The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

I. THE LORENTZ TRANSFORMATION

The Lorentz Transform is defined by

$$(x, y, z, t) \to (x', y', z', t') \text{ such that}$$
$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

If the transformation preserves the orientation of the spatial axes then is it called a proper Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also If $t \geq 0 \Rightarrow t' \geq 0$ then it is called an orthochronous Lorentz transforamtion. This ensures that the time direction is preserved. In this project the "Lorentz transforamation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as follows (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation:

$$x' = \gamma(x - vt) \ \text{, where } \gamma = (1 - v^2)^{-1/2}$$
$$t' = \gamma(t - vx)$$

Rearranging:

$$x' - t' = \gamma(1 + v)(x - t)$$
$$x' + t' = \gamma(1 - v)(x + t)$$

It is clear that:

$$x = \pm t \leftrightarrow x' = \pm t'$$

Thus there are two null directions (are null directions by definition invarient???)in this space-time at $x = \pm t$. It can be shown that all Lorentz transformations have two invarient null directions except the singular Lorentz transformation which has only one fixed null direcion.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

(what do we want the Kasner vacuum solution for???)
Starting with the Schwarschild Solution of the vacuum field equations:

$$\epsilon ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right)dt^2 \tag{1}$$

Now make the Eddington-Finkelstein coordinate transformation:

$$u = t - r - 2m \ln(r - 2m)$$

and write 1 in terms of u to obtain:

$$\epsilon ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2dudr - du^2 + \frac{2m}{r}du^2$$

See

Appendix A: Calculations

II. ACKNOWLEDGEMENTS