

HEPP-CPV-project

John Ronayne (), Kevin Maguire (10318135), Sinead Hales(), Dudley Grant (10275291)
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I. INTRODUCTION

Short intro here

II. \hat{P} , \hat{C} AND $\hat{C}\hat{P}\hat{T}$

III. CP VIOLATION

CP violation was first observed in the mixing of neutral K-mesons by Christenson, Cronin, Fitch and Turlay in 1964 [1]. They observed the $\hat{C}\hat{P} = -1$ state K_L^0 decaying to 2 pions, a state with $\hat{C}\hat{P} = 1$. Although the fraction of K_L^0 decays violating $\hat{C}\hat{P}$ in this way is tiny, the discovery was significant.

IV. $\hat{C}\hat{P}V$ IN KAON SYSTEM

A. Neutral Kaon Mixing

As mentioned $\hat{C}\hat{P}V$ was first observed in the neutral kaon system. Direct and indirect $\hat{C}\hat{P}V$ have been observed but it is found that the process is entirely dominated by the indirect method [Zeng, need better reference]. Essential to these mechanisms is the mixing between the neutral Kaon and its anti-particle, corresponding to the states $|K^0\rangle$ and $|\bar{K}^0\rangle$. These have quark compositions of $d\bar{s}$ and $s\bar{d}$, respectively.

In interactions involving the strong or EM force, the quantum number strangeness, which tells us the number of strange quarks in a particle, must be conserved. For the weak force it is found that, like parity, this symmetry is not conserved. Due to this many processes forbidden for the strong and EM interactions are allowed through the weak force. This violation is what makes mixing possible. Mixing is the decay of a particle into its anti-particle and can only take place when a particle is its own anti-particle, or if the particles differ by a quantum number which is not conserved by some interaction. This is the case in neutral Kaon mixing, also known as Kaon oscillations. The neutral Kaon and its anti-particle have opposite strangeness but can decay into each other through the strangeness violating weak force. See Fig.(add in feynman diagram of M+S pg 289)

Analogous to the mixing of mass eigenstate quarks to different quark flavours, it is found that the neutral Kaon flavour eigenstates do not correspond to eigenstates of the $\hat{C}\hat{P}$ operator. To show this we first operate on the Kaon states with the \hat{C} operator. We first assume that there is no $\hat{C}\hat{P}V$, then neglecting phase throughout we obtain:

$$\begin{aligned}\hat{C}|K^0(d\bar{s})\rangle &= (1)(-1)|\bar{K}^0(s\bar{d})\rangle = -|\bar{K}^0(s\bar{d})\rangle \\ \hat{C}|\bar{K}^0(s\bar{d})\rangle &= (1)(-1)|K^0(d\bar{s})\rangle = -|K^0(d\bar{s})\rangle\end{aligned}$$

Where we have used the convention that $\hat{C}(q) = 1$ and $\hat{C}(\bar{q}) = -1$. Also, the action of the \hat{P} is given by:

$$\begin{aligned}\hat{P}|K^0(d\bar{s})\rangle &= \hat{P}(d)\hat{P}(\bar{s})(-1)^l|K^0(d\bar{s})\rangle = (1)(-1)(-1)^0|K^0(d\bar{s})\rangle = -|K^0(d\bar{s})\rangle \\ \hat{P}|\bar{K}^0(s\bar{d})\rangle &= \hat{P}(s)\hat{P}(\bar{d})(-1)^l|\bar{K}^0(s\bar{d})\rangle = (1)(-1)(-1)^0|\bar{K}^0(s\bar{d})\rangle = -|\bar{K}^0(s\bar{d})\rangle\end{aligned}$$

Where we have used the convention $\hat{P}(fermion) = 1$ and $\hat{P}(anti-fermion) = -1$ as well as $l = 0$ because the Kaon is the lowest energy combination of these quarks and itself has a J^P of 0^- . Now we are in a position to determine the eigenstates of $\hat{C}\hat{P}$:

$$\begin{aligned}\hat{C}\hat{P}|K^0\rangle &= |\bar{K}^0\rangle \\ \hat{C}\hat{P}|\bar{K}^0\rangle &= |K^0\rangle\end{aligned}$$

So we can see that any eigenfunction of the $\hat{C}\hat{P}$ operator will be a linear combination of the two Kaon states:

$$|K_1^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \quad (1)$$

$$|K_2^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \quad (2)$$

Where 1 and 2 are the usual labels given to these states. Now we investigate the action of $\hat{C}\hat{P}$ on these linear combinations

$$\begin{aligned}\hat{C}\hat{P}|K_1^0\rangle &= \frac{1}{2}(\hat{C}\hat{P}|K^0\rangle + \hat{C}\hat{P}|\bar{K}^0\rangle) = \frac{1}{2}(|\bar{K}^0\rangle + |K^0\rangle) = |K_1^0\rangle \\ \hat{C}\hat{P}|K_2^0\rangle &= \frac{1}{2}(\hat{C}\hat{P}|K^0\rangle - \hat{C}\hat{P}|\bar{K}^0\rangle) = \frac{1}{2}(|\bar{K}^0\rangle - |K^0\rangle) = -|K_2^0\rangle\end{aligned}$$

In experiment, two Kaon states are observed, a short lived state denoted by $|K_S^0\rangle$ and a relatively long lived state, $|K_L^0\rangle$. The lifetimes of these particles are $8.954 \pm 0.004 \times 10^{11}$ s and $5.116 \pm 0.021 \times 10^8$ s [2]. We make the natural assumption that these are the $\hat{C}\hat{P}$ eigenstates just derived. We make the identifications $|K_S^0\rangle = |K_1^0\rangle$ and $|K_L^0\rangle = |K_2^0\rangle$ and see what this predicts. If $\hat{C}\hat{P}$ is conserved then all the decays of the $|K_S^0\rangle$ ($CP = 1$) state must be to final products with $CP = 1$, and similarly, the decays of $|K_L^0\rangle$ ($CP = -1$) must be to final products with $CP = -1$. The observed decays for these states are as follows [3, pg. 292]:

$$\begin{aligned}K_S^0 &\rightarrow \pi^0\pi^0 (B = 0.31), \quad K_S^0 \rightarrow \pi^+\pi^- (B = 0.69) \\ K_L^0 &\rightarrow \pi^0\pi^0\pi^0 (B = 0.20), \quad K_L^0 \rightarrow \pi^+\pi^-\pi^0 (B = 0.13)\end{aligned}$$

The reason for the difference in lifetimes of these two Kaon states is that the mass of the K_L^0 is not much bigger than the mass of three pions, thus it is relatively unlikely for it to undergo decay, compared to the K_S^0 which must only create energy to make two pions. We now determine the CP of these final states. This is easy for the two pion final states. We find:

$$P(\pi^0\pi^0) = (-1)(-1)(-1)^{l=0} = +1 \quad \Rightarrow P = 1 \quad (3)$$

$$C(\pi^0\pi^0) = 1 \quad \Rightarrow C = 1 \quad (4)$$

$$P(\pi^+\pi^-) = (-1)(-1)(-1)^{l=0} = +1 \quad \Rightarrow P = 1 \quad (5)$$

$$C(\pi^+\pi^-) = (-1)^{l=0} \quad \Rightarrow C = 1 \quad (6)$$

Thus $\hat{C}\hat{P}|\pi\pi\rangle = 1$. Now for the three pion final state we must take account of the second orbital angular momentum introduced by the third pion. The general formula for such a system is $\hat{P}(ABC) = \hat{P}(A)\hat{P}(B)\hat{P}(C)(-1)^{\mathbf{L}_{AB}}(-1)^{\mathbf{L}_{(AB)C}}$ where \mathbf{L}_{AB} is the orbital angular momentum of the first two pions and $\mathbf{L}_{(AB)C}$ is the orbital angular momentum of the third pion with respect to the mutual centre of mass of the first two pions. The J^P of the Kaon is 0^- , thus the overall orbital angular momentum must be zero: $\mathbf{L} = \mathbf{L}_{AB} + \mathbf{L}_{(AB)C} = 0$. As this is angular momentum addition and \mathbf{L} can only take positive values, we conclude that $L_{AB} = L_{(AB)C}$ so $L_{AB} + L_{(AB)C} = 2L$, which is an even number:

$$P(\pi^0\pi^0\pi^0) = (-1)(-1)(-1)(-1)^{2L=even} = -1 \quad \Rightarrow P = -1$$

$$C(\pi^0\pi^0\pi^0) = (1)(1)(1) = 1 \quad \Rightarrow C = +1$$

$$CP(\pi^0\pi^0\pi^0) = -1$$

For the $|\pi^+\pi^-\pi^0\rangle$ final state the parity is also -1, but the charge conjugation picks up an extra factor of $(-1)^l$ as in Eqn.(6). So if we take the centre of mass of pions A and B to be the centre of mass between the π^+ and π^- we obtain:

$$\begin{aligned} C(\pi^+\pi^-\pi^0) &= C(\pi^0)(-1)^{L_{AB}} = 1 & \Rightarrow C = +1 \\ CP(\pi^0\pi^0\pi^0) &= -1 \end{aligned}$$

Where $L_{AB} = 0$ is an experimentally determined quantity [Verify: “Measurement of the $1H(\gamma, \pi^0)$ cross section near threshold. II. Pion angular distributions” - J. C. Bergstrom, R. Igarashi, and J. M. Vogt, Phys. Rev. C 55, 20162023 (1997)]. Thus as long as K_L^0 decay to final states with three pions or other $CP = -1$ states and K_S^0 only decay to two pion final states or other $CP = 1$ states, then CP is conserved.

This was thought to be the case until in 1964 when Christenson et al discovered the decay mode $K_L^0(CP = -1) \rightarrow \pi^+\pi^-(CP = 1)$ with a branching ratio of $(2.3 \pm 0.3) \times 10^{-3}$, thus discovering CP violation for the first time [1]. The experiment exploits the difference in lifetimes between K_S^0 and K_L^0 . A 30GeV proton beam is incident on a metal target which creates a secondary beam of many different particles. The centre of mass energy for such an arrangement is 787 MeV, which is more than enough energy to produce a neutral Kaon having about a 497 MeV rest mass. The secondary beam is passed through a magnetic field to remove any charged particles and through a 4 cm thick block of lead to remove photons. At this point the beam contains both K_S^0 and K_L^0 . The detecting apparatus is placed 18 m away from the metal target, so by the time the beam reaches it, all of the K_S^0 have decayed and only K_L^0 remain. The beam is further collimated and then undergoes collisions in a helium filled bag. Two arms containing a series of detectors are mounted symmetrically around the helium bag, so they both make the same angle with the horizontal. These arms consist of a spark chamber and magnet to determine the momentum and direction of an incident particle. Water Cherenkov and scintillation detectors act as a trigger by only recording events with two oppositely charged particles and a velocity of 0.75 c to eliminate background, see Fig.(1). The aim of the experiment is to measure the angular distribution of produced particles. The results of the experiment are shown in Fig.(2) where N is the number of counts and θ is the angle between the net momentum of the detected particles and the initial beam direction. These measurements were taken in various mass ranges, two are shown. If $K_L^0 \rightarrow \pi^+\pi^-$ is observed, the detected particles will have opposite signs, their invariant mass will match that of K_L^0 (497) and their net momentum will be in the same direction as the incident beam, hence the measured angle will be zero. We see from the results that a peak occurs at an angle of 0° in the correct mass range. This is clear evidence of the CP violating decay $K_L^0 \rightarrow \pi^+\pi^-$.

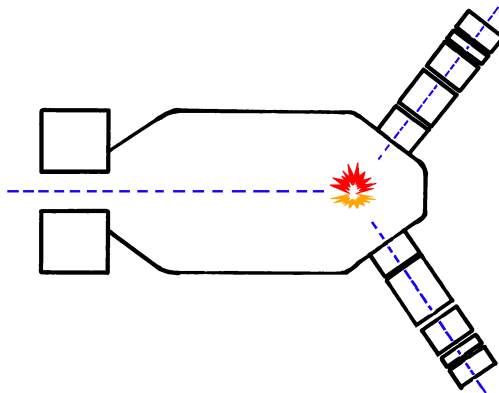


FIG. 1: Apparatus used in the Christenson et al experiment [4]

The results of the Christensen et al experiment implies, that the weak eigenstates $|K_S^0\rangle$ and $|K_L^0\rangle$ are not aligned with the true CP eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$. As in Eqn.(??) we write:

$$|K_S^0\rangle = a |K_1^0\rangle + b |K_2^0\rangle \quad (7)$$

$$|K_L^0\rangle = a |K_1^0\rangle - b |K_2^0\rangle \quad (8)$$

Where a and b are complex numbers. We can determine the degree to which the states are not aligned using the CP violation decay amplitudes and corresponding CP conserving amplitudes [5]:

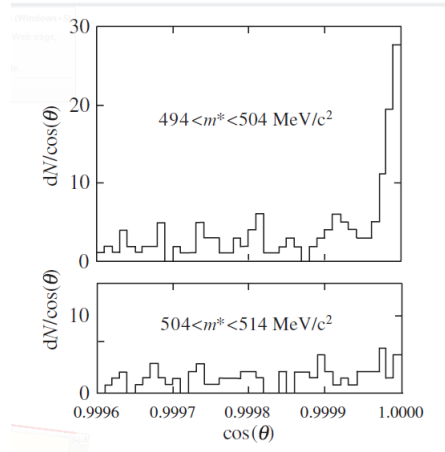


FIG. 2: Results of the Christenson et al experiment [1]

$$\eta_{+-} := \frac{A(K_L^0 \rightarrow \pi^+\pi^-)}{A(K_S^0 \rightarrow \pi^+\pi^-)} = \epsilon + \epsilon'$$

$$\eta_{00} := \frac{A(K_L^0 \rightarrow \pi^0\pi^0)}{A(K_S^0 \rightarrow \pi^0\pi^0)} = \epsilon - 2\epsilon'$$

The two complex parameters ϵ and ϵ' determine the amount of indirect and direct CPV , respectively. The indirect CPV is due to the CP conserving decay of the $K_1^0(CP = 1)$ component of the $K_L^0(CP = -1)$ to $CP = 1$ final states, this is possible because of Kaon oscillations. The direct CPV is due to the CP violating decay of the $K_2^0(CP = -1)$ component of the $K_L^0(CP = -1)$ to $CP = 1$ final states, this is possible due to interference between different decay methods with the same final state, as in Fig.(3).

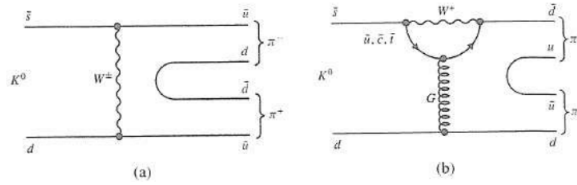


FIG. 3: Two possible decay modes for $K^0 \rightarrow \pi^+\pi^-$. (a) Tree diagram for decay by exchanging W boson (b) Penguin diagram for decay via quark states [6]

However it is found that the direct CPV contribution is much smaller in this case. The indirect CPV almost completely dominates as can be seen from the similarity of the experimental values for $|\eta_{+-}|$ and $|\eta_{00}|$ [2]:

$$|\eta_{00}| = 0.002220 \pm 0.000011$$

$$|\eta_{+-}| = 0.002232 \pm 0.000011$$

If these values were significantly different it would suggest the amount of direct CPV would be comparable to the amount of indirect CPV , this of course is not the case. An experimentally determined value which illustrates this is the real part of the ratio of ϵ' to ϵ [2]:

$$\Re\left(\frac{\epsilon'}{\epsilon}\right) = (1 - |\frac{\eta_{00}}{\eta_{+-}}|)/3 = 0.00166 \pm 0.00023$$

We can also determine $|\epsilon|$ using:

$$|\epsilon| = (2|\eta_{+-}| + |\eta_{00}|)/3 = 0.002228 \pm 0.000011$$

If we ignore the direct CPV contributions we can write Eqn.(7) and (8) in terms of ϵ :

$$|K_L^0\rangle = \frac{1}{(1 + |\epsilon|^2)^{1/2}} [\epsilon |K_1^0\rangle + |K_2^0\rangle] \quad (9)$$

$$|K_S^0\rangle = \frac{1}{(1 + |\epsilon|^2)^{1/2}} [|K_1^0\rangle - \epsilon |K_2^0\rangle] \quad (10)$$

Thus we have a linear combination which shows the non-zero amplitude for a state with definite CP to oscillate and decay into final states with the opposite CP .

B. Semi-leptonic decays

Decays of neutral Kaons to products containing leptons can be used to verify Eqn.(9) and (10) as well as finding the asymmetry in the Kaon oscillation $K^0 \leftrightarrow \bar{K}^0$. First we must discuss the selection rules that play an important role in these decays.

The $\Delta S = \Delta Q$ selection rule is an empirical rule backed up by some theoretical approximations. This rule states that in decays involving strangeness(S) and leptons, the change in the charge(Q) of the hadrons must be the same as the change in strangeness which must have a value of ± 1 . We first look at semi-leptonic decays of the charged Σ baryon. Two semi-leptonic decays of this baryon are:

$$\Sigma^-(dds) \rightarrow n(udd) + e^- + \bar{\nu}_e \quad (11)$$

$$\Sigma^+(uus) \rightarrow n(udd) + e^+ + \nu_e \quad (12)$$

The feynmann diagram for decay (11) can be drawn as in Fig.(pg232 M+S[use jaxo or others know how feynmann latex??]), while decay (12) requires a diagram such as Fig.(Two W decay). It is clear that the diagram for Σ^- is quite likely as it contains the Cabbibo favoured quark coupling V_{ud} while the digram for Σ^+ is very unlikely. In fact, the decay (12) must always contain at least two W bosons, as there are two quark flavour changes in the decay. For this reason it is highly suppressed and has a braching ratio of $< (5 \times 10^{-6})$, which is consitent with it not existing in nature [2]. In comparrison the decay (11) has a branching ratio of $(1.017 \pm 0.034) \times 10^{-3}$. As there is no selection rule forbidding this decay, the $\Delta S = \Delta Q$ rule was introduced to identify these types of decays. The change in strangeness and hadron charge for these decays can be found in Table.(I).

TABLE I: $\Delta S = \Delta Q$ selection rule table for the decays (11) and (12)

Shown decay of	ΔS	ΔQ	$\Delta S = \Delta Q$
Σ^-	+1	+1	yes
Σ^+	+1	-1	no

Where ΔA as the difference between the final and initial states of A such that $\Delta A = A_{final} - A_{initial}$. Also remember that the definition of strangeness assigns the strange quark a value of -1 and the anti-strange quark a value of $+1$. Thus we see that the decay (12) violates the $\Delta S = \Delta Q$ selection rule. Similarly, a decay with $\Delta S = \pm 2$ will contain two W bosons and as a result will be very suppressed. In conclusion, $\Delta S = \Delta Q = \pm 1$ for an allowed process.

This can now apply this rule to semi-leptonic decays of Kaons of the form $K \rightarrow \pi l \nu_l$. There are four Kaon decays that have this form:

$$K^0(d\bar{s}) \rightarrow \pi^- l^+ \nu_l \quad (13)$$

$$\bar{K}^0(s\bar{d}) \rightarrow \pi^+ l^- \bar{\nu}_l \quad (14)$$

$$K^0(d\bar{s}) \rightarrow \pi^+ l^- \bar{\nu}_l \quad (15)$$

$$\bar{K}^0(s\bar{d}) \rightarrow \pi^- l^+ \nu_l \quad (16)$$

TABLE II: $\Delta S = \Delta Q$ selection rule table for the decays (13) - (16)

Shown decay of	ΔS	ΔQ	$\Delta S = \Delta Q$
(13)	-1	-1	yes
(14)	+1	+1	yes
(15)	-1	+1	no
(16)	+1	-1	no

A similar table as before can no be constructed. See Table.(II), where the number shown refers to the equations above. Thus it is clear that the only possible semi-leptonic decays of this form for K^0 and \bar{K}^0 are (13) and (14). As there is only one way for these processes to occur we determine that there can be no direct CP violation in the semi-leptonic decays of Kaons as there is no interference between processes, because there is only one [7, pg. 10]. So it is clear that the amplitudes of the $\Delta S = \Delta Q$ violating decays are:

$$A(K^0 \rightarrow \pi^+ l^- \bar{\nu}_l) = A(\bar{K}^0 \rightarrow \pi^- l^+ \nu_l) = 0$$

It is possible now to write these amplitudes in terms of K_S^0 and K_L^0 . Using Eqn.(1),(2),(9) and (10) we obtain[7, pg. 11]:

$$\begin{aligned} A(K_S^0 \rightarrow \pi^+ l^- \bar{\nu}_l) &= -A(K_L^0 \rightarrow \pi^+ l^- \bar{\nu}_l) = \frac{1-\epsilon}{\sqrt{2}} A(\bar{K}^0 \rightarrow \pi^+ l^- \bar{\nu}_l) \\ A(K_S^0 \rightarrow \pi^- l^+ \nu_l) &= A(K_L^0 \rightarrow \pi^- l^+ \nu_l) = \frac{1+\epsilon}{\sqrt{2}} A(K^0 \rightarrow \pi^- l^+ \nu_l) \end{aligned}$$

Where terms of order $|\epsilon|^2$ have been neglected. The quantities δ_L, S can now be defined, which show the tendency for the oscillations $K^0 \leftrightarrow \bar{K}^0$ to favour the matter particle state. Thus making a very small contribution to the matter anti-matter assymetry.

$$\delta_L, S = \frac{A(K_{L,S}^0 \rightarrow \pi^- l^+ \nu_l) - A(K_{L,S}^0 \rightarrow \pi^+ l^- \bar{\nu}_l)}{A(K_{L,S}^0 \rightarrow \pi^- l^+ \nu_l) + A(K_{L,S}^0 \rightarrow \pi^+ l^- \bar{\nu}_l)} := 2\Re(\epsilon)$$

The experimental value for this quantity is $\delta_L = (3.27 \pm 0.12) \times 10^3$. Which clearly indicates a small tendency to favour the matter particle in oscillations. This can also be illustrated graphically by measuring the relative number(N) of K^0 and \bar{K}^0 particles over time in a beam consisting initially of K^0 . See Fig.(4)

C. Testing CPT conservation through Strangeness Oscillation

The CPT theorem links conservation of CPT with Lorentz invariance. Thus to preserve the fundamental Lorentz symmetry we desperately hope CPT is conserved. Here is presented some evidence that this is indeed the case. The theorem requires particles and their anti-particles to have the same masses and lifetimes. It has been stated previously that the K_L^0 and K_S^0 particles have very different lifetimes, but thankfully these are not a particle anti-particle pair. From earlier it is clear that K^0 and \bar{K}^0 are such a pair. Thus we aim to test CPT conservation by measuring their mass difference.

By investigating the time evolution of Kaon oscillations it is possible to measure their mass difference. By inverting the linear combinations in Eqn.(1) and (1) and neglecting the very small contribution of ϵ the flavour eigenstates are described by:

$$\begin{aligned} |K^0\rangle &= \frac{1}{\sqrt{2}}(|K_S^0\rangle + |K_L^0\rangle) \\ |\bar{K}^0\rangle &= \frac{1}{\sqrt{2}}(|K_S^0\rangle - |K_L^0\rangle) \end{aligned}$$

The time evolution of the states are then written in terms of the mass and the lifetimes of the particles

$$|K_S^0(t)\rangle = |K_S^0(0)\rangle \exp(-(im_S + \Gamma_s/2)t) |K_L^0(t)\rangle = |K_L^0(0)\rangle \exp(-(im_L + \Gamma_L/2)t)$$

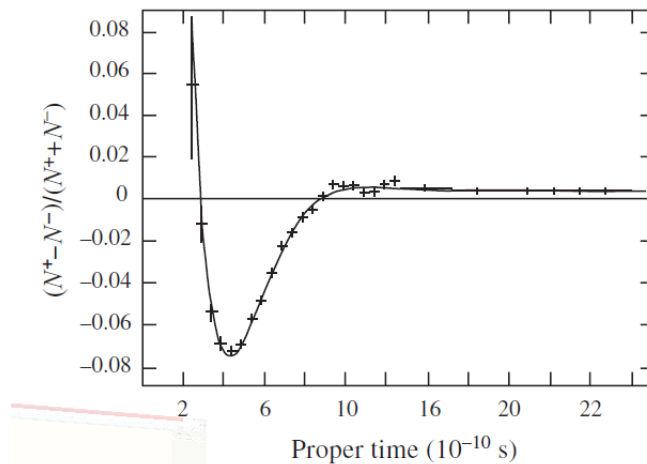


FIG. 4: Measurements of observed Kaon semi-leptonic decays from a beam initially consisting of K^0 mesons which shows the oscillation between the states K^0 and \bar{K}^0 as well as the small asymmetry, favouring the matter particle [8]

V. SPONTANEOUS CHARGE PARITY VIOLATION

A major motivation for creating models with new CPV mechanisms is to explain Baryon asymmetry. This is a huge research area, a simple search on arXiv.org reveals over 500 papers. The most prevalent theories are based around Super-Symmetry (SUSY) and Spontaneous Charge Parity Violation (SCPV).

Typically these models are created by looking for gauge symmetries which coincide with Lagrangians similar to the Standard Model, but with extra terms that may account for the preference of antimatter to decay into matter.

SCPV is praised for its “naturalness” in comparison to regular CPV[9]. It supposes the possibility to have spontaneous CPV by the vacuum, as opposed to explicit CPV from the CKM matrices. That is, the vacuum is no longer invariant under CP. In a sense this is nice as it may not introduce as many new particles. Indeed, the Minimal SUSY Standard Model is not compatible with SCPV, and in general it is difficult to incorporate SCPV in SUSY models[9], but it has been done. For example in SUSY SO(10) [10].

There are two primary goals to this section. First, to elucidate the use of groups in physics, particularly particle physics. Second, to use this knowledge to understand various SCPV models beyond the insufficient description given in this introduction.

A. SCPV. Group Theory for Physics

Some physicists take pride in never having learned group theory and still understanding its applications. This is not an unreasonable point of view, an engineer can launch a rocket without knowing real analysis.

Undergraduate group theory modules quite commonly focus entirely on groups useful for pure mathematics. This is understandable as it is taught by the math department, but the picture of group theory young physicists may end up with is often quite different to its use in physics.

Part of the reasons groups can be so abstract is that they have very little structure. Even basic physics requires complicated structure. Simply changing coordinates in classical mechanics requires differential geometry, which requires analysis and topology. A **Group** is a set G with a function $\cdot : G^2 \rightarrow G$, having the following properties $\forall g, h, k \in G$

- | | | |
|-------|---|---------------|
| (GA1) | $g \cdot h \in G$ | closure |
| (GA2) | $\exists e \in G, e \cdot g = g \cdot e = g$ | identity |
| (GA3) | $\exists g^{-1} \in G, g^{-1} \cdot g = g \cdot g^{-1} = e$ | inverse |
| (GA4) | $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ | associativity |

Abstract definitions will be avoided in general. To understand this proceed with a useful example. Let $GL(n, \mathbb{R})$ be the set of all $n \times n$ matrices with real coefficients and non-zero determinant. This forms a group under matrix

multiplication.

(GA1) The product of two $n \times n$ matrices is an $n \times n$ matrix, so it is closed.

(GA2) The identity matrix I satisfies $IM = MI = M$.

(GA3) Non-zero determinant matrices are invertible $M^{-1}M = MM^{-1} = I$.

(GA4) As real multiplication and addition are associative $((1 + 2) + 3 = 1 + (2 + 3))$ Matrix multiplication inherits this property.

NOTE:

- The set of all real matrices would fail, as zero determinant matrices do not have inverses.
- Associativity is usually trivial by the definition and not checked.

The purpose of this proof was to give some familiar meaning to the abstraction. Detailed proofs shall be avoided in favour of intuition. Note that already a more powerful structure is present than a group, the field of real numbers, $(\mathbb{R}, +, \times)$. This is essentially just two groups glued together, addition and multiplication.

Similarly, every vector space is a group under vector addition. With this it could be said that all of physics uses groups, however a secondary school student does not use differential geometry with Newton's laws. At this stage groups are still useless to the practical physicist.

A **Transformation Group** is a more useful idea. Let X be a set, $\text{Transf}(X)$ is the set of all one-to-one functions from X to X . This is the set of all ways of rearranging X . In the context of a finite group $S = \{1, 2, \dots, n\}$, $\text{Transf}(S)$ is just the set of all permutations.

If 3-dimensional space was modelled by \mathbb{R}^3 the transformation group would not be useful, it would contain unnatural discontinuous functions that do not relate to intuition about space. What would be useful is:

The transformations that preserve a property of a space form a group. It is known as the **invariance group** or the **symmetry group**.

This is an extremely general and potent idea. It is also not difficult to prove, so it shall be after the following motivation.

Model space again as \mathbb{R}^3 , transformations should preserve distance. Where Euclidean distance is defined by

$$d(\mathbf{x}, \mathbf{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

The desired transformation R has form such that $d(R\mathbf{x}, R\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. From intuition there are some obvious transformations $R = I$, the identity matrix, or $R\mathbf{x} = \mathbf{x} + \mathbf{a}$ a translation in space

$$\begin{aligned} d(R\mathbf{x}, R\mathbf{y}) &= d(\mathbf{x} + \mathbf{a}, \mathbf{y} + \mathbf{a}) \\ &= \sqrt{\sum_{i=1}^3 ((x_i + a_i) - (y_i + a_i))^2} \\ &= \sqrt{\sum_{i=1}^3 (x_i + a_i - y_i - a_i)^2} \\ &= \sqrt{\sum_{i=1}^3 (x_i - y_i)^2} \\ &= d(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Now note

$$d(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$$

Where \cdot is the regular dot product. Calling $\Delta\mathbf{x} := \mathbf{x} - \mathbf{y}$ and using serif font to signify matrix representation

$$\begin{aligned} \Delta\mathbf{x} \cdot \Delta\mathbf{x} &= \Delta\mathbf{x}^T \Delta\mathbf{x} \\ &= (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \end{aligned}$$

In other words dot product in an orthonormal basis is a row vector times a column vector. Apply the transformation and assume it preserves distance

$$(\mathbf{Rx} - \mathbf{Ry})^T(\mathbf{Rx} - \mathbf{Ry}) = (\mathbf{x} - \mathbf{y})^T(\mathbf{x} - \mathbf{y})$$

The only way this can hold for any \mathbf{x} and \mathbf{y} is if \mathbf{R} is affine, $\mathbf{Rx} = \mathbf{Ax} + \mathbf{a}$ where \mathbf{A} is linear, i.e. a Matrix,

$$\begin{aligned} (\mathbf{Rx} - \mathbf{Ry})^T(\mathbf{Rx} - \mathbf{Ry}) &= (\mathbf{x} - \mathbf{y})^T(\mathbf{x} - \mathbf{y}) \\ (\mathbf{Ax} + \mathbf{a} - \mathbf{Ay} - \mathbf{a})^T(\mathbf{Ax} + \mathbf{a} - \mathbf{Ay} - \mathbf{a}) &= (\mathbf{x} - \mathbf{y})^T(\mathbf{x} - \mathbf{y}) \\ (\mathbf{A}(\mathbf{x} - \mathbf{y}))^T(\mathbf{A}(\mathbf{x} - \mathbf{y})) &= (\mathbf{x} - \mathbf{y})^T(\mathbf{x} - \mathbf{y}) \\ (\mathbf{A}\Delta\mathbf{x})^T(\mathbf{A}\Delta\mathbf{x}) &= \Delta\mathbf{x}^T\Delta\mathbf{x} \\ \Delta\mathbf{x}^T\mathbf{A}^T\mathbf{A}\Delta\mathbf{x} &= \Delta\mathbf{x}^T\Delta\mathbf{x} \end{aligned}$$

This can only hold if $\mathbf{AA}^T = \mathbf{I}$, so $\mathbf{A}^T = \mathbf{A}^{-1}$. Immediately taking the determinant of both sides gives

$$\begin{aligned} \det \mathbf{AA}^T &= \det \mathbf{I} \\ \det \mathbf{A} \det \mathbf{A}^T &= 1 \\ \det \mathbf{A} \det \mathbf{A} &= 1 \\ \det \mathbf{A}^2 &= 1 \\ \det \mathbf{A} &\in \{-1, 1\} \end{aligned}$$

This relates to intuition. The determinant of a matrix corresponds to how much it changes a volume. If the determinant of a matrix is ± 7 , it will turn a 1 m^3 cube into a 7 m^3 parallelepiped, in order to preserve distance the determinant must have absolute value 1.

What sort of matrices have determinant -1 and $+1$? Consider the examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The left matrix represents a reflection in through the xz -plane. The right represents a rotation by $\pi/2$ in the xy -plane. It turns out that all matrices with $\mathbf{A}^T = \mathbf{A}^{-1}$ are either rotations or reflections, together they make the orthogonal group $\text{O}(3)$. A transformation that is an orthogonal matrix plus a translation preserves distance, and the set of all such transformations form a group. Just the usual isometrics of Euclidean space.

Denote the set of rotations, which preserve length and handedness (reflection changes the right hand rule to the left hand rule), as $\text{SO}(3)$, this is also a group.

So far this may seem over simplified but now it is already possible to define groups which preserve much more interesting and relevant properties

- Galilean Transformations. Preserve Newton's Laws.
- Unitary Operators. Preserve normalisation in quantum mechanics. Propagators should have this.
- Lorentz Transformations. Preserve the light cone. Equivalently the Minkowski metric.

Intuitively, because these preserve something about a space, they form a group. This can greatly simplify some things, they are always invertible, a composition of two transformations still preserves.

There are many other aspects of group theory that can be useful here. Consider similar matrices, in the context of groups these are called conjugate elements. In the context of rotations they rotate by the same angle (but not the same direction), in the context of Lorentz Boosts they have the same speed (but not the same direction).

If the reader is interested there is a lot of literature furthering this area, particularly physics-focused is [11]. In this paper the theory shall not be explained further, aside from the symmetry group theorem proved below.

THEOREM:

Consider a set X with a function $f : X \rightarrow Y$, the set of all transformations of X that preserve the function at x ,
 $S_f := \{T \in \text{Transf}(X) : f(Tx) = f(x), \forall x \in X\}$,
 form a group called the **symmetry group** of f .

For the example of distance in Euclidean space, $f(Tx) = f(x)$ corresponded to $d(\mathbf{Rx}, \mathbf{Ry}) = d(\mathbf{x}, \mathbf{y})$.

PROOF:

(GA1) Let $S, T \in S_f$, then

$$f((ST)x) = f(S(Tx)) = f(Sx) = f(x)$$

So ST also preserves f , $ST \in S_f$.

(GA2) The identity element e of $\text{Transf}(X)$ satisfies $f(ex) = f(x)$ by definition of the identity function, so $e \in S_f$.

(GA3) Let $S \in S_f$. Since it is a bijection, an inverse S^{-1} exists.

$$f(S^{-1}x) = f(SS^{-1}x) = f(ex) = f(x)$$

S was introduced as it satisfies $f(Sy) = f(y)$ and pick $y = S^{-1}x$. $S^{-1} \in S_f$.

(GA4) is inherited from $\text{Transf}(X)$. □

NOTE

- The proof of the Euclidean transformations is incomplete but trivial to extend. Assume R is affine, show it preserves distance.
- Strictly speaking symmetry groups should be considered group actions. They act on a set X , as matrices act on column vectors.

B. SCPV. Useful Physical Groups

In the language of symmetry groups, useful physical transformations including the Gauge-symmetry of the Standard Model are investigated.

example 1.0 The Euclidean Isometries.

As was shown in the previous section any combination of rotation, reflection and translation preserve Euclidean distance. This can be extended to n -dimensional Euclidean space \mathbb{E}^n , here the group of orthogonal transformations and rotations are denoted respectively as $O(n)$ and $SO(n)$.

example 1.1 Galilean Transformations.

Solely preserving distance does not say much about physics. Consider Newton's laws in Cartesian coordinates in an inertial frame with column vector representation

$$m\ddot{\mathbf{x}} = \mathbf{F}$$

What sort of transformations $\mathbf{T}\mathbf{x} = \mathbf{R}\mathbf{x} + \mathbf{a}$ preserve this equation? Here R is a function of time, for example transforming to a rotating frame

$$\begin{aligned} m \frac{d^2}{dt^2}(\mathbf{R}\mathbf{x} + \mathbf{a}) &= \mathbf{R}\mathbf{F} \\ m \frac{d}{dt}(\dot{\mathbf{R}}\mathbf{x} + \mathbf{R}\dot{\mathbf{x}}) + m\ddot{\mathbf{a}} &= \mathbf{R}\mathbf{F} \\ m \frac{d}{dt}(\dot{\mathbf{R}}\mathbf{x} + \mathbf{R}\dot{\mathbf{x}}) + m\ddot{\mathbf{a}} &= \mathbf{R}\mathbf{F} \\ m(\ddot{\mathbf{R}}\mathbf{x} + 2\dot{\mathbf{R}}\dot{\mathbf{x}} + \mathbf{R}\ddot{\mathbf{x}}) + m\ddot{\mathbf{a}} &= \mathbf{R}\mathbf{F} \end{aligned}$$

To have the same form this requires, $\dot{R} = 0$ and $\ddot{a} = 0$. This is equivalent to saying R is a constant change of basis, such as a rotation, and $\mathbf{a} = \mathbf{v}t + \mathbf{b}$, a translation and a constant velocity. In other words

$$\mathbf{T}\mathbf{x} = \mathbf{R}\mathbf{x} + \mathbf{a}t + \mathbf{b}$$

A Galilean transformation is a symmetry group which preserves Newton's laws. It can be readily extended to also preserve distance and time intervals.

The goal is to explain the Quantum Field Theory models, Euclidean distance and Newton's Laws are not applicable. They act as an analogy for the Minkowski metric and Quantum Mechanics.

example 2.0 Lorentz Transformations.

Suppose one wishes to preserve the Minkowski Metric, or equivalently, the light-cone. The transformations that do so are called Lorentz transformations, essentially defined by

$$\mathbf{L}^T \eta \mathbf{L} = \eta$$

Extending the transformations to allow for a translation $P_\nu^\mu x^\nu = L_\nu^\mu x^\nu + a^\mu$, these are called Poincar transformations. The essence of Special Relativity is that the laws of physics are invariant under Poincar transformation.

example 2.1 Unitary Transformations.

In quantum mechanics the total probability of a time evolving state must always be 1, otherwise it is simply not a probability. In the context of Hilbert spaces (complex inner product spaces complete under the induced norm), the norm of a state gives its probability, so the norm must be preserved.

Recall in Euclidean space Orthogonal Transformations preserve the norm of a vector, as it preserves the dot product. The complex result is analogous but in the case of operators extends far more, for example

$$|\Psi(t)\rangle = e^{-it\hat{\mathcal{H}}/\hbar} |\Psi(0)\rangle$$

The formal solution to the Schrödinger equation, the exponential operator on the right is unitary. In general this has quite complicated form.

If the Hilbert Space is spanned by a finite set $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ it is isometrically isomorphic to the inner product space \mathbb{C}^n . To preserve the norm here is similar to \mathbb{R}^n

$$|U\mathbf{u}|^2 = (\mathbf{U}\mathbf{u})^{*T} \mathbf{U}\mathbf{u} = \mathbf{u}^{*T} \mathbf{U}^{*T} \mathbf{U}\mathbf{u} = \mathbf{u}^{*T} \mathbf{u} = |\mathbf{u}|^2$$

This holds $\forall \mathbf{u} \in \mathbb{C}^n$ if and only if $U^{*T}U = I$. This is similar to orthogonal matrices except there is a complex conjugate. The group of all such matrices is denoted $U(n)$ and those which have positive determinant form a group $SU(n)$.

example 3.0 Classical Electrodynamics.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \frac{1}{\mu} \nabla \times \mathbf{B} = \mathbf{j} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

As $\nabla \cdot \mathbf{B} = 0$ a theorem shows $\exists \mathbf{A}, \mathbf{B} = \nabla \times \mathbf{A}$ this together with the third equation is

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t(\nabla \times \mathbf{A}) \\ \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) &= 0 \end{aligned}$$

Which in turn gives $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$, where in the time independent case ϕ reduced to the electrostatic potential $\phi := -\int \mathbf{E} \cdot d\mathbf{x}$

Consider the following transformation

$$\begin{pmatrix} \mathbf{A} \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{A} + \nabla\lambda \\ \phi - \partial_t\lambda \end{pmatrix}$$

These transformed potentials are also valid

$$\begin{aligned} \nabla \times (\mathbf{A} + \nabla\lambda) &= \nabla \times \mathbf{A} + \nabla \times \nabla \cdot \lambda = \mathbf{B} + 0 \\ -\partial_t(\mathbf{A} + \nabla\lambda) - \nabla(\phi - \partial_t\lambda) &= -\partial_t(\mathbf{A}) - \nabla\phi = \mathbf{E} \end{aligned}$$

As these equations are preserved the gauge transformations form a symmetry group. Just as preserving Newton's laws formed a symmetry group. This is an example of a **gauge transformation**.

More generally a gauge transformation is defined as a *local* transformation that preserves the Lagrangian of a theory. What local means exactly is yet to be defined.

C. SCPV. Gauge Theory

The reader may be having a hard time seeing how this relates to particle physics. With the benefit of the language being developed some light may be shed on what the remainder of this section will contain.

Most of the groups discussed have their place in modern particle physics. Lorentz invariance and unitarity of symmetry operators obviously extend from special relativity and quantum mechanics. That is not all, the gauge-symmetry group of the Standard Model is $SU(3) \times SU(2) \times U(1)$. Many of the popular extensions to the Standard Model are also defined by gauge symmetries such as $SU(5)$, $SO(10)$ and $O(16)$ [12].

It is possible to state group conditions used for either (a) deducing that a theory can allow SCPV or (b) creating theories which allow for it[?]. Explaining how to use this with the mathematics being developed is the primary goal of the section. Although, the hierarchy of knowledge required for theoretical physics is unfortunately large. Despite that the material has attempted to have been presented in a condensed and tangible manner, inevitably some details will be avoided.

In Quantum Mechanics overall phase is unimportant, that is $|\psi\rangle \sim |\psi\rangle e^{i\xi}$ where $\xi \in [0, 2\pi)$. When making any physical measurement the phase cancels out in the complex inner product. It forms a *global* gauge Symmetry. Phase is still important in Quantum mechanics in the form of phase difference. Consider two energy eigenstates with $a, b \in \mathbb{R}$

$$\begin{aligned} |\psi\rangle &= a|E_1\rangle e^{-iE_1/\hbar} + b|E_2\rangle e^{-iE_2/\hbar} \\ \rightarrow |\psi\rangle e^{iE_1/\hbar} &= a|E_1\rangle + b|E_2\rangle e^{i(E_1-E_2)/\hbar} \\ \langle \mathbf{x} \rangle &= a^2 \langle E_1 | \mathbf{x} | E_1 \rangle + 2ab \langle E_1 | \mathbf{x} | E_2 \rangle \cos \theta + b^2 \langle E_2 | \mathbf{x} | E_2 \rangle \end{aligned}$$

So here the phase *difference* $\theta := (E_1 - E_2)/\hbar$ appears explicitly in a measurable quantity. Compare to the phase of the CKM matrix, both phases have physical consequences. It will be shown the case is similar for SCPV. Consider $U(1)$, a "matrix" that acts on one-dimensional complex vectors, i.e., complex numbers. In this case then, both the "matrix" and "vector" can both be represented as complex numbers

$$|uz|^2 = (uz)^* uz = u^* z^* uz = |z|^2 |u|^2 = |z|^2$$

This holds for all z if and only if u has length 1, so $u = e^{i\xi}$ where $\xi \in [0, 2\pi)$. So it can be said that the global phase symmetry of Quantum Mechanics can be represented as $U(1)$. Note that $SU(1) = U(1)$ in this case, as the norm of a single complex number must be positive. Likewise as $U(1)$ represents a rotation in the complex plane it must have the same group structure as $SO(2)$. In mathematical language the groups are isomorphic, written as $U(1) \cong SO(2)$.

Thus it is clearer to say that the global phase invariance group of quantum mechanics is isomorphic to $U(1)$. Frequently it is the case that there is a deeper physical meaning behind symmetries. Merely stating a theory is invariant under $U(1)$ is not sufficient as one must define how $U(1)$ acts. For example, it can act globally or locally.

Consider Quantum Electrodynamics (QED) the Lagrangian of the theory is invariant under **local gauge symmetry** of $U(1)$.

$$\psi(x^0, x^1, x^2, x^3) \mapsto e^{i\xi(x^0, x^1, x^2, x^3)} \psi(x^0, x^1, x^2, x^3)$$

Where in cartesian coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. In shorthand

$$\psi(x) \mapsto \tilde{\psi}(x) = e^{i\xi(x)} \psi(x)$$

Where $x := (ct, x, y, z)$ This inherently insists that phase is irrelevant at any point. It was noted that more structure than merely a group is being used concerning physical groups. Fields, vector spaces, and vector calculus to name a few. Primarily calculus, a fundamentally geometric structure, appears everywhere in physics. The real scalar function $\xi(x)$ must be differentiable.

This differentiability calls for the structure of a **Lie Group** (acting on a set). A Lie group is simply a group where the group operation and taking inverses are differentiable. For a simple example consider the group of addition on \mathbb{R} .

$$\cdot (x, y) = x + y \qquad \text{inv}(x) = -x$$

Both of these functions are differentiable in their arguments so $(\mathbb{R}, +)$ forms a Lie Group. Likewise other familiar groups $SU(n)$, $O(n)$ and $GL(n, \mathbb{R})$ all form Lie groups.

To define calculus on non-Euclidean spaces the structure of differential manifolds is required. The details of which will have to be omitted but some intuitive and useful properties are worth discussing. The dimension of a differential manifold is related to the intuitive understanding of spatial dimension. The surface of a sphere is a two-dimensional space and as expected forms a two-dimensional manifold. This means, for the most part, the sphere can be parametrised by two numbers (θ, ϕ) .

$O(n)$, $U(n)$ and $SU(n)$ have manifold dimension $\frac{1}{2}n(n-1)$, n^2 and $n^2 - 1$ respectively. So these objects are no longer purely to be considered as groups but also as abstract smooth geometric spaces.

Returning the example of QED's local gauge symmetry and comparing with the global gauge symmetry of QM

$$\begin{aligned} |\psi\rangle &\mapsto |\tilde{\phi}\rangle = e^{i\xi} |\psi(x)\rangle \\ \psi(x) &\mapsto \tilde{\psi}(x) = e^{i\xi(x)} \psi(x) \end{aligned}$$

Both symmetries are isomorphic to $U(1)$ but in very different contexts. In QM it may be regarded simply as a group, but in QED the differentiability effects the Lagrangian non-trivially.

The local gauge symmetry of the strong force is $SU(3)$ and for the weak force it is $SU(2)$. Like $U(1)$ was represented by one parameter which was a function of space-time position $\xi(x)$, $SU(2)$ and $SU(3)$ have $2^2 - 1 = 3$ and $3^2 - 1 = 8$ respectively. Notice peculiarly these are the amount of force carriers in each theory.

What do these symmetries mean physically? According to [14], they are like the potentials \mathbf{A}, ϕ for classical electromagnetism; merely a tool to make the mathematics easier. In the same way as one could use \mathbf{E} and \mathbf{B} without the degeneracy of \mathbf{A} and ϕ , in principle one could formulate QED completely independent of phase, even without complex numbers. However the difference between usability would be extreme.

A full treatment of local gauge theory for physics can be found here [14] and more on Lie-algebras and differential manifolds related to physics can be found at [11].

The gauge symmetries of the Standard Model are written as the product group $SU(3) \times SU(2) \times U(1)$. A product group is formed in the same way \mathbb{R}^2 with vector addition is a product group of $\mathbb{R} \times \mathbb{R}$. This can also be considered a project manifold and indeed it tells us the number of force carriers there are in each theory. Knowing the dimension of the SUSY theory $SU(5)$ as $5^2 - 1 = 24$ then tells us the number of force carriers in this theory.

One must call it at a day at some point, actually dealing with Lagrangians is a much more time consuming and worth-while task, but the language developed shall suffice to investigate SCPV models.

D. SCPV. Group Theoretic Conditions.

The **vacuum expectation value** (VEV) of an operator is the expected value of the operator acting on the vacuum state. A vacuum state is analogous to the lowest energy eigenstate of the quantum harmonic oscillator (QHO). For the QHO, the VEV of \hat{H} is $\hbar\omega/2$. Considering the ladder operators for the QHO analogously to the creation and annihilation operators in QED, this lowest energy state is equivalent to the 'no quanta' state. For the remainder of the section the vacuum state for any system shall be denoted $|0\rangle$.

In the Standard Model vacuum state in general may refer to the vacuum state of different fields. For example in the process of vacuum polarization photons may exist while the electron-positron field is in vacuum state. In order to most generally consider new models abstract scalar fields will be considered. Meaning, ϕ need not represent the wave function solely for electromagnetism. It simply represents some interesting field to a new theory.

The following is a sketch of a more rigorous development found in [13]. This paper assumes familiarity with modern Lie Group/Algebra theory and new physics models. Presented below is no substitute, but may be viewed as an intuition-based introduction through the symmetry group language developed.

Overall a theory will have a total Lagrangian \mathcal{L} form that depends on every field, just like the total Lagrangian of the Standard Model. Consider first a generalised CP-transformation (GCP), denoted \hat{G}_{CP} , on a theory with scalar fields $\phi_i(\mathbf{x}, t)$, $i \in \{1, \dots, n\}$.

$$\phi_i(\mathbf{x}, t) \mapsto U_{ij} \phi_j^*(-\mathbf{x}, t)$$

Where $[U_{ij}] =: \mathbf{U} \in U(n)$, a symmetric unitary matrix. This is related to regular CP as $\hat{C}\hat{P}\phi(x) = \hat{C}\phi(-x) = c\phi^*(-x)$, where $c \in \mathbb{C}$ and ϕ is a scalar field. In column vector form we obtain

$$\begin{pmatrix} \phi_1(\mathbf{x}, t) \\ \vdots \\ \phi_n(\mathbf{x}, t) \end{pmatrix} =: \Phi(\mathbf{x}, t) \mapsto \hat{G}_{CP}\Phi(\mathbf{x}, t) = \mathbf{U}\Phi^*(-\mathbf{x}, t)$$

If the total Lagrangian is invariant under this transformation it is GCP invariant. Now the interest lies in determining the difference between explicit CPV (XCPV) and spontaneous CPV (SCPV). First it is useful to derive conditions where a model is GCP invariant.

So suppose \mathcal{L} is \hat{G}_{CP} invariant. Consider the Φ as a n -dimensional complex vector. By changing the basis $\Phi \mapsto C\Phi$, the coordinate transformation for U is

$$U \mapsto \tilde{U} = CUC^\dagger$$

Consider U temporarily as the metric for a complex inner product, by Gram-Schmidt orthonormalisation a change of basis C may be found such that $\tilde{U} = I$. In the context of group theory. In this new basis, then, \hat{G}_{CP} reduces to

$$\Phi(\mathbf{x}, t) \mapsto \Phi^*(-\mathbf{x}, t)$$

Merely charge conjugation and reflection of space. Theories are always coordinate invariant, else they are non-physical. Hence the coordinate system $C\Phi$ may be used. It has thus been shown if a theory is GCP invariant, then coordinates exist where $U = I$.

Explicit CP Violation:

Suppose that \mathcal{L} is invariant under \hat{G}_{CP} . As was just shown, there exists a basis where $U = I$. Conversely, if for no $U \in U(n)$ is \mathcal{L} invariant under $\Phi(\mathbf{x}, t) \mapsto U\Phi^*(-\mathbf{x}, t)$ then the theory is explicitly CP violating. Now, this may seem to cover all cases, but, SCPV is not dependent on the GCPV condition. In fact, it is only possible to have SCPV *without* XCPV, as will be explained soon. This must immediately mean that any CPV already observed and predicted by the Standard Model must be explainable in the context of SCPV also.

Spontaneous CP Violation:

Suppose that \mathcal{L} is invariant under \hat{G}_{CP} , this means XCPV cannot happen, however consider the following.

Pick a basis such that $U = I$. In this basis suppose that the VEVs (of the GCP operator) are not all real numbers. Depending on the VEV values it may or may not be possible to change basis again such that all the VEVs are real *and* U remains equal to I . If it is possible to find such a basis transformation, the model is totally CP-conserving. if it is not, the model has SCPV.

For a somewhat contrived but pedagogic example suppose a model has two scalar fields ϕ_1 and ϕ_2 . Suppose the Lagrangian they obey is invariant under the following transformation

$$\begin{pmatrix} \phi_1(\mathbf{x}, t) \\ \phi_2(\mathbf{x}, t) \end{pmatrix} \mapsto \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \phi_1^*(-\mathbf{x}, t) \\ \phi_2^*(-\mathbf{x}, t) \end{pmatrix}$$

This has the form of a GCP, as can be verified by checking the matrix is unitary. One can now change basis by

$$\begin{pmatrix} \phi_1(\mathbf{x}, t) \\ \phi_2(\mathbf{x}, t) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} \phi_1(\mathbf{x}, t) \\ \phi_2(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} \phi_1(\mathbf{x}, t) \\ \phi_2(\mathbf{x}, t) + i\phi_1(\mathbf{x}, t) \end{pmatrix}$$

This gives the new U by

$$\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So by changing the basis of the fields there is a simpler representation of GCP, it may make the overall Lagrangian more complicated, but one may simply solve that in the first coordinates for the vacuum state, then write that in the GCP preferred coordinates after.

Assume that \mathcal{L} is invariant under \hat{G}_{CP} , that is: The Lagrangian does *not* have XCPV, unlike the Standard Model. Whether SCPV is present or not depends on the VEVs of the \hat{G}_{CP} operator. To obtain these one must explicitly solve \mathcal{L} and find the vacuum state, doing this abstractly one may simply choose example VEVs.

case 1 No SCPV, no XCPV.

Suppose, using the coordinate system obtained where $U = I$, that

$$\langle 0 | \hat{G}_{CP} | 0 \rangle = \begin{pmatrix} \hbar\omega_1 \\ \hbar\omega_2 \end{pmatrix}$$

In this case, there is no SCPV. A coordinate basis has been founded with real VEVs and $U = I$.

case 2 SCPV.

Suppose, using the coordinate system obtained where $U = I$, that

$$\langle 0 | \hat{G}_{CP} | 0 \rangle = \begin{pmatrix} i\hbar\omega_1 \\ \hbar\omega_2 \end{pmatrix}$$

Here it is impossible to remove the complex number and keep U as the identity matrix. So there is SCPV. So what is SCPV exactly beyond the mathematical formulation? It means the vacuum is not invariant under CP, but it is not the same CP one is used to as in the Standard Model, as for SCPV to exist XCPV must not. Returning to the earlier discussion about phase invariance in QM, for SCPV to occur one needs a *physical* phase to occur in the VEVs (for example $i = e^{i\frac{\pi}{2}}$ in the previous example). Just as the physical phase difference occurs when measuring, say, the expected value of position. So these phases do give rise to real physical meaning. Described this way, the phase of the CKM matrix is an easy analogy to understand.

In quantum physics one must pay attention to which phases have physical consequence. Such as: (1) phase-difference between eigenstates in a quantum mechanical state, (2) the phase of the CKM matrix and (3) the relative phases of VEVs in SCP violating theories.

Through the language of symmetry groups the results just obtained are powerful tools for assessing the behaviour of CP in various theories. These initial results are summarised below.

- (A) Invariance of \mathcal{L} under \hat{G}_{CP} .
- (B) Having coordinates in which: (i) \hat{G}_{CP} reduces to spatial reflection and conjugation and (ii) the VEVs of \hat{G}_{CP} are real valued.

This leaves three interesting possibilities (as $\neg A \Rightarrow \neg B$)

- (i) (Neither A nor B) XCPV.
- (ii) (A and B) SCPV.
- (iii) (A and not B) Total CP invariance.

What is more interesting is painting this in more of a particle physics light, though it is difficult to do in general as the criterion is so general, it can be done by noting possible force carriers as “suprions” [13]. A simplified example will be taken.

E. SCPV. The Aspon Model

The theme of this section has been unravelling abstraction through examples. Continuing this trend the aspon model is explored [15]. It is an illustrative example because it is so close to the Standard Model. The local gauge symmetry of the aspon model is

$$SU(3) \times SU(2) \times U(1) \times U(1)$$

compared to the Standard Model

$$SU(3) \times SU(2) \times U(1)$$

If two differential manifolds \mathcal{M} and \mathcal{N} have dimension m and n respectively, their product manifold $\mathcal{M} \times \mathcal{N}$ has dimension $m + n$. A product manifold is constructed as one would expect with the example $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This means the only difference between the Standard Model’s gauge symmetry group’s dimension and the aspon model’s is the dimension of $U(1)$, which is $1^2 = 1$. As noted before, this corresponds to the number of force carriers in the theory which implies the aspon model has one more force carrier than the Standard Model. This force carrier is called the **aspon** and denoted A^0 .

The regular Standard Model particles all have aspon charge 0. This model allows additional possible states, of most important are two complex Higgs scalar particles denoted χ^α where $\alpha \in \{1, 2\}$, these have aspon charge 1. These are the cause of the VEV values with non-removable complex phase. In other words, here lies the SCPV.

Writing the VEVs of each new particle as

$$\langle \chi^\alpha \rangle = r_\alpha e^{i\theta_\alpha} \quad r_\alpha \in \mathbb{R}$$

SCPV violation occurs by symmetry breaking in a manner analogous to the Higgs. These being Higgs particles allows the Higgs symmetry-breaking mechanism to be used, which causes the CP violation. In turn this also gives the aspon mass.

A lot has been said in the previous paragraphs. Despite knowing the mathematical symmetry conditions required for SCPV, explicit intuition about how CP occurs was not given. Now, with the non-removable phases appearing on Higgs particles, by symmetry breaking through the Higgs mechanism this can cause measurable CP violation. Without the possibility of this symmetry breaking in SCPV theories, they would be worthless. At the very least the theories need to explain $K^0 - \bar{K}^0$ mixing and as XCPV is incompatible with SCPV, it *must* use the SCPV mechanism.

The Feynman Diagram for the leading order term for $K^0 - \bar{K}^0$ mixing is then



FIG. 5: Feynman Diagram for Kaon mixing in aspon theory

Generalisations of the aspon model are easy to construct and are the de-facto presentation in [13]. It is possible to understand what was meant by SCPV being more “natural” than XCPV in [9], simply by thinking that the symmetry breaking mechanism of the Higgs is more natural than the CKM matrix.

These generalisations simply rely on creating new fields and observing whether or not they have non-removable complex phases in VEVs and their physical effect after symmetry breaking. To be totally general the symmetry breaking need not be Higgs, but for all intents and purposes it usually will be.

In these generalised theories, the particles generated are called “spurions” of which the χ^α higgs particles are specific examples of. SCPV is always related to symmetry breaking of the U(1) lie group, otherwise, despite VEV having complex phase, it will not show physically. Another important point is the requirement for at least two spurions, for even if there were a non-removable vacuum value like

$$\langle 0 | \hat{G}_{CP} | 0 \rangle = r e^{i\theta}$$

As there is only one, this phase will never turn up measurably, one requires a phase difference (just as in quantum mechanics) for it to be physical.

F. SCPV. Conclusion

Despite in general having quite an abstract formulation, SCPV is quite an interesting competitor for explaining CPV. Unfortunately, the mathematical details remain beyond the comprehension of the writer’s current abilities. Hence, interesting ideas like exactly how new states are predicted have not been attempted. Still, seeing how complex ideas in particle physical physics can be represented by simple symmetry conditions, just as complex ideas can be represented by Feynman diagrams, makes it a far more approachable subject for an undergraduate.

As what has been discussed is such a general condition it is quite robust to how long it may remain experimentally unfalsifiable. One condition for falsification would be if XCPV is found to be necessary, as that would contradict the result of part D. What are perhaps more interesting are explicit SCPV theories which make falsifiable predictions. For example the aspon model makes the prediction that for K^0 mixing the CP parameters must $Re(\frac{\epsilon'}{\epsilon}) \leq 10^{-5}$ [15]. Comparing this to the experimental result given in the Neutral Kaon Mixing section, $Re(\frac{\epsilon'}{\epsilon}) = 0.00166 \pm 0.00023$ it is seen that the aspon method *has* been falsified.

Quite a few modern theories are difficult if not impossible to falsify. For example, the weak anthropic principle. It is refreshing to have dealt with an idea that has been proven wrong. That is science. Still, SCPV remains in many other prominent models. Aspon theory is introduced purely as an explanatory example.

Appendix A: Appendix

Difficult calculations in here.

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- [1] “Evidence for the 2π Decay of the K_2^0 Meson” - J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay (1964) Phys. Review letters, vol. 13, issue 4
 - [2] “2013 Review of Particle Physics” - J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012) and 2013 partial update for the 2014 edition.
 - [3] “Particle Physics, Third Edition” - B.R Martin, G. Shaw, Wiley(2008)
 - [4] <http://large.stanford.edu/courses/2008/ph204/coleman1/>
 - [5] “Measurements of Direct CP Violation, CPT Symmetry, and Other Parameters in the Neutral Kaon System” - KTeV Collaboration, arXiv:hep-ex/0208007v1 6 Aug 2002
 - [6] “Introduction to High Energy Physics” - Donald H. Perkins, Cambridge University Press, 4th edition.
 - [7] “CP violation in Kaon Decays” - Giancarlo D’Ambrosio and Gino Isidori, arXiv:hep-ph/9611284v1 8 Nov 1996
 - [8] “A Measurement of the $K(L)$ - $K(S)$ Mass Difference from the Charge Asymmetry in Semileptonic Kaon Decays” - Gjesdal, S. et al. Phys.Lett. B52 (1974) 113 Print-74-1358 (CERN)
 - [9] “Spontaneous CP Violation”- François Goffinet, CP3 Seminar, Unité de Physique Théorique et Mathématique, U.C.L., December 2003
 - [10] “Spontaneous CP Violation in SUSY” - Yoav Achiman, Physics Letters B, March 2007
 - [11] “A Course in Modern Mathematical Physics” - Peter Szekeres, Cambridge University Press, 2004
 - [12] “The Algebra of Grand Unified Theories” - John Baez and John Huerta, Bulletin of the American Mathematical Society, vol 47, May 4 2010
 - [13] “Group-theoretic Condition for Spontaneous CP Violation” - Howard E. Haber and Ze’ev Surujon, Physical Review D, Volume 86, Issue 7, October 2012
 - [14] “Field Theory in Particle Physics” Lecture Notes - Bernard de Wit and Eric Laenen, Universiteit Utrecht, 2009 <http://www.staff.science.uu.nl/~wit00103/ftip/Ch11.pdf>
 - [15] “Kaon Spontaneous CP Violation Reevaluated” - Paul H. Frampton and Masayasu Harada, Physical Review D, Volume 59, Issue 1, March 1998