The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

I. THE LORENTZ TRANSFORMATION

In this section the two null directions inherent in all Lorentz transformations, except the singular Lorentz transformations, are illustrated. The Lorentz Transform is defined by $(x, y, z, t) \to (x', y', z', t')$ such that

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2.$$

If the transformation preserves the orientation of the spatial axes then is it called a *proper* Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also if $t \geq 0$ implies that time is always positive then it is called an *orthochronous* Lorentz transformation, which ensures that the time direction is preserved. In this project the "Lorentz transformation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as in Fig.() (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx), \end{aligned}$$

where $\gamma = (1 - v^2)^{-1/2}$. Rearrange to obtain

$$\begin{aligned} x' - t' &= \gamma(1 + v)(x - t), \\ x' + t' &= \gamma(1 - v)(x + t). \end{aligned}$$

It is clear that $x = \pm t$ implies $x' = \pm t'$. Thus there are two null directions in this space-time at $x = \pm t$, as null directions are by definition invariant under a Lorentz transformation. It can be shown that all Lorentz transformations have two invariant null directions except the singular Lorentz transformation which has only one fixed null direction.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

In this section the Kasner solution of the vacuum field equations is derived from the Schwarzschild solution by taking the limit as the mass goes to infinity. It is then shown that the special case of the Kasner solution with no mass is equivalent to a novel form of Minkowskian space-time. (what do we want the Kasner vacuum solution for???). Start with the Schwarzschild Solution of the vacuum field equations given by

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2. \tag{1}$$

A. Eddington-Finkelstein Coordinate Transformation

First, make the Eddington-Finkelstein coordinate transformation

$$u = t - r - 2m \ln(r - 2m). \tag{2}$$

Calculate the differentials

$$\begin{aligned} \mathrm{d}u &= \mathrm{d}t - \mathrm{d}r - \frac{2m\mathrm{d}r}{r - 2m}, \\ &= \mathrm{d}t - \mathrm{d}r\left(1 - \frac{2m}{r}\right)^{-1}, \\ \mathrm{d}t &= \mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r, \end{aligned}$$

and sub them into Eqn.(1).

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right)\left(\mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r\right)^2,$$

which gives the result,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - 2\mathrm{d}u\mathrm{d}r - \mathrm{d}u^2 + \frac{2m}{r}\mathrm{d}u^2. \tag{3}$$

Note that if $m = 0$ the space-time becomes Minkowskian, as expected. If $r = 0$ in this Minkowskian space-time the line element becomes

$$\epsilon \mathrm{d}s^2 = -\mathrm{d}u^2,$$

which implies that $\epsilon = -1$ and the first integral of this trajectory is also equal to -1. Thus $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u.(ERROR CHECK ITS A GEODESIC???NEED TO CHECK ANSWER WITH HOGAN)

The limit of the Schwarzschild solution as $m \to \infty$ must be calculated to find the Kasner solution. In its current form the limit cannot be taken, so two suitable coordinate transformations must be made to get it in a more useful form. First Set

$$u = \mu u',$$
$$r = \mu^{-1} r',$$

where $\mu =$ const so that

$$\mathrm{d}u = \mu \mathrm{d}u',$$
$$\mathrm{d}r = \mu^{-1}\mathrm{d}r'.$$

The product of the differentials is then invariant

$$\mathrm{d}u\mathrm{d}r = \mathrm{d}u'\mathrm{d}r'.$$

When the new coordinates are subbed into Eqn.(3) the Schwarzschild solution becomes

$$\epsilon \mathrm{d}s^2 = r'^2 \sin^2\theta \left\{ \frac{\mathrm{d}\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2}\mathrm{d}\phi^2 \right\} - 2\mathrm{d}u'\mathrm{d}r' - \left(\mu^2 - \frac{2m\mu^3}{r'}\right)\mathrm{d}u'^2. \tag{4}$$

Now set $m\mu^3 = k$ or $m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by

$$\sin\theta = \frac{1}{\cosh(\mu\xi)}, \mu^{-1}\phi = \eta.$$

The second of these transformations gives simply $\mu^{-2}\mathrm{d}\phi^2 = \mathrm{d}\eta^2$. To rewrite the first coordinate transformation, first differentiate.

$$\cos\theta \mathrm{d}\theta = \frac{-1}{(\cosh(\mu\xi))^2}\sinh(\mu\xi)\mu \mathrm{d}\xi,$$
$$= -\sin^2\theta \sinh(\mu\xi)\mu \mathrm{d}\xi.$$

Use the forumla $\cosh^2 A - \sinh^2 A = 1$, divide by $\mathrm{d}\xi$ and simplfy using trigonometric identities

$$\cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\mu \sin^2\theta \sqrt{\frac{1}{\sin^2\theta} - 1},$$
$$= -\mu \sin\theta \cos\theta.$$

Finally rewrite in terms of $\mathrm{d}\xi$

$$\mathrm{d}\xi^2 = \left(\frac{\mathrm{d}\theta}{\mu \sin\theta}\right)^2.$$

Subbing these transformations into Eqn.(4) gives

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right)\mathrm{d}u^2,$$

where the primes have been dropped for convenience.

B. The Kasner Solution

This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$, to obtain

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \frac{2k}{r}\mathrm{d}u^2 \tag{5}$$

This is still a solution of the field equations but it is no longer the Schwarzschild solution. In this section it is shown to be the Kasner Solution(READ UP ABOUT THIS), which by definition is given by

$$\epsilon \mathrm{d}s^2 = T^{2p}\mathrm{d}X^2 + T^{2q}\mathrm{d}Y^2 + T^{2r}\mathrm{d}Z^2 - \mathrm{d}T^2, \tag{6}$$

such that

$$p + q + r = 1 = p^2 + q^2 + r^2.$$

To write Eqn.(5) in this form first make the transformation

$$\xi' = \lambda^{-1}\xi,\ \eta' = \lambda^{-1}\eta,$$
$$r' = \lambda r,\ u' = \lambda^{-1}u,$$

with $\lambda := k^{-1/3}$. Subbing in these new coordinates gives

$$\epsilon \mathrm{d}s^2 = {r'}^2(\mathrm{d}{\xi'}^2 + \mathrm{d}{\eta'}^2) - 2\mathrm{d}u'\mathrm{d}r' + \frac{2}{r'}\mathrm{d}{u'}^2.$$

Now add and subtract $(r'/2)\mathrm{d}{r'}^2$ to complete the square as follows

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\left(\mathrm{d}u - \frac{r}{2}\mathrm{d}r\right)^2 - \frac{r}{2}\mathrm{d}r^2,$$

where the primes have again been dropped for convenience. Now set

$$\bar{X} = u - \frac{r^2}{4},$$

so the differential of $\bar{X}$ is

$$\mathrm{d}\bar{X} = \mathrm{d}u - \frac{r}{2}\mathrm{d}r,$$

and the line element can be rewritten in terms of $\bar{X}$,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\mathrm{d}\bar{X}^2 - \left(\frac{r^{1/2}}{\sqrt{2}}\mathrm{d}r\right)^2.$$

Now define T such that

$$T = \frac{\sqrt{2}}{3}r^{3/2}, \text{ and } r = \left(\frac{3}{\sqrt{2}}\right)^{2/3} T^{2/3},$$

which results in the new line element

$$\epsilon \mathrm{d}s^2 = \left(\frac{3}{\sqrt{2}}\right)^{4/3} T^{4/3}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2\left(\frac{\sqrt{2}}{3}\right)^{2/3} T^{-2/3}\mathrm{d}\bar{X}^2 - \mathrm{d}T^2.$$

Then a final coordinate transformation can be made to remove the unwanted constants, given by

$$X = \left[2\left(\frac{\sqrt{2}}{3}\right)\right]^{1/2} \bar{X},$$
$$Y = \left(\frac{3}{\sqrt{2}}\right)^{2/3} \xi,$$
$$Z = \left(\frac{3}{\sqrt{2}}\right)^{2/3} \eta,$$

to obtain

$$\epsilon \mathrm{d}s^2 = T^{-2/3}\mathrm{d}X^2 + T^{4/3}\left(\mathrm{d}Y^2 + \mathrm{d}Z^2\right) - \mathrm{d}T^2. \tag{7}$$

Comparing this result to the general form of the Kasner solution in Eqn.(6) it is clear that they have the same form with $p = -1/3$ and $q = r = 2/3$. Thus the solution obtained by taking the limit of the Schwarzschild solution as $m \to \infty$ is indeed the Kasner Solution.

C. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting $k = 0$ in Eqn.(5), which is equivalent to $m = 0$.

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r \tag{8}$$

Setting $r = 0$ gives $\epsilon \mathrm{d}s^2 = 0$. In this section it is demonstrated that $r = 0$ is a null geodesic with u an affine parameter along it and that Eqn.(8) is indeed the Minkowskian space-time line element. To verify these properties first let $x^i = (x, y, z, t)$ be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element

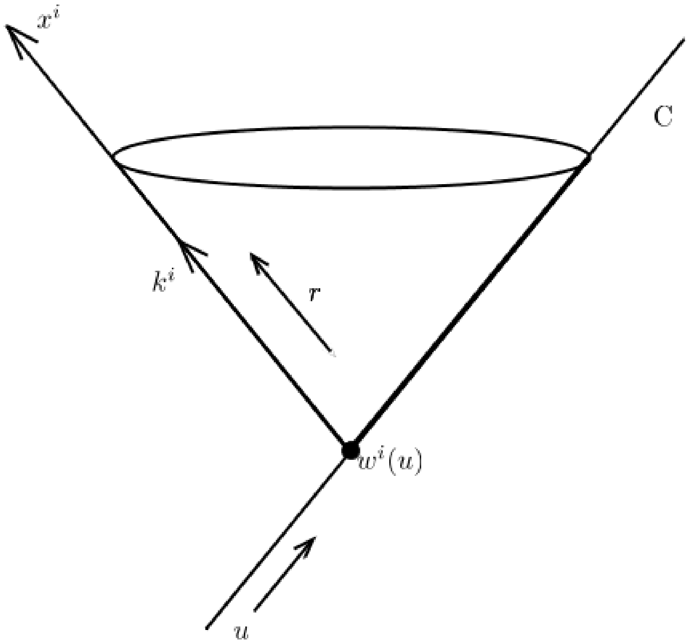
$$\epsilon \mathrm{d}{s_0}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2.$$

We note that the trajectory C defined by $x = 0$, $y = 0$, $z = t$ is a null geodesic as it will lie on the light cone of Minkowskian space-time. If C is writen parametrically as $x^i = w^i(u)$ such that $w^i = (0, 0, u, u)$ then u is an affine parameter along it. The tangent to C is then computed as

$$v^i(u) = \frac{\mathrm{d}w^i}{\mathrm{d}u} = (0, 0, 1, 1).$$

As C is a null geodesic the first integral will be $v_i v^i = 0$ and thus $v_i = (0, 0, 1, -1)$ where we have chosen the convention $(+, +, +, -)$.

FIG. 1: *Minkowskian space-time illustrating the new parameter r which is the shortest distance between some point x^i and the trajectory C along the k^i direction. The parameter u which determines the distance travelled along C is also shown*



The position vector of a point in Minkowskian space time can be written in the form

$$x^i - w^i(u) = rk^i,$$
$$\text{or } x^i = w^i(u) + rk^i.$$

Thus r is a new parameter which tell us the shortest distance between C and some point x^i, and k^i is the unit vector in that direction, see Fig.(1). As k^i is a unit vector it satisfies the relations

$$k^i k_i = 0, \tag{9}$$
$$k^i v_i = -1. \tag{10}$$

Thus k^i is normalized so that k^i and v^i are both future pointing (HOW DOES THIS MAKE THEM BOTH FUTURE POINTING?). Making the parameterisation

$$k^i = (\xi, \eta, A, B),$$
$$k_i = (\xi, \eta, A, -B).$$

We can choose any variable for the first two slots of k^i so we choose ξ and η from before for convenience. Using the relation (9) it is clear that

$$\xi^2 + \eta^2 + A^2 - B^2 = 0,$$

and using the relation (10) it is found that

$$A - B = -1, \tag{11}$$
$$\Rightarrow A^2 - B^2 = (A+B)(A-B) = -(A+B). \tag{12}$$

Which implies

$$\xi^2 + \eta^2 = A + B. \tag{13}$$

So expressions for A and B are found using Eqn.(11) and Eqn.(13):

$$A = \frac{1}{2}(-1 + \xi^2 + \eta^2)$$
$$B = \frac{1}{2}(1 + \xi^2 + \eta^2)$$

In summary so far we have

$$x^i = w^i(u) + rk^i, \tag{14}$$
$$w^i = (0, 0, u, u), \tag{15}$$
$$k^i = (\xi, \eta, \frac{1}{2}(-1 + \xi^2 + \eta^2), \frac{1}{2}(1 + \xi^2 + \eta^2)), \tag{16}$$
$$x^i = (x, y, z, t). \tag{17}$$

Consider Eqn.(14) as a coordinate transformation from (x, y, z, t) to (ξ, η, r, u) such that

$$\begin{aligned} x &= r\xi, \\ y &= r\eta, \\ z &= u + \frac{r}{2}(-1 + \xi^2 + \eta^2), \\ t &= u + \frac{r}{2}(1 + \xi^2 + \eta^2), \end{aligned} \tag{18}$$

which is clear from Eqns.(14) - (17). Now this is applied to the Minkowskian line element of Eqn.(8). First, the x and y differentials are

$$dx = r\mathrm{d}\xi + \xi\mathrm{d}r,$$
$$\mathrm{d}y = r\mathrm{d}\eta + \eta\mathrm{d}r.$$

Which gives

$$\mathrm{d}x^2 + \mathrm{d}y^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2r\xi\mathrm{d}\xi\mathrm{d}r + 2r\eta\mathrm{d}\eta\mathrm{d}r + (\xi^2 + \eta^2)\mathrm{d}r^2. \tag{19}$$

Next, the z and t differentials

$$\begin{aligned} z + t &= 2u + r(\xi^2 + \eta^2), \\ z - t &= -r, \\ \mathrm{d}z + \mathrm{d}t &= 2\mathrm{d}u + (\xi^2 + \eta^2)\mathrm{d}r + 2r\xi\mathrm{d}\xi + 2r\eta\mathrm{d}\eta, \\ \mathrm{d}z - \mathrm{d}t &= -\mathrm{d}r. \end{aligned}$$

Then using difference of two squares to obtain

$$\mathrm{d}z^2 - \mathrm{d}t^2 = -2\mathrm{d}u\mathrm{d}r - (\xi^2 + \eta^2)\mathrm{d}r^2 - 2r\xi\mathrm{d}\xi\mathrm{d}r - 2r\eta\mathrm{d}\eta\mathrm{d}r. \tag{20}$$

Combining Eqn.(19) and (20) to get:

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r$$

and from this it is clear that Eqn.(8) is the line element of Minkowskian space-time with $r = 0$ a null geodesic with affine parameter u along it as stated at the beginning of the section.(ERROR. REFER TO GEODESIC CALC FROM EARLIER)

Thus it has been shown that the Kasner solution to the vacuum filed equations can be obtained from the Schwarzschild solution of the vacuum field equations by taking the special case where the mass goes to infinity. (ERROR. ISNT THIS A CONTRADICTION???) Then the Minkowskian space-time line element in a particular set of coordinates (ξ, η, r, u), can be derived from the Kasner solution by setting the mass to zero. It has been shown that the special case where $r = 0$ in this Minkowskian space-time is then a null geodesic with u an affine parameter along it.

III. THE SINGULAR LORENTZ TRANSFORMATION

In this section a Lorentz transformation that leaves our line element Eqn.(8) invariant is constructed. This transformation is then expressed in terms of (x, y, z, t) and examined to see what form it has. The subgroup of the Lorentz group that it makes is then examined. First we define an arbitrary complex parameter by $\zeta = \xi + i\eta$ so that the differentials are given by

$$\begin{aligned} d\zeta &= d\xi + id\eta, \\ d\bar{\zeta} &= d\bar{\xi} - id\bar{\eta}, \end{aligned}$$

and the line element can be rewritten as

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

In this form the transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$, is trivial. It leaves the line element unchanged and the null geodesic $r = 0$ trivially invariant. This is a Lorentz transformation which leaves one null direction invariant. Therefore it is a two real parameter, singular Lorentz transformation, where the two parameters come from the complex variable w. With this form of the line element the transformation is obviously trivially invariant, we now want to see what this transformation looks like in terms of the usual coordinates (x, y, z, t).

First invert the transformation (18) and use the new variable ζ

$$\begin{aligned} x + iy &= r(\xi + i\eta) = r\zeta \\ z &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta}) \\ t &= u + \frac{r}{2}(1 + \zeta\bar{\zeta}) \end{aligned} \tag{21}$$

From this it is clear that

$$
\begin{aligned}
t - z &= r, \\
t + z &= 2u + r\zeta\bar{\zeta}.
\end{aligned}
$$

So finally

$$
\begin{aligned}
r &= t - z, \\
\zeta &= \frac{x + iy}{t - z}, \\
u &= \frac{1}{2}(t + z) - \frac{(x^2 + y^2)}{2(t - z)}.
\end{aligned} \tag{22}
$$

Now make the desired transformation $(\zeta', \bar{\zeta}', r', u') \rightarrow (\zeta + w, \bar{\zeta} + \bar{w}, r, u)$ by first replacing these new quantities into transformation (21)

$$
\begin{aligned}
x' + iy' &= r'\zeta' = r(\zeta + w), \\
z' &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}), \\
t' &= u + \frac{r}{2}(1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}).
\end{aligned}
$$

So the transformed Cartesian coordinates have been written in terms of the untransformed particular coordinates, (ζ, r, u). Next, using the relations (22), write the transformed Cartesian coordinates in terms of the untransformed Cartesian coordinates.

$$
x' + iy' = x + iy + w(t - z), \tag{23}
$$

$$
z' - t' = -r = z - t, \tag{24}
$$

$$
z' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \tag{25}
$$

It is necessary to show that this is indeed a Lorentz transformation by verifying the usual Lorentz invariant quadratic form. First Eqn.(23) implies

$$
\begin{aligned}
{x'}^2 + {y'}^2 &= (x + iy + w(t - z))(x - iy + \bar{w}(t - z)) \\
&= x^2 + y^2 + \bar{w}(t - z)(x + iy) + w(t - z)(x - iy) + w\bar{w}(t - z)^2.
\end{aligned}
$$

Then Eqn.(24) and Eqn.(25) imply

$$
\begin{aligned}
{z'}^2 - {t'}^2 &= (z' + t')(z' - t') \\
&= z^2 - t^2 + (z - t)w(x - iy) + (z - t)\bar{w}(x + iy) + (z - t)w\bar{w}(t - z)
\end{aligned}
$$

Thus the extra terms cancel and Lorentz invariant quadratic form in the primed frame is the same as that of the unprimed frame,

$$
{x'}^2 + {y'}^2 + {z'}^2 - {t'}^2 = x^2 + y^2 + z^2 - t^2.
$$

It is also clear from Eqn.(24) that the null direction $z = t$ is invariant under this Lorentz transformation.

In conclusion, this transformation involves one complex parameter and thus two real parameters. In the usual Cartesian coordinates it is described by Eqns.(23) - (25) and in the coordinates (ξ, η, r, u), also denoted by (ζ, r, u) derived in previous sections, it is expressed simply as

$$\zeta' = \zeta + w,$$
$$r' = r,$$
$$u' = u.$$

Note that the operation of addition of complex numbers is commutative so that if $\zeta' = \zeta + w_1$ and $\zeta'' = \zeta' + w_2$ then

$$\zeta'' = \zeta + w_1 + w_2 = \zeta + w_3.$$

Thus these transformations form a 2-parameter abelian subgroup of the Lorentz group with the binary operation of addition of complex numbers. (ERROR. SHOW ALL 2 PARAM ABELIAN SUBGROUPS OF THE LORENTZ GROUP ARE SINGULAR TRANSFORMATIONS???)

So a Lorentz transformation that preserves the line element Eqn.(8) has been constructed. It is found that this transformation is a singular Lorentz transformation as it keeps the null direction $r = 0$ fixed. Thus it is shown that all two parameter abelian subgroups of the Lorentz group are singular Lorentz transformations. (ERROR. HAVE WE DONE ENOUGH TO SAY THIS)

IV. SPECIAL LINEAR MATRICES OF THE LORENTZ TRANSFORMATION

(ERROR. NEED MORE INTRO)Let $\vec{x} = (x, y, z, t)$ be the position vector of a point in Minkowskian space-time. Knowing $\vec{x}$ we can construct the following 2×2 Hermitian matrix

$$A = \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix}, \tag{26}$$

with $A^\dagger(\vec{x}) = A(\vec{x})$. This is useful as its determinant is the same as the Lorentz invariant quadratic form, up to an arbitrary sign.

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

Consider any 2×2 Hermitian matrix H.

$$H = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \ H^\dagger = \begin{pmatrix} \bar{p} & \bar{r} \\ \bar{q} & \bar{s} \end{pmatrix}$$

It is know that $H^\dagger(\vec{x}) = H(\vec{x})$ so it is clear that $p = \bar{p}$ and $s = \bar{s}$ and thus p and s are real numbers. Also $q = \bar{r}$ and then of course $\bar{q} = r$. Hence knowing p, q, r and s is equivalent to knowing 4 real numbers, one from p, one from s and two from q. From these parameters the coordinates (x, y, z, t) of a point in Minkowskian space-time can be constructed as

$$x + iy = q = \bar{r},$$
$$t - z = p,$$
$$t + z = s.$$

by comparing with matrix A above. Hence it is true that there is a one to one correspondence between points in Minkowskian space-time and 2×2 Hermitian matrices.

Construct the following matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$