Singular Lorentz Transformations and Pure Radiation Fields

Kevin Maguire

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1. Introduction: Lorentz Transformations
2. Strange Minkowskian Line Element
3. Singular Lorentz transformation
4. $SL(2, \mathbb{C})$ Matrices of the Lorentz Transformation
5. The Fractional Linear Transformation
6. Infinitesimal Lorentz Transformation
7. Pure Radiation Conditions

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Introduction: Lorentz Transformations

- A Lorentz transformation is defined by the preservation of the quadratic form

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2,$$

in the transformation $(x, y, z, t) \to (x', y', z', t')$

- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group $SO^+(1,3)$
- In general Lorentz transformations have two invariant null directions

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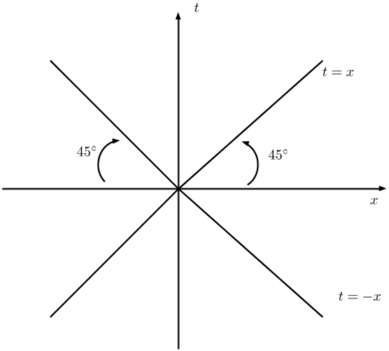

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Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2.$$

- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

- Make further coordinate transformations to obtain

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right) \mathrm{d}u^2.$$

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Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta := \xi + i\eta$, to get the new line element[3]
$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$
- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic $r = 0$ invariant.
- In Cartesian coordinates this transformation becomes

$$\begin{aligned} x' + iy' &= x + iy + w(t - z), \\ z' - t' &= -r = z - t, \\ z' + t' &= z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \end{aligned}$$

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$SL(2,\mathbb{C})$ Matrices of the POLT

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Construct the following matrix

$$A = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix},$$

- This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

- Construct the transformation $A(\vec{x}') = UA(\vec{x})U^{\dagger}$, where

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

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- $A(\vec{x}')$ and $A(\vec{x})$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus it is a Lorentz transformation.
- Write this transformation component wise

$$\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix},$$
$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha}+(x+iy)\bar{\beta} & (t-z)\bar{\gamma}+(x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha}+(t+z)\bar{\beta} & (x-iy)\bar{\gamma}+(t+z)\bar{\delta} \end{pmatrix}.$$

- Thus the general relations

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Example: Singular Lorentz transformation

- Take the singular Lorentz transformation from earlier

$$\begin{aligned} t' - z' &= t - z, \\ x' + iy' &= x + iy + w(t - z), \\ t' + z' &= t + z + w(x - iy) + \bar{w}(x + iy) + w\bar{w}(t - z). \end{aligned}$$

- Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$\alpha = \pm 1, \qquad \beta = 0,$$
$$\gamma = \bar{w}\alpha, \qquad \delta = \alpha.$$

- So there are always two possible choices of U

$$U = \pm \begin{pmatrix} 1 & 0 \\ \bar{w} & 1 \end{pmatrix}$$

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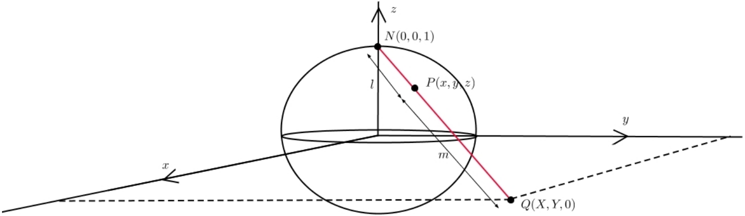
Fractional Linear Transformation: Stereographic Projection

- Use Stereographic Projection to map $\mathbb{S}^2$ to the extended complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



- The algebraic relation for a unit vector is

$$(x, y, z) = \left(\frac{\bar{\zeta} + \zeta}{\bar{\zeta}\zeta + 1}, i\frac{\bar{\zeta} - \zeta}{\bar{\zeta}\zeta + 1}, \frac{\bar{\zeta}\zeta - 1}{\bar{\zeta}\zeta + 1} \right),$$

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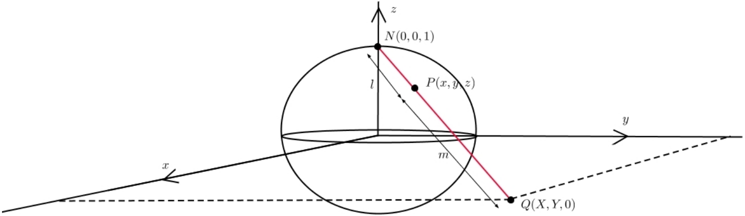

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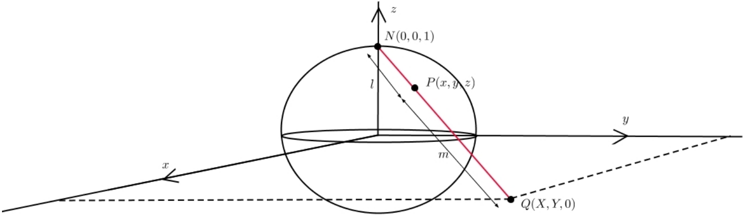

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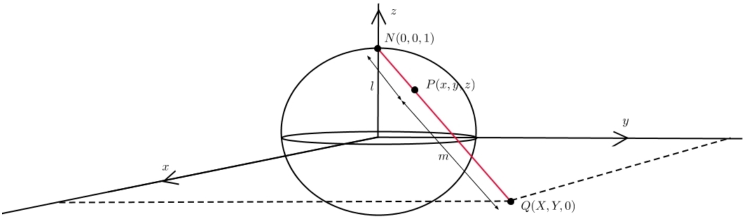

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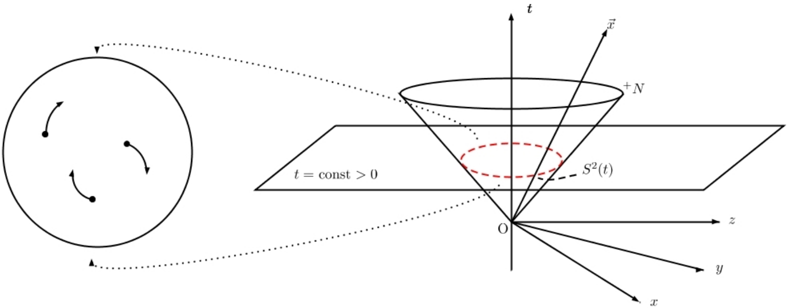

- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$
- So points in Minkowskian space-time can be written in the form

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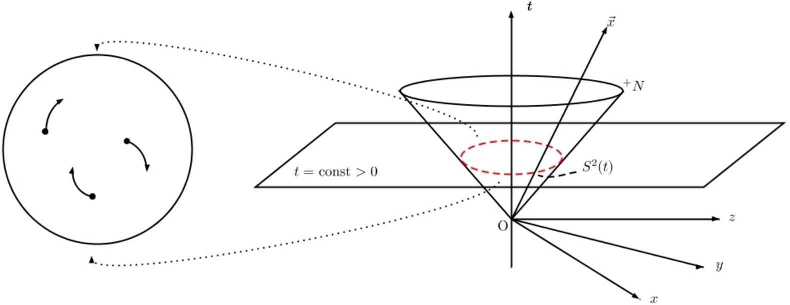

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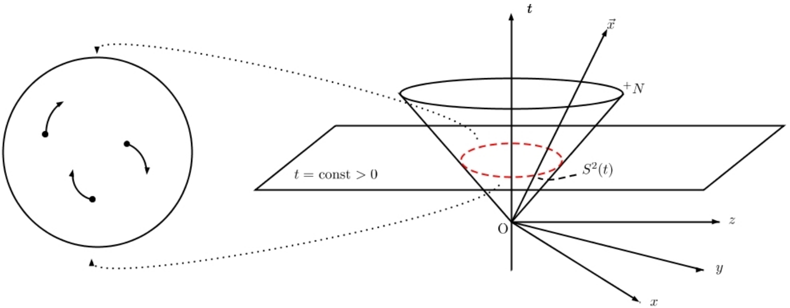
Fractional Linear Transformation: Stereographic Projection

- Extend this to Minkowskian space-time[1]



- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$
- So points in Minkowskian space-time can be written in the form

$$\vec{x} = t\left(\frac{\bar{\zeta}+\zeta}{\bar{\zeta}\zeta+1}, i\frac{\bar{\zeta}-\zeta}{\bar{\zeta}\zeta+1}, \frac{\bar{\zeta}\zeta-1}{\bar{\zeta}\zeta+1}, 1\right).$$

Fractional Linear Transformation

- Make the transformation $\zeta \to \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U.

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix},$$

$${c_0}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

- Solve for ζ' to get the fractional linear transformation

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)},$$

- There is a one to one correspondence between POLTs and fractional linear transformations

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Layout

1. Introduction: Lorentz Transformations
2. Strange Minkowskian Line Element
3. Singular Lorentz transformation
4. $SL(2, \mathbb{C})$ Matrices of the Lorentz Transformation
5. The Fractional Linear Transformation
6. Infinitesimal Lorentz Transformation
7. Pure Radiation Conditions

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix},$$

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2),$$

where

$$L^i{}_j = \begin{pmatrix} 0 & -2a_2 & (b_1 - c_1) & (b_1 + c_1) \\ 2a_2 & 0 & (b_2 + c_2) & (b_2 - c_2) \\ -(b_1 - c_1) & -(b_2 + c_2) & 0 & -2a_1 \\ (b_1 + c_1) & (b_2 - c_2) & -2a_1 & 0 \end{pmatrix}$$

$$\frac{d^2 x^i}{ds^2} = L^i{}_j(s) \frac{dx^j}{ds}.$$

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Infinitesimal Lorentz Transformation: Lorentz Force

- Can rewrite this equation in terms of the particles 3-velocity $\vec{u}$, in component form

$$\begin{aligned}
\frac{d}{dt}(\gamma(u)u^{(1)}) &= -2a_2 u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1, \\
\frac{d}{dt}(\gamma(u)u^{(2)}) &= 2a_2 u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2, \\
\frac{d}{dt}(\gamma(u)u^{(3)}) &= -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1, \\
\frac{d\gamma(u)}{dt} &= (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1 u^{(3)}.
\end{aligned}$$

- Define the 3-vectors

$$\begin{aligned}
\vec{P} &= (b_1 + c_1, b_2 - c_2, -2a_1), \\
\vec{Q} &= (b_2 + c_2, -(b_1 - c_1), -2a_2).
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Infinitesimal Lorentz Transformation: Lorentz Force

- Writing the equations in terms of these

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q},$$

- This is the same form as the Lorentz force
- Make the Identification

$$\vec{P} = \frac{q}{m}\vec{E}, \qquad \vec{Q} = \frac{q}{m}\vec{B}, \tag{1}$$

- To be compatible with special relativity the Lorentz force must depend on $\vec{u}$ in this way. So the Lorentz force is a special case of a charged particle moving along a world line in Minkowskian space-time generated by an infinitesimal Lorentz transformation.

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Layout

Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions
- With this condition solve the fractional linear transformation for ζ

$$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0.$$

- A quadratic means it has two roots in general
- Interested in the singular root case so take the discriminant equal to zero to get

$$a^2 + bc = 0.$$

refer to this as the quadratic condition.

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Pure Radiation Conditions

- The a and b are related to $\vec{E}$ and $\vec{B}$ through the Lorentz force as

$$a_1 = -\frac{1}{2}E^3, \qquad b_2 = \frac{1}{2}(E^2 + B^1), \qquad c_1 = \frac{1}{2}(E^1 + B^2),$$
$$a_2 = -\frac{1}{2}B^3, \qquad b_1 = \frac{1}{2}(E^1 - B^2), \qquad c_2 = \frac{1}{2}(B^1 - E^2).$$

- Then the real and imaginary parts of the quadratic condition give us the relations

$$|\vec{E}|^2 = |\vec{B}|^2,$$
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- These are the familiar pure radiation conditions. Thus if the world line of a charged particle is generated by an infinitesimal singular Lorentz transformation then the particle is moving in a pure radiation EM field.

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References

1 J.L. Synge - "Relativity: The Special Theory" - North Holland Publishing Company (1965)

2 D. Finkelstein - "Past-Future Asymmetry of the Gravitational Field of a Point Particle" - Phys.Rev.Vol 110, (1958) - `http://journals.aps.org/pr/pdf/10.1103/PhysRev.110.965`

3 P.A. Hogan, C.Barrabès - "Advanced General Relativity: Gravity Waves, Spinning Particles and Black Holes" - Oxford University Press (May 2013)

4 I. Robinson "Spherical Gravitational Waves" - Phys.Rev.Lett. 4 (1960) 431-432 - `http://journals.aps.org/prl/pdf/10.1103/PhysRevLett.4.431`

5 P.A Hogan, C. Barrabès - "Singular Null Hypersurfaces" - World Scientific Pub Co Inc (April 2004)

6 R. Penrose, W. Rindler - "Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields" - Cambridge University Press, (Feb 1987)

7 Tristan Needham - "Visual Complex Analysis" - Clarendon Press, Oxford (1997)

8 Various Authors - "Space-Time and Geometry: The Alfred Schild Lectures" - University of Texas Press (March 21, 2012)