Singular Lorentz Transformations and Pure Radiation Fields

Kevin Maguire

April 23, 2014

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1. Introduction: Lorentz Transformations
2. Strange Minkowskian Line Element
3. Singular Lorentz transformation
4. $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation
5. The Fractional Linear Transformation
6. Infinitesimal Lorentz Transformation
7. Pure Radiation Conditions

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- –Proper is det 1 . preserves the orientation of spacial axes, preserves handedness
- –orthochronous means time is always positive and the direction of time is preserved
- –Think of the standard Lorentz transformation, always two null directions at $x \pm t$

Introduction: Lorentz Transformations

- A Lorentz transformation is defined by the preservation of the quadratic form

  $$x'^2+y'^2+z'^2-t'^2 = x^2+y^2+z^2-t^2,$$

  in the transformation $(x, y, z, t) \rightarrow (x', y', z', t')$
- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group $SO^+(1,3)$
- In general Lorentz transformations have two invariant null directions

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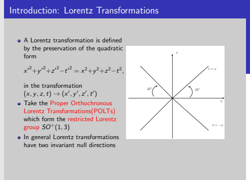

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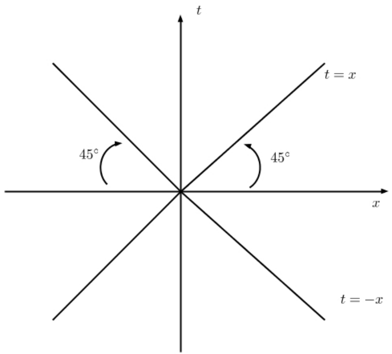

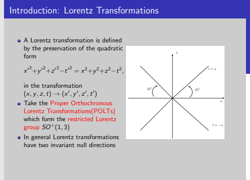

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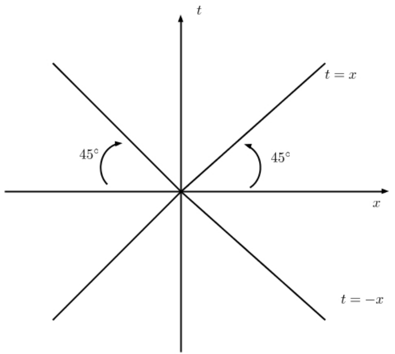

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- derive a strange Minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial

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- to remove the coordinate singularity in the Schwarzschild solution
- These transformations put the line element in a form where we can take the limit as the energy goes to 0
- It is easily shown with further coordinate transforms that this is Kasner, but it wont be done here

Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$\epsilon ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right) dt^2.$$

- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m \ln(r - 2m).$$

- Make further coordinate transformations to obtain

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 \mu\xi}(d\xi^2 + d\eta^2) - 2dudr - \left(\mu^2 - \frac{2k}{r}\right) du^2.$$

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Then with $m = 0$ the strange Minkowskian line element is obtained
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- Its easily shown with suitable coordinate transforms that this is Minkowskian line element
- This is best shown by calculating the geodesic equations after the Eddington-Finkelstein coordinate transforms, all zero if u is proper time along the geodesic

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LTs that leave one null invariant direction are constructed

Layout

Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta := \xi + i\eta$, to get the new line element[3]
$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

- This is what we want, An LT which leaves one null invariant.
- The use in the previous coord transforms was to make this transformation look trivial
- So this is what the seemingly trivial transformation looks like in Cartesian
- Again its clear that $r = 0$ keeps one direction fixed, as then z=t
- but it doesn't work both ways, not all 2 parameter abelian subgroups are singular Lorentz transformations

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- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic $r = 0$ invariant.
- In Cartesian coordinates this transformation becomes

$$\begin{aligned} x' + iy' &= x + iy + w(t - z), \\ z' - t' &= -r = z - t, \\ z' + t' &= z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \end{aligned}$$

- Addition of complex numbers is commutative, and w has two parameters, so the singular Lorentz transformations form a 2-parameter abelian subgroup of the Lorentz group

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- shown here that there is a 2 to 1 correspondence between SL(2,C) and POLTs

Layout

- Complex Hermitian matrices have 4 independent components, so the element of such a matrix can be used to represent points in Minkowskian space-time.
- where $\alpha, \beta, \gamma, \delta$ are complex its an element of the special linear group. This means it has determinant 1. **write it on the board**

$SL(2,\mathbb{C})$ Matrices of the POLT

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Construct the following matrix

$$A = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix},$$

- This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

- Construct the transformation $A(\vec{x}') = UA(\vec{x})U^\dagger$, where

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Example: Singular Lorentz transformation

- Take the singular Lorentz transformation from earlier

$$t' - z' = t - z,$$
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- Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$\alpha = \pm 1, \qquad \beta = 0,$$
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- So there are always two possible choices of U

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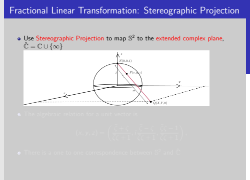
- Connect Minkowskian space to the 2-sphere by stereographic projection, so we can use points on a 2 sphere to think about LTs

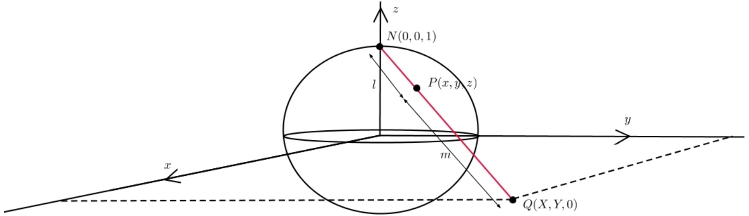
Layout



- As we know, stereographic projection doesn't map the point N at the top of the circle, so that's why we map N to infinity and need to consider the extended complex plane
- It can also be written in terms of θ and ϕ

Fractional Linear Transformation: Stereographic Projection

- Use Stereographic Projection to map $\mathbb{S}^2$ to the extended complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



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$$(x, y, z) = \left(\frac{\bar{\zeta} + \zeta}{\bar{\zeta}\zeta + 1}, i\frac{\bar{\zeta} - \zeta}{\bar{\zeta}\zeta + 1}, \frac{\bar{\zeta}\zeta - 1}{\bar{\zeta}\zeta + 1} \right),$$

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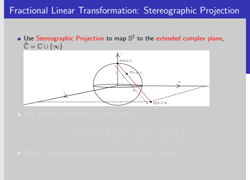

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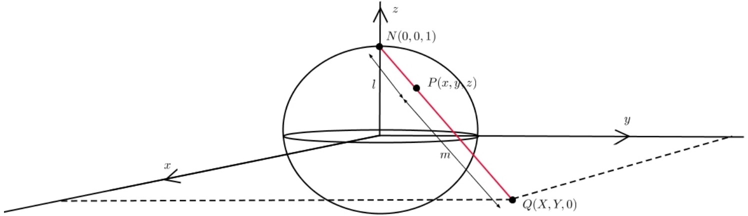

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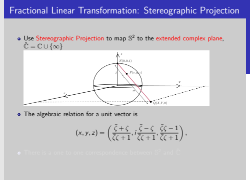

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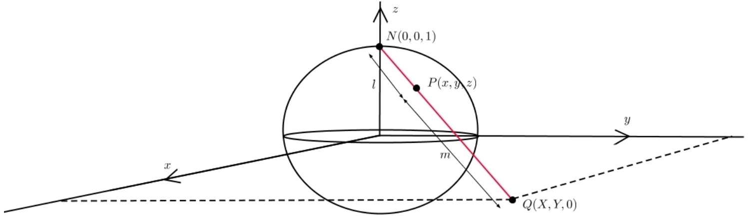

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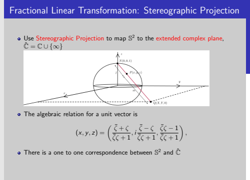

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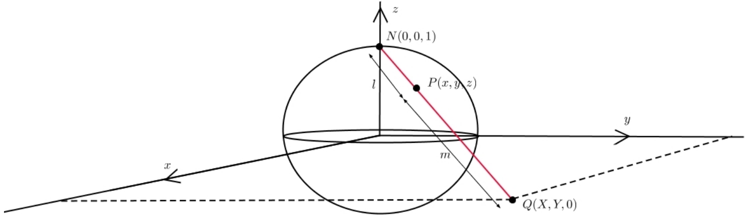

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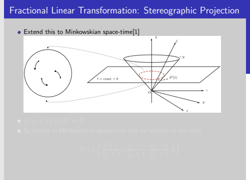
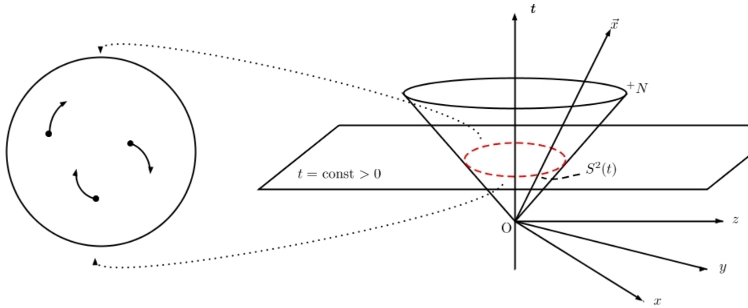


- all the points on the 2 sphere are generators of the future null cone in Minkowskian space time
- Can denote an LT by moving three arbitrary points along the surface of the sphere as the generators have dimension two, so to match the dim of the LT (it's 6) we need three of them
- Extra coord is because we take time into account now, now ζ has two parameters so x is in terms of two parameters, t just defines the direction

Fractional Linear Transformation: Stereographic Projection

- Extend this to Minkowskian space-time[1]



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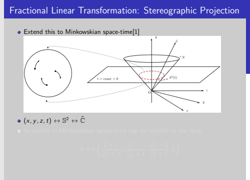

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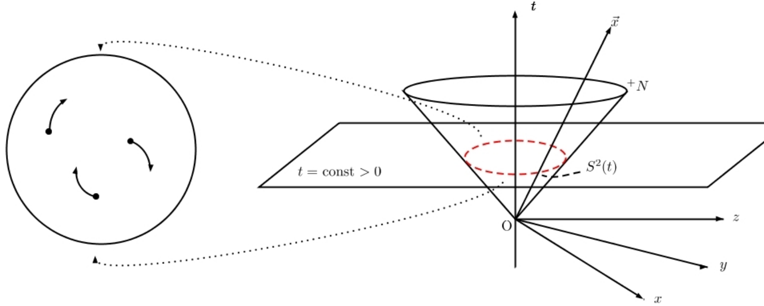

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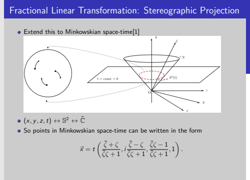

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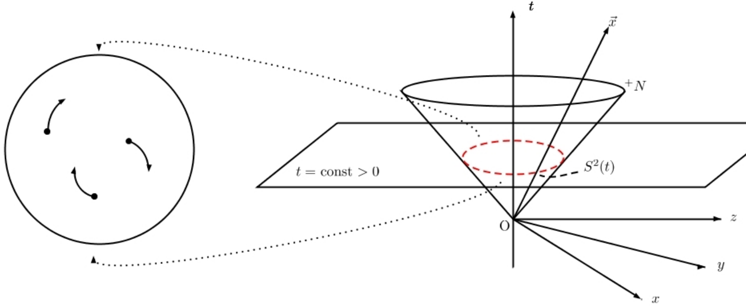

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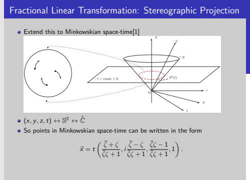

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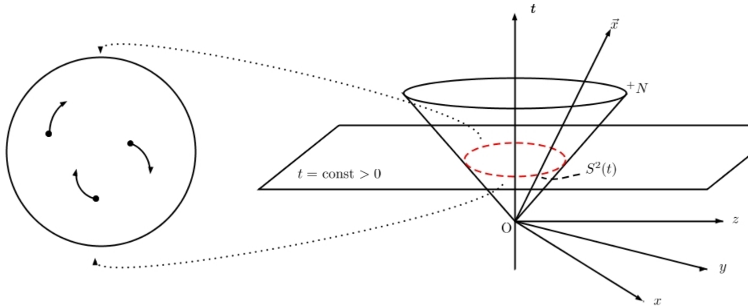
Fractional Linear Transformation: Stereographic Projection

- Extend this to Minkowskian space-time[1]



- $(x, y, z, t) \leftrightarrow \mathbb{S}^2 \leftrightarrow \hat{\mathbb{C}}$
- So points in Minkowskian space-time can be written in the form

$$\vec{x} = t\left(\frac{\bar{\zeta}+\zeta}{\bar{\zeta}\zeta+1}, i\frac{\bar{\zeta}-\zeta}{\bar{\zeta}\zeta+1}, \frac{\bar{\zeta}\zeta-1}{\bar{\zeta}\zeta+1}, 1\right).$$

- These are null directions
- Refer to eqn (27) which should be on the board
- AS we did in the previous example, determine U
- Remember we had $\pm U$ now the signs will cancel in the denominator and numerator

Fractional Linear Transformation

- Make the transformation $\zeta \to \zeta'$ by constructing the matrix $A(\vec{x})$ and determining the matrix U.

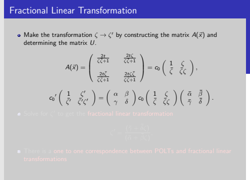
$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \zeta\bar{\zeta} \end{pmatrix},$$

$$c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \zeta\bar{\zeta} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

- Solve for ζ' to get the fractional linear transformation

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)},$$

- There is a one to one correspondence between POLTs and fractional linear transformations



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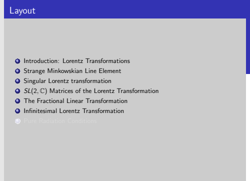

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Layout

- This was done in a recent lecture(Relativistic QM) so I wont do it
- Where a,b,c,d are complex
- Take many infinitesimal LT steps along a particles trajectory and let ϵ go to zero

Infinitesimal Lorentz Transformation

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix},$$

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2),$$

where

$$L^i{}_j = \begin{pmatrix} 0 & -2a_2 & (b_1 - c_1) & (b_1 + c_1) \\ 2a_2 & 0 & (b_2 + c_2) & (b_2 - c_2) \\ -(b_1 - c_1) & -(b_2 + c_2) & 0 & -2a_1 \\ (b_1 + c_1) & (b_2 - c_2) & -2a_1 & 0 \end{pmatrix}$$

$$\frac{d^2 x^i}{ds^2} = L^i{}_j(s) \frac{dx^j}{ds}.$$

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- Clear that these equations can be expressed in terms of P and Q

Infinitesimal Lorentz Transformation: Lorentz Force

- Can rewrite this equation in terms of the particles 3-velocity $\vec{u}$, in component form

$$\begin{aligned}
\frac{d}{dt}(\gamma(u)u^{(1)}) &= -2a_2u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1, \\
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\frac{d\gamma(u)}{dt} &= (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1u^{(3)}.
\end{aligned}$$

- Define the 3-vectors

$$\begin{aligned}
\vec{P} &= (b_1 + c_1, b_2 - c_2, -2a_1), \\
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- Writing the equations in terms of these

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q},$$

- This is the same form as the Lorentz force
- Make the Identification

$$\vec{P} = \frac{q}{m}\vec{E}, \qquad \vec{Q} = \frac{q}{m}\vec{B}, \tag{1}$$

- To be compatible with special relativity the Lorentz force must depend on $\vec{u}$ in this way. So the Lorentz force is a special case of a charged particle moving along a world line in Minkowskian space-time generated by an infinitesimal Lorentz transformation.

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1. Introduction: Lorentz Transformations
2. Strange Minkowskian Line Element
3. Singular Lorentz transformation
4. $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation
5. The Fractional Linear Transformation
6. Infinitesimal Lorentz Transformation
7. Pure Radiation Conditions

Layout

- This is derived from the form of the infinitesimal U and from the fractional linear transformation formula
- This is one of the things I said I would show earlier
- THIS WILL BE VERY IMPORTANT, **write on the board**

Pure Radiation Conditions

- The fractional linear transformation of the infinitesimal transformation is

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions
- With this condition solve the fractional linear transformation for ζ

$$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0.$$

- A quadratic means it has two roots in general
- Interested in the singular root case so take the discriminant equal to zero to get

$$a^2 + bc = 0.$$

refer to this as the quadratic condition.

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2014-04-23

Pure Radiation Conditions

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- The fractional linear transformation of the infinitesimal transformation is $$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$
- Fixed points of the system are given by $\zeta = \zeta'$ and correspond to null directions
- With this condition solve the fractional linear transformation for ζ $$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0.$$
- A quadratic means it has two roots in general
- Interested in the singular root case so take the discriminant equal to zero to get $$a^2 + bc = 0.$$ refer to this as the quadratic condition.

- This is derived from the form of the infinitesimal U and from the fractional linear transformation formula
- This is one of the things I said I would show earlier
- THIS WILL BE VERY IMPORTANT, **write on the board**

Pure Radiation Conditions

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refer to this as the quadratic condition.

- q/m factors suppressed for convenience

Pure Radiation Conditions

- The a and b are related to $\vec{E}$ and $\vec{B}$ through the Lorentz force as

$$\begin{aligned} a_1 &= -\frac{1}{2}E^3, & b_2 &= \frac{1}{2}(E^2 + B^1), & c_1 &= \frac{1}{2}(E^1 + B^2), \\ a_2 &= -\frac{1}{2}B^3, & b_1 &= \frac{1}{2}(E^1 - B^2), & c_2 &= \frac{1}{2}(B^1 - E^2). \end{aligned}$$

- Then the real and imaginary parts of the quadratic condition give us the relations

$$\begin{aligned} \left|\vec{E}\right|^2 &= \left|\vec{B}\right|^2, \\ \vec{E} \cdot \vec{B} &= 0. \end{aligned}$$

- These are the familiar pure radiation conditions. Thus if the world line of a charged particle is generated by an infinitesimal singular Lorentz transformation then the particle is moving in a pure radiation EM field.

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$$|\vec{E}|^2 = |\vec{B}|^2,$$
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References

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