The Lorentz Group and Singular Lorentz Transformations

Kevin Maguire (10318135)
(Dated: January 25, 2014)

abstract

Contents

I. The Lorentz Transformation 2

II. Reparameterisation of the Schwarzschild Solution 2

A. Calculations 4
1. Eddington-Finkelstein Coordinate Transform 4

II. Acknowledgements 4

References 4

I. THE LORENTZ TRANSFORMATION

The Lorentz Transform is defined by

$$(x, y, z, t) \to (x', y', z', t') \text{ such that}$$
$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

If the transformation preserves the orientation of the spatial axes then is it called a proper Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also If $t \geq 0 \Rightarrow t' \geq 0$ then it is called an orthochronous Lorentz transforamtion. This ensures that the time direction is preserved. In this project the "Lorentz transforamation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as follows (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation:

$$x' = \gamma(x - vt) \ \ \text{, where } \gamma = (1 - v^2)^{-1/2}$$
$$t' = \gamma(t - vx)$$

Rearranging:

$$x' - t' = \gamma(1 + v)(x - t)$$
$$x' + t' = \gamma(1 - v)(x + t)$$

It is clear that:

$$x = \pm t \leftrightarrow x' = \pm t'$$

Thus there are two null directions (are null directions by definition invarient???)in this space-time at $x = \pm t$. It can be shown that all Lorentz transformations have two invarient null directions except the singular Lorentz transformation which has only one fixed null direcion.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

(what do we want the Kasner vacuum solution for???)
Starting with the Schwarschild Solution of the vacuum field equations:

$$\epsilon ds^2 = \big(1 - \frac{2m}{r}\big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \big(1 - \frac{2m}{r}\big)dt^2 \tag{1}$$

Now make the Eddington-Finkelstein coordinate transformation:

$$u = t - r - 2m \ln(r - 2m) \tag{2}$$

and write Eqn.(1) in terms of u to obtain:

$$\epsilon ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2dudr - du^2 + \frac{2m}{r} du^2$$

See A 1 for a detailed calculation. Note that if $m = 0$ the space-time becomes Minkowskian, as expected. Setting $r = 0$ in this Minkowskian space-time the line element becomes:

$$\epsilon ds^2 = -du^2 \Rightarrow t = -1$$

Which implies that $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u.(CHECK ITS A GEODESIC???)

We want to find the limit of the Schwarzschild solution as $m \to \infty$. In its current form, the limit cannot be calculated so first a suitable coordinate transformation must be made.

$$\begin{aligned} u &= \mu u' \quad \text{, where } \mu = \text{ const} \\ r &= \mu^{-1} r' \end{aligned}$$

So that

$$\begin{aligned} du &= \mu du' \\ dr &= \mu^{-1} dr' \\ \Rightarrow dudr &= du'dr' \end{aligned}$$

To obtain (SEE CALCS PG3):

$$\epsilon ds^2 = r'^2 \sin^2\theta \left\{ \frac{d\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2} d\phi^2 \right\} - 2du'dr' - \left(\mu^2 - \frac{2m\mu^3}{r'} du' \right)$$

Now set $m\mu^3 = k = \text{ const} \Rightarrow m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by:

$$\sin\theta = \frac{1}{\cosh(\mu\xi)} \quad , \mu^{-1}\phi = \eta \tag{3}$$

Which results in (SEE CALCS PG 3):

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 \mu\xi} (d\xi^2 + d\eta^2) - 2dudr - \left(\mu^2 - \frac{2k}{r} \right) du^2$$

Where the primes have been dropped for notational simplicity. This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$. This limit gives:

$$\epsilon ds^2 = r^2 (d\xi^2 + d\eta^2) - 2dudr - \frac{2k}{r} du^2 \tag{4}$$

This is still a solution of the field equations, but it is no longer the Schwarzschild solution. It is found that this is the Kasner solution. To see this (SEE CALCS PG ???):

$$\epsilon ds^2 = T^{-2/3} dX^2 + T^{4/3} \left(dY^2 + dZ^2 \right) - dT^2 \tag{5}$$

By definition the Kasner solution is given by:

$$\epsilon ds^2 = T^{2p} dX^2 + T^{2q} dY^2 + T^{2r} dZ^2 - dT^2$$

With:

$$p + q + r = 1 = p^2 + q^2 + r^2$$

So it is clear that Eqn.(5) is the Kasner solution with $p = -1/3$ and $q = r = 2/3$.

It is clear that Minkowskian space-time emerges again by setting $k = 0$ in Eqn.(4), which is equivalent to $m = 0$. Again setting $r = 0$ to find that $\epsilon ds^2 = 0$ in this case. Thus $r = 0$ is now a null geodesic with u an affine parameter along it (SHOW).

Appendix A: Calculations

1. Eddington-Finkelstein Coordinate Transform

Using the Eddington-Finkelstein coordinate transform Eqn.(??) and working out the differential:

$$du = dt - dr - \frac{2mdr}{r - 2m}$$

SEE CALCULATIONS PG1

II. ACKNOWLEDGEMENTS