Introduction: Lorentz Transformations

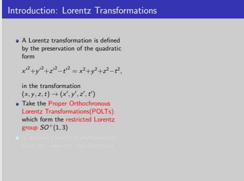
- A Lorentz transformation is defined by the preservation of the quadratic form

$$x'^2+y'^2+z'^2-t'^2 = x^2+y^2+z^2-t^2,$$

in the transformation $(x, y, z, t) \to (x', y', z', t')$

- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group $SO^+(1,3)$
- In general Lorentz transformations have two invariant null directions

- –Proper is det 1 . preserves the orientation of spacial axes, preserves handedness
- –orthochronous means time is always positive and the direction of time is preserve
- –Think of the standard Lorentz transformation, always two null directions at $x \pm$



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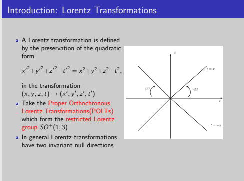

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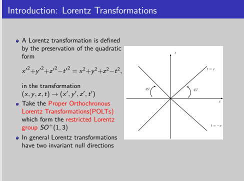

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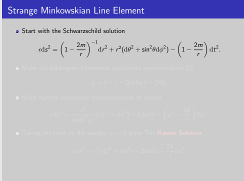
- derive a strange Minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial



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- to remove the coordinate singularity in the Schwarzschild solution
- These transformations put the line element in a form where we can take the limit the energy goes to 0
- It is easily shown with further coordinate transforms that this is Kasner, but it wo be done here

Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2.$$

- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

- Make further coordinate transformations to obtain

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right) \mathrm{d}u^2.$$

- Taking the limit as the energy, $\mu \to 0$ gives The Kasner Solution

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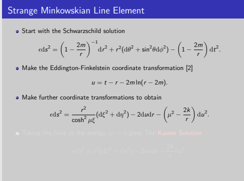

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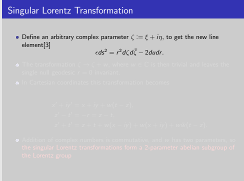
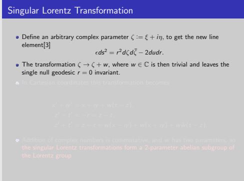
Layout

LTs that leave one null invariant direction are constructed



- This is what we want, An LT which leaves one null invariant.
- The use in the previous coord transforms was to make this transformation look trivial
- So this is what the seemingly trivial transformation looks like in Cartesian
- Again its clear that $r = 0$ keeps one direction fixed, as then z=t
- but it doesn't work both ways, not all 2 parameter abelian subgroups are singular Lorentz transformations



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Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta := \xi + i\eta$, to get the new line element[3]

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic $r = 0$ invariant.
- In Cartesian coordinates this transformation becomes

$$\begin{aligned} x' + iy' &= x + iy + w(t - z), \\ z' - t' &= -r = z - t, \\ z' + t' &= z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \end{aligned}$$

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- shown here that there is a 2 to 1 correspondence between SL(2,C) and POLTs

$SL(2,\mathbb{C})$ Matrices of the POLT

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Construct the following matrix

$$A = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix},$$

- This is useful as its determinant is the Lorentz quadratic form modulo a sign

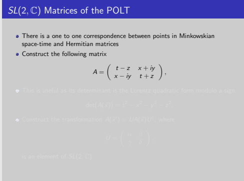
$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

- Construct the transformation $A(\vec{x}') = UA(\vec{x})U^\dagger$, where

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

is an element of $SL(2,\mathbb{C})$

- Complex Hermitian matrices have 4 independent components, so the element of such a matrix can be used to represent points in Minkowskian space-time.
- where $\alpha, \beta, \gamma, \delta$ are complex its an element of the special linear group. This mean it has determinant 1. **write it on the board**



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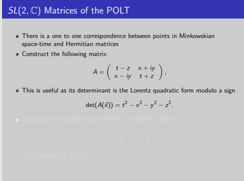

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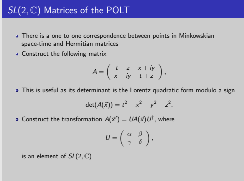

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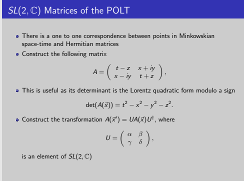

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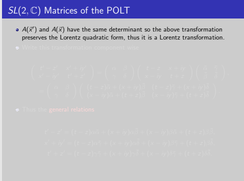

- This is because the determinant of U is 1
- I want to show you an example calculation of U, to do this we write in componen form

$SL(2,\mathbb{C})$ Matrices of the POLT

- $A(\vec{x}')$ and $A(\vec{x})$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus it is a Lorentz transformation.
- Write this transformation component wise

$$\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix},$$
$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar\alpha + (x+iy)\bar\beta & (t-z)\bar\gamma + (x+iy)\bar\delta \\ (x-iy)\bar\alpha + (t+z)\bar\beta & (x-iy)\bar\gamma + (t+z)\bar\delta \end{pmatrix}.$$

- Thus the general relations

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Example: Singular Lorentz transformation

- Take the singular Lorentz transformation from earlier

$$\begin{aligned} t' - z' &= t - z, \\ x' + iy' &= x + iy + w(t - z), \\ t' + z' &= t + z + w(x - iy) + \bar{w}(x + iy) + w\bar{w}(t - z). \end{aligned}$$

- Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$\begin{aligned} \alpha &= \pm 1, & \beta &= 0, \\ \gamma &= \bar{w}\alpha, & \delta &= \alpha. \end{aligned}$$

- So there are always **two** possible choices of U

$$U = \pm \begin{pmatrix} 1 & 0 \\ \bar{w} & 1 \end{pmatrix}$$

- Thus **there is a 2 to 1 correspondence between elements of $SL(2, \mathbb{C})$ and POLTs**

- Where we have also used $det(U) = 1$
- because the sign doesn't matter is still a solution Eqn(27)
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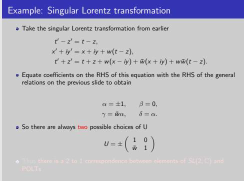

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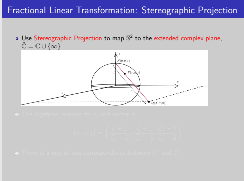
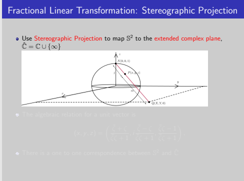
Layout

- Connect Minkowskian space to the 2-sphere by stereographic projection, so we ca
use points on a 2 sphere to think about LTs



- As we know, stereographic projection doesn't map the point N at the top of the circle, so that's why we map N to infinity and need to consider the extended complex plane
- It can also be written in terms of θ and ϕ



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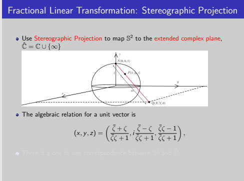

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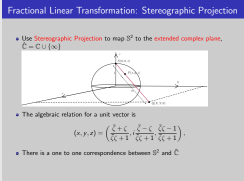

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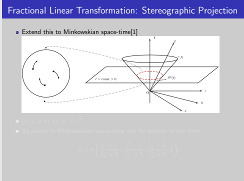

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- Can denote an LT by moving three arbitrary points along the surface of the spher as the generators have dimension two, so to match the dim of the LT (it's 6) we need three of them
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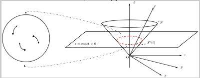
Fractional Linear Transformation: Stereographic Projection

- Extend this to Minkowskian space-time[1]



- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$

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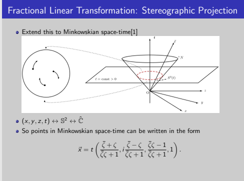

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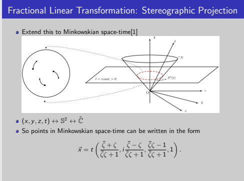

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Fractional Linear Transformation

- Make the transformation $\zeta \to \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U.

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\bar{\zeta}\zeta+1} & \frac{2t\zeta}{\bar{\zeta}\zeta+1} \\ \frac{2t\bar{\zeta}}{\bar{\zeta}\zeta+1} & \frac{2t\zeta\bar{\zeta}}{\bar{\zeta}\zeta+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix},$$

$$c_0' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

- Solve for ζ' to get the fractional linear transformation

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)}.$$

- There is a one to one correspondence between POLTs and fractional linear transformations

---

- These are null directions
- Refer to eqn (27) which should be on the board
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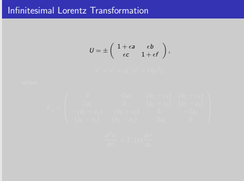

- This was done in a recent lecture(Relativistic QM) so I wont do it
- Where a,b,c,d are complex
- Take many infinitesimal LT steps along a particles trajectory and let ϵ go to zero

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix},$$

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2),$$

where

$$L^i{}_j = \begin{pmatrix} 0 & -2a_2 & (b_1 - c_1) & (b_1 + c_1) \\ 2a_2 & 0 & (b_2 + c_2) & (b_2 - c_2) \\ -(b_1 - c_1) & -(b_2 + c_2) & 0 & -2a_1 \\ (b_1 + c_1) & (b_2 - c_2) & -2a_1 & 0 \end{pmatrix}$$

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Infinitesimal Lorentz Transformation: Lorentz Force

- Can rewrite this equation in terms of the particles 3-velocity $\vec{u}$, in component form

$$\begin{aligned}
\frac{d}{dt}(\gamma(u)u^{(1)}) &= -2a_2u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1,\\
\frac{d}{dt}(\gamma(u)u^{(2)}) &= 2a_2u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2,\\
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- Define the 3-vectors

$$\begin{aligned}
\vec{P} &= (b_1 + c_1, b_2 - c_2, -2a_1),\\
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Infinitesimal Lorentz Transformation: Lorentz Force

- Writing the equations in terms of these

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q},$$

- This is the same form as the Lorentz force
- Make the Identification

$$\vec{P} = \frac{q}{m}\vec{E}, \qquad \vec{Q} = \frac{q}{m}\vec{B}, \tag{1}$$

- To be compatible with special relativity the Lorentz force must depend on $\vec{u}$ in this way. So the Lorentz force is a special case of a charged particle moving along a world line in Minkowskian space-time generated by an infinitesimal Lorentz transformation.

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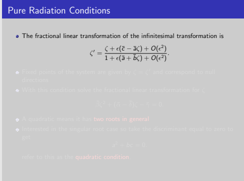

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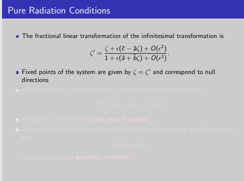

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Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions
- With this condition solve the fractional linear transformation for ζ

$$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0.$$

- A quadratic means it has two roots in general
- Interested in the singular root case so take the discriminant equal to zero to get

$$a^2 + bc = 0.$$

refer to this as the quadratic condition.

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- Then the real and imaginary parts of the quadratic condition give us the relations

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- The a and b are related to $\vec{E}$ and $\vec{B}$ through the Lorentz force as

$$a_1 = -\frac{1}{2}E^3, \qquad b_2 = \frac{1}{2}(E^2 + B^1), \qquad c_1 = \frac{1}{2}(E^1 + B^2),$$
$$a_2 = -\frac{1}{2}B^3, \qquad b_1 = \frac{1}{2}(E^1 - B^2), \qquad c_2 = \frac{1}{2}(B^1 - E^2).$$

- Then the real and imaginary parts of the quadratic condition give us the relations

$$|\vec{E}|^2 = |\vec{B}|^2,$$
$$\vec{E} \cdot \vec{B} = 0.$$

- These are the familiar pure radiation conditions. Thus if the world line of a charged particle is generated by an infinitesimal singular Lorentz transformation then the particle is moving in a pure radiation EM field.

- q/m factors suppressed for convenience