Appendix

This appendix contains the proofs of the theorems presented in the paper entitled: *A new generic method to improve machine learning applications in official statistics.* Recall that we have assumed a population of size N in which a fraction $\alpha := N_{1+}/N$ belongs to the class of interest, referred to as the class labelled as 1. We assume that a binary classification algorithm has been trained that correctly classifies a data point that belongs to class $i \in \{0, 1\}$ with probability $p_{ii} > 0.5$, independently across all data points. In addition, we assume that a test set of size $n \ll N$ is available and that it can be considered a simple random sample from the population. The classification probabilities p_{00} and p_{11} are estimated on that test set as described in Section **??**. Finally, we assume that the classify-and-count estimator $\hat{\alpha}^*$ is distributed independently of $\hat{p}_{00}$ and $\hat{p}_{11}$, which is reasonable (at least as an approximation) when $n \ll N$.

It may be noted that the estimated probabilities $\hat{p}_{11}$ and $\hat{p}_{00}$ cannot be computed if $n_{1+} = 0$ or $n_{0+} = 0$. Similarly, the calibration probabilities c_{11} and c_{00} cannot be estimated if $n_{+1} = 0$ or $n_{+0} = 0$. We assume here that these events occur with negligible probability. This will be true when n is sufficiently large so that $n\alpha \gg 1$ and $n(1 - \alpha) \gg 1$.

Preliminaries

Many of the proofs presented in this appendix rely on the following two mathematical results. First, we will use univariate and bivariate Taylor series to approximate the expectation of non-linear functions of random variables. That is, to estimate $E[f(X)]$ and $E[g(X, Y)]$ for sufficiently differentiable functions f and g, we will insert the Taylor series for f and g at $x_0 = E[X]$ and $y_0 = E[Y]$ up to terms of order 2 and utilize the linearity of the expectation. Second, we will use the following conditional variance decomposition for the variance of a random variable X:

$$V(X) = E[V(X \mid Y)] + V(E[X \mid Y]). \tag{4}$$

The conditional variance decomposition follows from the tower property of conditional expectations Knottnerus (2003). Before we prove the theorems presented in the paper, we begin by proving the following lemma.

Lemma 1. *The variance of the estimator $\hat{p}_{11}$ for p_{11} estimated on the test set is given by*

$$V(\hat{p}_{11}) = \frac{p_{11}(1 - p_{11})}{n\alpha} \left[1 + \frac{1 - \alpha}{n\alpha} \right] + O\left(\frac{1}{n^3}\right). \tag{5}$$

Similarly, the variance of $\hat{p}_{00}$ is given by

$$V(\hat{p}_{00}) = \frac{p_{00}(1 - p_{00})}{n(1 - \alpha)} \left[1 + \frac{\alpha}{n(1 - \alpha)} \right] + O\left(\frac{1}{n^3}\right). \tag{6}$$

Moreover, $\hat{p}_{11}$ and $\hat{p}_{00}$ are uncorrelated: $C(\hat{p}_{11}, \hat{p}_{00}) = 0$.

of Lemma 1. We approximate the variance of $\hat{p}_{00}$ using the conditional variance decomposition and a second-order Taylor series, as follows:

$$
\begin{aligned}
V(\hat{p}_{00}) &= V\left(\frac{n_{00}}{n_{0+}}\right) \\
&= E_{n_{0+}}\left[V\left(\frac{n_{00}}{n_{0+}} \mid n_{0+}\right)\right] + V_{n_{0+}}\left[E\left(\frac{n_{00}}{n_{0+}} \mid n_{0+}\right)\right] \\
&= E_{n_{0+}}\left[\frac{1}{n_{0+}^2}V(n_{00} \mid n_{0+})\right] + V_{n_{0+}}\left[\frac{1}{n_{0+}}E(n_{00} \mid n_{0+})\right] \\
&= E_{n_{0+}}\left[\frac{n_{0+}p_{00}(1-p_{00})}{n_{0+}^2}\right] + V_{n_{0+}}\left[\frac{n_{0+}p_{00}}{n_{0+}}\right] \\
&= E_{n_{0+}}\left[\frac{1}{n_{0+}}\right]p_{00}(1-p_{00}) \\
&= \left[\frac{1}{E[n_{0+}]} + \frac{1}{2}\frac{2}{E[n_{0+}]^3}\times V[n_{0+}]\right]p_{00}(1-p_{00}) + O\left(\frac{1}{n^3}\right) \\
&= \frac{p_{00}(1-p_{00})}{E[n_{0+}]}\left[1 + \frac{V[n_{0+}]}{E[n_{0+}]^2}\right] + O\left(\frac{1}{n^3}\right) \\
&= \frac{p_{00}(1-p_{00})}{n(1-\alpha)}\left[1 + \frac{\alpha}{n(1-\alpha)}\right] + O\left(\frac{1}{n^3}\right).
\end{aligned}
$$

The variance of $\hat{p}_{11}$ is approximated in the exact same way.

Finally, to evaluate $C(\hat{p}_{11}, \hat{p}_{00})$ we use the analogue of (4) for covariances:

$$
\begin{aligned}
C(\hat{p}_{11}, \hat{p}_{00}) &= C\left(\frac{n_{11}}{n_{1+}}, \frac{n_{00}}{n_{0+}}\right) \\
&= E_{n_{1+},n_{0+}}\left[C\left(\frac{n_{11}}{n_{1+}}, \frac{n_{00}}{n_{0+}} \mid n_{1+}, n_{0+}\right)\right] \\
&\quad + C_{n_{1+},n_{0+}}\left[E\left(\frac{n_{11}}{n_{1+}} \mid n_{1+}, n_{0+}\right), E\left(\frac{n_{00}}{n_{0+}} \mid n_{1+}, n_{0+}\right)\right] \\
&= E_{n_{1+},n_{0+}}\left[\frac{1}{n_{1+}n_{0+}}C(n_{11}, n_{00} \mid n_{1+}, n_{0+})\right] \\
&\quad + C_{n_{1+},n_{0+}}\left[\frac{1}{n_{1+}}E(n_{11} \mid n_{1+}), \frac{1}{n_{0+}}E(n_{00} \mid n_{0+})\right].
\end{aligned}
$$

The second term is zero as before. The first term also vanishes because, conditional on the row totals n_{1+} and n_{0+}, the counts n_{11} and n_{00} follow independent binomial distributions, so $C(n_{11}, n_{00} \mid n_{1+}, n_{0+}) = 0$. $\qquad\square$

Note: in the remainder of this appendix, we will not add explicit subscripts to expectations and variances when their meaning is unambiguous.

Mixed estimator

In this section, we will prove the bias and the variance of the mixed estimator under concept drift. The mixed estimator is dependent on the calibration estimator at time 0, the misclassification estimator on time 0 and the misclassification estimator on time t.

Proof of Theorem 1. First, we will make a proof for the bias of the Mixed Estimator. The expression for the Mixed Estimator is:

$$
\begin{aligned}
\hat{\alpha}'_m &= \hat{\alpha}_c + (\hat{\alpha}'_p - \hat{\alpha}_p) \\
&= \hat{\alpha}_c + [(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}.
\end{aligned}
\tag{7}
$$

The bias is defined as the difference between the expected value of the estimator minus the true value of the target variable:

$$
B[\hat{\alpha}'_m] = E[\hat{\alpha}'_m] - \alpha'
\tag{8}
$$

Using Equation 7, we can write out the expected value of the Mixed estimator.

$$
\begin{aligned}
E[\hat{\alpha}'_m] &= E\left[\hat{\alpha}_c + [(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] \\
&= E\left[\hat{\alpha}_c\right] + E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]
\end{aligned}
\tag{9}
$$

From Kloos et al. (2021), we already know that:

$$
E\left[\hat{\alpha}_c\right] = E\left[E\left[\hat{\alpha}_c \mid \boldsymbol{N}\right]\right] = \alpha
\tag{10}
$$

$E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]$ can be computed by conditioning on the Classify-and-count estimators $(\hat{\alpha}')^*$ and $\hat{\alpha}^*$.

$$
\begin{aligned}
E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] &= E\left[E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]\right] \\
&= E\left[((\hat{\alpha}')^* - \hat{\alpha}^*) \times E\left[\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]\right] \\
&= E\left[((\hat{\alpha}')^* - \hat{\alpha}^*) \times E\left[\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]\right]
\end{aligned}
\tag{11}
$$

From Kloos et al. (2021), we used Taylor Series to approximate the expected value of $\frac{1}{\hat{p}_{00}+\hat{p}_{11}-1}$.

$$E\left[\frac{1}{\hat{p}_{00}+\hat{p}_{11}-1}\right] = \frac{1}{p_{00}+p_{11}-1} + \frac{V(\hat{p}_{00})+V\hat{p}_{11}}{(p_{00}+p_{11}-1)^3} + O(n^{-2}) \qquad (12)$$

Now it only remains to calculate the expected values of the Classify-and-count estimators.

$$E\left[(\hat{\alpha}')^* - \hat{\alpha}^*\right] = E\left[(\hat{\alpha}')^*\right] - E\left[\hat{\alpha}^*\right] \qquad (13)$$
$$E\left[(\hat{\alpha}')^*\right] = \alpha' p_{11} + (1-\alpha')(1-p_{00}) = \alpha'(p_{00}+p_{11}-1)+(1-p_{00}) \qquad (14)$$
$$E\left[\hat{\alpha}^*\right] = \alpha p_{11} + (1-\alpha)(1-p_{00}) = \alpha(p_{00}+p_{11}-1)+(1-p_{00}) \qquad (15)$$

Combining these expressions, $E\left[(\hat{\alpha}')^* - \hat{\alpha}^*\right]$ can be simplified towards the following expression.

$$E\left[(\hat{\alpha}')^* - \hat{\alpha}^*\right] = (\alpha'-\alpha)(p_{00}+p_{11}-1) \qquad (16)$$

Combining (12) and (16) gives the expression that should be in the big expectation of (11).

$$E\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00}+\hat{p}_{11}-1}\right] = E\left[((\hat{\alpha}')^* - \hat{\alpha}^*) \times E\left[\frac{1}{\hat{p}_{00}+\hat{p}_{11}-1}\right]\right]$$
$$= E\left[(\hat{\alpha}')^* - \hat{\alpha}^*\right] \times E\left[\frac{1}{\hat{p}_{00}+\hat{p}_{11}-1}\right]$$
$$= (\alpha'-\alpha)(p_{00}+p_{11}-1) \times \left[\frac{1}{p_{00}+p_{11}-1} + \frac{V(\hat{p}_{00})+V(\hat{p}_{11})}{(p_{00}+p_{11}-1)^3}\right] + O(n^{-2})$$
$$= \alpha'-\alpha + \frac{(\alpha'-\alpha)(V(\hat{p}_{00})+V(\hat{p}_{11}))}{(p_{00}+p_{11}-1)^2} + O(n^{-2}) \qquad (17)$$

Finalizing the proof given the equations (8), (10) and (17).

$$B[\hat{\alpha}'_m] = E[\hat{\alpha}'_m] - \alpha'$$
$$= \alpha + \alpha' - \alpha + \frac{(\alpha'-\alpha)(V(\hat{p}_{00})+V(\hat{p}_{11}))}{(p_{00}+p_{11}-1)^2} - \alpha' + O(n^{-2})$$
$$= \frac{(\alpha'-\alpha)(V(\hat{p}_{00})+V(\hat{p}_{11}))}{(p_{00}+p_{11}-1)^2} + O(n^{-2}) \qquad (18)$$

Now it only remains to proof the variance of the mixed estimator. Recall that the mixed estimator can be written as

$$\hat{\alpha}'_m = \hat{\alpha}_c + [(\hat{\alpha}')^* - \hat{\alpha}^*] \times \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}. \tag{19}$$

It clearly follows from (19) that the variance of this mixed estimator can be written as

$$V[\alpha'_m] = V[\hat{\alpha}_c] + V\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] + 2C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]. \tag{20}$$

From Kloos et al. (2021), we already know that the variance of the calibration estimator is equal to

$$\begin{aligned}
V(\hat{\alpha}_c) = {}& \left[\frac{(1-\alpha)(1-p_{00}) + \alpha p_{11}}{n} + \frac{(1-\alpha)p_{00} + \alpha(1-p_{11})}{n^2}\right] \\
& \times \left[\frac{\alpha p_{11}}{(1-\alpha)(1-p_{00}) + \alpha p_{11}}\left(1 - \frac{\alpha p_{11}}{(1-\alpha)(1-p_{00}) + \alpha p_{11}}\right)\right] \\
& + \left[\frac{(1-\alpha)p_{00} + \alpha(1-p_{11})}{n} + \frac{(1-\alpha)(1-p_{00}) + \alpha p_{11}}{n^2}\right] \\
& \times \left[\frac{(1-\alpha)p_{00}}{(1-\alpha)p_{00} + \alpha(1-p_{11})}\left(1 - \frac{(1-\alpha)p_{00}}{(1-\alpha)p_{00} + \alpha(1-p_{11})}\right)\right] \\
& + O\left(\max\left[\frac{1}{n^3}, \frac{1}{Nn}\right]\right). \tag{21}
\end{aligned}$$

The second term in equation (20) makes use of previous assumptions in this thesis. We can say that $\hat{p}_{00}$ and $\hat{p}_{11}$ are independent of our Classify-and-count estimators $\hat{\alpha}^*$ and $(\hat{\alpha}')^*$. Furthermore, a well-known result on variances states that for two independent random variables A and B, it holds that $V(AB) = E[A]^2 V(B) + E[B]^2 V(A) + V(A)V(B)$. Combining these statements gives

$$\begin{aligned}
V\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] = {}& [E((\hat{\alpha}')^* - \hat{\alpha}^*)]^2 V\left[\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] \\
& + \left[E\left(\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right)\right]^2 V[(\hat{\alpha}')^* - \hat{\alpha}^*] \\
& + V[(\hat{\alpha}')^* - \hat{\alpha}^*] V\left[\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]. \tag{22}
\end{aligned}$$

Assuming that $N \gg n$, we can make the statement that $V[(\hat{\alpha}')^* - \hat{\alpha}^*]$ is of $O(\frac{1}{N})$.

$$V\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] = \left[E\left((\hat{\alpha}')^* - \hat{\alpha}^*\right)\right]^2 V\left[\frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] + O(\frac{1}{N}) \qquad (23)$$

The expected value of the differences between the classify-and-count estimators is already computed in (16) and the variance term in (23) is already proven in Theorem **??**. This eases the derivation of the second term in (20).

$$V\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] = (\alpha' - \alpha)^2 \times \frac{V(\hat{p}_{00}) + V(\hat{p}_{11})}{(p_{00} + p_{11} - 1)^2} + O\left(\max\left[\frac{1}{N}, \frac{1}{n^2}\right]\right) \qquad (24)$$

Thus it remains to evaluate the covariance term in (20). By conditioning on the classify-and-count estimators $\hat{\alpha}^*$ and $(\hat{\alpha}')^*$, we obtain:

$$C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] = E\left[C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]\right]$$
$$+ C\left[E\left[\hat{\alpha}_c \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right], E\left[\frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]\right]$$
$$(25)$$

It can be proven that the second term of (25) is equal to zero. The expectation of the calibration estimator, given Classify-and-count estimators, is equal to α. This is a constant and the covariance with a constant is equal to zero.

$$E\left[\hat{\alpha}_c \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right] = E\left[\frac{n_{10}}{n_{+0}}(1 - \hat{\alpha}^*) + \frac{n_{11}}{n_{+1}}\hat{\alpha}^* \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]$$
$$= E\left[\frac{n_{10}}{n_{+0}}(1 - \hat{\alpha}^*) \mid \hat{\alpha}^*\right] + E\left[\frac{n_{11}}{n_{+1}}\hat{\alpha}^* \mid \hat{\alpha}^*\right]$$
$$= (1 - \hat{\alpha}^*)E\left[\frac{n_{10}}{n_{+0}} \mid \hat{\alpha}^*\right] + \hat{\alpha}^* E\left[\frac{n_{11}}{n_{+1}} \mid \hat{\alpha}^*\right]$$
$$= (1 - \hat{\alpha}^*)E\left[E\left[\frac{n_{10}}{n_{+0}} \mid n_{+0}, \mathbf{N}\right] \mid \hat{\alpha}^*\right] + \hat{\alpha}^* E\left[E\left[\frac{n_{11}}{n_{+1}} \mid n_{+1}, \mathbf{N}\right] \mid \hat{\alpha}^*\right]$$
$$= (1 - \hat{\alpha}^*)E\left[\frac{1}{n_{+0}}E\left[n_{10} \mid n_{+0}, \mathbf{N}\right] \mid \hat{\alpha}^*\right] + \alpha^* E\left[\frac{1}{n_{+1}}E\left[n_{11} \mid n_{+1}, \mathbf{N}\right] \mid \hat{\alpha}^*\right]$$
$$= \frac{N_{+0}}{N}E\left[\frac{1}{n_{+0}}n_{+0}\frac{N_{10}}{N_{+0}} \mid \hat{\alpha}^*\right] + \frac{N_{+1}}{N}E\left[\frac{1}{n_{+1}}n_{+1}\frac{N_{11}}{N_{+1}} \mid \hat{\alpha}^*\right]$$
$$= E\left[\frac{N_{10}}{N} \mid \hat{\alpha}^*\right] + E\left[\frac{N_{11}}{N} \mid \hat{\alpha}^*\right] = E\left[\frac{N_{10}}{N} + \frac{N_{11}}{N} \mid \hat{\alpha}^*\right] = E\left[\frac{N_{1+}}{N} \mid \hat{\alpha}^*\right] = \alpha$$
$$(26)$$

Therefore, the covariance term can also be written as:

$$C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] = E\left[C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right]\right]. \qquad (27)$$

We can derive an expression for the inner covariance, which is written as

$$C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right] = [(\hat{\alpha}')^* - \hat{\alpha}^*] C\left[\hat{\alpha}_c, \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid \hat{\alpha}^*\right]. \qquad (28)$$

The terms in (28) can be written in terms of the test set $(n_{00}, n_{01}, n_{10}, n_{11})$. This eases the computation further on. Note that the elements of this test set do not depend on the Classify-and-count estimator $\hat{\alpha}^*$.

$$
\begin{aligned}
C\left[\hat{\alpha}_c, \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right] &= C\left[\frac{n_{10}}{n_{+0}}(1 - \hat{\alpha}^*) + \frac{n_{11}}{n_{+1}}\hat{\alpha}^*, \frac{1}{\frac{n_{00}}{n_{0+}} + \frac{n_{11}}{n_{1+}} - 1} \mid \hat{\alpha}^*\right] \\
&= C\left[\frac{n_{10}}{n_{+0}}(1 - \hat{\alpha}^*) + \frac{n_{11}}{n_{+1}}\hat{\alpha}^*, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid \hat{\alpha}^*\right] \\
&= (1 - \hat{\alpha}^*)C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right] \\
&\quad + \hat{\alpha}^*C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right] \qquad (29)
\end{aligned}
$$

We are able to evaluate both covariance terms with the same methods. We can condition on one of the row totals. Note that the other row total is also fixed, because we work with binary classifiers ($n_{1+} = n - n_{0+}$). Furthermore, we are able to write as many variables as possible in terms of n_{0+} and n_{1+}. This helps with the Taylor Series that we apply to approximate the covariances.

$$
\begin{aligned}
&C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right] \\
&= E\left[C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right]\right] + C\left[E\left[\frac{n_{10}}{n_{+0}} \mid n_{1+}\right], E\left[\frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right]\right] \\
&= E\left[C\left[\frac{n_{1+} - n_{11}}{n_{1+} + n_{00} - n_{11}}, \frac{n_{0+}n_{1+}}{n_{0+}n_{11} + n_{1+}n_{00} - n_{0+}n_{1+}} \mid n_{1+}\right]\right] \\
&\quad + C\left[E\left[\frac{n_{1+} - n_{11}}{n_{1+} + n_{00} - n_{11}} \mid n_{1+}\right], E\left[\frac{n_{0+}n_{1+}}{n_{0+}n_{11} + n_{1+}n_{00} - n_{0+}n_{1+}} \mid n_{1+}\right]\right] \\
&\hspace{11cm} (30)
\end{aligned}
$$

While we condition on the row totals, the other variables in the covariance functions are n_{00} and n_{11}. Say $\frac{n_{10}}{n_{+0}} = f(n_{00}, n_{11})$ and $\frac{n_{0+}n_{1+}}{n_{00}n_{11}-n_{01}n_{10}} = g(n_{00}, n_{11})$, with

$$f(x, y) = \frac{n_{1+} - y}{n_{1+} + x - y} \tag{31}$$

$$g(x, y) = \frac{n_{0+}n_{1+}}{n_{1+}x + n_{0+}y - n_{0+}n_{1+}} \tag{32}$$

we are able to compute first-order Taylor series approximations for these terms to obtain an approximation for $C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11}-n_{01}n_{10}}\right]$.

$$\frac{\partial f}{\partial x} = \frac{(n_{1+} + x - y) \cdot 0 - (n_{1+} - y) \cdot 1}{(n_{1+} + x - y)^2} = \frac{y - n_{1+}}{(n_{1+} + x - y)^2} \tag{33}$$

$$\frac{\partial f}{\partial y} = \frac{(n_{1+} + x - y) \cdot -1 - (n_{1+} - y) \cdot -1}{(n_{1+} + x - y)^2} = \frac{-x}{(n_{1+} + x - y)^2} \tag{34}$$

$$\frac{\partial g}{\partial x} = \frac{-(n_{0+}n_{1+})n_{1+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^2} = \frac{-n_{1+}^2 n_{0+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^2} \tag{35}$$

$$\frac{\partial g}{\partial y} = \frac{-(n_{0+}n_{1+})n_{0+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^2} = \frac{-n_{0+}^2 n_{1+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^2} \tag{36}$$

The approximation can be made with substituting $x = E[n_{00} \mid n_{1+}]$ and $y = E[n_{11} \mid n_{1+}]$ and applying the approximation rules for covariance. Given that n_{00} and n_{11} are independent from each other given the row totals, we can cross out $C(n_{00}, n_{11})$.

$$
\begin{aligned}
C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] &\approx \frac{E[n_{11} \mid n_{1+}] - n_{1+}}{(n_{1+} + E[n_{00} \mid n_{1+}] - E[n_{11} \mid n_{1+}])^2} \\
&\times \frac{-n_{1+}^2 n_{0+}}{(n_{0+}E[n_{11} \mid n_{1+}] + n_{1+}E[n_{00} \mid n_{1+}] - n_{0+}n_{1+})^2} V(n_{00} \mid n_{1+}) \\
&+ \frac{-E[n_{00} \mid n_{1+}]}{(n_{1+} + E[n_{00} \mid n_{1+}] - E[n_{11} \mid n_{1+}])^2} \\
&\times \frac{-n_{0+}^2 n_{1+}}{(n_{0+}E[n_{11} \mid n_{1+}] + n_{1+}E[n_{00} \mid n_{1+}] - n_{0+}n_{1+})^2} V(n_{11} \mid n_{1+})
\end{aligned}
\tag{37}
$$

In order to use this approximation, we can use the following properties:

$$E(n_{00} \mid n_{1+}) = n_{0+}p_{00}$$
$$V(n_{00} \mid n_{1+}) = n_{0+}p_{00}(1 - p_{00})$$
$$E(n_{11} \mid n_{1+}) = n_{1+}p_{11}$$
$$V(n_{11} \mid n_{1+}) = n_{1+}p_{11}(1 - p_{11})$$

Substituting these elements gives

$$
C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] \approx \frac{(n_{1+}p_{11}) - n_{1+}}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^2}
$$
$$
\times \frac{-n_{1+}^2 n_{0+}}{(n_{0+}(n_{1+}p_{11}) + n_{1+}(n_{0+}p_{00}) - n_{0+}n_{1+})^2}n_{0+}p_{00}(1 - p_{00})
$$
$$
+ \frac{-n_{0+}p_{00}}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^2}
$$
$$
\times \frac{-n_{0+}^2 n_{1+}}{(n_{0+}(n_{1+}p_{11}) + n_{1+}(n_{0+}p_{00}) - n_{0+}n_{1+})^2}n_{1+}p_{11}(1 - p_{11}).
$$
$$(38)$$

This expression simplifies to

$$
C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] \approx \frac{n_{1+}p_{00}(1 - p_{00})(1 - p_{11}) + n_{0+}p_{00}p_{11}(1 - p_{11})}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^2(p_{00} + p_{11} - 1)^2}
$$
$$(39)$$

Now that the inner covariance of (30) is computed, we can move on and calculate the inner expectations of (30). This can be done with a second order Taylor series approximation.

$$\frac{\partial^2 f}{\partial x^2} = 2 \times \frac{n_{1+} - y}{(n_{1+} + x - y)^3} \tag{40}$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \times \frac{-x}{(n_{1+} + x - y)^3} \tag{41}$$

$$\frac{\partial^2 g}{\partial x^2} = 2 \times \frac{n_{1+}^3 n_{0+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^3} \tag{42}$$

$$\frac{\partial^2 g}{\partial y^2} = 2 \times \frac{n_{0+}^3 n_{1+}}{(n_{0+}y + n_{1+}x - n_{0+}n_{1+})^3} \tag{43}$$

Applying the Taylor rules for approximating an expected value and substituting $x = E[n_{00} \mid n_{1+}]$ and $y = E[n_{11} \mid n_{1+}]$ into the formulas gives:

$$E\left[\frac{n_{10}}{n_{+0}} \mid n_{1+}\right] \approx \frac{n_{1+} - E[n_{11} \mid n_{1+}]}{n_{1+} + E[n_{00} \mid n_{1+}] - E[n_{11} \mid n_{1+}]}$$
$$+ \frac{n_{1+} - E[n_{11} \mid n_{1+}]}{(n_{1+} + E[n_{00} \mid n_{1+}] - E[n_{11} \mid n_{1+}])^3} V[n_{00} \mid n_{1+}]$$
$$- \frac{E[n_{00} \mid n_{1+}]}{(n_{1+} + E[n_{00} \mid n_{1+}] - E[n_{11} \mid n_{1+}])^3} V[n_{11} \mid n_{1+}]$$
$$\tag{44}$$

$$= \frac{n_{1+} - n_{1+}p_{11}}{n_{1+} + n_{0+}p_{00} - n_{1+}p_{11}}$$
$$+ \frac{n_{1+} - n_{1+}p_{11}}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^3} n_{0+}p_{00}(1 - p_{00})$$
$$- \frac{n_{0+}p_{00}}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^3} n_{1+}p_{11}(1 - p_{11}) \tag{45}$$

$$= \frac{n_{1+}(1 - p_{11})}{n_{1+} + n_{0+}p_{00} - n_{1+}p_{11}} + \frac{n_{0+}n_{1+}p_{00}(p_{11} - 1)(p_{00} + p_{11} - 1)}{(n_{1+} + n_{0+}p_{00} - n_{1+}p_{11})^3}$$
$$\tag{46}$$

$$E\left[\frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] \approx \frac{n_{0+}n_{1+}}{n_{0+}E[n_{11} \mid n_{1+}] + n_{1+}E[n_{00} \mid n_{1+}] - n_{0+}n_{1+}}$$
$$+ \frac{n_{1+}^3 n_{0+}}{(n_{0+}E[n_{11} \mid n_{1+}] + n_{1+}E[n_{00} \mid n_{1+}] - n_{0+}n_{1+})^3} V[n_{00} \mid n_{1+}]$$
$$+ \frac{n_{0+}^3 n_{1+}}{(n_{0+}E[n_{11} \mid n_{1+}] + n_{1+}E[n_{00} \mid n_{1+}] - n_{0+}n_{1+})^3} V[n_{11} \mid n_{1+}]$$
$$\tag{47}$$

$$= \frac{n_{0+}n_{1+}}{n_{0+}n_{1+}p_{11} + n_{1+}n_{0+}p_{00} - n_{0+}n_{1+}}$$
$$+ \frac{n_{1+}^3 n_{0+}}{(n_{0+}n_{1+}p_{11} + n_{1+}n_{0+}p_{00} - n_{0+}n_{1+})^3} n_{0+}p_{00}(1 - p_{00})$$
$$+ \frac{n_{0+}^3 n_{1+}}{(n_{0+}n_{1+}p_{11} + n_{1+}n_{0+}p_{00} - n_{0+}n_{1+})^3} n_{1+}p_{11}(1 - p_{11})$$
$$\tag{48}$$

$$= \frac{1}{p_{00} + p_{11} - 1} + \frac{n_{1+}p_{00}(1 - p_{00}) + n_{0+}p_{11}(1 - p_{11})}{(n_{0+}n_{1+})(p_{00} + p_{11} - 1)^3}$$
$$\tag{49}$$

The next step is computing the outer expectation and the outer covariance of (30). The

outer expectation can be approximated with a zero-order Taylor series.

$$E\left[C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right]\right] \approx \frac{n\alpha p_{00}(1 - p_{00})(1 - p_{11}) + n(1 - \alpha)p_{00}p_{11}(1 - p_{11})}{(n\alpha + n(1 - \alpha)p_{00} - n\alpha p_{11})^2(p_{00} + p_{11} - 1)^2}$$
$$= \frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))^2(p_{00} + p_{11} - 1)^2} \tag{50}$$

Furthermore, it can be proven that the outer covariance of the two expectations is of $O\left(\frac{1}{n^2}\right)$ and can therefore be neglected in (30). In general, we can say that

$$C[f(X), g(X)] \approx f'(E[X]) \times g'(E[X]) \times V(X) \tag{51}$$

Let $f(x)$ and $g(x)$ be the expectations of equation (46) and (49), with $x = n_{1+}$. Taking the derivative with respect to x gives:

$$f(x) = \frac{x(1 - p_{11})}{x + (n - x)p_{00} - xp_{11}} + \frac{(n - x)xp_{00}(p_{11} - 1)(p_{00} + p_{11} - 1)}{(x + (n - x)p_{00} - xp_{11})^3}$$
$$f'(x) = \frac{np_{00}(p_{11} - 1)}{(np_{00} - x(p_{00} + p_{11} - 1))^2}$$
$$+ \frac{[p_{00}(1 - p_{11})(p_{00} + p_{11} - 1)][(2x - n) + 3(x^2 - nx)(np_{00} - x(p_{00} + p_{11} - 1))^2(p_{00} + p_{11} - 1)}{(np_{00} - x(p_{00} + p_{11} - 1)^6} \tag{52}$$

$$g(x) = \frac{1}{p_{00} + p_{11} - 1} + \frac{xp_{00}(1 - p_{00}) + (n - x)p_{11}(1 - p_{11})}{((n - x)x)(p_{00} + p_{11} - 1)^3}$$
$$g'(x) = \frac{(nx - x^2)(p_{00}(1 - p_{00}) - p_{11}(1 - p_{11})) + (2x - n)[xp_{00}(1 - p_{00}) + (n - x)p_{11}(1 - p_{11})])}{(nx - x^2)^2(p_{00} + p_{11} - 1)^3} \tag{53}$$

If we substitute $x = E[n_{1+}] = n\alpha$ in the derivatives, we obtain the following expressions:

$$f'(E[n_{1+}]) = \frac{p_{00}(p_{11} - 1)}{n((1 - \alpha)p_{00} + \alpha(1 - p_{11}))^2}$$
$$+ p_{00}(1 - p_{11})(p_{00} + p_{11} - 1)$$
$$\times \frac{n(2\alpha - 1) + 3n^4(1 - \alpha)((1 - \alpha)p_{00} + \alpha(1 - p_{11}))^2(p_{00} + p_{11} - 1)}{n^6((1 - \alpha)p_{00} + \alpha(1 - p_{11}))^6}$$

$$(54)$$

$$g'(E[n_{1+}]) = \frac{(\alpha - \alpha^2)(p_{00}(1 - p_{00}) - p_{11}(1 - p_{11}) + (2\alpha - 1)(\alpha p_{00}(1 - p_{00}) + (1 - \alpha)p_{11}(1 - p_{11}))}{n^2(\alpha - \alpha^2)(p_{00} + p_{11} - 1)^3}$$

$$(55)$$

It can be clearly seen that $f'(E[n_{1+}]) = O(\frac{1}{n})$, $g'(E[n_{1+}]) = O(\frac{1}{n^2})$ and that $V(n_{1+}) = O(n)$. Therefore, the whole covariance term is small enough to be negligible ($O(\frac{1}{n}) \cdot O(\frac{1}{n^2}) \cdot O(n) = O(\frac{1}{n^2})$, see (51)) and that the covariance term can be written as:

$$C\left[\frac{n_{10}}{n_{+0}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right] \approx \frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))^2(p_{00} + p_{11} - 1)^2}.$$

$$(56)$$

Similarly, $C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right]$ can be computed. First, $C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{+1}\right]$ can be computed with a first-order Taylor series approximation. Because we condition on the row-totals, we rewrite $\frac{n_{11}}{n_{+1}}$ as

$$\frac{n_{11}}{n_{+1}} = \frac{n_{11}}{n - n_{00} - n_{10}} = \frac{n_{11}}{n - n_{00} - (n_{1+} - n_{11})} = \frac{n_{11}}{n_{0+} - n_{00} + n_{11}}$$

and make a function dependent on $x = n_{00}$ and $y = n_{11}$, which we can derive.

$$h(x, y) = \frac{y}{n_{0+} - x + y}$$

$$\frac{\partial h}{\partial x} = \frac{y}{(n_{0+} - x + y)^2} \tag{57}$$

$$\frac{\partial h}{\partial y} = \frac{n_{0+} - x}{(n_{0+} - x + y)^2} \tag{58}$$

Accordingly, we can borrow the expectations from the previous covariance term. Therefore we end up with the following term:

$$C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] \approx \frac{n_{1+}p_{11}}{(n_{0+}(1 - p_{00}) + n_{1+}p_{11})^2}$$

$$\times \frac{-n_{1+}^2 n_{0+}}{(n_{0+}(n_{1+}p_{11}) + n_{1+}(n_{0+}p_{00}) - n_{0+}n_{1+})^2} n_{0+}p_{00}(1 - p_{00})$$

$$+ \frac{n_{0+}(1 - p_{00})}{(n_{0+}(1 - p_{00}) + n_{1+}p_{11})^2}$$

$$\times \frac{-n_{0+}^2 n_{1+}}{(n_{0+}(n_{1+}p_{11}) + n_{1+}(n_{0+}p_{00}) - n_{0+}n_{1+})^2} n_{1+}p_{11}(1 - p_{11}).$$

(59)

This simplifies to:

$$C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right] \approx -\frac{n_{1+}p_{00}(1 - p_{00})p_{11} + n_{0+}(1 - p_{00})p_{11}(1 - p_{11})}{(n_{0+}(1 - p_{00}) + n_{1+}p_{11})^2(p_{00} + p_{11} - 1)^2}$$

(60)

The next step is computing the expected value of this expression.

$$E\left[C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}} \mid n_{1+}\right]\right] \approx -\frac{n\alpha p_{00}(1 - p_{00})p_{11} + n(1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{(n(1 - \alpha)(1 - p_{00}) + n\alpha p_{11})^2(p_{00} + p_{11} - 1)^2}$$

$$= -\frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})^2(p_{00} + p_{11} - 1)^2}$$

(61)

The covariance between the expectations is again of a negligible low order, so the covariance term can be written as:

$$C\left[\frac{n_{11}}{n_{+1}}, \frac{n_{0+}n_{1+}}{n_{00}n_{11} - n_{01}n_{10}}\right] \approx -\frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})^2(p_{00} + p_{11} - 1)^2}$$

(62)

Now that we have obtained the two conditional covariance in (56) and (62), we can substitute these terms in (29).

$$C\left[\hat{\alpha}_c, \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid (\hat{\alpha}')^*, \hat{\alpha}^*\right] \approx (1 - \hat{\alpha}^*) \times \frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n((1 - \alpha)p_{00} + \alpha(1 - p_{11}))^2(p_{00} + p_{11} - 1)^2}$$

$$- \hat{\alpha}^* \times \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})^2(p_{00} + p_{11} - 1)^2}$$

(63)

Combining (27), (28) and (63), we can compute $C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right]$ by taking the expected value of the difference between the Classify-and-count estimators multiplied by the expected value of (63). Note that the first part of both denominators are equal to respectively the expected value of $(1 - \hat{\alpha}^*)$ and $\hat{\alpha}^*$ squared.

$$
\begin{aligned}
C\left[\hat{\alpha}_c, \frac{(\hat{\alpha}')^* - \hat{\alpha}^*}{\hat{p}_{00} + \hat{p}_{11} - 1}\right] &= E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*] C\left[\hat{\alpha}_c, \frac{1}{\hat{p}_{00} + \hat{p}_{11} - 1} \mid \hat{\alpha}^*\right]\right] \\
&\approx E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*]\left[(1 - \hat{\alpha}^*)\frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))^2(p_{00} + p_{11} - 1)^2}\right.\right. \\
&\qquad \left.\left. - \hat{\alpha}^* \times \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})^2(p_{00} + p_{11} - 1)^2}\right]\right] \\
&\approx E\left[[(\hat{\alpha}')^* - \hat{\alpha}^*]\left[\frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))(p_{00} + p_{11} - 1)^2}\right.\right. \\
&\qquad \left.\left. - \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})(p_{00} + p_{11} - 1)^2}\right]\right] \\
&= [(\alpha') - \alpha](p_{00} + p_{11} - 1)\left[\frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))(p_{00} + p_{11} - 1)^2}\right. \\
&\qquad \left. - \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})(p_{00} + p_{11} - 1)^2}\right] \\
&= [(\alpha') - \alpha]\left[\frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))(p_{00} + p_{11} - 1)}\right. \\
&\qquad \left. - \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})(p_{00} + p_{11} - 1)}\right]
\end{aligned}
$$
(64)

Combining all elements gives the total variance of the mixed estimator.

$$
\begin{aligned}
V(\hat{\alpha}'_m) &= \frac{\alpha p_{11}}{n} \times \left(1 - \frac{\alpha p_{11}}{(1 - \alpha)(1 - p_{00}) + \alpha p_{11}}\right) \\
&\quad + \frac{(1 - \alpha)p_{00}}{n} \times \left(1 - \frac{(1 - \alpha)p_{00}}{(1 - \alpha)p_{00} + \alpha(1 - p_{11})}\right) \\
&\quad + (\alpha' - \alpha)^2 \times \frac{V(\hat{p}_{00}) + V(\hat{p}_{11})}{(p_{00} + p_{11} - 1)^2} \\
&\quad + (\alpha' - \alpha) \times \left[\frac{\alpha p_{00}(1 - p_{00})(1 - p_{11}) + (1 - \alpha)p_{00}p_{11}(1 - p_{11})}{n(p_{00} - \alpha(p_{00} + p_{11} - 1))(p_{00} + p_{11} - 1)}\right. \\
&\quad \left. - \frac{\alpha p_{00}(1 - p_{00})p_{11} + (1 - \alpha)(1 - p_{00})p_{11}(1 - p_{11})}{n((1 - \alpha)(1 - p_{00}) + \alpha p_{11})(p_{00} + p_{11} - 1)}\right] + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$
(65)

□

This concludes the proof of the bias and variance of the mixed estimator. Note that all terms of $O\left(\frac{1}{n^2}\right)$ are excluded.