

Introduction: Lorentz Transformations

▼ A Lorentz transformation is defined by the preservation of the quadratic form

$$x^2 + y^2 + z^2 - t^2 = x'^2 + y'^2 + z'^2 - t'^2,$$

in the transformation
 $(x, y, z, t) \rightarrow (x', y', z', t')$

► [Lorentz Transformations](#)
 Lorentz Transformations (P20.14)
 which form the restricted Lorentz group $SO^+(1, 3)$.

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- –Proper is $\det 1$. preserves the orientation of spacial axes, preserves handedness
- –orthochronous means time is always positive and the direction of time is preserved
- –Think of the standard Lorentz transformation, always two null directions at $x \pm t$

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• Take the **Proper Orthochronous Lorentz Transformations** (POTs) which form the **restricted Lorentz group** $SO^+(1, 3)$

• The **proper orthochronous Lorentz group** is the group of Lorentz transformations that preserve the orientation of space and time.

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- –Think of the standard Lorentz transformation, always two null directions at $x \pm t = 0$

Introduction: Lorentz Transformations

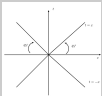
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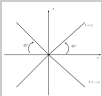
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 Layout

- derive a strange Minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial

Strange Minkowskian Line Element

Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right) dr^2.$$
- Make the coordinate transformation $r = r(u, v)$

$$r = \frac{1}{2}(v - u), \quad t = \frac{1}{2}(v + u)$$
- Make another coordinate transformation $u = u(\rho, \sigma)$

$$u = \frac{1}{2} \ln \frac{1 - \rho}{1 - \sigma}, \quad v = \frac{1}{2} \ln \frac{1 + \rho}{1 + \sigma}$$
- Doing the above transformations gives the **Kasner Solution**

$$ds^2 = -d\rho^2 + d\sigma^2 + \frac{d\rho^2 d\sigma^2}{\rho^2 \sigma^2}$$

- First we are going to derive a strange form of the Minkowskian line element.. of the vacuum field equations, which will be familiar to most of us
- to remove the coordinate singularity in the Schwarzschild solution
- These transformations put the line element in a form where we can take the limit the energy goes to 0
- It is easily shown with further coordinate transforms that this is Kasner, but it won't be done here

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- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$
- Make another coordinate transformation to obtain

$$v = t + r, \quad \rho = r - 2m, \quad \theta = \theta, \quad \phi = \phi.$$
- Convert the line element to (v, ρ, θ, ϕ) coordinates. The **Kasner Solution**

$$ds^2 = -\frac{1}{4\rho} dv^2 + \rho dv d\rho + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

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Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

Make further coordinate transformations to obtain

$$ds^2 = \frac{r^2}{\cosh^2 \rho \csc^2 \theta} (dz^2 + dy^2) - 2dzdr - \left(r^2 - \frac{2k}{r}\right) dr^2.$$

Conformal transformation to obtain the Kasner Solution

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Strange Minkowskian Line Element

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$$cds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right) dr^2.$$

Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

Make further coordinate transformations to obtain

$$cds^2 = \frac{r^2}{\cosh^2 \rho} (du^2 + dv^2) - 2du dv - \left(r^2 - \frac{2k}{r}\right) dr^2.$$

Taking the limit as the energy $\mu \rightarrow 0$ gives The **Kasner Solution**

$$cds^2 = r^2(du^2 + dv^2) - 2du dv - \frac{2k}{r} dr^2.$$

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$$u = t - r - 2m \ln(r - 2m).$$

Make further coordinate transformations to obtain

$$cds^2 = \frac{r^2}{\cosh^2 \rho} (d\zeta^2 + d\eta^2) - 2d\zeta dr - \left(r^2 - \frac{2k}{r}\right) dr^2.$$

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Strange Minkowskian Line Element

Then with $m = 0$ the strange Minkowskian line element is obtained

$$c^2 ds^2 = r^2 (dt^2 + d\eta^2) - 2du dr.$$

See also [1] and [2].

$$\frac{1}{r} \frac{dr}{dt} = \frac{1}{r} \frac{dr}{d\eta}$$

and this is zero for $r = 0$ and $r = \infty$.

- Its easily shown with suitable coordinate transforms that this is Minkowskian line element
- This is best shown by calculating the geodesic equations after the Eddington-Finkelstein coordinate transforms, all zero if u is proper time along the geodesic

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Set $r = 0$ to give

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 Layout

- Introduction: Lorentz Transformations
- Strange Minkowskian Line Element
- Singular Lorentz transformation
- Lightlike Lorentz transformations
- Subluminal Lorentz transformations
- Superluminal Lorentz transformations
- Null Rotations

LTs that leave one null invariant direction are constructed

Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta := \xi + i\eta$, to get the new line element[1]

$$e d\bar{s}^2 = r^2 d\zeta d\bar{\zeta} - 2d\zeta dr$$

- The transformation $\zeta \rightarrow \bar{\zeta}$ corresponds to a time-reversal and leaves the singular direction $\zeta = \bar{\zeta}$ invariant

- For a general coordinate the transformation becomes

$$x' = x \cosh \eta - y \sinh \eta, \quad y' = -x \sinh \eta + y \cosh \eta$$

$$z' = z, \quad t' = t$$

$$x' = x \cosh \eta - y \sinh \eta, \quad y' = -x \sinh \eta + y \cosh \eta$$

- The singular Lorentz transformations are a 2-parameter subgroup of the singular Lorentz transformations, form a 2-parameter abelian subgroup of the Lorentz group

- This is what we want, An LT which leaves one null invariant.
- The use in the previous coord transforms was to make this transformation look trivial
- So this is what the seemingly trivial transformation looks like in Cartesian
- Again its clear that $r = 0$ keeps one direction fixed, as then $z=t$
- but it doesn't work both ways, not all 2 parameter abelian subgroups are singular Lorentz transformations

Singular Lorentz Transformation

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$$e^2 d\bar{s}^2 = r^2 d\zeta d\bar{\zeta} - 2d\zeta dr.$$

- The transformation $\zeta \rightarrow \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic $r = 0$ invariant.

[1] For a detailed derivation of the transformation element

$$e^2 d\bar{s}^2 = r^2 d\zeta d\bar{\zeta} - 2d\zeta dr$$

$$\zeta = \frac{1}{2} \ln \frac{1 + iu + v}{1 - iu + v}, \quad \bar{\zeta} = \frac{1}{2} \ln \frac{1 - iu + v}{1 + iu + v}$$

$$r = \frac{1}{2} \ln \frac{1 + iu + v}{1 - iu + v} - \frac{1}{2} \ln \frac{1 - iu + v}{1 + iu + v}$$

[2] The singular Lorentz transformations in two dimensions form a 2 parameter abelian subgroup of the Lorentz group.

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- In Cartesian coordinates this transformation becomes

$$\begin{aligned} x' + iy' &= x + iy + w(t - z), \\ x' - t' &= -t - z, \\ x' + t' &= x + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \end{aligned}$$

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 Layout

- shown here that there is a 2 to 1 correspondence between $SL(2, \mathbb{C})$ and POLTs

$SL(2, \mathbb{C})$ Matrices of the POLT

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- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices

$$X = \begin{pmatrix} t & \mathbf{x} \\ \mathbf{x} & t \end{pmatrix}$$
- For a vector \mathbf{a} we associate to it the corresponding Hermitian matrix

$$A = \begin{pmatrix} a_0 & \mathbf{a} \\ \mathbf{a} & a_0 \end{pmatrix}$$
- Consider the transformation $X \rightarrow AXA^\dagger$

$$X' = \begin{pmatrix} t' & \mathbf{x}' \\ \mathbf{x}' & t' \end{pmatrix}$$

is a Lorentz transformation

- Complex Hermitian matrices have 4 independent components, so the element of such a matrix can be used to represent points in Minkowskian space-time.
- where $\alpha, \beta, \gamma, \delta$ are complex its an element of the special linear group. This means it has determinant 1. **write it on the board**

$SL(2, \mathbb{C})$ Matrices of the POLT

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Construct the following matrix:

$$A = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix},$$

- For a given point in Minkowskian space-time, there is a unique Hermitian matrix A associated with it.
- Conversely, for a given Hermitian matrix A , there is a unique point in Minkowskian space-time associated with it.

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$$A = \begin{pmatrix} t-x & x+iy \\ x-iy & t+x \end{pmatrix},$$

This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(x)) = t^2 - x^2 - y^2 - z^2.$$

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$$\det(A(x)) = t^2 - x^2 - y^2 - z^2.$$
- Construct the transformation $A(\vec{x}) = UA(\vec{z})U^\dagger$, where

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$
 is an element of $SL(2, \mathbb{C})$

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└ $SL(2, \mathbb{C})$ Matrices of the POLT

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✓ $A(p)$ and $A(i)$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus it is a Lorentz transformation.

$$A(p) = \begin{pmatrix} 1 + \frac{p^2}{2} & p \\ p & 1 - \frac{p^2}{2} \end{pmatrix}$$

$$A(i) = \begin{pmatrix} 1 - \frac{p^2}{2} & ip \\ ip & 1 + \frac{p^2}{2} \end{pmatrix}$$

• This is a general relation

$$A(p) = \frac{1}{2} \left(\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$A(i) = \frac{1}{2} \left(\begin{pmatrix} 1 & ip \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

- This is because the determinant of U is 1
- I want to show you an example calculation of U, to do this we write in component form

$SL(2, \mathbb{C})$ Matrices of the POLT

$SL(2, \mathbb{C})$ Matrices of the POLT

- $A(\vec{v})$ and $A(\vec{u})$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus it is a Lorentz transformation.
- Write this transformation component wise

$$\begin{pmatrix} t' - x' & x' + iy' \\ x' - iy' & t' + x' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t - x & x + iy \\ x - iy & t + x \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} \\ = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-x)\bar{\alpha} + (x+iy)\bar{\beta} & (t-x)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha} + (t+x)\bar{\beta} & (x-iy)\bar{\gamma} + (t+x)\bar{\delta} \end{pmatrix}.$$

• This is the general relation

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- Thus the **general relations**

$$\begin{aligned} t' - x' &= (t-x)\alpha\bar{\alpha} + (x+iy)\alpha\bar{\beta} + (x-iy)\bar{\alpha}\bar{\beta} + (t+x)\bar{\beta}\bar{\beta}, \\ x' + iy' &= (t-x)\alpha\bar{\gamma} + (x+iy)\alpha\bar{\delta} + (x-iy)\bar{\alpha}\bar{\gamma} + (t+x)\bar{\beta}\bar{\delta}, \\ t' + x' &= (t-x)\gamma\bar{\alpha} + (x+iy)\gamma\bar{\beta} + (x-iy)\bar{\gamma}\bar{\alpha} + (t+x)\bar{\delta}\bar{\beta}. \end{aligned}$$

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Example: Singular Lorentz transformation

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Take the singular Lorentz transformation from earlier

$$t' - z' = t - z,$$

$$x' + y' = x + y + w(t - z),$$

$$t' + z' = t + z + w(x - y) + w(x + y) + w(t - z).$$

Express coefficients in the first 2 equations in terms of the given relations in the second equation alone

$$\begin{aligned} w(t - z) &= \frac{1}{2}(t' - z' - t + z) \\ w(t - z) &= \frac{1}{2}(t' - z' - t + z) \end{aligned}$$

So there are **two** possible choices of w

$$w = \pm \frac{1}{2} \left(\frac{t' - z' - t + z}{t - z} \right)$$

Thus, there is a 2 to 1 correspondence between elements of $SL(2, \mathbb{C})$ and POLTs

- Where we have also used $\det(U) = 1$
- because the sign doesn't matter is still a solution Eqn(27)
- A, A' are points in Minkowskian space time, and $\pm U$ are POLTs

Example: Singular Lorentz transformation

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Take the singular Lorentz transformation from earlier

$$t' - z' = t - z,$$

$$x' + y' = x + y + w(t - z),$$

$$t' + z' = t + z + w(x - y) + \tilde{w}(x + y) + w\tilde{w}(t - z).$$

Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$\alpha = \pm 1, \quad \beta = 0,$$

$$\gamma = \tilde{w}\alpha, \quad \delta = \alpha.$$

So there are always **two** possible choices of α

$$U = \begin{pmatrix} \alpha & 0 & \tilde{w}\alpha \\ 0 & 1 & 0 \\ \tilde{w}\alpha & 0 & \alpha \end{pmatrix}$$

Thus, there is a 2 to 1 correspondence between elements of $SL(2, \mathbb{C})$ and POLTs

- Where we have also used $\det(U) = 1$
- because the sign doesn't matter is still a solution Eqn(27)
- A, A' are points in Minkowskian space time, and $\pm U$ are POLTs

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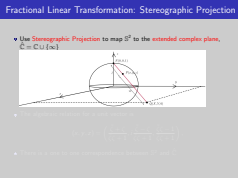
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- Connect Minkowskian space to the 2-sphere by stereographic projection, so we can use points on a 2 sphere to think about LTs

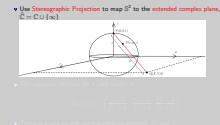
Fractional Linear Transformation: Stereographic Projection



- As we know, stereographic projection doesn't map the point N at the top of the circle, so that's why we map N to infinity and need to consider the extended complex plane
- It can also be written in terms of θ and ϕ

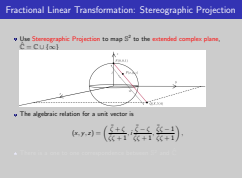
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Fractional Linear Transformation: Stereographic Projection



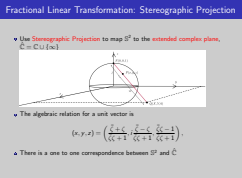
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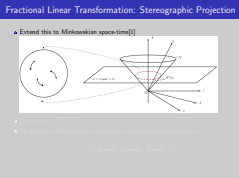
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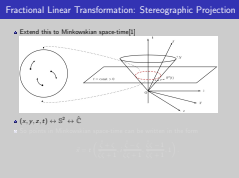
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└ Fractional Linear Transformation: Stereographic Projection



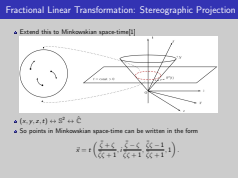
- all the points on the 2 sphere are generators of the future null cone in Minkowski space time
- Can denote an LT by moving three arbitrary points along the surface of the sphere as the generators have dimension two, so to match the dim of the LT (it's 6) we need three of them
- Extra coord is because we take time into account now, now ζ has two parameters so x is in terms of two parameters, t just defines the direction

└ Fractional Linear Transformation: Stereographic Projection



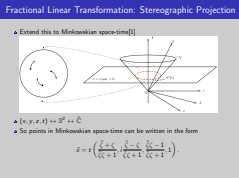
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Fractional Linear Transformation

- Make the transformation $\zeta \rightarrow \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U .

$$A(\vec{x}) = \begin{pmatrix} \frac{\partial \zeta'}{\partial \zeta} & \frac{\partial \zeta'}{\partial \bar{\zeta}} \\ \frac{\partial \bar{\zeta}'}{\partial \zeta} & \frac{\partial \bar{\zeta}'}{\partial \bar{\zeta}} \end{pmatrix} = \alpha \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & 1 \end{pmatrix}.$$

$$\alpha = \frac{1}{2} \left(\frac{\partial \zeta'}{\partial \zeta} + \frac{\partial \bar{\zeta}'}{\partial \bar{\zeta}} \right) = \frac{1}{2} \left(\frac{\partial \zeta'}{\partial \zeta} + \frac{\partial \bar{\zeta}'}{\partial \bar{\zeta}} \right)$$

- Take the α to get the fractional linear transformation

$$\frac{\zeta' - \alpha}{\bar{\zeta}' - \bar{\alpha}}$$

- There is a one to one correspondence between PGL's and fractional linear transformations

- These are null directions
- Refer to eqn (27) which should be on the board
- AS we did in the previous example, determine U
- Remember we had $\pm U$ now the signs will cancel in the denominator and numerator

Fractional Linear Transformation

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$$\alpha' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & 1 \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & \bar{a} \end{pmatrix} \alpha \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & 1 \end{pmatrix} \begin{pmatrix} a & \beta \\ \gamma & \bar{a} \end{pmatrix}.$$

- Now let's compare to fractional linear transformations

$$\frac{\zeta' - \gamma}{\zeta' - \bar{\gamma}} = \frac{a\zeta + \beta}{\gamma\zeta + \bar{a}}$$

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Fractional Linear Transformation

- Make the transformation $\zeta \rightarrow \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U .

$$A(\vec{x}) = \begin{pmatrix} \frac{\partial}{\partial \zeta'^1} & \frac{\partial}{\partial \zeta'^2} \\ \frac{\partial}{\partial \zeta'^3} & \frac{\partial}{\partial \zeta'^4} \end{pmatrix} = \alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix}.$$

$$\alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}.$$

- Solve for ζ' to get the **fractional linear transformation**

$$\zeta' = \frac{\gamma + \delta \zeta}{\delta + \beta \zeta}$$

- Then use ζ' to see correspondence between PGL's and fractional linear transformations

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Fractional Linear Transformation

▲ Make the transformation $\zeta \rightarrow \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U .

$$A(\vec{x}) = \begin{pmatrix} \frac{\partial}{\partial \zeta'^4} & \frac{\partial}{\partial \zeta'^3} \\ \frac{\partial}{\partial \zeta'^2} & \frac{\partial}{\partial \zeta'^1} \end{pmatrix} = \alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix}.$$

$$\alpha' \begin{pmatrix} 1 & \zeta' \\ \zeta' & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}.$$

▲ Solve for ζ' to get the **fractional linear transformation**

$$\zeta' = \frac{(\gamma + \delta \zeta)}{(\alpha + \beta \zeta)}$$

▼ There is a **one to one correspondence** between PGLTs and fractional linear transformations

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Fractional Linear Transformation

▲ Make the transformation $\zeta \rightarrow \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U .

$$A(\vec{x}) = \begin{pmatrix} \frac{\partial}{\partial \zeta'^1} & \frac{\partial}{\partial \zeta'^2} \\ \frac{\partial}{\partial \zeta'^3} & \frac{\partial}{\partial \zeta'^4} \end{pmatrix} = \alpha \begin{pmatrix} 1 & \zeta \\ \zeta & 1 \end{pmatrix}.$$

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 Layout

- Introduction: Lorentz Transformations
- Strange Minkowskian Line Element
- Singular Lorentz transformation
- $SL(2, \mathbb{C})$ Matrices of the Lorentz Transformation
- The Fractional Linear Transformation
- Infinitesimal Lorentz Transformation



- temp

2014-04-23

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1 + i\epsilon & -ib \\ ic & 1 + i\epsilon \end{pmatrix},$$

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$$U = \pm \begin{pmatrix} 1 + i\epsilon & -ib & -ic & 0 \\ i\epsilon & 1 + i\epsilon & 0 & 0 \\ 0 & 0 & 1 + i\epsilon & -ib \\ 0 & 0 & ic & 1 + i\epsilon \end{pmatrix}$$

- This was done in a recent lecture(Relativistic QM) so I wont do it
- Where a,b,c,d are complex
- Take many infinitesimal LT steps along a particles trajectory and let ϵ go to zero

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1 + i\epsilon & -ib \\ i\epsilon & 1 + i\epsilon \end{pmatrix},$$

$$x' = x' + iL_j x^j + O(\epsilon^2),$$

where

$$L_j = \begin{pmatrix} 0 & -2a_j & (b_j - c_j) & (b_j + c_j) \\ 2a_j & 0 & (b_j + c_j) & (b_j - c_j) \\ -(b_j - c_j) & -(b_j + c_j) & 0 & -2a_j \\ (b_j + c_j) & (b_j - c_j) & -2a_j & 0 \end{pmatrix}$$

$$a_j = \frac{1}{2} \epsilon_{jkl} \omega_k x_l$$

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Singular Lorentz Transformations and Pure Radiation Fields

2014-04-23

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1 + i\epsilon & -\epsilon b \\ \epsilon c & 1 + i\epsilon f \end{pmatrix},$$

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$$\frac{d^2 x^j}{ds^2} = L_j^i(x) \frac{dx^i}{ds}.$$

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Infinitesimal Lorentz Transformation: Lorentz Force

Can rewrite this equation in terms of the particle's 3-velocity \vec{u} , in component form

$$\frac{d}{dt}(\gamma(u)u^{(1)}) = -2a_1 u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1,$$

$$\frac{d}{dt}(\gamma(u)u^{(2)}) = 2a_1 u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2,$$

$$\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1,$$

$$\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1 u^{(3)}.$$

• Lorentz force

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

- Clear that these equations can be expressed in terms of P and Q

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$$\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1,$$

$$\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1 u^{(3)}.$$

Define the 3-vectors

$$\vec{P} = (b_1 + c_1, b_2 - c_2, -2a_1),$$

$$\vec{Q} = (b_2 + c_2, -(b_1 - c_1), -2a_2).$$

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- Clear that these equations can be expressed in terms of \vec{P} and \vec{Q}

Infinitesimal Lorentz Transformation: Lorentz Force

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▲ Writing the equations in terms of these

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{E} + \vec{u} \times \vec{B},$$

- This is the relativistic version of **Lorentz force**
- More on acceleration

$$\vec{u} = \frac{d\vec{x}}{dt}, \quad \gamma(u) = \frac{1}{\sqrt{1-u^2}}$$

▲ To be compatible with special relativity the Lorentz force must depend on **u** in this way. In the Lorentz force is a relativistic version of a classical equation, having along a field of pure electromagnetic fields, generated by a non-relativistic Lorentz transformation.

└ Infinitesimal Lorentz Transformation: Lorentz Force

- temp

Infinitesimal Lorentz Transformation: Lorentz Force

▲ Writing the equations in terms of these

$$\frac{d}{ds}(\gamma(u)\dot{x}) = \vec{P} + \vec{a} \times \vec{Q},$$

▼ This is the same form as the **Lorentz force**

$$\vec{F} = \frac{q}{c} \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right)$$

▲ To be compatible with special relativity the Lorentz force must depend on \vec{v} in this way. In the Lorentz force is a natural case of a charged particle moving along a world line in electromagnetic fields which gives rise to a relativistic Lorentz transformation.

Infinitesimal Lorentz Transformation: Lorentz Force

- Writing the equations in terms of these

$$\frac{d}{ds}[\gamma(u)\vec{u}] = \vec{P} + \vec{u} \times \vec{Q},$$

- This is the same form as the **Lorentz force**
- Make the identification

$$\vec{P} = \frac{q}{m} \vec{E}, \quad \vec{Q} = \frac{q}{m} \vec{B}, \quad (1)$$

- To be compatible with special relativity the Lorentz force must depend on \vec{u} in this way. In the Lorentz force is a natural choice to interpret particles moving along a world line in Minkowski space-time, associated to a 4-dimensional Lorentz transformation.

- temp

Infinitesimal Lorentz Transformation: Lorentz Force

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- To be compatible with special relativity the Lorentz force must depend on \vec{u} in this way. So the Lorentz force is a special case of a charged particle moving along a world line in Minkowskian space-time generated by an infinitesimal Lorentz transformation.

- temp



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- temp

Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + a(\zeta - b\zeta) + O(\zeta^2)}{1 + a(\zeta + b\zeta) + O(\zeta^2)}$$

- From which we find the transformation of the infinitesimal transformation
- From which we find the transformation of the infinitesimal transformation
- In particular, we find the transformation of the infinitesimal transformation
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which is the condition

- This is derived from the form of the infinitesimal U and from the fractional linear transformation formula
- This is one of the things I said I would show earlier
- THIS WILL BE VERY IMPORTANT, **write on the board**

Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + a(\zeta - b\zeta') + O(\zeta'^2)}{1 + a(\zeta + b\zeta') + O(\zeta'^2)}$$

- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions

- The fixed points are given by the roots of the quadratic equation

$$\zeta^2 - 2a\zeta + b = 0$$

- In particular, when $a = 0$, the roots are given by

$$\zeta = 0, \zeta = b$$

- In general, the fixed points are given by the roots of the quadratic equation

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- With this condition solve the fractional linear transformation for ζ

$$\beta\zeta^2 + (\delta - \beta)\zeta - \gamma = 0.$$

- In particular, choose $\gamma = 0$. This leads to general

- conditions in the singular case. Using the transformation from (1) to (2)

$$\zeta' = \frac{\zeta + a}{1 + b\zeta}$$

- one obtains the pure radiation condition

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Pure Radiation Conditions

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- With this condition solve the fractional linear transformation for ζ

$$\beta\zeta^2 + (\delta - \beta)\zeta - \gamma = 0.$$

- A quadratic means it has **two roots in general**

← The two roots are the null directions of the radiation field. The condition $\beta\zeta^2 + (\delta - \beta)\zeta - \gamma = 0$ is the **quadratic condition**.

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Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + a(\zeta - b\zeta) + O(\zeta^2)}{1 + a(\zeta + b\zeta) + O(\zeta^2)}$$

- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions

- With this condition solve the fractional linear transformation for ζ

$$\beta\zeta^2 + (\delta - \beta)\zeta - \gamma = 0.$$

- A quadratic means it has **two roots in general**

- Interested in the singular root case so take the discriminant equal to zero to get

$$a^2 + bc = 0.$$

refer to this as the **quadratic condition**.

- This is derived from the form of the infinitesimal U and from the fractional linear transformation formula
- This is one of the things I said I would show earlier
- THIS WILL BE VERY IMPORTANT, **write on the board**

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▼ The a and b are related to \vec{E} and \vec{B} through the Lorentz force as

$$\begin{aligned} a_1 &= -\frac{1}{2}E^2, & b_2 &= \frac{1}{2}(E^2 + B^2), & c_1 &= \frac{1}{2}(E^2 + B^2), \\ a_2 &= -\frac{1}{2}B^2, & b_1 &= \frac{1}{2}(E^2 - B^2), & c_2 &= \frac{1}{2}(B^2 - E^2). \end{aligned}$$

▼ Using the two-dimensional basis of the radiation conditions on the radiation

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} + \begin{pmatrix} \vec{B} \\ -\vec{E} \end{pmatrix}$$

▼ Therefore, the Lorentz force is null, i.e., $\vec{F} = 0$. If the single force is a charged particle is generated by an infinitesimal singular Lorentz transformation then the particle is moving in a pure radiation EM field.

- q/m factors suppressed for convenience

Pure Radiation Conditions

✓ The a and b are related to \vec{E} and \vec{B} through the Lorentz force as

$$\begin{aligned} a_0 &= -\frac{1}{2}E^2, & b_0 &= \frac{1}{2}(E^2 + B^2), & c_0 &= \frac{1}{2}(E^2 + B^2), \\ a_1 &= -\frac{1}{2}B^2, & b_1 &= \frac{1}{2}(E^2 - B^2), & c_1 &= \frac{1}{2}(B^2 - E^2). \end{aligned}$$

✓ Then the real and imaginary parts of the quadratic condition give us the relations

$$\begin{aligned} |\vec{E}|^2 &= |\vec{B}|^2, \\ \vec{E} \cdot \vec{B} &= 0. \end{aligned}$$

✦ If the right-hand side of the quadratic condition is zero, then the Lorentz force on a charged particle is generated by an infinitesimal singular Lorentz transformation that the particle is moving in a pure radiation EM field.

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Pure Radiation Conditions

✓ The a and b are related to \vec{E} and \vec{B} through the Lorentz force as

$$\begin{aligned} a_0 &= -\frac{1}{2}E^2, & b_0 &= \frac{1}{2}(E^2 + B^2), & c_0 &= \frac{1}{2}(E^2 + B^2), \\ a_1 &= -\frac{1}{2}B^2, & b_1 &= \frac{1}{2}(E^2 - B^2), & c_1 &= \frac{1}{2}(B^2 - E^2). \end{aligned}$$

✓ Then the real and imaginary parts of the quadratic condition give us the relations

$$\begin{aligned} |\vec{E}|^2 &= |\vec{B}|^2, \\ \vec{E} \cdot \vec{B} &= 0. \end{aligned}$$

■ These are the familiar pure radiation conditions. Thus if the world line of a charged particle is generated by an infinitesimal singular Lorentz transformation then the particle is moving in a pure radiation EM field.

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