The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

I. THE LORENTZ TRANSFORMATION

The Lorentz Transform is defined by

$$(x, y, z, t) \rightarrow (x', y', z', t') \text{ such that}$$
$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

If the transformation preserves the orientation of the spatial axes then is it called a proper Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also If $t \geq 0 \Rightarrow t' \geq 0$ then it is called an orthochronous Lorentz transforamtion. This ensures that the time direction is preserved. In this project the "Lorentz transforamation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as follows (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation:

$$x' = \gamma(x - vt) \quad \text{, where } \gamma = (1 - v^2)^{-1/2}$$
$$t' = \gamma(t - vx)$$

Rearranging:

$$x' - t' = \gamma(1 + v)(x - t)$$
$$x' + t' = \gamma(1 - v)(x + t)$$

It is clear that:

$$x = \pm t \leftrightarrow x' = \pm t'$$

Thus there are two null directions (are null directions by definition invarient???)in this space-time at $x = \pm t$. It can be shown that all Lorentz transformations have two invarient null directions except the singular Lorentz transformation which has only one fixed null direcion.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

(what do we want the Kasner vacuum solution for???)
Starting with the Schwarschild Solution of the vacuum field equations:

$$\epsilon ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right)dt^2 \tag{1}$$

Now make the Eddington-Finkelstein coordinate transformation:

$$u = t - r - 2m \ln(r - 2m) \tag{2}$$

Working out the differential:

$$du = dt - dr - \frac{2mdr}{r - 2m}$$

SEE CALCULATIONS PG1
Then write Eqn.(1) in terms of u to obtain:

$$\epsilon ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2dudr - du^2 + \frac{2m}{r}du^2$$

Note that if $m = 0$ the space-time becomes Minkowskian, as expected. Setting $r = 0$ in this Minkowskian space-time the line element becomes:

$$\epsilon ds^2 = -du^2 \Rightarrow t = -1$$

Which implies that $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u.(CHECK ITS A GEODESIC???)

We want to find the limit of the Schwarzschild solution as $m \to \infty$. In its current form, the limit cannot be calculated so first a suitable coordinate transformation must be made.

$$\begin{aligned} u &= \mu u' \quad \text{, where } \mu = \text{ const} \\ r &= \mu^{-1} r' \end{aligned}$$

So that

$$\begin{aligned} du &= \mu du' \\ dr &= \mu^{-1} dr' \\ \Rightarrow dudr &= du'dr' \end{aligned}$$

To obtain (SEE CALCS PG3):

$$\epsilon ds^2 = r'^2 \sin^2\theta \left\{ \frac{d\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2} d\phi^2 \right\} - 2du'dr' - \left(\mu^2 - \frac{2m\mu^3}{r'} du' \right)$$

Now set $m\mu^3 = k = \text{ const} \Rightarrow m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by:

$$\sin\theta = \frac{1}{\cosh(\mu\xi)} \quad , \mu^{-1}\phi = \eta \tag{3}$$

Which results in (SEE CALCS PG 3):

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 \mu\xi} (d\xi^2 + d\eta^2) - 2dudr - \left(\mu^2 - \frac{2k}{r} \right) du^2$$

Where the primes have been dropped for notational simplicity. This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$. This limit gives:

$$\epsilon ds^2 = r^2 (d\xi^2 + d\eta^2) - 2dudr - \frac{2k}{r} du^2 \tag{4}$$

This is still a solution of the field equations, but it is no longer the Schwarzschild solution. It is found that this is the Kasner solution. To see this (SEE CALCS PG ???):

$$\epsilon ds^2 = T^{-2/3} dX^2 + T^{4/3} \left(dY^2 + dZ^2 \right) - dT^2 \tag{5}$$

By definition the Kasner solution is given by:

$$\epsilon ds^2 = T^{2p} dX^2 + T^{2q} dY^2 + T^{2r} dZ^2 - dT^2$$

With:

$$p + q + r = 1 = p^2 + q^2 + r^2$$

So it is clear that Eqn.(5) is the Kasner solution with $p = -1/3$ and $q = r = 2/3$.

A. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting $k = 0$ in Eqn.(4), which is equivalent to $m = 0$.

$$\epsilon ds^2 = r^2(d\xi^2 + d\eta^2) - 2dudr \tag{6}$$

Setting $r = 0$ it can be shown that $\epsilon ds^2 = 0$ in this case. Then $r = 0$ is a null geodesic with u an affine parameter along it. To demonstrate these properties first let $x^i = (x, y, z, t)$ be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element:

$$\epsilon {ds_0}^2 = dx^2 + dy^2 + dz^2 - dt^2$$

We note that $C : x = 0, y = 0, z = t$ is a null geodesic as it will lie in the light cone of Minkowskian space-time. it we write it parametrically as $x^i = w^i(u)$ such that $w^i = (0, 0, u, u)$ then u is an affine parameter along C. The tangent to C is then computed as:

$$v^i(u) = \frac{dw^i}{du} = (0, 0, 1, 1)$$

As C is a null geodesic the first integral will be $v_i v^i = 0$ (IS THIS CALLED THE FIRST INTEGRAL?) and thus $v_i = (0, 0, 1, -1)$ where we have chosen the convention $(+, +, +, -)$.

The position vector of a point in Minkowskian space time can be written in the form(DO THE PICTURE FROM 2:1):

$$\begin{aligned} x^i &= w^i(u) = rk^i \\ \text{or } x^i &= w^i(u) + rk^i \end{aligned}$$

Thus r is a new parameter which tell us the shortest distance between C and some point x^i, and k^i is the unit vector in that direction. As k^i is a unit vector it satisfies the realtions:

$$k^i k_i = 0 \tag{7}$$
$$k^i v_i = -1 \tag{8}$$

Thus k^i is normalized so that k^i and v^i are both future pointing (HOW DOES THIS MAKE THEM BOTH FUTURE POINTING?). Making the parameterisation:

$$\begin{aligned} k^i &= (\xi, \eta, A, B) \\ \Rightarrow k_i &= (\xi, \eta, A, -B) \end{aligned}$$

We can choose any variable for the first two slots of k^i so we choose ξ and η from before for conveinience. Using the relation (7) it is clear that:

$$\xi^2 + \eta^2 + A^2 - B^2 = 0$$

and using the relation (8) it is found that:

$$A - B = -1 \tag{9}$$
$$\Rightarrow A^2 - B^2 = (A + B)(A - B) = -(A + B) \tag{10}$$

Which implies:

$$\xi^2 + \eta^2 = A + B \tag{11}$$

So expressions for A and B are found using Eqn.(9) and Eqn.(11):

$$
\begin{aligned}
A &= \frac{1}{2}(-1+\xi^2+\eta^2) \\
B &= \frac{1}{2}(1+\xi^2+\eta^2)
\end{aligned}
$$

In summary so far we have:

$$x^i = w^i(u) + rk^i \tag{12}$$
$$w^i = (0,0,u,u) \tag{13}$$
$$k^i = (\xi, \eta, \frac{1}{2}(-1+\xi^2+\eta^2), \frac{1}{2}(1+\xi^2+\eta^2)) \tag{14}$$
$$x^i = (x,y,z,t) \tag{15}$$

Consider Eqn.(12) as a coordinate transformation from (x,y,z,t) to (ξ,η,r,u) such that:

$$
\begin{aligned}
x &= r\xi \\
y &= r\eta \\
z &= u + \frac{r}{2}(-1+\xi^2+\eta^2) \\
t &= u + \frac{r}{2}(1+\xi^2+\eta^2)
\end{aligned}
$$

as is clear from Eqn.(12) - Eqn.(15). Now this is applied to the Minkowskian line element Eqn.(6). First, the x and y differentials are:

$$
\begin{aligned}
dx &= rd\xi + \xi dr \\
dy &= rd\eta + \eta dr
\end{aligned}
$$

Which gives:

$$dx^2 + dy^2 = r^2(d\xi^2 + d\eta^2) + 2r\xi d\xi dr + 2r\eta d\eta dr + (\xi^2+\eta^2)dr^2 \tag{16}$$

Next, the z and t differentials:

$$
\begin{aligned}
z + t &= 2u + r(\xi^2+\eta^2) \\
z - t &= -r \\
dz + dt &= 2du + (\xi^2+\eta^2)dr + 2r\xi d\xi + 2r\eta d\eta \\
dz - dt &= -dr
\end{aligned}
$$

Using difference of two squares to obtai:

$$dz^2 - dt^2 = -2dudr - (\xi^2+\eta^2)dr^2 - 2r\xi d\xi dr - 2r\eta d\eta dr \tag{17}$$

Combining Eqn.(**??**) and (**??**) to get:

$$dx^2 + dy^2 + dz^2 - dt^2 = r^2(d\xi^2 + d\eta^2) - 2dudr$$

and from this it is clear that Eqn.(6) is the line element of Minkowskian space-time with $r = 0$ a null geodesic with affine parameter u along it as before.

III. ACKNOWLEDGEMENTS