Cabibbo-Kobayashi-Maskawa matrix: parameterizations and rephasing invariants

H. Pérez R.,[1, a)] P. Kielanowski,[1, b)] and S.R. Juárez W.[2, c)]

[1)] *Departamento de Física, Centro de Investigación y de Estudios Avanzados, 07000 México D.F., Mexico*

[2)] *Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, U.P. "Adolfo López Mateos". C.P. 07738, México D.F.*[d)]

The paper is devoted to a discussion of general properties of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. First we propose a general method of a recursive construction of the CKM matrix for any number of generations. This allows to set up a parameterization with desired properties. As an application we generalize the Wolfenstein parameterization to the case of 4 generations and obtain restrictions on the CKM suppression of the fourth generation. Motivated by the rephasing invariance of the CKM observables we next consider the general phase invariant monomials built out of the CKM matrix elements and their conjugates. We show, that there exist 30 fundamental phase invariant monomials and 18 of them are a product of 4 CKM matrix elements and 12 are a product of 6 CKM matrix elements. In the Main Theorem we show that all rephasing invariant monomials can be expressed as a product of at most 5 factors: 4 of them are fundamental phase invariant monomials and the fifth factor consists of powers of squares of absolute values of the CKM matrix elements.

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I. INTRODUCTION

The Standard Model has 18 adjustable parameters and 13 of these parameters have their origin in the Yukawa couplings of the Higgs boson with fermions. These parameters are: 3 lepton masses, 6 quark masses and 4 parameters of the CKM matrix[1]. The quark masses are the running masses and they are obtained from the eigenvalues of the Yukawa couplings in the process of diagonalization by the bi-unitary transformation. The CKM matrix is obtained from the *left* diagonalizing matrices of the up- and down quark mass matrices. The CKM matrix is a 3×3 unitary matrix, which has additional properties stemming from the rephasing freedom of the quark fields. This is the reason why the CKM matrix has only 4 real parameters and one of them is a phase. The non vanishing phase is the source of the CP violation in the Standard Model.

The CKM matrix has 9 complex matrix elements, which are parameterized by 4 real parameters. The choice of the parameters is not unique. There exist various equivalent parameterizations, which were chosen to fulfill various needs. Let us start with the standard parameterization of the PDG[2]. This parameterization is exact and uses 3 angles and 1 phase and can be represented as the product of 3 real rotation matrices and the diagonal matrices with phase terms.

Another widely used parameterization is the one proposed by Wolfenstein[3]. Initially it was considered to be an approximate representation of the CKM matrix, because it was chosen in such a way as to reproduce the suppression for the weak transitions of quarks between the generations. Later, it was improved[4–6] and also it was made exact[7].

The particular choice of parameters in principle is not important, because all exact parameterizations are mathematically equivalent. But an appropriate choice of parameterization for a given physical situation can lead to natural relationship between the parameters and physical observables. Fo example, in the parameterization of Wolfenstein the Jarlskog's invariant[8] $J = A\eta\lambda^6 + \mathcal{O}(\lambda^8)$ has simple form, while in the standard parameterization of the CKM matrix[2] it is more complicated.

In Section II we discuss the recursive construction of the CKM matrix for a general case of n generations, assuming that the CKM matrix for $(n-1)$ quark generations is known. There exist various approaches to the problem of parameterization of the CKM matrix, which fulfill various requirements. The parameterization of Harari and Leuler[9] stresses the simplicity

and a possibility of generalization to an arbitrary number of quark generations. Jarlskog proposed the recursive parameterization also for an arbitrary number of quark generations that represents the CKM matrix as a product of special unitary matrices[10,11]. Bjorken and Dunietz[12] construct the CKM matrix out of rephasing invariants: the absolute values of the CKM matrix elements $|V_{i\alpha}|$, $i < \alpha$ and the phase of "plaquette" $\arg(V_{i\alpha}V_{j\beta}V_{i\beta}^*V_{j\alpha}^*)$. There are many more parameterizations[13–24] which help to illustrate various features of the CKM matrix, but their discussion is beyond the scope of our paper.

Our algorithm for the construction of the parameterization of the CKM matrix has the flexibility that allows to adjust the parameterization to the required physical situation. It can be used to generalize the standard and Wolfenstein parameterizations to the case of 4 quark generations. From the Wolfenstein-type parameterization for 4 generations we then obtain the restrictions on the suppression factors for the 4-th generation, which are compatible with the present data for the CKM matrix.

The next topic, discussed in Section III, is the quark fields rephasing properties of the CKM matrix. The Yukawa couplings (mass matrices) of the standard model are not invariant under the rephasing of the quark fields. From this it follows that the CKM matrix is not invariant either. Since observable effects cannot depend on the choice of the phase of the quark fields it means that all observables including those containing the CKM matrix must not depend on arbitrary phases of the quark fields. This is the reason of the reduction of the number of the physically significant parameters of the unitary CKM matrix from 9 to 4.

The general observables of CKM matrix are usually monomials built out of the CKM matrix elements V_{ij} and its conjugates V_{ij}^*. The most important observables obtained from the rephasing invariant monomials of the CKM matrix[25] are the Jarlskog invariant[8] [26] J

$$J = \text{Im}(V_{11}V_{22}V_{12}^*V_{21}^*),\tag{1}$$

and the unitarity triangle angles:

$$\begin{aligned}\alpha &= \arg\left(-\frac{V_{31}V_{33}^*}{V_{11}V_{13}^*}\right) = \arg\left(-V_{13}V_{31}V_{11}^*V_{33}^*\right),\\\beta &= \arg\left(-\frac{V_{21}V_{23}^*}{V_{31}V_{33}^*}\right) = \arg\left(-V_{21}V_{33}V_{23}^*V_{31}^*\right),\\\gamma &= \arg\left(-\frac{V_{11}V_{13}^*}{V_{21}V_{23}^*}\right) = \arg\left(-V_{11}V_{23}V_{13}^*V_{21}^*.\right)\end{aligned}\tag{2}$$

In Section III we derive the conditions for the rephasing invariance of an arbitrary monomial built from the CKM matrix elements and then demonstrate in the Main Theorem

that an arbitrary rephasing monomial can be expressed as the product of five factors: powers of absolute values of the CKM matrix elements multiplied by powers of 4 fundamental rephasing monomials, which are built out of 4 and 6 CKM matrix elements.

The last section of the paper contains a summary and conclusions.

II. PARAMETERIZATIONS OF THE CKM MATRIX

A. Introductory remarks

The CKM matrix is unitary, but the rephasing freedom for the quark fields reduces the number of the physically significant parameters. The $(n \times n)$ unitary matrix is described by n^2 parameters. The up- and down-quarks rephasing freedom reduces the number of the parameters by $(2n - 1)$, so the CKM matrix for n quark generations is described by $(n-1)^2$ parameters. These parameters are divided into two classes: angle-like and phases. Angle-like parameters are derived from the $(n \times n)$ *real* unitary matrix[27] (rotation or orthogonal matrix) and there are $\frac{n(n-1)}{2}$ such parameters. The remaining $\frac{(n-1)(n-2)}{2}$ parameters are phases. One can observe that if the number of quark generations is incremented from $(n-1)$ to n then the number of angle-like parameters increases by $(n - 1)$ and the number of phases by $(n - 2)$.

We will present here the recursive construction of the $(n \times n)$ CKM matrix $V^{(n)}$, assuming that the $(n - 1) \times (n - 1)$ CKM matrix $V^{(n-1)}$ is known. Let us introduce the notation, where the parameters of the CKM matrix (angle-like and phases) are labeled according to the generation number to which they belong:

$$
\begin{aligned}
&\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{k-1,k} \text{ - angle-like parameters for the } k\text{-th generation} \\
&\delta_{1,k}, \delta_{2,k}, \ldots, \delta_{k-2,k} \text{ - phases for the } k\text{-th generation}
\end{aligned}
\tag{3}
$$

In such a way the following hierarchy of the parameters has been formed:

generation	parameters
2	$\theta_{1,2}$
3	$\theta_{1,3}, \theta_{2,3}, \delta_{1,3}$
4	$\theta_{1,4}, \theta_{2,4}, \theta_{3,4}, \delta_{1,4}, \delta_{2,4}$
...	...

$$\tag{4}$$

The continuation of the Table in Eq. (4) is obvious. The $(n \times n)$ CKM matrix contains the parameters from the Table in Eq. (4) that are in the n-th row and above.

B. Recursive construction of the CKM matrix

In this section we outline how to construct and parameterize the $(n \times n)$ CKM matrix $V^{(n)}$, if we know the CKM matrix $V^{(n-1)}$. The presented algorithm does not impose any conditions on the parameterization of the matrix $V^{(n-1)}$, so this method allows to introduce such properties of the CKM matrix that are required by the physical conditions for each generation separately.

Let us first introduce the necessary notation. We write the CKM matrix $V^{(n)}$ in terms of column vectors

$$V^{(n)} = \left(\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_n^{(n)} \right) \tag{5}$$

i.e., the vectors $\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_n^{(n)}$ are constructed from the elements of the matrix $V^{(n)}$

$$\mathbf{v}_k^{(n)} = \begin{pmatrix} V_{1k}^{(n)} \\ V_{2k}^{(n)} \\ \vdots \\ V_{nk}^{(n)} \end{pmatrix}. \tag{6}$$

We assume that the explicit form of the matrix $V^{(n-1)}$ is known, so according to Eqs. (3) and (4) it is a function of the following parameters:

$$\theta_{1,2}, \theta_{1,3}, \theta_{2,3}, \ldots, \theta_{n-2,n-1},$$
$$\delta_{1,3}, \delta_{1,4}, \delta_{2,4}, \ldots, \delta_{n-3,n-1}$$

The CKM matrix $V^{(n)}$ is built from $V^{(n-1)}$ in two steps:

a. We construct n *real* column vectors with n rows

$$\mathbf{e}_1, \ldots, \mathbf{e}_n \tag{7}$$

that depend on $(n-1)$ angle-like independent parameters $\theta_{1,n}, \ldots, \theta_{n-1,n}$, are normalized to 1 and orthogonal:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \tag{8}$$

b. The columns of the CKM matrix $V^{(n)}$ are obtained from the vectors $\mathbf{e}_k$ and the elements of the matrix $V_{ij}^{(n-1)}$ in the following way

$$\begin{aligned} \mathbf{v}_k^{(n)} &= V_{1k}^{(n-1)} \mathbf{e}_1 + \sum_{l=2}^{n-1} V_{lk}^{(n-1)} e^{-i\delta_{l-1,n}} \mathbf{e}_l, \quad k = 1, \ldots, n-1, \\ \mathbf{v}_n^{(n)} &= \mathbf{e}_n, \end{aligned} \qquad n \geq 2 \tag{9}$$

and this completes the construction of the CKM matrix $V^{(n)}$ (see Eq. (5)). The matrix $V^{(n)}$ constructed in such a way has following properties

a. It is unitary. This follows from the unitarity of the matrix $V^{(n-1)}$ and the orthogonality (8) of the vectors $\mathbf{e}_l$.

b. It depends on parameters of the matrix $V^{(n-1)}$ and on $(n-1)$ parameters of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and on $(n-2)$ phase factors $\mathrm{e}^{-i\delta_{l,n}}$ from (9). Altogether it depends on $\frac{n(n-1)}{2}$ angle-like variables and $\frac{(n-1)(n-2)}{2}$ phases.

The resulting parameterization of the matrix $V^{(n)}$ depends on the parametrization of $V^{(n-1)}$ and that of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and on the phase factors $\mathrm{e}^{-i\delta_{l,n}}$. If we additionally assume that the vectors $\mathbf{e}_i$ fulfill the conditions

$$
(\mathbf{e}_i)_j|_{\substack{\theta_{k,n}=0 \\ \delta_{l,n}=0}} = \begin{cases} 1 & \text{if } i = j \qquad k = 1, \ldots, n-1 \\ 0 & \text{otherwise} \qquad l = 1, \ldots, n-2. \end{cases} \tag{10}
$$

then one obtains

$$
V^{(n)}\big|_{\substack{\theta_{k,n}=0 \\ \delta_{l,n}=0}} = \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} k = 1, \ldots, n-1 \\ l = 1, \ldots, n-2. \end{array} \tag{11}
$$

Example 1. *Standard parameterization for 3 quark generations*

We will show here how one can obtain the CKM matrix $V^{(3)}$ in the standard parameterization using the procedure outlined above.

The matrix $V^{(2)}$ depends on one angle $\theta_{1,2}$ (it is the 2×2 rotation matrix)

$$
V^{(2)} = \begin{pmatrix} \cos\theta_{1,2} & \sin\theta_{1,2} \\ -\sin\theta_{1,2} & \cos\theta_{1,2} \end{pmatrix} \equiv \begin{pmatrix} c_{1,2} & s_{1,2} \\ -s_{1,2} & c_{1,2} \end{pmatrix}, \tag{12}
$$

where we use the notation $s_{i,j} = \sin\theta_{i,j}$ and $c_{i,j} = \cos\theta_{i,j}$. We now choose the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in the following way

$$
\mathbf{e}_1 = \begin{pmatrix} c_{1,3} \\ -s_{1,3}s_{2,3} \\ -s_{1,3}c_{2,3} \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ c_{2,3} \\ -s_{2,3} \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} s_{1,3} \\ c_{1,3}s_{2,3} \\ c_{1,3}c_{2,3} \end{pmatrix}. \tag{13}
$$

The vectors in Eq. (13) fulfill the condition (8).

Now, according to Eq. (9) we construct the columns of the matrix $V^{(3)}$

$$\mathbf{v}_1^{(3)} = V_{11}\mathbf{e}_1 + V_{21}\mathrm{e}^{-i\delta_{1,3}}\mathbf{e}_2 = \begin{pmatrix} c_{1,2}c_{1,3} \\ -c_{1,2}s_{1,3}s_{2,3} - \mathrm{e}^{-i\delta_{1,3}}s_{1,2}c_{2,3} \\ -c_{1,2}s_{1,3}c_{2,3} + \mathrm{e}^{-i\delta_{1,3}}s_{1,2}s_{2,3} \end{pmatrix}, \tag{14a}$$

$$\mathbf{v}_2^{(3)} = V_{12}\mathbf{e}_1 + V_{22}\mathrm{e}^{-i\delta_{1,3}}\mathbf{e}_2 = \begin{pmatrix} s_{1,2}c_{1,3} \\ -s_{1,2}s_{1,3}s_{2,3} + \mathrm{e}^{-i\delta_{1,3}}c_{1,2}c_{2,3} \\ -s_{1,2}s_{1,3}c_{2,3} - \mathrm{e}^{-i\delta_{1,3}}c_{1,2}s_{2,3} \end{pmatrix}, \tag{14b}$$

$$\mathbf{v}_3^{(3)} = \mathbf{e}_3 = \begin{pmatrix} s_{1,3} \\ c_{1,3}s_{2,3} \\ c_{1,3}c_{2,3} \end{pmatrix}, \tag{14c}$$

so the matrix $V^{(3)}$ is equal

$$V^{(3)} = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_3) = \begin{pmatrix} c_{1,2}c_{1,3} & s_{1,2}c_{1,3} & s_{1,3} \\ -c_{1,2}s_{1,3}s_{2,3} - \mathrm{e}^{-i\delta_{1,3}}s_{1,2}c_{2,3} & -s_{1,2}s_{1,3}s_{2,3} + \mathrm{e}^{-i\delta_{1,3}}c_{1,2}c_{2,3} & c_{1,3}s_{2,3} \\ -c_{1,2}s_{1,3}c_{2,3} + \mathrm{e}^{-i\delta_{1,3}}s_{1,2}s_{2,3} & -s_{1,2}s_{1,3}c_{2,3} - \mathrm{e}^{-i\delta_{1,3}}c_{1,2}s_{2,3} & c_{1,3}c_{2,3} \end{pmatrix}. \tag{15}$$

The form of the matrix (15) is not exactly the same as that of the standard parameterization[2], but both forms are equivalent, because after rephasing of the matrix $V^{(3)}$ in Eq. (15) by multiplying the first and second column by $\mathrm{e}^{i\delta_{1,3}}$ and the first row by $\mathrm{e}^{-i\delta_{1,3}}$ one obtains exactly the standard parameterization.

Example 2. *Standard parameterization for 4 quark generations*

In this example we will construct an analogue of the standard parameterization for 4 generations using the standard form of the CKM matrix $V^{(3)}$ from Ref.[2]. First we construct the basis vectors $\mathbf{e}_i$ for the 4-dimensional case

$$\mathbf{e}_1 = \begin{pmatrix} c_{1,4} \\ -s_{1,4}s_{2,4} \\ -s_{1,4}c_{2,4}s_{3,4} \\ -s_{1,4}c_{2,4}c_{3,4} \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ c_{2,4} \\ -s_{2,4}s_{3.4} \\ -s_{2,4}c_{3.4} \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ c_{3,4} \\ -s_{3,4} \end{pmatrix}, \ \mathbf{e}_4 = \begin{pmatrix} s_{1,4} \\ c_{1,4}s_{2,4} \\ c_{1,4}c_{2,4}s_{3,4} \\ c_{1,4}c_{2,4}c_{3,4} \end{pmatrix}. \tag{16}$$

Then, using Eqs. (9) we construct the vectors $\mathbf{v}_i^{(4)}$ and the matrix $V^{(4)}$, which we call the standard parameterization of the CKM matrix for 4 generations. The explicit form of the

matrix elements of $V_{ij}^{(4)}$ is rather lengthy and reads

$$V_{11}^{(4)} = c_{1,2}c_{1,3}c_{1,4},$$

$$V_{21}^{(4)} = - e^{-i\delta_{1,4}}c_{2,3}c_{2,4}s_{1,2} - e^{i(\delta_{1,3}-\delta_{1,4})}c_{1,2}c_{2,4}s_{1,3}s_{2,3} - c_{1,2}c_{1,3}s_{1,4}s_{2,4},$$

$$V_{31}^{(4)} = - e^{i(\delta_{1,3}-\delta_{2,4})}c_{1,2}c_{2,3}c_{3,4}s_{1,3} + e^{i(\delta_{1,3}-\delta_{1,4})}c_{1,2}s_{2,3}s_{2,4}s_{3,4}s_{1,3}$$

$$+ e^{-i\delta_{2,4}}c_{3,4}s_{1,2}s_{2,3} - c_{1,2}c_{1,3}c_{2,4}s_{1,4}s_{3,4} + e^{-i\delta_{1,4}}c_{2,3}s_{1,2}s_{2,4}s_{3,4},$$

$$V_{41}^{(4)} = - c_{1,2}c_{1,3}c_{2,4}c_{3,4}s_{1,4} + e^{-i\delta_{1,4}}c_{2,3}c_{3,4}s_{1,2}s_{2,4} - e^{-i\delta_{2,4}}s_{1,2}s_{2,3}s_{3,4}$$

$$+ e^{i(\delta_{1,3}-\delta_{1,4})}c_{1,2}c_{3,4}s_{1,3}s_{2,3}s_{2,4} + e^{i(\delta_{1,3}-\delta_{2,4})}c_{1,2}c_{2,3}s_{1,3}s_{3,4},$$

$$V_{12}^{(4)} = c_{1,3}c_{1,4}s_{1,2},$$

$$V_{22}^{(4)} = e^{-i\delta_{1,4}}c_{1,2}c_{2,3}c_{2,4} - e^{i(\delta_{1,3}-\delta_{1,4})}s_{1,2}s_{1,3}s_{2,3}c_{2,4} - c_{1,3}s_{1,2}s_{1,4}s_{2,4},$$

$$V_{32}^{(4)} = - e^{i(\delta_{1,3}-\delta_{2,4})}c_{2,3}c_{3,4}s_{1,2}s_{1,3} + e^{i(\delta_{1,3}-\delta_{1,4})}s_{1,2}s_{2,3}s_{2,4}s_{3,4}s_{1,3}$$

$$- e^{-i\delta_{2,4}}c_{1,2}c_{3,4}s_{2,3} - c_{1,3}c_{2,4}s_{1,2}s_{1,4}s_{3,4} - e^{-i\delta_{1,4}}c_{1,2}c_{2,3}s_{2,4}s_{3,4},$$

$$V_{42}^{(4)} = - c_{1,3}c_{2,4}c_{3,4}s_{1,2}s_{1,4} - e^{-i\delta_{1,4}}c_{1,2}c_{2,3}c_{3,4}s_{2,4} + e^{-i\delta_{2,4}}c_{1,2}s_{2,3}s_{3,4}$$

$$+ e^{i(\delta_{1,3}-\delta_{1,4})}c_{3,4}s_{1,2}s_{1,3}s_{2,3}s_{2,4} + e^{i(\delta_{1,3}-\delta_{2,4})}c_{2,3}s_{1,2}s_{1,3}s_{3,4},$$

$$V_{13}^{(4)} = e^{-i\delta_{1,3}}c_{1,4}s_{1,3},$$

$$V_{23}^{(4)} = e^{-i\delta_{1,4}}c_{1,3}c_{2,4}s_{2,3} - e^{-i\delta_{1,3}}s_{1,3}s_{1,4}s_{2,4},$$

$$V_{33}^{(4)} = e^{-i\delta_{2,4}}c_{1,3}c_{2,3}c_{3,4} - e^{-i\delta_{1,3}}c_{2,4}s_{1,3}s_{1,4}s_{3,4} - e^{-i\delta_{1,4}}c_{1,3}s_{2,3}s_{2,4}s_{3,4},$$

$$V_{43}^{(4)} = - e^{-i\delta_{1,3}}c_{2,4}c_{3,4}s_{1,3}s_{1,4} - e^{-i\delta_{1,4}}c_{1,3}c_{3,4}s_{2,3}s_{2,4} - e^{-i\delta_{2,4}}c_{1,3}c_{2,3}s_{3,4},$$

$$V_{14}^{(4)} = s_{14}, \quad V_{24}^{(4)} = c_{1,4}s_{2,4}, \quad V_{34}^{(4)} = c_{1,4}c_{2,4}s_{3,4}, \quad V_{44}^{(4)} = c_{1,4}c_{2,4}c_{3,4}.$$

$$(17)$$

C.   Wolfenstein parameterization for 4 quark generations

The other widely used parameterization of the CKM matrix is the one proposed by Wolfenstein[3] in which the matrix elements are expressed as powers of the parameter λ

$$\lambda \approx |V_{12}|^2. \tag{18}$$

For 3 generations the CKM matrix in the Wolfenstein parameterization has the form

$$V^{(3)} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{19}$$

8

The parameterization given in Eq. (19) is approximate. It can be given an exact meaning by assuming a one to one correspondence between the Wolfenstein parameters A, λ, ρ and η and the parameters of the standard parameterization $s_{12} \equiv \lambda$, $s_{23} \equiv A\lambda^2$ and $s_{13}e^{i\delta_{13}} \equiv A\lambda^3(\rho+i\eta)$.[4] We shall generalize the parameterization (19) to the case of 4 quark generations using the method outlined earlier.

First we have to construct the vectors $\mathbf{e}_i$, which are expressed in the spirit of the Wolfenstein parameterization in terms of the powers of λ. The vectors $\mathbf{e}_i$ are real and are chosen in the following way

$$\mathbf{e}_1 = N_1 \begin{pmatrix} 1 + z_2^2 + z_3^2 \\ -z_1 z_2 \\ -z_1 z_3 \\ -z_1 \end{pmatrix}, \; \mathbf{e}_2 = N_2 \begin{pmatrix} 0 \\ 1 + z_3^2 \\ -z_2 z_3 \\ -z_2 \end{pmatrix}, \mathbf{e}_3 = N_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -z_3 \end{pmatrix}, \; \mathbf{e}_4 = N_4 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{pmatrix}. \tag{20}$$

Here $z_i = A_i^{(4)}\lambda^{k_i}$, $A_i^{(4)} \sim 1$, $k_i \geq 1$ are integers and N_i are suitable normalization factors. The powers k_i are considered to be constants (suppression factors) and $A_i^{(4)}$, $i = 1, 2, 3$ are free parameters. It is easy to verify that the vectors $\mathbf{e}_i$ fulfill Eq. (8), i.e., are orthogonal and normalized to 1. The columns of the CKM matrix for 4 generations are then equal

$$\begin{aligned} \mathbf{v}_1^{(4)} &= V_{11}^{(3)}\mathbf{e}_1 + e^{-i\delta_{1,4}}V_{21}^{(3)}\mathbf{e}_2 + e^{-i\delta_{2,4}}V_{31}^{(3)}\mathbf{e}_3, \\ \mathbf{v}_2^{(4)} &= V_{12}^{(3)}\mathbf{e}_1 + e^{-i\delta_{1,4}}V_{22}^{(3)}\mathbf{e}_2 + e^{-i\delta_{2,4}}V_{32}^{(3)}\mathbf{e}_3, \\ \mathbf{v}_3^{(4)} &= V_{13}^{(3)}\mathbf{e}_1 + e^{-i\delta_{1,4}}V_{23}^{(3)}\mathbf{e}_2 + e^{-i\delta_{2,4}}V_{33}^{(3)}\mathbf{e}_3, \\ \mathbf{v}_4^{(4)} &= \mathbf{e}_4, \end{aligned} \tag{21}$$

and the CKM matrix is equal

$$V^{(4)} = \left(\mathbf{v}_1^{(4)}, \mathbf{v}_2^{(4)}, \mathbf{v}_3^{(4)}, \mathbf{v}_4^{(4)} \right). \tag{22}$$

The matrix $V^{(4)}$ in Eq. (22) is described by 9 parameters:

λ, A, ρ, η of the matrix (19), introduced by Wolfenstein,
$A_1^{(4)}$, $A_2^{(4)}$, $A_3^{(4)}$, $\delta_{1,4}$, $\delta_{2,4}$ of the vectors in (21).

Not all these parameters can be determined from the experimental data. On the other hand we can derive some restrictions on the powers k_i of the supression factors for the 4-th generation from the experimental information for the CKM matrix for 3 generations. We

have the following information[28]

$$|V_{12}| \sim \lambda, \quad |V_{21}| \sim \lambda, \quad |V_{23}| \sim \lambda^2, \quad |V_{32}| \sim \lambda^2, \quad |V_{13}| \sim \lambda^3,$$
$$|V_{31}| \sim \lambda^3, \quad (|V_{12}| - |V_{21}|) \sim \lambda^3, \quad (|V_{23}| - |V_{32}|) \sim \lambda^4. \tag{23}$$

Now, using the information in Eq. (23) with the explicit representation of the 4×4 CKM matrix in Eqs. (21) and (22) we obtain the following restrictions on the powers k_i

$$k_i \geq 1, \quad k_1 + k_2 \geq 3, \quad k_2 + k_3 \geq 4, \quad k_1 + k_3 \geq 3, \tag{24}$$

which can be resolved and give

$$k_1 \geq 1, \quad k_2 \geq 2, \quad k_3 \geq 2. \tag{25}$$

The vector $\mathbf{v}_4$ for the minimal values of k_i in Eq. (25) has the following form

$$\mathbf{v}_4 = N_4 \begin{pmatrix} A_1^{(4)} \lambda \\ A_2^{(4)} \lambda^2 \\ A_3^{(4)} \lambda^2 \\ 1 \end{pmatrix}. \tag{26}$$

This result is rather surprising, because in the case of 3 generations the suppression has totally different structure. The full analysis of the 4×4 CKM matrix based on Eq. (22) will be published elsewhere. However, we would like to note that the real suppression may be different, because we have only obtained the *lower limits* of the suppression powers.

To conclude this section we will compare the values of two following rephasing invariants

$$\begin{aligned} J_A &= \mathrm{Im}(V_{12} V_{23} V_{13}^* V_{22}^*) \\ J_B &= \mathrm{Im}(V_{21} V_{33} V_{23}^* V_{31}^*) \end{aligned}, \tag{27}$$

that describe the CP violation effects in the strange and bottom sectors. For 3 generations of quarks from the unitarity of the CKM matrix it follows that $J_A = J$ and $J_B = -J$, where J is the Jarlskog invariant. We thus have

$$J_A + J_B = 0, \text{ for 3 quark generations.} \tag{28}$$

For 4 generations Eq. (28) does not hold and we have

$$J_A + J_B \approx -\mathrm{Im}(\mathrm{e}^{-i\delta_{2,4}} V_{21}^{(3)} V_{33}^{(3)} (V_{23}^{(3)} V_{11}^{(3)})^*) z_1 z_3. \tag{29}$$

10

Experimentally J_A is of the order λ^6 and J_B is rather poorly known, because it contains the CKM matrix elements that are known with large errors. The sum $J_A + J_B$ gives the information how the CP violation effects differ in the strange and bottom sectors. If $J_A + J_B = 0$, which holds exactly for 3 generations, then the CP violation parameters obtained from both sectors should be the same. If $J_A + J_B \sim \lambda^6$, then there is no cancellation between J_A and J_B and the CP violation in the strange and bottom sectors are not correlated. The estimated value of $J_A + J_B$ for 4 generations depends on the powers of the suppression factors through the sum $k_1 + k_3$ and we obtain the following dependence of $J_A + J_B$ on this sum[29]

$$
\begin{array}{c|c|l}
k_1 + k_3 & J_A + J_B & \\
\hline
3 & \sim \lambda^6 & \text{CP violation not correlated in the strange and} \\
& & \text{bottom sectors} \\
4 & \sim \lambda^7 & \text{CP violation weakly correlated in the strange} \\
& & \text{and bottom sectors (20\% difference)} \\
5 & \sim \lambda^8 & \text{CP violation strongly correlated in the strange} \\
& & \text{and bottom sectors (4\% difference)}
\end{array}
\tag{30}
$$

To conclude this section let us note that the presence of the 4-th generation in the CKM matrix $V^{(3)}$ might be detected through the violation of the unitarity of $V^{(3)}$. This can be done by experimental verification of the asymmetries of the CKM matrix

$$
|V_{12}|^2 - |V_{21}|^2 = |V_{23}|^2 - |V_{32}|^2 = |V_{31}|^2 - |V_{13}|^2
\tag{31}
$$

which hold *only* for 3 generations. Thus the experimental violation of Eq. (31) or observation that $J_A + J_B \neq 0$ would be an experimental proof of the presence of 4-th generation.

III.   REPHASING MONOMIALS OF THE CKM MATRIX

All observables of the CKM matrix are invariant under the rephasing of the quark fields. This invariance has important consequences on the properties of the CKM matrix and observables, like the reduction of the number of independent parameters of the CKM matrix. The rephasing invariant monomials built from the CKM matrix elements and its conjugates have been used in the discussion of various properties of the standard model related to the CKM matrix. The best known application of such a formalism is the Jarlskog's condition for

CP symmetry[30], and other applications also include the rephasing invariant parameterization of the CKM matrix and the discussion of the CP violation[6,8,11,12,31–34].

In this section we will present the systematic study of the most general rephasing invariant monomials that can be built out of the CKM matrix elements and its conjugates. These monomials can be considered as building blocks of general observables of the CKM matrix. Next we will show that such invariant monomials can be expressed as powers of a finite number of elementary rephasing monomials.

The discussion of the rephasing invariant monomials depends on the number of generations and we will discuss here in detail the case of 3 generations. The generalization to 4 or more generations is in most cases simple, but may be tedious.

Let us denote by $P(m,n)$ the most general monomial constructed from the CKM matrix elements and its conjugates:

$$P(m,n) = \Pi_{ij}(V_{ij})^{m_{ij}} \Pi_{kl}(V_{kl}^*)^{n_{kl}}. \tag{32}$$

Here m and n are 3×3 matrices with integer[35], non negative matrix elements and $[m]_{ij} = m_{ij}$ and $[n]_{ij} = n_{ij}$. The mapping between the monomial $P(m,n)$ and the matrices m and n is one to one[36].

The monomials $P(m,n)$ fulfill the following properties

$$P(m_1,n_1) \cdot P(m_2,n_2) = P(m_1 + m_2, n_1 + n_2), \tag{33a}$$

$$(P(m,n))^* = P(n,m). \tag{33b}$$

The monomial $P(m,n)$ in general is not rephasing invariant. Suppose that we make the following phase transformation of the CKM matrix

$$V_{\text{CKM}} \to \text{diag}(e^{i\phi_1}, 0, 0)V_{\text{CKM}}, \tag{34}$$

then the monomial $P(m,n)$ is transformed in the following way

$$P(m,n) \to e^{i\phi_1(m_{11}+m_{12}+m_{13}-n_{11}-n_{12}-n_{13})}P(m,n), \tag{35}$$

so we see that $P(m,n)$ is invariant under the transformation in Eq. (34) only if

$$m_{11} + m_{12} + m_{13} = n_{11} + n_{12} + n_{13}, \tag{36}$$

i.e., if the sum of the elements of the first row of the matrices m and n are equal. From this one obtains

Theorem 1. *The monomial $P(m, n)$ is rephasing invariant if the sums of the elements of the corresponding rows and columns of the matrices m and n are equal. It means that for the rephasing invariant monomial $P(m, n)$ the matrices m and n fulfill the following conditions*

$$\sum_{j=1}^{3} m_{ij} = \sum_{j=1}^{3} n_{ij}, \quad \sum_{j=1}^{3} m_{ji} = \sum_{j=1}^{3} n_{ji}, \quad i = 1, 2, 3. \tag{37}$$

We want to consider here the rephasing invariants that carry the information about the phases of the CKM matrix elements, e.g., the Jarlskog rephasing invariant. The squares of the CKM matrix elements $|V_{ij}|^2$ are rephasing invariant but they do not contain any phase information and if we multiply any rephasing invariant by $|V_{ij}|^{2k}$ then it does not change the phase information in any way. We therefore introduce the notion of the pure rephasing invariant

Definition 1. The rephasing invariant monomial of the CKM matrix which cannot be factored out into the product of the absolute values of the elements of the CKM matrix and other invariant is called the *pure rephasing invariant* (PRI).

Example 3. *Rephasing invariant and pure rephasing invariant (PRI)*
The rephasing invariant

$$V_{11}^2 V_{22} V_{11}^* V_{12}^* V_{21}^*$$

is not a *pure rephasing invariant*, because it is equal to

$$|V_{11}|^2 V_{11} V_{22} V_{12}^* V_{21}^*,$$

i.e., one can factor out the square $|V_{11}|^2$ out of it. On the other hand the remaining part $V_{11} V_{22} V_{12}^* V_{21}^*$ is the pure rephasing invariant.

The PRIs can be represented by two matrices m and n, as in Eq. (32), but they can also be represented by one 3×3 matrix p with the following properties:

1. The matrix elements of p are integers (positive, negative or 0).

2. The sum of the elements of p in each row and column is equal to 0.

3. A permutation of the rows and columns of the p matrix is reversible and the resulting matrix is also the p matrix of pure rephasing invariant.

The PRI, which we call $B(p)$, corresponding to the matrix p is constructed in the following way:

$$B(p) = \Pi_{p_{ij}>0}(V_{ij})^{p_{ij}} \cdot \Pi_{p_{kl}<0}(V_{kl}^*)^{-p_{kl}}. \tag{38}$$

It is easy to show that $B(p)$ constructed in such a way is rephasing invariant and that it cannot be factored out into the squares of the CKM matrix elements and other rephasing invariant, so it is indeed the PRI.

The one to one mapping between the p matrix and PRI $B(p)$ has the following additional properties

$$(p_1 + p_2) \rightarrow B(p_1 + p_2) = B(p_1) \cdot B(p_2), \quad n \cdot p \rightarrow (B(p))^n, \quad n \text{ integer},$$
$$\text{if } p \rightarrow B(p), \text{ then } (-p) \rightarrow (B(p))^*. \tag{39}$$

Example 4. *Analytic expression of PRI for a given matrix p*

Suppose that the matrix p is equal

$$p = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{pmatrix}$$

then the rephasing invariant defined by p is equal

$$V_{11}V_{13}^2V_{21}^2V_{22}V_{32}^2V_{33}(V_{12}^3V_{23}^3V_{31}^3)^*$$

and it fulfills all the properties of a PRI.

Let us introduce now the notion of the *fundamental rephasing invariant* (FRI).

Definition 2. The *fundamental rephasing invariant (FRI)* is such a pure rephasing invariant monomial that is the product of 4 or 6 CKM matrix elements and its complex conjugates.

It turns out that there are only 30 FRIs. 18 of them are products of 4 CKM matrix elements and their complex conjugates and 12 are the products of 6 CKM matrix elements and their complex conjugates. Their explicit form is the following

4-th order fundamental rephasing invariants $(J_1, J_2, \ldots, J_{18})$ [37]

$$J_1 = V_{11}V_{22}V_{12}^*V_{21}^*, \quad J_5 = V_{11}V_{33}V_{13}^*V_{31}^*,$$

$$J_2 = V_{11}V_{23}V_{13}^*V_{21}^*, \quad J_6 = V_{12}V_{33}V_{13}^*V_{32}^*,$$

$$J_3 = V_{12}V_{23}V_{13}^*V_{22}^*, \quad J_7 = V_{21}V_{32}V_{22}^*V_{31}^*,$$

$$J_4 = V_{11}V_{32}V_{12}^*V_{31}^*, \quad J_8 = V_{21}V_{33}V_{23}^*V_{31}^*,$$

$$J_9 = V_{22}V_{33}V_{23}^*V_{32}^*$$

$$J_{9+i} = (J_i)^*, \quad i = 1, \ldots, 9. \tag{40}$$

6-th order fundamental rephasing invariants $(I_1, I_2, \ldots, I_{12})$

$$I_1 = V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^*, \quad I_4 = V_{11}V_{23}V_{32}V_{13}^*V_{22}^*V_{31}^*,$$

$$I_2 = V_{11}V_{22}V_{33}V_{12}^*V_{23}^*V_{31}^*, \quad I_5 = V_{12}V_{23}V_{31}V_{13}^*V_{21}^*V_{32}^*,$$

$$I_3 = V_{11}V_{23}V_{32}V_{12}^*V_{21}^*V_{33}^*, \quad I_6 = V_{12}V_{21}V_{33}V_{13}^*V_{22}^*V_{31}^*$$

$$I_{6+i} = (I_i)^*, \quad i = 1, \ldots, 6. \tag{41}$$

For each FRI in Eqs. (40) and (41) there corresponds a p matrix, e.g.,

$$J_1 \to p_{J_1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad I_1 \to p_{I_1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{etc.} \tag{42}$$

All the matrices p_i corresponding to the invariants in Eqs. (40) and (41) can be obtained by the permutations of the rows and columns the p matrices of J_1 and I_1 that are given in Eq. (42). This means that an arbitrary permutation of the rows and columns of a p_J matrix maps it into another p_J matrix. The same applies to the p_I matrices. The operation of the permutation of the rows and columns of the p_J and p_I matrices is reversible.

We can now formulate the following

Theorem 2. *Any pure rephasing invariant can be expressed in a unique way as the product of positive powers of at most 4 fundamental rephasing invariants. Not more than one of these invariants can be from the set (41) and the remaining are from the set (40).*

The important condition in Theorem 2 is that the powers of the invariants are *positive*. The next important point is that in the decomposition there may be no more than 1 fundamental rephasing invariant of the 6-th order. Without these conditions the decomposition

of a pure rephasing invariant into the fundamental invariants is not unique. The proof of this theorem is given in the Appendix.

The inverse theorem is not true, the product of two or more FRIs is rephasing invariant, but it does not have to be the PRI.

From Theorem 2 follows

Theorem 3 (Main Theorem for the Rephasing Invariants). *Any rephasing invariant monomial of the CKM matrix for 3 generations is the product of no more than 5 factors: 4 fundamental rephasing invariants taken to positive powers and the product of the squares of the absolute values of the CKM matrix elements also taken to positive powers. Only one fundamental invariant is from the set* (41).

The Main Theorem has important consequences. From this theorem follows that the imaginary part of any rephasing invariant monomial is proportional to the Jarlskog invariant or equal to 0.

From the unitarity of the CKM matrix it follows that the 6-th order FRIs in Eq. (41) can be expressed by the 4-th order FRIs from Eq. (40) and the squares of the CKM matrix elements[38]. We have for example,

$$I_1 = V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^* = |V_{22}|^2 V_{12}V_{33}V_{13}^*V_{32}^* - |V_{13}|^2 V_{22}V_{33}V_{23}^*V_{32}^* = |V_{22}|^2 J_6 - |V_{13}|^2 J_9 \quad (43)$$

and there are analogous formulas for the remaining I_i's.

To conclude this section let us briefly discuss some properties of the rephasing invariants for 4 generations of quarks.

The notion of the fundamental rephasing invariant is generalized to contain no more than 8 CKM matrix elements and there are 3 classes of FRIs, with 4, 6 and 8 CKM matrix elements, respectively. The notion of the pure rephasing invariant remains the same and the Main Theorem is modified.

Let us also briefly discuss the unitarity properties of the FRIs with 4 CKM matrix elements. From simple considerations one can find out that there are 36 such invariants (and its conjugates), e.g.,

$$V_{11}V_{24}V_{14}^*V_{21}^*. \quad (44)$$

Unitarity gives 48 relations between them, e.g.,

$$|V_{11}|^2|V_{21}|^2 + V_{12}V_{21}V_{11}^*V_{22}^* + V_{13}V_{21}V_{11}^*V_{23}^* + V_{14}V_{21}V_{11}^*V_{24}^* = 0. \quad (45)$$

If we take the imaginary parts of all these unitarity relations, then we obtain 48 linear homogeneous equations for 36 variables. Not all these equations are linearly independent and eventually it turns out that only 9 of these equations are linearly independent. In Ref.[8] it has been shown that from unitarity of the CKM matrix one can obtain further linear relations between these 9 imaginary parts and only 3 imaginary parts are sufficient to express the remaining ones. The coefficients of the relations in the latter step depend on the real parts of the invariants and the squares of the CKM matrix elements, so the final expressions are complicated. We will address this problem and discuss the consequences of the unitarity of the CKM matrix for 4 generations elsewhere.

IV. CONCLUSIONS

We have discussed here two important topics concerning the CKM matrix: parameterizations and the rephasing invariants of the CKM matrix.

From the theoretical point of view all exact parameterizations of the CKM matrix are equivalent. From the practical point of view the situation is less obvious, because certain experimental facts can be presented in a more transparent way in one parameterization than in the other. The scheme of the construction of the parameterization of the CKM matrix presented in this paper allows to adjust properties of the parameterization according to the needs. Such an approach has not been discussed before and it can facilitate the discussion of the properties of the CKM in the Standard Model or its extensions.

The next topic discussed in the paper was the rephasing invariance of the CKM matrix and the properties of the rephasing invariant monomials, built out of the CKM matrix. The Jarlskog invariant and unitarity triangle angles are examples of such monomials. Let us note that general rephasing invariant monomials constructed from the CKM matrix elements appear at higher orders of the renormalization group equations for the CKM matrix elements. The systematic analysis of such equations based on the results obtained in this paper will be published elsewhere.

The most important result concerning the rephasing invariance is stated in Theorem 3 (Main Theorem) and it is mathematically a strong result. It tells that any rephasing invariant monomial of the CKM matrix can be expressed as the product of 5 factors which are functions of a finite, small number of the fundamental rephasing invariant monomials. The unitarity of

the CKM matrix allows to express the 6-th order rephasing invariant monomials by the 4-th monomials and the 4-th order monomials can be expressed by the squares of the absolute values of the CKM matrix element, but such an approach involves the subtractions of the terms that are very close (e.g., $(|V_{12}|^2 - |V_{12}|^2) \sim \lambda^3$), so it may lead to instability of the final result.

ACKNOWLEDGMENTS

APPENDIX: PROOF OF THEOREM 2

From Definition 1 and the discussion afterwords we know that there is one to one mapping between the *pure rephasing invariants* and the p matrices with the property that the sum of the elements of each row and column is equal to 0. To prove the theorem we will analyze these matrices and we will show that the matrix corresponding to a PRI can be decomposed in a unique way as a linear combination with positive coefficients of at most 4 p matrices corresponding to the fundamental rephasing invariants defined in Eqs. (40) and (41). Then using the property (39) of the p matrices one obtains the Main Theorem.

The elements of a general p matrix in (38) are positive or negative integers and zeros. In Table I we list all possible distributions of the number of the elements of the p matrix which are positive, negative or zero. The Type 3, 4, 6 and 7 contain two subtypes of distributions, which are related by complex conjugation and thus do not require separate proofs. As we know one can make permutations of rows and columns of the p matrix and this operation has an inverse. In such a way we can simplify the proof by organizing the elements of the p matrix in the standard form without losing the generality.

We will consider now each type of the p matrix listed in Table I.

Type 1. After a suitable permutation the p matrix of the Type 1 has the following standard

Type	1	2	3		4		5	6		7	
k_+	2	3	3	4	5	3	4	3	6	4	5
k_-	2	3	4	3	3	5	4	6	3	5	4
k_0	5	3	2	2	2	1	1	0	6	0	0

TABLE I. All possible distributions of the number of the elements of the 3×3 p matrix which are positive, negative or equal to 0. k_+ is the number of the elements that are positive, k_- is the number of the elements that are negative and k_0 is the number of the elements that are equal to 0.

form

$$p^1 = \begin{pmatrix} n & -n & 0 \\ -n & n & 0 \\ 0 & 0 & 0 \end{pmatrix} = n \times \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n > 0. \tag{A.1}$$

Then from Eqs. (39), (40) and (42) one sees, that p^1 matrix corresponds to the J_1 fundamental invariant taken to the power n, that is

$$p^1 \to B(p^1) = (J_1)^n = (V_{11}V_{22}V_{12}^*V_{21}^*)^n. \tag{A.2}$$

Type 2. Here we have two types of nonequivalent p matrices. The first one in the standard form reads

$$p^{2A} = \begin{pmatrix} n_1 & n_2 & -(n_1+n_2) \\ -n_1 & -n_2 & (n_1+n_2) \\ 0 & 0 & 0 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + n_2 \times \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad n_1, n_2 > 0 \tag{A.3}$$

and $B(p^{2A})$ has the representation

$$p^{2A} \to B(p^{2A}) = (J_2)^{n_1} \cdot (J_3)^{n_2}. \tag{A.4}$$

The second nonequivalent p matrix of the Type 2 has the form

$$p^{2B} = \begin{pmatrix} n & 0 & -n \\ -n & n & 0 \\ 0 & -n & n \end{pmatrix} = n \times \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad n > 0 \tag{A.5}$$

and $B(p^{2B})$ has the representation

$$p^{2B} \to B(p^{2B}) = (I_1)^n. \tag{A.6}$$

Type 3. The p^3 matrix in the standard form is equal

$$p^3 = \begin{pmatrix} n_1 + n_2 & 0 & -(n_1 + n_2) \\ -n_2 & n_2 & 0 \\ -n_1 & -n_2 & n_1 + n_2 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + n_2 \times \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$n_1, n_2 > 0 \quad \text{(A.7)}$$

and $B(p^3)$ has the representation

$$p^3 \rightarrow B(p^3) = (J_5)^{n_1} \cdot (I_1)^{n_2}. \tag{A.8}$$

Type 4. The p^4 matrix after a suitable permutation of rows and columns takes the following standard form

$$p^4 = \begin{pmatrix} n_1 & n_2 & -(n_1 + n_2) \\ n_3 & -(n_1 + n_2 + n_3) & (n_1 + n_2) \\ -(n_1 + n_3) & (n_1 + n_3) & 0 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$+ n_2 \times \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + n_3 \times \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad n_1, n_2, n_3 > 0 \quad \text{(A.9)}$$

and $B(p^4)$ has the representation

$$p^4 \rightarrow B(p^4) = (I_4)^{n_1} \cdot (J_3)^{n_2} \cdot (J_7)^{n_3}. \tag{A.10}$$

Type 5. The standard form of the matrix p^5 is equal

$$p^5 = \begin{pmatrix} n_1 + n_3 & n_2 & -(n_1 + n_2 + n_3) \\ -n_3 & n_3 & 0 \\ -n_1 & -(n_2 + n_3) & n_1 + n_2 + n_3 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$+ n_2 \times \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} + n_3 \times \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad n_1, n_2, n_3 > 0 \quad \text{(A.11)}$$

and $B(p^5)$ has the representation

$$p^5 \rightarrow B(p^5) = (J_5)^{n_1} \cdot (J_6)^{n_2} \cdot (I_1)^{n_3}. \tag{A.12}$$

Type 6. The standard form of the matrix p^6 is equal

$$
p^6 = \begin{pmatrix} n_1 + n_3 + n_4 & -(n_1 + n_4) & -n_3 \\ -n_1 & n_1 + n_2 + n_4 & -(n_2 + n_4) \\ -(n_3 + n_4) & -n_2 & n_2 + n_3 + n_4 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
+ n_2 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + n_3 \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + n_4 \times \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix},
$$

$$
n_1, n_2, n_3, n_4 > 0 \quad \text{(A.13)}
$$

and $B(p^6)$ has the representation

$$
p^6 \to B(p^6) = (J_1)^{n_1} \cdot (J_9)^{n_2} \cdot (J_5)^{n_3} \cdot (I_2)^{n_4}. \tag{A.14}
$$

Type 7. The standard form of the matrix p^7 reads

$$
p^7 = \begin{pmatrix} n_1 + n_4 & n_3 & -(n_1 + n_3 + n_4) \\ -n_4 & n_2 + n_4 & -n_2 \\ -n_1 & -(n_2 + n_3 + n_4) & n_1 + n_2 + n_3 + n_4 \end{pmatrix} = n_1 \times \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}
$$

$$
+ n_2 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + n_3 \times \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} + n_4 \times \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},
$$

$$
n_1, n_2, n_3, n_4 > 0 \quad \text{(A.15)}
$$

and $B(p^7)$ has the representation

$$
p^7 \to B(p^7) = (J_5)^{n_1} \cdot (J_9)^{n_2} \cdot (J_6)^{n_3} \cdot (I_1)^{n_4}. \tag{A.16}
$$

We have considered here all possible types of pure monomial rephasing invariants and thus Theorem 2 follows from explicit calculation. This completes the proof.

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[25]The squares of the absolute values of the CKM matrix elements $|V_{ij}|^2$ are clearly rephasing invariant observables, but we are focused on observables sensitive to the phases of the CKM matrix elements.

[26]Also the following formula holds

$$\text{Im}(V_{\alpha i}V_{\beta j}V_{\alpha j}^*V_{\beta i}^*) = J\sum_{\gamma,k}\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk} = J(3(\delta_{\alpha i}\delta_{\beta j} - \delta_{\alpha j}\delta_{\beta i}) + \delta_{\alpha i} + \delta_{\beta j} - \delta_{\alpha j} - \delta_{\beta i}).$$

.

[27]For example the parameter λ for the 2×2 rotation matrix

$$\begin{pmatrix} \sqrt{1 - \lambda^2}, & \lambda \\ -\lambda, & \sqrt{1 - \lambda^2} \end{pmatrix}$$

will be called the angle-like variable.

[28]If one uses unitarity of the 3×3 CKM matrix then one has $(|V_{12}| - |V_{21}|) \sim \lambda^5$, but the element $|V_{21}|$ is not measured with such a precision and experimentally we have $(|V_{12}| - |V_{21}|) \sim \lambda^3$ as in (23).

[29]Note that according to Eqs. (24) and (25) we have $k_1 + k_3 \geq 3$.

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[35] The condition that the elements of the matrices m and n are integers may be relaxed, but the CKM observables are monomials that contain only integer powers.

[36] For example, if the monomial $P(m, n)$ is equal to

$$P(m, n) = V_{11} V_{22} V_{12}^* V_{21}^*,$$

then the matrices m and n are equal to

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

.

[37] Here, in the analogy to the Jarlskog invariant we use the same letter J to denote the 4-th order monomials, but note that the Jarlskog invariant contains the imaginary part and for monomials we do not take the imaginary part.

[38] It should be emphasized that without the unitarity of the CKM matrix there are no simple relations between the invariants of the 4-th and 6-th order. Thus relation (43) is also a test of the unitarity of the CKM matrix.