HEPP-CPV-project

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I. INTRODUCTION

Short intro here

II. $\hat{P}$, $\hat{C}$ AND $\hat{C}\hat{P}\hat{T}$

III. CP VIOLATION

CP violation was first observed in the mixing of neutral K-mesons by Christenson, Cronin, Fitch and Turlay in 1964 [1]. They observed the $\hat{C}\hat{P} = -1$ state K_L^0 decaying to 2 pions, a state with $\hat{C}\hat{P} = 1$. Although the fraction of K_L^0 decays violating $\hat{C}\hat{P}$ in this way is tiny, the discovery was significant.

IV. $\hat{C}\hat{P}$V IN KAON SYSTEM

A. Neutral Kaon Mixing

As mentioned $\hat{C}\hat{P}$V was first observed in the neutral kaon system. Direct and indirect $\hat{C}\hat{P}$V have been observed but it is found that the process is entirely dominated by the indirect method [Zeng, need better reference]. Essential to these mechanisms is the mixin g between the neutral Kaon and its anti-particle, corresponding to the states $|K^0\rangle$ and $|\bar{K}^0\rangle$. These have quark compositions of $d\bar{s}$ and $s\bar{d}$, respectively.

In interactions involving the strong or EM force, the quantum number strangeness, which tells us the number of strange quarks in a particle, must be conserved. For the weak force it is found that, like parity, this symmetry is not conserved. Due to this many processes forbidden for the strong and EM interactions are allowed through the weak force. This violation is what makes mixing possible. Mixing is the decay of a particle into its anti-particle and can only take place when a particle is its own anti-particle, or if the particles differ by a quantum number which is not conserved by some interaction. This is the case in neutral Kaon mixing, also know as Kaon oscillations. The neutral Kaon and its anti-particle have opposite strangeness but can decay into each other through the strangeness violating weak force. See Fig.(add in feynman digram of M+S pg 289)

Analogous to the mixing of mass eigenstate quarks to different quark flavours, it is found that the neutral Kaon flavour eigenstates do not correspond to eigenstates of the $\hat{C}\hat{P}$ operator. To show this we first operate on the Kaon states with the $\hat{C}$ operator. We first assume that there is no $\hat{C}\hat{P}$V, then neglecting phase throughout we obtain:

$$\hat{C}\,|K^0(d\bar{s})\rangle = (1)(-1)\,|\bar{K}^0(s\bar{d})\rangle = -\,|\bar{K}^0(s\bar{d})\rangle$$
$$\hat{C}\,|\bar{K}^0(s\bar{d})\rangle = (1)(-1)\,|K^0(d\bar{s})\rangle = -\,|K^0(d\bar{s})\rangle$$

Where we have used the convention that $\hat{C}(q) = 1$ and $\hat{C}(\bar{q}) = -1$. Also, the action of the $\hat{P}$ is given by:

$$\hat{P}\,|K^0(d\bar{s})\rangle = \hat{P}(d)\hat{P}(\bar{s})(-1)^l\,|K^0(d\bar{s})\rangle = (1)(-1)(-1)^0\,|K^0(d\bar{s})\rangle = -\,|K^0(d\bar{s})\rangle$$
$$\hat{P}\,|\bar{K}^0(s\bar{d})\rangle = \hat{P}(s)\hat{P}(\bar{d})(-1)^l\,|\bar{K}^0(s\bar{d})\rangle = (1)(-1)(-1)^0\,|\bar{K}^0(s\bar{d})\rangle = -\,|\bar{K}^0(s\bar{d})\rangle$$

Where we have used the convention $\hat{P}(fermion) = 1$ and $\hat{P}(anti-fermion) = -1$ as well as $l = 0$ because the Kaon is the lowest energy combination of these quarks and itself has a J^P pf 0^-. Now we are in a position to determine the eigenstates of $\hat{C}\hat{P}$:

$$\hat{C}\hat{P}\,|K^0\rangle = |\bar{K}^0\rangle$$
$$\hat{C}\hat{P}\,|\bar{K}^0\rangle = |K^0\rangle$$

So we can see that any eigenfunction of the $\hat{C}\hat{P}$ operator will be a linear combination of the two Kaon states:

$$|K_1^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \tag{1}$$
$$|K_2^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \tag{2}$$

Where 1 and 2 are the usual labels given to these states. Now we investigate the action of $\hat{C}\hat{P}$ on these linear combinations

$$\hat{C}\hat{P}\,|K_1^0\rangle = \frac{1}{2}(\hat{C}\hat{P}\,|K^0\rangle + \hat{C}\hat{P}\,|\bar{K}^0\rangle) = \frac{1}{2}(|\bar{K}^0\rangle + |K^0\rangle) = |K_1^0\rangle$$
$$\hat{C}\hat{P}\,|K_2^0\rangle = \frac{1}{2}(\hat{C}\hat{P}\,|K^0\rangle - \hat{C}\hat{P}\,|\bar{K}^0\rangle) = \frac{1}{2}(|\bar{K}^0\rangle - |K^0\rangle) = -\,|K_2^0\rangle$$

In experiment, two Kaon states are observed, a short lived state denoted by $|K_S^0\rangle$ and a relatively long lived state, $|K_L^0\rangle$. The lifetimes of these particles are $8.954 \pm 0.004 \times 10^{11}$ s and $5.116 \pm 0.021 \times 10^8$ s [2]. We make the natural assumption that these are the $\hat{C}\hat{P}$ eigenstates just derived. We make the identifications $|K_S^0\rangle = |K_1^0\rangle$ and $|K_L^0\rangle = |K_2^0\rangle$ and see what this predicts. If $\hat{C}\hat{P}$ is conserved then all the decays of the $|K_S^0\rangle$ ($CP = 1$) state must be to final products with $CP = 1$, and similarly, the decays of $|K_L^0\rangle$ ($CP = -1$) must be to final products with $CP = -1$. The observed decays for these states are as follows [3, pg. 292]:

$$K_S^0 \to \pi^0\pi^0 (B = 0.31), \quad K_S^0 \to \pi^+\pi^- (B = 0.69)$$
$$K_L^0 \to \pi^0\pi^0\pi^0 (B = 0.20), \quad K_L^0 \to \pi^+\pi^-\pi^0 (B = 0.13)$$

The reason for the difference in lifetimes of these two Kaon states is that the mass of the K_L^0 is not much bigger than the mass of three pions, thus it is relatively unlikey for it to undergo decay, compared to the K_S^0 which must only create energy to make two pions. We now determine the CP of these final states. This is easy for the two pion final states. We find:

$$P(\pi^0\pi^0) = (-1)(-1)(-1)^{l=0} = +1 \qquad \Rightarrow P = 1 \tag{3}$$
$$C(\pi^0\pi^0) = 1 \qquad \Rightarrow C = 1 \tag{4}$$
$$P(\pi^+\pi^-) = (-1)(-1)(-1)^{l=0} = +1 \qquad \Rightarrow P = 1 \tag{5}$$
$$C(\pi^+\pi^-) = (-1)^{l=0} \qquad \Rightarrow C = 1 \tag{6}$$

Thus $\hat{C}\hat{P}\,|\pi\pi\rangle = 1$. Now for the three pion final state we must take account of the second orbital angular momentum introduced by the third pion. The general formula for such a system is $\hat{P}(ABC) = \hat{P}(A)\hat{P}(B)\hat{P}(C)(-1)^{\mathbf{L}_{AB}}(-1)^{\mathbf{L}_{(AB)C}}$ where $\mathbf{L}_{AB}$ is the orbital angular momentum of the first two pions and $\mathbf{L}_{(AB)C}$ is the orbital angular momentum of the third pion with respect to the mutual centre of mass of the first two pions. The J^P of the Kaon is 0^-, thus the overall orbital angular momentum must be zero: $\mathbf{L} = \mathbf{L}_{AB} + \mathbf{L}_{(AB)C} = 0$. As this is angular momentum addition and $\mathbf{L}$ can only take positive values, we conclude that $L_{AB} = L_{(AB)C}$ so $L_{AB} + L_{(AB)C} = 2L$, which is an even number:

$$P(\pi^0\pi^0\pi^0) = (-1)(-1)(-1)(-1)^{2L=even} = -1 \qquad \Rightarrow P = -1$$
$$C(\pi^0\pi^0\pi^0) = (1)(1)(1) = 1 \qquad \Rightarrow C = +1$$
$$CP(\pi^0\pi^0\pi^0) = -\,1$$

For the $|\pi^+\pi^-\pi^0\rangle$ final state the parity is also -1, but the charge conjugation picks up an extra factor of $(-1)^l$ as in Eqn.(6). So if we take the centre of mass of pions A and B to be the centre of mass between the π^+ and π^- we obtain:

$$\begin{aligned} C(\pi^+\pi^-\pi^0) =& C(\pi^0)(-1)^{L_{AB}} = 1 \qquad\qquad \Rightarrow C = +1 \\ CP(\pi^0\pi^0\pi^0) =& -1 \end{aligned}$$

Where $L_{AB} = 0$ is an experimentally determined quantity [Verify: "Measurement of the $1H(\gamma, \pi^0)$ cross section near threshold. II. Pion angular distributions" - J. C. Bergstrom, R. Igarashi, and J. M. Vogt , Phys. Rev. C 55, 20162023 (1997)]. Thus as long as K_L^0 deacy to final states with three pions or other $CP = -1$ states and K_S^0 only decay to two pion final states or other $CP = 1$ states, then CP is conserved.

This was thought to be the case until in 1964 when Christenson et al discovered the decay mode $K_L^0(CP = -1) \rightarrow \pi^+\pi^-(CP = 1)$ with a branching ratio of $(2.3 \pm 0.3) \times 10^{-3}$, thus discovering CP violation for the first time [1]. The experiment exploits the difference in lifetimes between K_S^0 and K_L^0. A 30GeV proton beam is incident on a metal target which creates a secondary beam of many different particles. The centre of mass energy for such an arrangement is 787 MeV, which is more than enough energy to produce a neutral Kaon having about a 497 MeV rest mass. The secondary beam is passed through a magnetic field to remove any charged particles and through a 4 cm thick block of lead to remove photons. At this point the beam contains both K_S^0 and K_L^0. The detecting aparatus is placed 18 m away from the metal target, so by the time the beam reaches it, all of the K_S^0 have decayed and only K_L^0 remain. The beam is further collimated and then undergoes collisions in a helium filled bag. Two arms containing a series of detectors are mounted symmetrically around the helium bag, so they both make the same angle with the horizontal. These arms consist of a spark chamber and magnet to determine the momentum and direction of an incident particle. Water Cherenkov and scintillation detectos act as a trigger by only recording events with two oppositely charged particles and a velocity of 0.75 c to eliminate background, see Fig.(1). The aim of the experiment is to measure the angular distribution of produced particles. The results of the experiment are shown in Fig.(2) where N is the number of counts and θ is the angle between the net momentum of the detected particles and the initial beam direction. These measurements were taken in various mass ranges, two are shown. If $K_L^0 \rightarrow \pi^+\pi^-$ is observed, the detected particles will have opposite signs, their invariant mass will match that of $K_L^0(497)$ and their net momentum will be in the same direction as the incident beam, hence the measured angle will be zero. We see from the results that a peak occurs at an angle of 0° in the correct mass range. This is clear evidence of the CP violating decay $K_L^0 \rightarrow \pi^+\pi^-$.

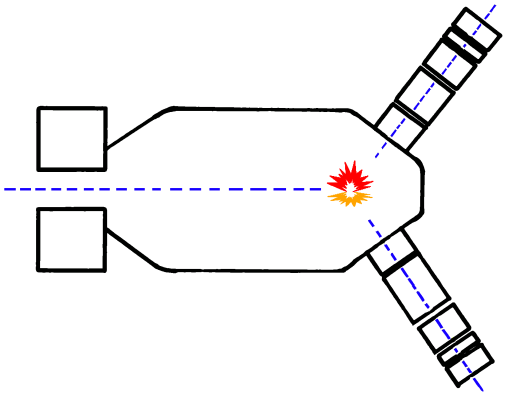

FIG. 1: *Apparatus used in the Christenson et al experiment [4]*

The results of the Christensen et al experiment implies, that the weak eigenstates $|K_S^0\rangle$ and $|K_L^0\rangle$ are not aligned with the true CP eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$. As in Eqn.(1) we write:

$$|K_S^0\rangle = a\,|K_1^0\rangle + b\,|K_2^0\rangle \tag{7}$$
$$|K_L^0\rangle = a\,|K_1^0\rangle - b\,|K_2^0\rangle \tag{8}$$

Where a and b are complex numbers. We can determine the degree to which the states are not aligned using the CP violation decay amplitudes and corresponding CP conserving amplitudes [5]:

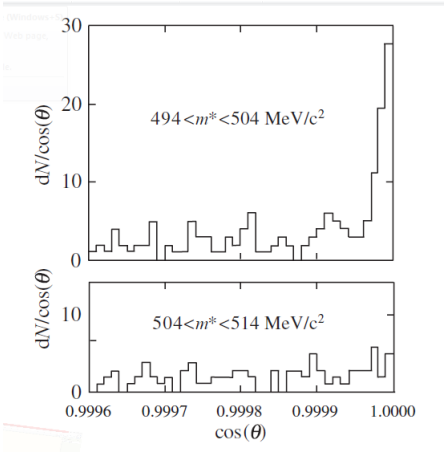

FIG. 2: *Results of the Christenson et al experiment [1]*

$$\eta_{+-} := \frac{A(K_L^0 \to \pi^+\pi^-)}{A(K_S^0 \to \pi^+\pi^-)} = \epsilon + \epsilon'$$
$$\eta_{00} := \frac{A(K_L^0 \to \pi^0\pi^0)}{A(K_S^0 \to \pi^0\pi^0)} = \epsilon - 2\epsilon'$$

The two complex parameters ϵ and ϵ' determine the amount of indirect and direct CPV, respectively. The indirect CPV is due to the CP conserving decay of the $K_1^0(CP = 1)$ component of the $K_L^0(CP = -1)$ to $CP = 1$ final states, this is possible because of Kaon oscillations. The direct CPV is due to the CP violating decay of the $K_2^0(CP = -1)$ component of the $K_L^0(CP = -1)$ to $CP = 1$ final states, this is possible due to interference between different decay methods with the same final state, as in Fig.(3).

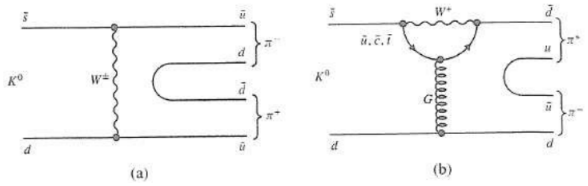

FIG. 3: *Two possible decay modes for $K^0 \to \pi^+\pi^-$. (a) Tree diagram for decay by exchanging W boson (b) Penguin diagram for decay via quark states [6]*

However it is found that the direct CPV contribution is much smaller in this case. The indirect CPV almost completely dominates as can be seen from the similarity of the experimental values for $|\eta_{+-}|$ and $|\eta_{00}|$ [2]:

$$|\eta_{00}| = 0.002220 \pm 0.000011 \qquad |\eta_{+-}| = 0.002232 \pm 0.000011$$

If these values were significantly different it would suggest the amount of direct CPV would be comparable to the amount of indirect CPV, this of course is not the case. An experimentally determined value which illustrates this is the real part of the ratio of ϵ' to *epsilon* [2]:

$$\mathbb{R}\left(\frac{\epsilon'}{\epsilon}\right) = \left(1 - \left|\frac{\eta_{00}}{\eta_{+-}}\right|\right)/3 = 0.00166 \pm 0.00023$$

We can also determine $|\epsilon|$ using:

$$|\epsilon| = (2|\eta_{+-}| + |\eta_{00}|)/3 = 0.002228 \pm 0.000011$$

If we ignore the direct CPV contributions we can write Eqn.(7) and (8) in terms of ϵ:

$$|K_L^0\rangle = \frac{1}{(1+|\epsilon|^2)^{1/2}}\left[\epsilon\,|K_1^0\rangle + |K_2^0\rangle \right. \tag{9}$$

$$|K_S^0\rangle = \frac{1}{(1+|\epsilon|^2)^{1/2}}\left[\,|K_1^0\rangle - \epsilon\,|K_2^0\rangle \right. \tag{10}$$

Thus we have a linear combination which shows the non-zero amplitude for a state with definite CP to osciallate and decay into final states with the opposite CP.

B. Semi-leptonic decays

Decays of neutral Kaons to products containing leptons can be used to verify Eqn.(9) and (10) as well as finding the asymmetry in the Kaon oscillation $K^0 \leftrightarrow \bar{K}^0$. First we must discuss the selection rules that play an important role in these decays.

The $\Delta S = \Delta Q$ selection rule is an empirical rule backed up by some theoretical approximations. This rules states that in decays involving strangeness(S) and leptons, the change in the charge(Q) of the hadrons must be the same as the change in strangeness which must have a value of ± 1. We first look at semi-leptonic decays of the charged Σ baryon. Two semi-leptonic decays of this baryon are:

$$\Sigma^-(dds) \to n(udd) + e^- + \bar{\nu}_e \tag{11}$$

$$\Sigma^+(uus) \to n(udd) + e^+ + \nu_e \tag{12}$$

The feynmann diagram for decay (11) can be drawn as in Fig.(pg232 M+S[use jaxo or others know how feynmann latex??]), while decay (12) requires a diagram such as Fig.(Two W decay). It is clear that the diagram for Σ^- is quite likely as it contains the Cabbibo favoured quark coupling V_{ud} while the digram for Σ^+ is very unlikely. In fact, the decay (12) must always contain at least two W bosons, as there are two quark flavour changes in the decay. For this reason it is highly suppressed and has a braching ratio of $< (5 \times 10^{-6})$, which is consitent with it not existing in nature [2]. In comparrison the decay (11) has a branching ratio of $(1.017 \pm 0.034) \times 10^{-3}$. As there is no selection rule forbidding this decay, the $\Delta S = \Delta Q$ rule was introduced to identify these types of decays. The change in strangeness and hadron charge for these decays can be found in Table.(I).

TABLE I: $\Delta S = \Delta Q$ *selection rule table for the decays (11) and (12)*

Shown decay of	ΔS	ΔQ	$\Delta S = \Delta Q$
Σ^-	+1	+1	yes
Σ^+	+1	-1	no

Where ΔA as the difference between the final and initial states of A such that $\Delta A = A_{final} - A_{initial}$. Also remember that the definition of strangeness assigns the strange quark a value of -1 and the anti-strange quark a value of $+1$. Thus we see that the decay (12) violates the $\Delta S = \Delta Q$ selection rule. Similarly, a decay with $\Delta S = \pm 2$ will contain two W bosons and as a result will be very suppressed. In conclusion, $\Delta S = \Delta Q = \pm 1$ for an allowed process.

This can now apply this rule to semi-leptonic decays of Kaons of the form $K \to \pi l \nu_l$. There are four Kaon decays that have this form:

$$K^0(d\bar{s}) \to \pi^- l^+ \nu_l \tag{13}$$

$$\bar{K}^0(s\bar{d}) \to \pi^+ l^- \bar{\nu}_l \tag{14}$$

$$K^0(d\bar{s}) \to \pi^+ l^- \bar{\nu}_l \tag{15}$$

$$\bar{K}^0(s\bar{d}) \to \pi^- l^+ \nu_l \tag{16}$$

A similar table as before can no be constructed. See Table.(II), where the number shown refers to the equations above

TABLE II: $\Delta S = \Delta Q$ *selection rule table for the decays (13) - (16)*

Shown decay of	ΔS	ΔQ	$\Delta S = \Delta Q$
(13)	-1	-1	yes
(14)	+1	+1	yes
(15)	-1	+1	no
(16)	+1	-1	no

V. CKM MECHANISM

VI. SPONTANEOUS CHARGE PARITY VIOLATION

A major motivation for creating models with new CPV mechanisms is to explain Baryon asymmetry. This is a huge research area, a simple search on arXiv.org reveals over 500 papers. The most prevalent theories are based around Super-Symmetry (SUSY) and Spontaneous Charge Parity Violation (SCPV).

Typically these models are created by looking for gauge symmetries which coincide with Langrangians similar to the Standard Model, but with extra terms that may account for the preference of antimatter to decay into matter.

SCPV is praised for its "naturalness" in comparison to regular CPV[7]. It supposes the possibility to have spontaneous CPV by the vacuum, as opposed to explicit CPV from the CKM matrices. That is, the vacuum is no longer invariant under CP. In a sense this is nice as it may not introduce as many new particles. Indeed, the Minimal SUSY Standard Model is not compatible with SCPV, and in general it is difficult to incorporate SCPV in SUSY models[7], but it has been done. For example in SUSY SO(10) [8].

There are two primary goals to this section. First, to elucidate the use of groups in physics, particularly particle physics. Second, to use this knowledge to understand various SCPV models beyond the insufficient description given in this introduction.

A. SCPV. Group Theory for Physics

Some physicists take pride in never having learned group theory and still understanding its applications. This is not an unreasonable point of view, an engineer can launch a rocket without knowing real analysis.

Undergraduate group theory modules quite commonly focus entirely on groups useful for pure mathematics. This is understandable as it is taught by the math department, but the picture of group theory young physicists may end up with is often quite different to its use in physics.

Part of the reasons groups can be so abstract is that they have very little structure. Even basic physics requires complicated structure. Simply changing coordinates in classical mechanics requires differential geometry, which requires analysis and topology. A **Group** is a set G with a function $\cdot : G^2 \to G$, having the following properties $\forall g, h, k \in G$

(GA1)	$g \cdot h \in G$	closure
(GA2)	$\exists e \in G,\ \ e \cdot g = g \cdot e = g$	identity
(GA3)	$\exists g^{-1} \in G,\ \ g^{-1} \cdot g = g \cdot g^{-1} = e$	inverse
(GA4)	$(g \cdot h) \cdot k = g \cdot (h \cdot k)$	associativity

Abstract definitions will be avoided in general. To understand this proceed with a useful example. Let $\mathrm{GL}(n, \mathbb{R})$ be the set of all $n \times n$ matrices with real coefficients and non-zero determinant. This forms a group under matrix multiplication.

(GA1) The product of two $n \times n$ matrices is an $n \times n$ matrix, so it is closed.

(GA2) The identity matrix I satisfies $IM = MI = M$.

(GA3) Non-zero determinant matrices are invertible $M^{-1}M = MM^{-1} = I$.

(GA4) As real multiplication and addition are associative $((1+2)+3 = 1+(2+3))$ Matrix multiplication inherits this property.

NOTE:

- The set of all real matrices would fail, as zero determinant matrices do not have inverses.
- Associativity is usually trivial by the definition and not checked.

The purpose of this proof was to give some familiar meaning to the abstraction. Detailed proofs shall be avoided in favour of intuition. Note that already a more powerful structure is present than a group, the field of real numbers, $(\mathbb{R}, +, \times)$. This is essentially just two groups glued together, addition and multiplication.

Similarly, every vector space is a group under vector addition. With this it could be said that all of physics uses groups, however a secondary school student does not use differential geometry with Newton's laws. At this stage groups are still useless to the practical physicist.

A **Transformation Group** is a more useful idea. Let X be a set, $\text{Transf}(X)$ is the set of all one-to-one functions from X to X. This is the set of all ways of rearranging X. In the context of a finite group $S = \{1, 2, ..., n\}$, $\text{Transf}(S)$ is just the set of all permutations.

If 3-dimensional space was modelled by $\mathbb{R}^3$ the transformation group would not be useful, it would contain unnatural discontinuous functions that do not relate to intuition about space. What would be useful is:

The transformations that preserve a property of a space form a group. It is known as the **invariance group** or the **symmetry group**.

This is an extremely general and potent idea. It is also not difficult to prove, so it shall be after the following motivation.

Model space again as $\mathbb{R}^3$, transformations should preserve distance. Where Euclidean distance is defined by

$$d(\mathbf{x}, \mathbf{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

The desired transformation R has form such that $d(\mathrm{R}\mathbf{x}, \mathrm{R}\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. From intuition there are some obvious transformations $\mathrm{R} = I$, the identity matrix, or $\mathrm{R}\mathbf{x} = \mathbf{x} + \mathbf{a}$ a translation in space

$$\begin{aligned} d(\mathrm{R}\mathbf{x}, \mathrm{R}\mathbf{y}) &= d(\mathbf{x} + \mathbf{a}, \mathbf{y} + \mathbf{a}) \\ &= \sqrt{\sum_{i=1}^{3} ((x_i + a_i) - (y_i + a_i))^2} \\ &= \sqrt{\sum_{i=1}^{3} (x_i + a_i - y_i - a_i)^2} \\ &= \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2} \\ &= d(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Now note

$$d(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$$

Where $\cdot$ is the regular dot product. Calling $\Delta\mathbf{x} := \mathbf{x} - \mathbf{y}$ and using serif font to signify matrix representation

$$\begin{aligned} \Delta\mathbf{x} \cdot \Delta\mathbf{x} &= \Delta\mathsf{x}^T \Delta\mathsf{x} \\ &= (\mathsf{x} - \mathsf{y})^T (\mathsf{x} - \mathsf{y}) \end{aligned}$$

In other words dot product in an orthonormal basis is a row vector times a column vector. Apply the transformation and assume it preserves distance

$$(\mathrm{R}\mathsf{x} - \mathrm{R}\mathsf{y})^T (\mathrm{R}\mathsf{x} - \mathrm{R}\mathsf{y}) = (\mathsf{x} - \mathsf{y})^T (\mathsf{x} - \mathsf{y})$$

The only way this can hold for any $\mathbf{x}$ and $\mathbf{y}$ is if R is affine, $\mathrm{R}\mathbf{x} = \mathrm{A}\mathbf{x} + \mathbf{a}$ where A is linear, i.e. a Matrix,

$$\begin{aligned} (\mathrm{R}\mathsf{x} - \mathrm{R}\mathsf{y})^T (\mathrm{R}\mathsf{x} - \mathrm{R}\mathsf{y}) &= (\mathsf{x} - \mathsf{y})^T (\mathsf{x} - \mathsf{y}) \\ (\mathrm{A}\mathsf{x} + \mathsf{a} - \mathrm{A}\mathsf{y} - \mathsf{a})^T (\mathrm{A}\mathsf{x} + \mathsf{a} - \mathrm{A}\mathsf{y} - \mathsf{a}) &= (\mathsf{x} - \mathsf{y})^T (\mathsf{x} - \mathsf{y}) \\ (\mathrm{A}(\mathsf{x} - \mathsf{y}))^T (\mathrm{A}(\mathsf{x} - \mathsf{y})) &= (\mathsf{x} - \mathsf{y})^T (\mathsf{x} - \mathsf{y}) \\ (\mathrm{A}\Delta\mathsf{x})^T (\mathrm{A}\Delta\mathsf{x}) &= \Delta\mathsf{x}^T \Delta\mathsf{x} \\ \Delta\mathsf{x}^T \mathrm{A}^T \mathrm{A} \Delta\mathsf{x} &= \Delta\mathsf{x}^T \Delta\mathsf{x} \end{aligned}$$

This can only hold if $\mathsf{A}\mathsf{A}^T = I$, so $\mathsf{A}^T = \mathsf{A}^{-1}$. Immediately taking the determinant of both sides gives

$$\begin{aligned} \det \mathsf{A}\mathsf{A}^T &= \det I \\ \det \mathsf{A} \det \mathsf{A}^T &= 1 \\ \det \mathsf{A} \det \mathsf{A} &= 1 \\ \det \mathsf{A}^2 &= 1 \\ \det \mathsf{A} &\in \{-1, 1\} \end{aligned}$$

This relates to intuition. The determinant of a matrix corresponds to how much it changes a volume. If the determinant of a matrix is ± 7, it will turn a 1 m^3 cube into a 7 m^3 parallelepiped, in order to preserve distance the determinant must have absolute value 1.

What sort of matrices have determinant -1 and $+1$? Consider the examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The left matrix represents a reflection in through the x-z plane. The right represents a rotation by $\pi/2$ in the x-y plane. It turns out that all matrices with $\mathsf{A}^T = \mathsf{A}^{-1}$ are either rotations or reflections, together they make the orthogonal group O(3). A transformation that is an orthogonal matrix plus a translation preserves distance, and the set of all such transformations form a group. Just the usual isometrics of Euclidean space.

Denote the set of rotations, which preserve length and handedness (reflection changes the right hand rule to the left hand rule), as SO(3), this is also a group.

So far this may seem over simplified but now it is already possible to define groups which preserve much more interesting and relevant properties

- Galilean Transformations. Preserve Newton's Laws.
- Unitary Operators. Preserve normalisation in quantum mechanics. Propagators should have this.
- Lorrentz Transformations. Preserve the light cone. Equivalently the Minkowski metric.

Intuitively, because these preserve something about a space, they form a group. This can greatly simplify some things, they are always invertible, a composition of two transformations still preserves.

There are many other aspects of group theory that can be useful here. Consider similar matrices, in the context of groups these are called conjugate elements. In the context of rotations they rotate by the same angle (but not the same direction), in the context of Lorrentz Boosts they have the same speed (but not the same direction).

If the reader is interested there is a lot of literature furthering this area, particularly physics-focused is [9]. In this paper the theory shall not be explained further, aside from the symmetry group theorem proved below.

THEOREM:

Consider a set X with a function $f : X \to Y$, the set of all transformations of X that preserve the function at x,

$$S_f := \{T \in \text{Transf}(X) : f(Tx) = f(x), \forall x \in X\},$$

form a group called the ***symmetry group*** *of f.*

For the example of distance in Euclidean space, $f(Tx) = f(x)$ corresponded to $d(R\mathbf{x}, R\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$.

PROOF:

(GA1) Let $S, T \in S_f$, then

$$f((ST)x = f(S(Tx)) = f(Sx) = f(x)$$

So ST also preserves f, $ST \in S_f$.

(GA2) The identity element of e of Transf(X) satisfies $f(ex) = f(x)$ by definition of the identity function, so $e \in S_f$.

(GA3) Let $S \in S_f$. Since it is a bijection, an inverse S^{-1} exists.

$$f(S^{-1}x) = f(SS^{-1}x) = f(ex) = f(x)$$

S was introduced as it satisfies $f(Sy) = f(y)$ and pick $y = S^{-1}x$. $S^{-1} \in S_f$.

(GA4) is inherited from Transf(X). □

NOTE

- The proof of the Euclidean transformations is incomplete but trivial to extend. Assume R is affine, show it preserves distance.
- Strictly speaking symmetry groups should be considered group actions. They act on a set X, as matrices act on column vectors.

B. SCPV. Useful Physical Groups

In the language of symmetry groups, useful physical transformations including the Gauge-symmetry of the standard model are investigated.

example 1.0 The Euclidean Isometries.

As was shown in the previous section any combination of rotation, reflection and translation preserve Euclidean distance. This can be extended to n-dimensional Euclidean space $\mathbb{E}^n$, here the group of orthogonal transformations and rotations are denoted respectively as O(n) and SO(n).

example 1.1 Galilean Transformations.

Solely preserving distance does not say much about physics. Consider Newton's laws in Cartesian coordinates in an inertial frame with column vector representation

$$m\ddot{\mathsf{x}} = \mathsf{F}$$

What sort of transformations $\mathsf{Tx} = \mathsf{Rx} + \mathsf{a}$ preserve this equation? Here R is a function of time, for example transforming to a rotating frame

$$\begin{aligned} m\frac{d^2}{dt^2}(\mathsf{Rx} + \mathsf{a}) &= \mathsf{RF} \\ m\frac{d}{dt}(\dot{\mathsf{R}}\mathsf{x} + \mathsf{R}\dot{\mathsf{x}}) + m\ddot{\mathsf{a}} &= \mathsf{RF} \\ m\frac{d}{dt}(\dot{\mathsf{R}}\mathsf{x} + \mathsf{R}\dot{\mathsf{x}}) + m\ddot{\mathsf{a}} &= \mathsf{RF} \\ m(\ddot{\mathsf{R}}\mathsf{x} + 2\dot{\mathsf{R}}\dot{\mathsf{x}} + \mathsf{R}\ddot{\mathsf{x}}) + m\ddot{\mathsf{a}} &= \mathsf{RF} \end{aligned}$$

To have the same form this requires, $\dot{R} = 0$ and $\ddot{a} = 0$. This is equivalent to saying R is a constant change of basis, such as a rotation, and $\mathsf{a} = \mathsf{v}t + \mathsf{b}$, a translation and a constant velocity. In other words

$$\mathsf{Tx} = \mathsf{Rx} + \mathsf{a}t + \mathsf{b}$$

A Galilean transformation is a symmetry group which preserves Newton's laws. It can be readily extended to also preserve distance and time intervals.

The goal is to explain the Quantum Field Theory models, Euclidean distance and Newton's Laws are not applicable. They act as an analogy for the Minkowski metric and Quantum Mechanics.

example 2.0 Lorrentz Transformations.

Suppose one wishes to preserve the Minkowski Metric, or equivalently, the light-cone. The transformations that do so are called Lorrentz transformations, essentially defined by

$$\mathsf{L}^T\eta\mathsf{L} = \eta$$

Extending the transformations to allow for a translation $P^\mu_\nu x^\nu = L^\mu_\nu x^\nu + a^\mu$, these are called Poincar transformations. The essence of Special Relativity is that the laws of physics are invariant under Poincar transformation.

example 2.1 Unitary Transformations.

In quantum mechanics the total probability of a time evolving state must always be 1, otherwise it is simply not a probability. In the context of Hilbert spaces (complex inner product spaces complete under the induced norm), the norm of a state gives its probability, so the norm must be preserved.

Recall in Euclidean space Orthogonal Transformations preserve the norm of a vector, as it preserves the dot product. The complex result is analogous but in the case of operators extends far more, for example

$$|\Psi(t)\rangle = e^{-it\hat{\mathcal{H}}/\hbar}\,|\Psi(0)\rangle$$

The formal solution to the Schödinger equation, the exponential operator on the right is unitary. In general this has quite complicated form.

If the Hilbert Space is spanned by a finite set $\{|\psi_1\rangle, ..., |\psi_n\rangle\}$ it is isometrically isomorphic to the inner product space $\mathbb{C}^n$. To preserve the norm here is similar to $\mathbb{R}^n$

$$|U\mathbf{u}|^2 = (\mathsf{U}\mathsf{u})^{*T}\mathsf{U}\mathsf{u} = \mathsf{u}^{*T}\mathsf{U}^{*T}\mathsf{U}\mathsf{u} = \mathsf{u}^{*T}\mathsf{u} = |\mathbf{u}|^2$$

This holds $\forall u \in \mathbb{C}^n$ if and only if $U^{*T}U = I$. This is similar to orthogonal matrices except there is a complex conjugate. The group of all such matrices is denoted $\mathrm{U}(n)$ and those which have positive determinant form a group $\mathrm{SU}(n)$.

example 3.0 Classical Electrodynamics.

$$\nabla\cdot\mathbf{E} = \tfrac{\rho}{\varepsilon} \qquad \nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{E} = -\tfrac{\partial\mathbf{B}}{\partial t} \qquad \tfrac{1}{\mu}\nabla\times\mathbf{B} = \mathbf{j} + \varepsilon\tfrac{\partial\mathbf{E}}{\partial t}$$

As $\nabla\cdot\mathbf{B} = 0$ a theorem shows $\exists\mathbf{A}, \mathbf{B} = \nabla\times\mathbf{A}$ this together with the third equation is

$$\nabla\times\mathbf{E} = -\partial_t(\nabla\times\mathbf{E})$$
$$\nabla\times(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t}) = 0$$

Which in turn gives $\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi$, where in the time independent case ϕ reduced to the electrostatic potential $\phi := -\int\mathbf{E}\cdot\mathrm{d}\mathbf{x}$

Consider the following transformation

$$\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{A}+\nabla\lambda\\ \phi - \partial_t\lambda\end{pmatrix}$$

These transformed potentials are also valid

$$\nabla\times(\mathbf{A}+\nabla\lambda) = \nabla\times\mathbf{A} + \nabla\times\nabla\cdot\lambda = \mathbf{B} + 0$$
$$-\partial_t(\mathbf{A}+\nabla\lambda) - \nabla(\phi - \partial_t\lambda) = -\partial_t(\mathbf{A}) - \nabla\phi = \mathbf{E}$$

As these equations are preserved the gauge transformations form a symmetry group. Just as preserving Newton's laws formed a symmetry group. This is an example of a **Gauge Transformation**.

More generally a gauge transformation is defined as a *local* transformation that preserves the Lagrangian of a theory. What local means exactly is yet to be defined.

C. SCPV. Gauge Theory

The reader may be having a hard time seeing how this relates to particle physics. With the benefit of the language being developed some light may be shed on what the remainder of this section will contain.

Most of the groups discussed have their place in modern particle physics. Lorrentz invariance and unitarity of symmetry operators obviously extend from special relativity and quantum mechanics. That is not all, the gauge-symmetry group of the standard model is $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Many of the popular extensions to the standard Model are also defined by gauge symmetries such as SU(5), SO(10) and O(16) [10].

It is possible to state group conditions used for either (a) deducing that a theory can allow SCPV or (b) creating theories which allow for it[11]. Explaining how to use this with the mathematics being developed is the primary goal of the section. Although, the hierarchy of knowledge required for theoretical physics is unfortunately large. Despite that the material has attempted to have been presented in a condensed and tangible manner, inevitably some details will be avoided.

In Quantum Mechanics overall phase is unimportant, that is $|\psi\rangle \sim |\psi\rangle\, e^{i\xi}$ where $\xi \in [0, 2\pi)$. When making any physical measurement the phase cancels out in the complex inner product. It forms a *global* gauge Symmetry. Phase is still important in Quantum mechanics in the form of phase difference. Consider two energy eigenstates with $a, b \in \mathbb{R}$

$$\begin{aligned} |\psi\rangle &= a\,|E_1\rangle\, e^{-iE_1/\hbar} + b\,|E_2\rangle\, e^{-iE_2/\hbar} \\ \rightarrow |\psi\rangle\, e^{iE_1/\hbar} &= a\,|E_1\rangle + b\,|E_2\rangle\, e^{i(E_1-E_2)/\hbar} \\ \langle \mathbf{x}\rangle &= a^2\,\langle E_1|\mathbf{x}|E_1\rangle + 2ab\,\langle E_1|\mathbf{x}|E_2\rangle \cos\theta + b^2\,\langle E_2|\mathbf{x}|E_2\rangle \end{aligned}$$

So here the phase *difference* $\theta := (E_1 - E_2)/\hbar$ appears explicitly in a measurable quantity. Consider U(1), a "matrix" that acts on one-dimensional complex vectors, i.e., complex numbers. In this case then, both the "matrix" and "vector" can both be represented as complex numbers

$$|uz|^2 = (uz)^* uz = u^* z^* uz = |z^2||u|^2 = |z|^2$$

This holds for all z if and only if u has length 1, so $u = e^{i\xi}$ where $\xi \in [0, 2\pi)$. So it can be said that the global phase symmetry of Quantum Mechanics can be represented as U(1).

Note that SU(1) = U(1) in this case, as the norm of a single complex number must be positive. Likewise as U(1) represents a rotation in the complex plane it must have the same group structure as SO(2). In mathematical language the groups are isomorphic, written as U(1) $\cong$ SO(2).

Thus it is clearer to say that the global phase invariance group of quantum mechanics is isomorphic to U(1). Frequently it is the case that there is a deeper physical meaning behind symmetries. Merely stating a theory is invariant under U(1) is not sufficient as one must define how U(1) acts. For example, it can act globally or locally.

Consider Quantum Electrodynamics (QED) the Lagrangian of the theory is invariant under **local gauge symmetry** of U(1).

$$\psi(x^1, x^2, x^3, x^4) \mapsto e^{i\xi(x^1,x^2,x^3,x^4)}\psi(x^1, x^2, x^3, x^4)$$

Where in cartesian coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, ct)$. In shorthand

$$\psi(x) \mapsto \tilde{\psi}(x) = e^{i\xi(x)}\psi(x)$$

Where x:=(x,y,z,ct) This inherently insists that phase is irrelevant at any point. It was noted that more structure than merely a group is being used concerning physical groups. Fields, vector spaces, and vector calculus to name a few. Primarily calculus, a fundamentally geometric structure, appears everywhere in physics. The real scalar function $\xi(x)$ must be differentiable.

This differentiability calls for the structure of a **Lie Group** (acting on a set). A Lie group is simply a group where the group operation and taking inverses are differentiable. For a simple example consider the group of addition on $\mathbb{R}$.

$$\cdot\,(x, y) = x + y \qquad\qquad \mathrm{inv}(x) = -x$$

Both of these functions are differentiable in their arguments so $(\mathbb{R}, +)$ forms a Lie Group. Likewise other familiar groups SU(n), O(n) and GL(n,) all form Lie groups.

To define calculus on non-Euclidean spaces the structure of differential manifolds is required. The details of which will have to be omitted but some intuitive and useful properties are worth discussing. The dimension of a differential manifold is related to the intuitive understanding of spatial dimension. The surface of a sphere is a two-dimensional space and as expected forms a two-dimensional manifold. This means, for the most part, the sphere can be parametrised by two numbers (θ, ϕ).

O(n), U(n) and SU(n) have manifold dimension $\frac{1}{2}n(n-1)$, n^2 and $n^2 - 1$ respectively. So these objects are no longer purely to be considered as groups but also as abstract smooth geometric spaces.

Returning the example of QED's local gauge symmetry and comparing with the global gauge symmetry of QM

$$\begin{aligned} |\psi\rangle &\mapsto |\tilde{\phi}\rangle = e^{i\xi}\,|\psi(x)\rangle \\ \psi(x) &\mapsto \tilde{\psi}(x) = e^{i\xi(x)}\psi(x) \end{aligned}$$

Both symmetries are isomorphic to U(1) but in very different contexts. In QM it may be regarded simply as a group, but in QED the differentiability effects the Lagrangian non-trivially.

The local gauge symmetry of the strong force is SU(3) and for the weak force it is SU(2). Like $U(1)$ was represented by one parameter which was a function of space-time position $\xi(x)$, SU(2) and SU(3) have $2^2 - 1 = 3$ and $3^3 - 1 = 8$ respectively. Notice peculiarly these these are the amount of force carriers in each theory.

What do these symmetries mean physically? According to [12], they are like the the potentials $\mathbf{A}, \phi$ for classical electromagnetism; merely a tool to make the mathematics easier. In the same way as one could use $\mathbf{E}$ and $\mathbf{B}$ without the degeneracy of $\mathbf{A}$ and ϕ, in principle one could formulate QED completely independent of phase, even without complex numbers. However the difference between usability would be extreme.

A full treatment of local gauge theory for physics can be found here [12] and more on Lie-algebras and differential manifolds related to physics can be found at [9].

The gauge symmetries of the standard model are written as the product group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. A product group is formed in the same way $\mathbb{R}^2$ with vector addition is a product group of $\mathbb{R} \times \mathbb{R}$. This can also be considered a project manifold and indeed it tells us the number of force carriers there are in each theory. Knowing the dimension of the SUSY theory SU(5) as $5^2 - 1 = 24$ then tells us the number of force carriers in this theory.

One must call it at a day at some point, actually dealing with Lagrangians is a much more time consuming and worth-while task, but the language developed shall suffice to investigate SCPV models.

D. SCPV. Group Theoretic Conditions.

Appendix A: Appendix

Difficult calculations in here.

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