Singular Lorentz Transformations and Pure Radiation Fields

Kevin Maguire

April 21, 2014

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2. Strange Minkowskian Line Element
3. Singular Lorentz transformation

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Introduction: Lorentz Transformations

- A Lorentz transformation is defined by the preservation of the quadratic form

  $$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2,$$

  in the transformation $(x, y, z, t) \to (x', y', z', t')$
- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group $SO^+(1,3)$
- In general lorentz transformations have two invariant null directions

Introduction: Lorentz Transformations

- –Proper is det 1 . preserves the orientation of spacial axes, preserves handedness
- –orthochronous means time is always positive and the direction of time is preserve
- –Think of the standard Lornetz transformation, always two null directions at $x \pm$

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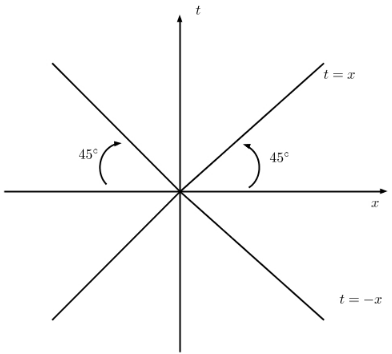

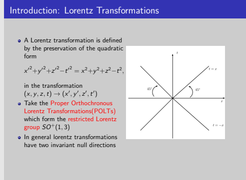

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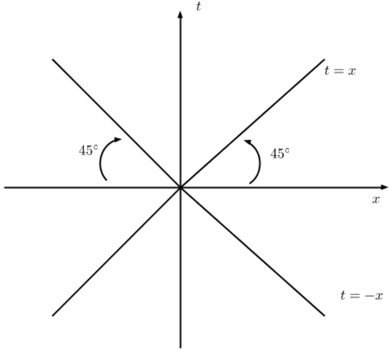

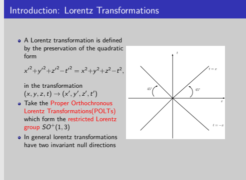

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- derive a strange minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial

Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2.$$

- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

- Make further coordinate transformations to obtain

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right) \mathrm{d}u^2.$$

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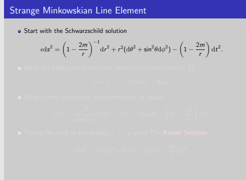

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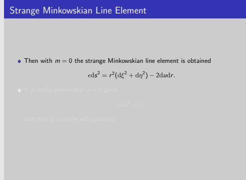
and thus is a single null geodesic.



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LTs that leave one null invariant direction are constructed

Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta = \xi + i\eta$, to get the new line element

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leave the single null geodesic $r = 0$ invariant.
- Now see what this looks like in Cartesian coordinates

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