Singular Lorentz Transformations and Pure Radiation Fields

Kevin Maguire

April 22, 2014



2014-04-22



- Introduction: Lorentz Transformations
- Strange Minkowskian Line Element
- Singular Lorentz transformation
- \bigcirc $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation
- The Fractional Linear Transformation
- Infinitesimal Lorentz Transformation
- Pure Radiation conditions

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Introduction: Lorentz Transformations

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 A Lorentz transformation is defined by the preservation of the quadratic form

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

in the transformation

$$(x,y,z,t) \rightarrow (x',y',z',t')$$

- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group SO⁺(1,3)
- In general lorentz transformations



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Introduction: Lorentz Transformations



stroduction: Lorentz Transformations

- -Proper is det 1 . preserves the orientation of spacial axes, preserves handedness
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Introduction: Lorentz Transformations



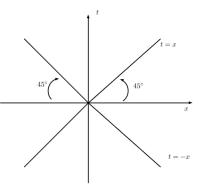
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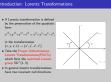




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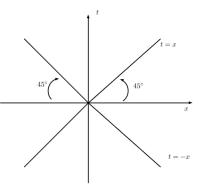
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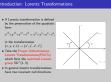




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- derive a strange minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial

Start with the Schwarzschild solution

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2.$$

• Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m)$$

• Make further coordinate transformations to obtain

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 u\xi} (d\xi^2 + d\eta^2) - 2dudr - \left(\mu^2 - \frac{2k}{r}\right) du^2.$$

ullet Taking the limit as the energy, $\mu o 0$ gives The Kasner Solution

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Strange Minkowskian Line Element

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Strange Minkowskian Line Element

 $\begin{aligned} & \text{there with the Schwarzschild extension} \\ & \text{district with the Schwarzschild extension} \\ & \text{all} z^2 \left((-\frac{1}{r_0^2})^2 \delta^2 + r^2(\delta^2 + u^2) \delta \delta^2 \right) - \left(1 - \frac{2r}{r_0^2} \right) dt^2. \\ & \text{a Make the Editing contribution constraints transformations of the Schwarzschild (solid for the constraints)} \\ & \text{while further constraints contribute to the data in the Schwarzschild (solid for the constraints)} \\ & \text{solid } z^2 = \frac{r_0^2}{\cos^2 r_0^2} (\delta^2 v^2 + dv^2 - 2\Delta i dv^2 - \left(x^2 - \frac{2r}{r} \right) du^2. \end{aligned}$

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• Then with m = 0 the strange Minkowskian line element is obtained

$$\epsilon ds^2 = r^2 (d\xi^2 + d\eta^2) - 2du dr.$$

• It is easily shown that r = 0 gives

$$\mathrm{d}s^2 = 0.$$

and thus is a null geodesic

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Strange Minkowskian Line Element

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Layout

otoduction: Lorentz Transformations trange Minkowskian Line Element ingular Lorentz transformation

LTs that leave one null invariant direction are constructed

• Define an arbitrary complex parameter $\zeta := \xi + i \eta,$ to get the new line element[3]

$$\epsilon ds^2 = r^2 d\zeta d\overline{\zeta} - 2dudr.$$

- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic r = 0 invariant.
- In Cartesian coordinates this transformation becomes

$$x' + iy' = x + iy + w(t - z),$$

 $x' - t' = -r = z - t,$
 $x' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z).$

• Addition of complex numbers is commutative, and *w* has two parameters, so the singular Lorentz transformations form a 2-parameter abelian subgroup of the Lorentz group



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—Singular Lorentz Transformation

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- This is what we want, An LT which leaves one null invariant.
- The use in the previous coord transforms was to make this transformation look trivial
- So this is what the seemingly trivial transformation looks like in cartesians
- Again its clear that r = 0 keeps one direction fixed, as then z=t
- but it doesn't work both ways, not all 2 parameter abelian subgroups are singular lorentz transformations

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Singular Lorentz Transformation

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Singular Lorentz Transformation $a \text{ Divin} \text{ is additive summing } parameter <math>\zeta = \xi + i \eta_1$ to get the new for size $m(\xi) = d^{-1} e^{-i \xi} e^{i \xi} e^{-i \xi} - 2 \sin \theta e^{-i \xi} e^{-$

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Singular Lorentz Transformation

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| June | Jun

ullet shown here that there is a 2 to 1 correspondence between SL(2,C) and POLTs

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Contruct the following matrix

$$A = \left(\begin{array}{cc} t - z & x + iy \\ x - iy & t + z \end{array}\right).$$

• This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

• Construct the transformation $A(\vec{x}') = UA(\vec{x})U^{\dagger}$, where

$$U = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right)$$

is an element of $SL(2,\mathbb{C})$



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 \subseteq $SL(2,\mathbb{C})$ Matrices of the POLT

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$SL(2,\mathbb{C})$ Matrices of the POLT
There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices

- Complex Hermitian matrices have 4 independant components, so the element of such a matrix can be used to represent points in Minkowskian space-time.
- where $\alpha, \beta, \gamma, \delta$ are complex its an element of the special linear group. This mean it has determinant 1. **write it on the board**

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Contruct the following matrix

$$A = \left(\begin{array}{cc} t - z & x + iy \\ x - iy & t + z \end{array}\right),$$

• This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

• Construct the transformation $A(\vec{x}') = UA(\vec{x})U^{\dagger}$, where

$$U = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right),$$

is an element of $SL(2,\mathbb{C})$



Singular Lorentz Transformations and Pure Radiation Fields

2014-04-22

 $-\mathit{SL}(2,\mathbb{C})$ Matrices of the POLT



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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields

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- $A(\vec{x}')$ and $A(\vec{x})$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus is a Lorentz transformation.
- Write this transformation component wise

$$\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix},$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha} + (x+iy)\bar{\beta} & (t-z)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha} + (t+z)\bar{\beta} & (x-iy)\bar{\gamma} + (t+z)\bar{\delta} \end{pmatrix}.$$

• Thus the general relations

$$t' - z' = (t - z)\alpha\bar{\alpha} + (x + iy)\alpha\bar{\beta} + (x - iy)\beta\bar{\alpha} + (t + z)\beta\bar{\beta},$$

$$x' + iy' = (t - z)\alpha\bar{\gamma} + (x + iy)\alpha\bar{\delta} + (x - iy)\beta\bar{\gamma} + (t + z)\beta\bar{\delta},$$

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Singular Lorentz Transformations and Pure Radiation Fields



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Singular Lorentz Transformations and Pure Radiation Fields

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 $^-SL(2,\mathbb{C})$ Matrices of the POLT

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SL(2 C) Matrices of the POLT

 $\begin{array}{c} x'+ij' \\ t'+x' \\ \delta \end{array} \right) \left(\begin{array}{ccc} t-x & x+ij' \\ x-ij' & t+x \end{array} \right) \left(\begin{array}{ccc} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\beta} & \tilde{\delta} \end{array} \right) \left(\begin{array}{ccc} (t-x)\tilde{\alpha} + (x+ij)\tilde{\beta} \\ (x-ij)\tilde{\alpha} + (x+ij)\tilde{\beta} \end{array} \right) \left(\begin{array}{ccc} (t-x)\tilde{\gamma} + (x+ij)\tilde{\beta} \\ (x-ij)\tilde{\alpha} + (t+x)\tilde{\beta} \end{array} \right) \left(\begin{array}{ccc} (t-x)\tilde{\alpha} + (x+ij)\tilde{\beta} \\ (x-ij)\tilde{\gamma} + (t+x)\tilde{\delta} \end{array} \right) .$

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Singular Lorentz Transformations and Pure Radiation Fields

 \subseteq $SL(2,\mathbb{C})$ Matrices of the POLT



SL(2 C) Matrices of the POLT

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$$t' - z' = t - z,$$

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• Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$\alpha = \pm 1, \qquad \beta = 0,$$
 $\gamma = \bar{w}\alpha, \qquad \delta = \alpha.$

So there are always two possible choices of U

$$U=\pm \left(egin{array}{cc} 1 & 0 \ ar{w} & 1 \end{array}
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Singular Lorentz Transformations and Pure Radiation Fields

Example: Singular Lorentz transformation

V Take the singular Lorentz transformation from softer t'-x'=t-x, x'+y'=t+y+w(t-x), $t'+x'=t+x+w(t-y)+\tilde{w}(x+\tilde{y})+w\tilde{w}(t-x).$

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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields

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Layout

- Introduction: Lorentz Transformations
- Strange Minkowskian Line Element
- Singular Lorentz transformation
- **3** $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation
- **1** The Fractional Linear Transformation
- Infinitesimal Lorentz Transformation
- Pure Radiation conditions



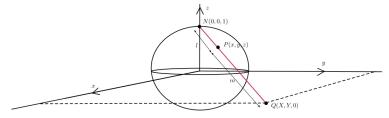
Singular Lorentz Transformations and Pure Radiation Fields

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O Introduction: Lorentz Transformations
O Strange Minkowskian Line Element
O Singular Lorentz transformation
O SiQ2_C) Matrices of the Lorentz Transformation
O The Fractional Linear Transformation

Connect Minkoswkian space to the 2-sphere by stereographic projection, so we cause points on a 2 sphere to think about LTs

• Use Stereographic Projection to map \mathbb{S}^2 to the extended complex plane, $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$



• The algebraic relation for a unit vector is

$$(x,y,z) = \left(\frac{\overline{\zeta} + \zeta}{\overline{\zeta}\zeta + 1}, i\frac{\overline{\zeta} - \zeta}{\overline{\zeta}\zeta + 1}, \frac{\overline{\zeta}\zeta - 1}{\overline{\zeta}\zeta + 1}\right),$$

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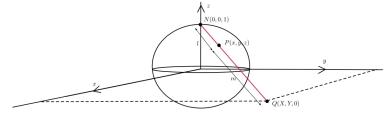
Singular Lorentz Transformations and Pure Radiation Fields

2014-04-22



- As we know, stereographic projection doesn't map the point N at the top of the circle, so thats why we map N to infinity and need to consider the extended complex plane
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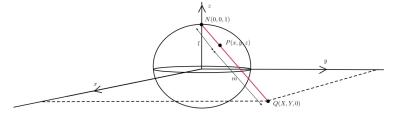
Singular Lorentz Transformations and Pure Radiation Fields



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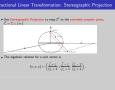
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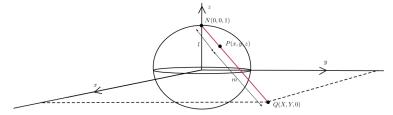
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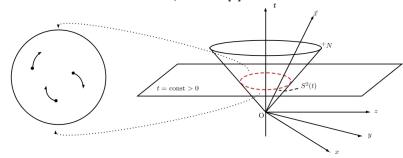
Singular Lorentz Transformations and Pure Radiation Fields

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• Extend this to Minkowskian space-time[1]

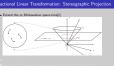


- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$
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$$\vec{x} = t \left(\frac{\vec{\zeta} + \zeta}{\vec{\zeta}\zeta + 1}, i \frac{\vec{\zeta} - \zeta}{\vec{\zeta}\zeta + 1}, \frac{\vec{\zeta}\zeta - 1}{\vec{\zeta}\zeta + 1}, 1 \right).$$

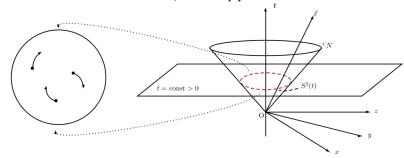


Singular Lorentz Transformations and Pure Radiation Fields



- all the points on the 2 sphere are generators of the future null cone in Minkowskia space time
- Can denote an LT by moving three arbitrary points along the surface of the spher
 as the generators have dimension two, so to match the dim of the LT (it's 6) we
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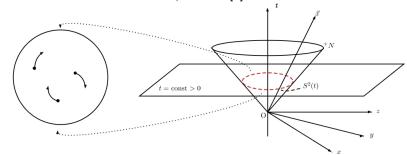


Singular Lorentz Transformations and Pure Radiation Fields

A formal thin is Minimized representation. Setting a prior A(x,y) is $A(x,y,x) \mapsto S^2 \mapsto \hat{\mathbb{C}}$

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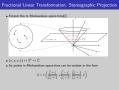


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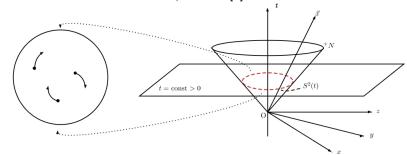


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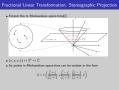


- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$
- So points in Minkowskian space-time can be written in the form

$$\vec{x} = t \left(\frac{\bar{\zeta} + \zeta}{\bar{\zeta}\zeta + 1}, i \frac{\bar{\zeta} - \zeta}{\bar{\zeta}\zeta + 1}, \frac{\bar{\zeta}\zeta - 1}{\bar{\zeta}\zeta + 1}, 1 \right).$$



Singular Lorentz Transformations and Pure Radiation Fields



- all the points on the 2 sphere are generators of the future null cone in Minkowskia space time
- Can denote an LT by moving three arbitrary points along the surface of the spher
 as the generators have dimension two, so to match the dim of the LT (it's 6) we
 need three of them
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• Make the transformation $\zeta \to \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U.

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} \frac{1}{\zeta} & \frac{\zeta}{\zeta\zeta} \\ \end{pmatrix},$$

$$c_0{'}\left(\begin{array}{cc} \frac{1}{\zeta'} & \zeta'\\ \overline{\zeta'} & \overline{\zeta'}\zeta' \end{array}\right) = \left(\begin{array}{cc} \alpha & \beta\\ \gamma & \delta \end{array}\right)c_0\left(\begin{array}{cc} \frac{1}{\zeta} & \zeta\\ \overline{\zeta} & \overline{\zeta}\zeta \end{array}\right)\left(\begin{array}{cc} \overline{\alpha} & \overline{\beta}\\ \overline{\gamma} & \overline{\delta} \end{array}\right).$$

• Solve for ζ' to get the fractional linear transformation

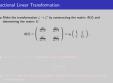
$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)}$$

 There is a one to one correspondence between POLTs and fractional linear transformations



Singular Lorentz Transformations and Pure Radiation Fields

Fractional Linear Transformation



These are null directions

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Singular Lorentz Transformations and Pure Radiation Fields

Fractional Linear Transformation

sectional Linear Transformation $Make the transformation <math>\xi - c^* \cdot b^* \cdot b$ constraints (the matrix A(t)) and determining the native $\xi^* - c^* \cdot b$ and $\xi^* - c^* \cdot b$. $A(t) = \begin{pmatrix} \frac{\partial t}{\partial t^2} & \frac{\partial t}{\partial t^2} \\ \frac{\partial t}{\partial t^2} & \frac{\partial t}{\partial t^2} \end{pmatrix} = \alpha \begin{pmatrix} \frac{t}{\xi} & \xi \\ \xi & \xi \end{pmatrix},$ $\alpha \begin{pmatrix} \frac{t}{\xi} & \xi \\ \xi & \xi \end{pmatrix} = \begin{pmatrix} \frac{t}{\xi} & \frac{t}{\xi} \\ \frac{t}{\xi} & \xi \end{pmatrix} = \begin{pmatrix} \frac{t}{\xi} & \xi \\ \frac{t}{\xi} & \xi \end{pmatrix}.$

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Singular Lorentz Transformations and Pure Radiation Fields

— Fractional Linear Transformation



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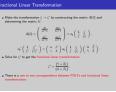
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Singular Lorentz Transformations and Pure Radiation Fields

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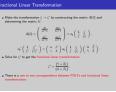
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Singular Lorentz Transformations and Pure Radiation Fields

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- Strange Minkowskian Line Element
- Singular Lorentz transformation
- **3** $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation
- **5** The Fractional Linear Transformation
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Singular Lorentz Transformations and Pure Radiation Fields

4-04-22

Layout

Introduction: Lorentz Transformations
 Strange Melanostian Line Element
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 SEC, Old Marion of the Lement Transformation
 The Transformation
 The Fractional Linear Transformation
 Infinitesimal Lorentz Transformation

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$$U=\pm\left(egin{array}{cc} 1+\epsilon a & \epsilon b \ \epsilon c & 1+\epsilon f \end{array}
ight),$$

$$\bar{x}^i = x^i + \epsilon L^i_{\ j} x^j + O(\epsilon^2),$$

where

$$L^{i}_{j} = \begin{pmatrix} 0 & -2a_{2} & (b_{1} - c_{1}) & (b_{1} + c_{1}) \\ 2a_{2} & 0 & (b_{2} + c_{2}) & (b_{2} - c_{2}) \\ -(b_{1} - c_{1}) & -(b_{2} + c_{2}) & 0 & -2a_{1} \\ (b_{1} + c_{1}) & (b_{2} - c_{2}) & -2a_{1} & 0 \end{pmatrix}$$

$$\frac{d^{2}x^{i}}{dc^{2}} = L^{i}_{j}(s)\frac{dx^{j}}{ds}.$$



Singular Lorentz Transformations and Pure Radiation Fields

 $U=\pm\left(\begin{array}{cc} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{array}\right),$

finitesimal Lorentz Transformation

14-04-22

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- Where a,b,c,d are complex
- ullet Take many infinitesimal LT steps along a particles trajectory and let ϵ go to zero

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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields



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Singular Lorentz Transformations and Pure Radiation Fields



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ullet Can rewrite this equation in terms of the particles 3-velocity $ec{u}$, in component form

$$\frac{d}{dt}(\gamma(u)u^{(1)}) = -2a_2u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1,
\frac{d}{dt}(\gamma(u)u^{(2)}) = 2a_2u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2,
\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1,
\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1u^{(3)}.$$

Define the 3-vectors

$$\vec{P} = (b_1 + c_1, b_2 - c_2, -2a_1),$$

 $\vec{Q} = (b_2 + c_2, -(b_1 - c_1), -2a_2)$



Singular Lorentz Transformations and Pure Radiation Fields

2014-04-22

Infinitesimal Lorentz Transformation: Lorentz Force

finitesimal Lorentz Transformation: Lorentz Force

Can rewrite this equation in terms of the particles 5-velocity d_i in composition $\frac{d}{dt}(\gamma_i(u)u^{(1)}) = -2s_2u^{(2)} + (b_1 - c_1)u^{(4)} + b_1 + c_1,$ $\frac{d}{dt}(\gamma_i(u)u^{(2)}) = 2s_2u^{(1)} + (b_2 + c_3)u^{(4)} + b_2 - c_2.$

 $\begin{aligned} & \frac{ds}{dt}(\gamma(u)u^{(2)}) &= 2s_2u^{(1)} + (b_2 + c_2)u^{(4)} + b_2 - c_2, \\ & \frac{ds}{dt}(\gamma(u)u^{(6)}) &= -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2s_2, \\ & \frac{d\gamma(u)}{dt} &= (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2s_1u^{(3)}. \end{aligned}$

temp

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Singular Lorentz Transformations and Pure Radiation Fields

Infinitesimal Lorentz Transformation: Lorentz Force

 $\begin{aligned} & \zeta \text{ on motive this equation in terms of the particle 3 solutiony } \vec{u}, \text{ is component term} \\ & \frac{d}{dt}(\zeta(a)d^{(i)}) = -2ae^{i\theta} + (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} + \mathbf{h}_i + \mathbf{a}_i, \\ & \frac{d}{dt}(\zeta(a)d^{(i)}) = 2ae^{i\theta} + (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} + \mathbf{h}_i - \mathbf{a}_i, \\ & \frac{d}{dt}(\zeta(a)d^{(i)}) = 2ae^{i\theta} + (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - \mathbf{h}_i - \mathbf{a}_i, \\ & \frac{d}{dt}(d^{(i)}) = -(\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - 2ae^{i\theta} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - 2ae^{i\theta} - 2ae^{i\theta} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - 2ae^{i\theta} - 2ae^{i\theta} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - (\mathbf{h}_i - \mathbf{a}_i)d^{(i)} - 2ae^{i\theta} - 2ae$

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afinitesimal Lorentz Transformation: Lorentz Force

temp

Writing the equations in terms of these

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q},$$

- This is the same form as the Lorentz force
- Make the Identification

$$\vec{P} = -\frac{q}{m}\vec{E}, \qquad \vec{Q} = -\frac{q}{m}\vec{B}, \tag{1}$$

• To be compatible with special relativity the Lorentz force must depend on \vec{u}



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Singular Lorentz Transformations and Pure Radiation Fields

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Singular Lorentz Transformations and Pure Radiation Fields

2014-04-22

Infinitesimal Lorentz Transformation: Lorentz Force

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the equations in terms of these $\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q},$ the same form as the Lorentz force

nfinitesimal Lorentz Transformation: Lorentz Force

• temp

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Singular Lorentz Transformations and Pure Radiation Fields

Infinitesimal Lorentz Transformation: Lorentz Force

 $\frac{d}{dt}(z(\phi)\bar{\theta}) = \bar{\theta} + \bar{x} \times \bar{Q},$ with is the same form as the Lowest force M Make the Identification $\bar{\theta} = \frac{q}{m}\bar{E}, \quad \bar{Q} = \frac{q}{m}\bar{B},$ (1) V To be compatible with specific indicately the Lowest force must depend on $\bar{\theta}$ in this way. So the Lowest force in a specific case of a charged particle in this way. So the Lowest force is a specific case of a charged particle

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Layout



• temp

• The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

- Fixed points of the system are given by $\zeta=\zeta'$ and correspond to null directions
- ullet With this condition solve the fractional linear transformation for ζ

$$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0.$$

- A quadratic means it has two roots in general
- Interested in the singular root case so take the descriminant equal to zero to get

$$a^2 + bc = 0$$
.

refer to this as the quadratic condition.



Singular Lorentz Transformations and Pure Radiation Fields

Pure Radiation Conditions



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Singular Lorentz Transformations and Pure Radiation Fields

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Pure Radiation Conditions

a The fractional linear transformation of the infinitesimal transformation in $C = \frac{1+(d-2A) \cdot O(r^2)}{r^2 + (d+2A) \cdot O(r^2)}$ a Final points of the system was given by $c \in C$ and correspond to null distribution. We have the contraction of the transformation $c \in C$ and correspond to $c \in C$ and $c \in C$ are a substitution of $c \in C$ and $c \in C$ and

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- Interested in the singular root case so take the descriminant equal to zero to get

$$a^2 + bc = 0.$$

refer to this as the quadratic condition.



Singular Lorentz Transformations and Pure Radiation Fields

Pure Radiation Conditions

2014-04-22

The functional linear transformation of the infoliational transformation in $\zeta = \frac{(-1-\epsilon)^2 - (-1-\epsilon)^2 - (-1-\epsilon)^2}{(-1-\epsilon)^2 - (-1-\epsilon)^2}$ a Fixed points of the system as spiken by $\zeta = \zeta^2$ and correspond to null directions. Within this condition under the functional linear transformation for ζ^2 . With this consideration under the functional linear transformation for ζ^2 . $\frac{1}{2}\zeta^2 + (\epsilon - \frac{1}{2}\zeta^2 - \zeta^2) = 0.$ A quantitative mass in the transformation of the singular rank case on the distributional equal to an entire parameter.

Pure Radiation Conditions

- This is one of the things I said I would show earlier
- THIS WILL BE VERY IMPORTANT, **write on the board**

• The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

- \bullet Fixed points of the system are given by $\zeta=\zeta'$ and correspond to null directions
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Singular Lorentz Transformations and Pure Radiation Fields

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Pure Radiation Conditions



Pure Radiation Conditions

- This is one of the things I said I would show earlier
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• The a and b are related to \vec{E} and \vec{B} through the Lorentz force as

$$a_1 = -\frac{1}{2}E^3,$$
 $b_2 = \frac{1}{2}(E^2 + B^1),$ $c_1 = \frac{1}{2}(E^1 + B^2),$ $a_2 = -\frac{1}{2}B^3,$ $b_1 = \frac{1}{2}(E^1 - B^2),$ $c_2 = \frac{1}{2}(B^1 - E^2).$

• Then the real and imaginary parts of the quadratic condition give us the relations

$$|\vec{E}|^2 = |\vec{B}|^2,$$

$$\vec{E} \cdot \vec{B} = 0.$$

• These are the familiar pure radiation conditions. Thus if the world line of a charged particle is generated by an infinitesimal Lorentz transformation then the particle is moving in a pure radiation EM field.



Singular Lorentz Transformations and Pure Radiation Fields

Pure Radiation Conditions

w The s and \$b\$ are related to \$E\$ and \$B\$ through the Lorentz force as
$$\begin{split} a_1 &= -\frac{1}{2}E^3, & b_2 = \frac{1}{2}(E^2+B^2), & c_1 = \frac{1}{2}(E^2+B^2), \\ a_2 &= -\frac{1}{2}B^3, & b_2 = \frac{1}{2}(E^3-B^2), & c_2 = \frac{1}{2}(B^3-E^2), \\ \end{split}$$

Pure Radiation Conditions

• q/m factors suppresed for convenience

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Singular Lorentz Transformations and Pure Radiation Fields

Pure Radiation Conditions

 $\sqrt{1}$ he s and b are related to E and B through the Lorentz force as $A = -\frac{1}{2}E^2, \quad b = \frac{1}{2}(E^2 + B^2), \quad c_1 = \frac{1}{2}(E^2 + B^2), \quad c_2 = \frac{1}{2}B^2, \quad b_1 = \frac{1}{2}(E^2 - B^2), \quad c_2 = \frac{1}{2}(B^2 - E^2).$ Then the air air disease years of the equationic conditions give us the relations $|B_1^2 - B_2^2|, \quad E_2 = \frac{1}{2}(B^2 - E^2).$ $|E_1^2 - B_2^2|, \quad E_3 = \frac{1}{2}(B^2 - E^2).$

Pure Radiation Conditions

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Singular Lorentz Transformations and Pure Radiation Fields

Pure Radiation Conditions

Pure Radiation Conditions

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 $\vec{E} \cdot \vec{B} \equiv 0$. familiar pure radiation conditions. Thus if the we is is generated by an infinitesimal Lorentz transfo

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Layout

add in the contents



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Layout

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 $\bullet\,$ MAny other things can be shown, I will finally show one small lemma of this resu

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