^2}{d^2t}(-M)$$

Equation 2

For simplicity we will now assume that the ground moves at a constant velocity relative to the inertial frame during an earthquake and as such the acceleration of the ground $\frac{dz^2}{d^2t} = 0$. Hence the following system of two first order equations can be obtain:

1) $u = x'$

Equation 3

2) $u'(M) + u(��) + x(k) = 0$

Equation 4

Now we need to rewrite the second equation in the form of u’=... Hence:

$$u'(M) = -u(��) - x(k)$$
$$u' = -(1/M)((u(��) + x(k))$$

Equation 5

The next step is to obtain the matrix A:

$$\begin{pmatrix} x \\ u \end{pmatrix}' = \begin{pmatrix} x' \\ -\mathrm{u}(\gamma) - \mathrm{x}(\mathrm{k}) \end{pmatrix} = \begin{pmatrix} u \\ -\mathrm{u}(\gamma) - \mathrm{x}(\mathrm{k}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/M & -\gamma/\mathrm{M} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = A * \begin{pmatrix} x \\ u \end{pmatrix}$$

Equation 6

Hence the matrix A is equal to:

$$A = \begin{pmatrix} 0 & 1 \\ -k/M & -\gamma/\mathrm{M} \end{pmatrix}$$

Another assumption that needs to be added are the initial conditions that will be used as our x_0.

$$x_0 = \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We also need to define the step size ‘h’. Since with the Forward Euler’s Method the local error is proportional to the square of the step size, and the global error is proportional to the step size, we will use relatively small h=0.1 to improve the accuracy of our results [5].

Now we can apply the Forward Euler’s Method and the values of x can be approximated using the following iteration process:

$$x_{i+1} = x_i + h * A * x_i$$

Equation 7

However, to make our solution more realistic, now we can modify the iteration equation to obtain the values of x by changing our initial assumption about the velocity of the ground relatively to the inertial frame being constant. If we instead assume that the ground is accelerating at some non-constant, unknown rate $\frac{dz^2}{d^2t}$, with the initial condition of z”(0)=0, then the following iteration process can be used to estimate the values of x and different times:

$$x_{i+1} = x_i + h * A * x_i + h * [0; \ -(1/M) * z(i)]$$

Equation 8

Hence, putting substituting the matrix A into the equation:

$$x_{i+1} = x_i + h * \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{\gamma}{M} \end{pmatrix} * \begin{pmatrix} x(i) \\ x'(i) \end{pmatrix} + h * \begin{pmatrix} 0 \\ -\frac{1}{M} * z(i) \end{pmatrix}$$

Equation 9

The iteration process shown above (Equation 9) was coded in MATLAB with the starting point of $x_0 = x(0) = 0$ and $(x_0)' = x'(0) = 1$. The obtained results were used for further analysis.

Description of the numerical experiments that will be carried out

The intended numerical experiment involves introducing variations in the input parameters - mainly the mechanical properties of the mass-spring system (such as mass, the spring constant and the damping constant) - in order to test what combination of parameters will result in our set-up being the most sensitive to a simulated earthquake and could be used to accurately measure the magnitude of a real earthquake. We will also vary the step size 'h' to observe how it affects the obtained output.

Numerical results

The function z(t) that was used to model the displacement of the ground relative to the inertial frame for each of the experiments:

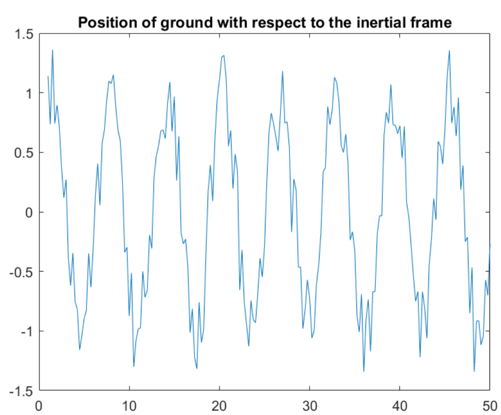

Figure 2: Displacement of the ground relative to the inertial frame - function z(t)

1) **Experiment 1 -** varying the value of the mass with $k = 4.3N/m$, $\gamma = 0.4\ N/\frac{m}{s}$, $h = 0.1$

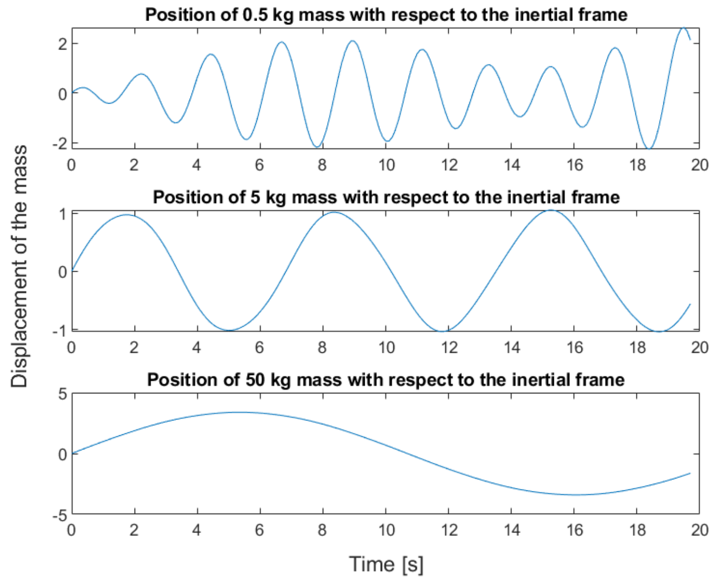

Figure 3: Displacement of different masses relative to the inertial frame

2) **Experiment 2 -** varying the value of the spring constant with $m = 0.5kg$, $\gamma = 0.4\ N/\frac{m}{s}$, $h = 0.1$

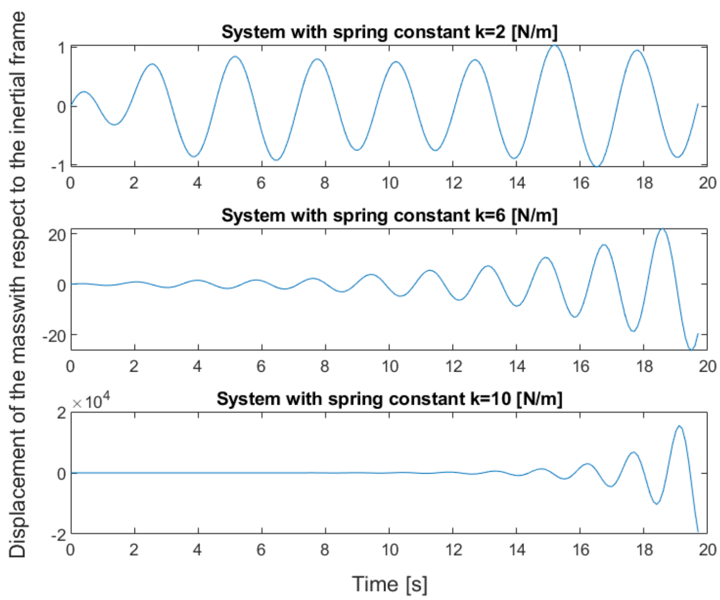

Figure 4: Effect of changing the spring constant (k)

3) **Experiment 3 -** varying the value of the damping constant with *m = 0.5kg, k = 2N/m, h = 0.1*

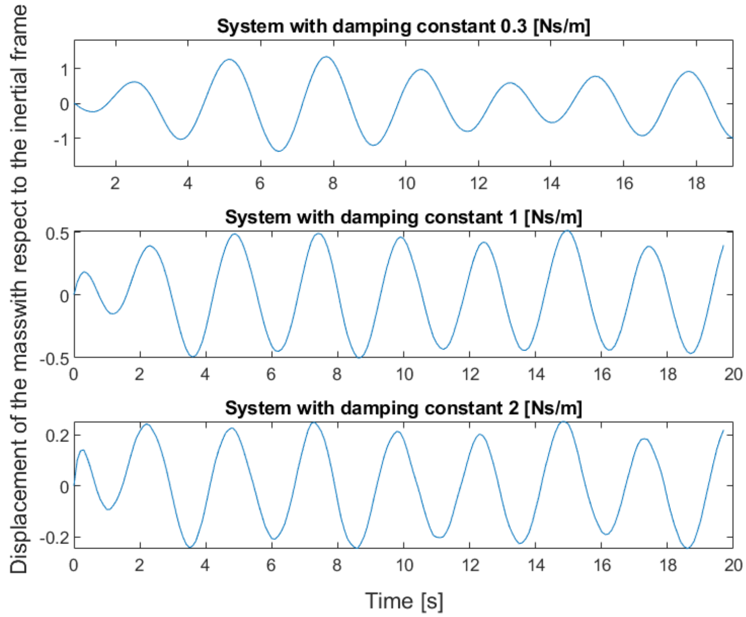

Figure 5: Effect of changing the damping constant (γ)

4) **Experiment 4 -** varying the value of the step size with *m = 0.5kg, k = 4.3N/m,* $\gamma = 0.4\ N/\frac{m}{s}$

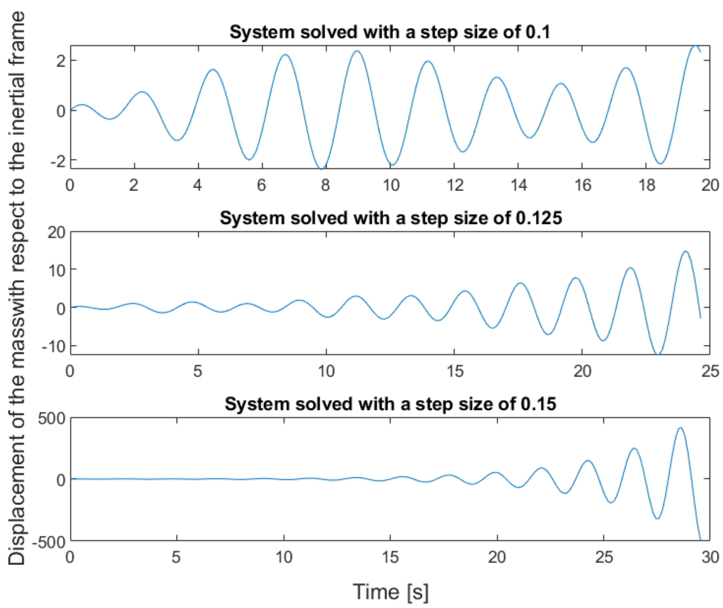

Figure 6: Effect of changing the step size (h)

Discussion of the results

From experiment 1, we can see in Figure 3 that as the mass of the system increases, the period in the displacement also increases. Furthermore, we can see from this direct relationship that when the mass is 0.5 and 50 kg, the amplitude is consistently greater than 1 m (for 50 kg, the amplitude is almost 5 m), but for the mass between these two, 5 kg, the amplitude only reaches 1 m. This may suggest that there is an optimum mass where the amplitude is minimized. Having a minimized amplitude may prove useful for seismographs that are designed to record the motion on a physically slender strip; using a mass far from this optimum mass may create amplitudes that go off the paper, and will prohibit the entire shape of motion to be modelled and recorded.

From experiment 2, it is observed in Figure 4 that as the spring constant increases, the amplitude grows overtime at a faster rate. In the first case where the spring constant is 2, the amplitude stays relatively constant at 1 m; but for the other two cases, there is a clear increase in amplitude as time increases, with the highest amplitude reached 20 m and 20,000 m when the spring constant was 6 N/m and 10 N/m, respectively. Selecting one of the latter two spring constants may not be the best choice for constructing a seismograph, as the amplitude will likely be too large and go off the data collection paper.

From experiment 3, when the damping constant varied, the results showed that the amplitudes in all cases remained relatively constant overtime, unlike the results in experiment 2. Having a constant range of amplitudes is ideal when constructing a seismograph, for the reasons explained above. When the damping constant was 0.3 Ns/m, the difference in amplitudes among each peak was the most defined, which allows the reader of the graph to see discrepancies among each peak. Having this is important so that the sensitivity of the movement can be fully recognized. For the other cases where the damping constant is 1 Ns/m and 2 Ns/m, the difference in amplitudes is not as easily legible, making it not an ideal selection for a seismograph.

From experiment 4, we can see that when the step size was increased uniformly from above a certain value, in this case 0.1, the amplitude of the wave increased almost 20 times the previous step size. We can also observe that the amplitude grew much slower as the step size increased. We can also notice that as the step size was increased, the results only became slightly more accurate. This method is not always the best to achieve more accuracy in computation as it requires more step size to achieve more accuracy.

Euler's method is first order convergent, i.e., the error of the computed solution is O(h), where h is the time step. The method is highly computation intensive and thus requires more steps to achieve accurate results. Seismographs are highly sensitive and important instruments which are required to be accurate and precise to understand and predict the possibility of earthquakes which can save human lives. Thus, we conclude that numerical methods such as the Forward Euler's methods might not be reliable nor accurate enough to be considered in seismological instruments.

References:

[1] *Why Are We Having so Many Earthquakes? Has Naturally Occurring Earthquake Activity Been Increasing? Does This Mean a Big One Is Going to Hit? OR We Haven't Had Any Earthquakes in a Long Time; Does This Mean That the Pressure Is Building up for a Big One?*, www.usgs.gov/faqs/why-are-we-having-so-many-earthquakes-has-naturally-occurring-earthquake-activity-been?qt-news_science_products=0.

[2] "Differential Equation - Modeling - Spring and Mass." *ShareTechnote*, www.sharetechnote.com/html/DE_Modeling_Example_SpringMass.html.

[3] *Numerical Methods--Euler's Method*, calculuslab.deltacollege.edu/ODE/7-C-1/7-C-1-h-a.html.

[4] Habbak, E., Nofal, H., Lotfy, E., El-Sabban, S. and Nossair, Z., 2016. *Design and simulation of a vertical broadband seismometer.* <https://link.springer.com/article/10.1007/s12517-016-2422-x>

[5] Brilliant.org. 2020. *Differential Equations - Euler's Method - Small Step Size | Brilliant Math*. <https://brilliant.org/wiki/differential-equations-eulers-method-small-step/#:~:text=One%20of%20the%20simplest%20and,proportional%20to%20the%20step%20size.>