The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

I. THE LORENTZ TRANSFORMATION

In this section the two null directions inherent in all Lorentz transformations, except the singular Lorentz transformations, are illustrated. The Lorentz Transform is defined by $(x, y, z, t) \to (x', y', z', t')$ such that

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2.$$

If the transformation preserves the orientation of the spatial axes then is it called a *proper* Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also if $t \geq 0$ implies that time is always positive then it is called an *orthochronous* Lorentz transformation, which ensures that the time direction is preserved. In this project the "Lorentz transformation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as in Fig.() (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx), \end{aligned}$$

where $\gamma = (1 - v^2)^{-1/2}$. Rearrange to obtain

$$\begin{aligned} x' - t' &= \gamma(1 + v)(x - t), \\ x' + t' &= \gamma(1 - v)(x + t). \end{aligned}$$

It is clear that $x = \pm t$ implies $x' = \pm t'$. Thus there are two null directions in this space-time at $x = \pm t$, as null directions are by definition invariant under a Lorentz transformation. It can be shown that all Lorentz transformations have two invariant null directions except the singular Lorentz transformation which has only one fixed null direction.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

In this section the Kasner solution of the vacuum field equations is derived from the Schwarzschild solution by taking the limit as the mass goes to infinity. It is then shown that the special case of the Kasner solution with no mass is equivalent to a novel form of Minkowskian space-time. (what do we want the Kasner vacuum solution for???). Start with the Schwarzschild Solution of the vacuum field equations given by

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2. \tag{1}$$

A. Eddington-Finkelstein Coordinate Transformation

First, make the Eddington-Finkelstein coordinate transformation

$$u = t - r - 2m\ln(r - 2m). \tag{2}$$

Calculate the differentials

$$\begin{aligned} \mathrm{d}u &= \mathrm{d}t - \mathrm{d}r - \frac{2m\mathrm{d}r}{r - 2m}, \\ &= \mathrm{d}t - \mathrm{d}r\left(1 - \frac{2m}{r}\right)^{-1}, \\ \mathrm{d}t &= \mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r, \end{aligned}$$

and sub them into Eqn.(1).

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right)\left(\mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r\right)^2,$$

which gives the result,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - 2\mathrm{d}u\mathrm{d}r - \mathrm{d}u^2 + \frac{2m}{r}\mathrm{d}u^2. \tag{3}$$

Note that if $m = 0$ the space-time becomes Minkowskian, as expected. If $r = 0$ in this Minkowskian space-time the line element becomes

$$\epsilon \mathrm{d}s^2 = -\mathrm{d}u^2,$$

which implies that $\epsilon = -1$ and the first integral of this trajectory is also equal to -1. Thus $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u.(ERROR CHECK ITS A GEODESIC???NEED TO CHECK ANSWER WITH HOGAN)

The limit of the Schwarzschild solution as $m \to \infty$ must be calculated to find the Kasner solution. In its current form the limit cannot be taken, so two suitable coordinate transformations must be made to get it in a more useful form. First Set

$$u = \mu u',$$
$$r = \mu^{-1} r',$$

where $\mu =$ const so that

$$\mathrm{d}u = \mu \mathrm{d}u',$$
$$\mathrm{d}r = \mu^{-1} \mathrm{d}r'.$$

The product of the differentials is then invariant

$$\mathrm{d}u\mathrm{d}r = \mathrm{d}u'\mathrm{d}r'.$$

When the new coordinates are subbed into Eqn.(3) the Schwarzschild solution becomes

$$\epsilon \mathrm{d}s^2 = r'^2 \sin^2\theta \left\{ \frac{\mathrm{d}\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2}\mathrm{d}\phi^2 \right\} - 2\mathrm{d}u'\mathrm{d}r' - \left(\mu^2 - \frac{2m\mu^3}{r'}\right)\mathrm{d}u'^2. \tag{4}$$

Now set $m\mu^3 = k$ or $m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by

$$\sin\theta = \frac{1}{\cosh(\mu\xi)}, \mu^{-1}\phi = \eta.$$

The second of these transformations gives simply $\mu^{-2}\mathrm{d}\phi^2 = \mathrm{d}\eta^2$. To rewrite the first coordinate transformation, first differentiate.

$$\cos\theta \mathrm{d}\theta = \frac{-1}{(\cosh(\mu\xi))^2}\sinh(\mu\xi)\mu \mathrm{d}\xi,$$
$$= -\sin^2\theta \sinh(\mu\xi)\mu \mathrm{d}\xi.$$

Use the forumla $\cosh^2 A - \sinh^2 A = 1$, divide by $\mathrm{d}\xi$ and simplfy using trigonometric identities

$$\cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\mu \sin^2\theta \sqrt{\frac{1}{\sin^2\theta} - 1},$$
$$= -\mu \sin\theta \cos\theta.$$

Finally rewrite in terms of $\mathrm{d}\xi$

$$\mathrm{d}\xi^2 = \left(\frac{\mathrm{d}\theta}{\mu \sin\theta}\right)^2.$$

Subbing these transformations into Eqn.(4) gives

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right)\mathrm{d}u^2,$$

where the primes have been dropped for convenience.

B. The Kasner Solution

This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$, to obtain

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \frac{2k}{r}\mathrm{d}u^2 \tag{5}$$

This is still a solution of the field equations but it is no longer the Schwarzschild solution. In this section it is shown to be the Kasner Solution(READ UP ABOUT THIS), which by definition is given by

$$\epsilon \mathrm{d}s^2 = T^{2p}\mathrm{d}X^2 + T^{2q}\mathrm{d}Y^2 + T^{2r}\mathrm{d}Z^2 - \mathrm{d}T^2, \tag{6}$$

such that

$$p + q + r = 1 = p^2 + q^2 + r^2.$$

To write Eqn.(5) in this form first make the transformation

$$\xi' = \lambda^{-1}\xi,\ \eta' = \lambda^{-1}\eta,$$
$$r' = \lambda r,\ u' = \lambda^{-1}u,$$

with $\lambda := k^{-1/3}$. Subbing in these new coordinates gives

$$\epsilon \mathrm{d}s^2 = r'^2(\mathrm{d}\xi'^2 + \mathrm{d}\eta'^2) - 2\mathrm{d}u'\mathrm{d}r' + \frac{2}{r'}\mathrm{d}u'^2.$$

Now add and subtract $(r'/2)\mathrm{d}r'^2$ to complete the square as follows

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\left(\mathrm{d}u - \frac{r}{2}\mathrm{d}r\right)^2 - \frac{r}{2}\mathrm{d}r^2,$$

where the primes have again been dropped for convenience. Now set

$$\bar{X} = u - \frac{r^2}{4},$$

so the differential of $\bar{X}$ is

$$\mathrm{d}\bar{X} = \mathrm{d}u - \frac{r}{2}\mathrm{d}r,$$

and the line element can be rewritten in terms of $\bar{X}$,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\mathrm{d}\bar{X}^2 - \left(\frac{r^{1/2}}{\sqrt{2}}\mathrm{d}r\right)^2.$$

Now define T such that

$$T = \frac{\sqrt{2}}{3}r^{3/2}, \text{ and } r = \left(\frac{3}{\sqrt{2}}\right)^{2/3} T^{2/3},$$

which results in the new line element

$$\epsilon \mathrm{d}s^2 = \left(\frac{3}{\sqrt{2}}\right)^{4/3} T^{4/3}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2\left(\frac{\sqrt{2}}{3}\right)^{2/3} T^{-2/3}\mathrm{d}\bar{X}^2 - \mathrm{d}T^2.$$

Then a final coordinate transformation can be made to remove the unwanted constants, given by

$$\begin{aligned} X &= \left[2\left(\frac{\sqrt{2}}{3}\right)\right]^{1/2} \bar{X}, \\ Y &= \left(\frac{3}{\sqrt{2}}\right)^{2/3} \xi, \\ Z &= \left(\frac{3}{\sqrt{2}}\right)^{2/3} \eta, \end{aligned}$$

to obtain

$$\epsilon \mathrm{d}s^2 = T^{-2/3}\mathrm{d}X^2 + T^{4/3}\left(\mathrm{d}Y^2 + \mathrm{d}Z^2\right) - \mathrm{d}T^2. \tag{7}$$

Comparing this result to the general form of the Kasner solution in Eqn.(6) it is clear that they have the same form with $p = -1/3$ and $q = r = 2/3$. Thus the solution obtained by taking the limit of the Schwarzschild solution as $m \to \infty$ is indeed the Kasner Solution.

C. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting $k = 0$ in Eqn.(5), which is equivalent to $m = 0$.

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r \tag{8}$$

Setting $r = 0$ gives $\epsilon \mathrm{d}s^2 = 0$. In this section it is demonstrated that $r = 0$ is a null geodesic with u an affine parameter along it and that Eqn.(8) is indeed the Minkowskian space-time line element. To verify these properties first let $x^i = (x, y, z, t)$ be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element

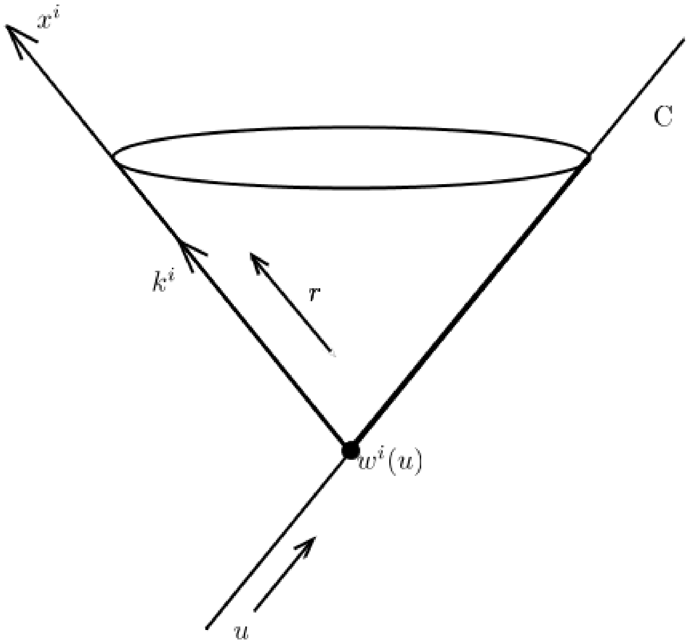
$$\epsilon \mathrm{d}{s_0}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2.$$

We note that the trajectory C defined by $x = 0,\ y = 0,\ z = t$ is a null geodesic as it will lie on the light cone of Minkowskian space-time. If C is writen parametrically as $x^i = w^i(u)$ such that $w^i = (0, 0, u, u)$ then u is an affine parameter along it. The tangent to C is then computed as

$$v^i(u) = \frac{\mathrm{d}w^i}{\mathrm{d}u} = (0, 0, 1, 1).$$

As C is a null geodesic the first integral will be $v_i v^i = 0$ and thus $v_i = (0, 0, 1, -1)$ where we have chosen the convention $(+, +, +, -)$.

FIG. 1: *Minkowskian space-time illustrating the new parameter r which is the shortest distance between some point x^i and the trajectory C along the k^i direction. The parameter u which determines the distance travelled along C is also shown*



The position vector of a point in Minkowskian space time can be written in the form

$$x^i - w^i(u) = rk^i,$$
$$\text{or } x^i = w^i(u) + rk^i.$$

Thus r is a new parameter which tell us the shortest distance between C and some point x^i, and k^i is the unit vector in that direction, see Fig.(1). As k^i is a unit vector it satisfies the relations

$$k^i k_i = 0, \tag{9}$$
$$k^i v_i = -1. \tag{10}$$

Thus k^i is normalized so that k^i and v^i are both future pointing (HOW DOES THIS MAKE THEM BOTH FUTURE POINTING?). Making the parameterisation

$$k^i = (\xi, \eta, A, B),$$
$$k_i = (\xi, \eta, A, -B).$$

We can choose any variable for the first two slots of k^i so we choose ξ and η from before for convenience. Using the relation (9) it is clear that

$$\xi^2 + \eta^2 + A^2 - B^2 = 0,$$

and using the relation (10) it is found that

$$A - B = -1, \tag{11}$$
$$\Rightarrow A^2 - B^2 = (A+B)(A-B) = -(A+B). \tag{12}$$

Which implies

$$\xi^2 + \eta^2 = A + B. \tag{13}$$

So expressions for A and B are found using Eqn.(11) and Eqn.(13):

$$\begin{aligned} A &= \frac{1}{2}(-1 + \xi^2 + \eta^2) \\ B &= \frac{1}{2}(1 + \xi^2 + \eta^2) \end{aligned}$$

In summary so far we have

$$x^i = w^i(u) + rk^i, \tag{14}$$
$$w^i = (0, 0, u, u), \tag{15}$$
$$k^i = (\xi, \eta, \frac{1}{2}(-1 + \xi^2 + \eta^2), \frac{1}{2}(1 + \xi^2 + \eta^2)), \tag{16}$$
$$x^i = (x, y, z, t). \tag{17}$$

Consider Eqn.(14) as a coordinate transformation from (x, y, z, t) to (ξ, η, r, u) such that

$$\begin{aligned} x &= r\xi, \\ y &= r\eta, \\ z &= u + \frac{r}{2}(-1 + \xi^2 + \eta^2), \\ t &= u + \frac{r}{2}(1 + \xi^2 + \eta^2), \end{aligned} \tag{18}$$

which is clear from Eqns.(14) - (17). Now this is applied to the Minkowskian line element of Eqn.(8). First, the x and y differentials are

$$\begin{aligned} dx &= r\mathrm{d}\xi + \xi\mathrm{d}r, \\ \mathrm{d}y &= r\mathrm{d}\eta + \eta\mathrm{d}r. \end{aligned}$$

Which gives

$$\mathrm{d}x^2 + \mathrm{d}y^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2r\xi\mathrm{d}\xi\mathrm{d}r + 2r\eta\mathrm{d}\eta\mathrm{d}r + (\xi^2 + \eta^2)\mathrm{d}r^2. \tag{19}$$

Next, the z and t differentials

$$\begin{aligned} z+t &= 2u + r(\xi^2+\eta^2), \\ z-t &= -r, \\ \mathrm{d}z+\mathrm{d}t &= 2\mathrm{d}u + (\xi^2+\eta^2)\mathrm{d}r + 2r\xi\mathrm{d}\xi + 2r\eta\mathrm{d}\eta, \\ \mathrm{d}z-\mathrm{d}t &= -\mathrm{d}r. \end{aligned}$$

Then using difference of two squares to obtain

$$\mathrm{d}z^2 - \mathrm{d}t^2 = -2\mathrm{d}u\mathrm{d}r - (\xi^2+\eta^2)\mathrm{d}r^2 - 2r\xi\mathrm{d}\xi\mathrm{d}r - 2r\eta\mathrm{d}\eta\mathrm{d}r. \tag{20}$$

Combining Eqn.(19) and (20) to get:

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r$$

and from this it is clear that Eqn.(8) is the line element of Minkowskian space-time with $r = 0$ a null geodesic with affine parameter u along it as stated at the beginning of the section.(ERROR. REFER TO GEODESIC CALC FROM EARLIER)

Thus it has been shown that the Kasner solution to the vacuum filed equations can be obtained from the Schwarzschild solution of the vacuum field equations by taking the special case where the mass goes to infinity. (ERROR. ISNT THIS A CONTRADICTION???) Then the Minkowskian space-time line element in a particular set of coordinates (ξ, η, r, u), can be derived from the Kasner solution by setting the mass to zero. It has been shown that the special case where $r = 0$ in this Minkowskian space-time is then a null geodesic with u an affine parameter along it.

III. THE SINGULAR LORENTZ TRANSFORMATION

In this section a Lorentz transformation that leaves our line element Eqn.(8) invariant is constructed. This transformation is then expressed in terms of (x, y, z, t) and examined to see what form it has. The subgroup of the Lorentz group that it makes is then examined. First we define an arbitrary complex parameter by $\zeta = \xi + i\eta$ so that the differentials are given by

$$\begin{aligned} d\zeta &= d\xi + id\eta, \\ d\bar{\zeta} &= d\bar{\xi} - id\bar{\eta}, \end{aligned}$$

and the line element can be rewritten as

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

In this form the transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$, is trivial. It leaves the line element unchanged and the null geodesic $r = 0$ trivially invariant. This is a Lorentz transformation which leaves one null direction invariant. Therefore it is a two real parameter, singular Lorentz transformation, where the two parameters come from the complex variable w. With this form of the line element the transformation is obviously trivially invariant, we now want to see what this transformation looks like in terms of the usual coordinates (x, y, z, t).

First invert the transformation (18) and use the new variable ζ

$$\begin{aligned} x + iy &= r(\xi + i\eta) = r\zeta \\ z &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta}) \\ t &= u + \frac{r}{2}(1 + \zeta\bar{\zeta}) \end{aligned} \tag{21}$$

From this it is clear that

$$
\begin{aligned}
t - z &= r, \\
t + z &= 2u + r\zeta\bar{\zeta}.
\end{aligned}
$$

So finally

$$
\begin{aligned}
r &= t - z, \\
\zeta &= \frac{x + iy}{t - z}, \\
u &= \frac{1}{2}(t + z) - \frac{(x^2 + y^2)}{2(t - z)}.
\end{aligned} \tag{22}
$$

Now make the desired transformation $(\zeta', \bar{\zeta}', r', u') \to (\zeta + w, \bar{\zeta} + \bar{w}, r, u)$ by first replacing these new quantities into transformation (21)

$$
\begin{aligned}
x' + iy' &= r'\zeta' = r(\zeta + w), \\
z' &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}), \\
t' &= u + \frac{r}{2}(1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}).
\end{aligned}
$$

So the transformed Cartesian coordinates have been written in terms of the untransformed particular coordinates, (ζ, r, u). Next, using the relations (22), write the transformed Cartesian coordinates in terms of the untransformed Cartesian coordinates.

$$
x' + iy' = x + iy + w(t - z), \tag{23}
$$
$$
z' - t' = -r = z - t, \tag{24}
$$
$$
z' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \tag{25}
$$

It is necessary to show that this is indeed a Lorentz transformation by verifying the usual Lorentz invariant quadratic form. First Eqn.(23) implies

$$
\begin{aligned}
{x'}^2 + {y'}^2 &= (x + iy + w(t - z))(x - iy + \bar{w}(t - z)) \\
&= x^2 + y^2 + \bar{w}(t - z)(x + iy) + w(t - z)(x - iy) + w\bar{w}(t - z)^2.
\end{aligned}
$$

Then Eqn.(24) and Eqn.(25) imply

$$
\begin{aligned}
{z'}^2 - {t'}^2 &= (z' + t')(z' - t') \\
&= z^2 - t^2 + (z - t)w(x - iy) + (z - t)\bar{w}(x + iy) + (z - t)w\bar{w}(t - z)
\end{aligned}
$$

Thus the extra terms cancel and Lorentz invariant quadratic form in the primed frame is the same as that of the unprimed frame,

$$
{x'}^2 + {y'}^2 + {z'}^2 - {t'}^2 = x^2 + y^2 + z^2 - t^2.
$$

It is also clear from Eqn.(24) that the null direction $z = t$ is invariant under this Lorentz transformation.

In conclusion, this transformation involves one complex parameter and thus two real parameters. In the usual Cartesian coordinates it is described by Eqns.(23) - (25) and in the coordinates (ξ, η, r, u), also denoted by (ζ, r, u) derived in previous sections, it is expressed simply as

$$\zeta' = \zeta + w,$$
$$r' = r,$$
$$u' = u.$$

Note that the operation of addition of complex numbers is commutative so that if $\zeta' = \zeta + w_1$ and $\zeta'' = \zeta' + w_2$ then

$$\zeta'' = \zeta + w_1 + w_2 = \zeta + w_3.$$

Thus these transformations form a 2-parameter abelian subgroup of the Lorentz group with the binary operation of addition of complex numbers. (ERROR. SHOW ALL 2 PARAM ABELIAN SUBGROUPS OF THE LORENTZ GROUP ARE SINGULAR TRANSFORMATIONS???)

So a Lorentz transformation that preserves the line element Eqn.(8) has been constructed. It is found that this transformation is a singular Lorentz transformation as it keeps the null direction $r = 0$ fixed. Thus it is shown that all two parameter abelian subgroups of the Lorentz group are singular Lorentz transformations. (ERROR. HAVE WE DONE ENOUGH TO SAY THIS)

IV. SPECIAL LINEAR MATRICES OF THE LORENTZ TRANSFORMATION

(ERROR. NEED MORE INTRO)Let $\vec{x} = (x, y, z, t)$ be the position vector of a point in Minkowskian space-time. Knowing $\vec{x}$ we can construct the following 2×2 Hermitian matrix

$$A = \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix}, \tag{26}$$

with $A^\dagger(\vec{x}) = A(\vec{x})$. This is useful as its determinant is the same as the Lorentz invariant quadratic form, up to an arbitrary sign.

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

Consider any 2×2 Hermitian matrix H.

$$H = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, H^\dagger = \begin{pmatrix} \bar{p} & \bar{r} \\ \bar{q} & \bar{s} \end{pmatrix}$$

It is know that $H^\dagger(\vec{x}) = H(\vec{x})$ so it is clear that $p = \bar{p}$ and $s = \bar{s}$ and thus p and s are real numbers. Also $q = \bar{r}$ and then of course $\bar{q} = r$. Hence knowing p, q, r and s is equivalent to knowing 4 real numbers, one from p, one from s and two from q. From these parameters the coordinates (x, y, z, t) of a point in Minkowskian space-time can be constructed as

$$x + iy = q = \bar{r},$$
$$t - z = p,$$
$$t + z = s.$$

by comparing with matrix A above. Hence it is true that there is a one to one correspondence between points in Minkowskian space-time and 2×2 Hermitian matrices.

Construct the following matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with α, β, γ, $\delta \in \mathbb{C}$, and the condition that $\det(U) = 1$. Such matrices U form a group called the special linear group, which is denoted by $SL(2,\mathbb{C})$. Given $A(\vec{x})$ consider $UA(\vec{x})U^\dagger$. This is a 2×2 Hermitian matrix since

$$\begin{aligned}(UA(\vec{x})U^\dagger)^\dagger &= (U^\dagger)^\dagger A^\dagger(\vec{x})U^\dagger \\ &= UA(\vec{x})U^\dagger,\end{aligned}$$

as $(U^\dagger)^\dagger = U$ and $A^\dagger = A$. Hence there exists a point $\vec{x}' = (x', y', z', t')$ in Minkowskian space-time for which

$$A(\vec{x}') = UA(\vec{x})U^\dagger. \tag{27}$$

Any U involves 6 real parameters, 2 each from the four complex components, with the condition $\det(U) = 1$ supplying two constraints, one on the real parts and one on the imaginary parts of the components. Now calculate the determinant of the matrix in the primed frame

$$\begin{aligned}\det(A(\vec{x}')) &= \det(UA(\vec{x})U^\dagger), \\ &= (\det(U))(\det(A(\vec{x}))(\det(U^\dagger)), \\ &= (\det(U))(\det(A(\vec{x}))(\bar{\det}(U)), \\ &= \det(A(\vec{x})).\end{aligned}$$

Thus we have the relation

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2.$$

Hence the transformation $\vec{x} \to \vec{x}'$ implicit in Eqn.(27) is a Lorentz transformation. This equation describes the most general proper, orthochronous Lorentz transformation.

It is useful to calculate the matrix U for some examples of Lorentz transformations. First, write Eqn.(27) in terms of its components:

$$\begin{aligned}\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix}\begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} (t-z)\bar{\alpha} + (x+iy)\bar{\beta} & (t-z)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha} + (t+z)\bar{\beta} & (x-iy)\bar{\gamma} + (t+z)\bar{\delta} \end{pmatrix}.\end{aligned}$$

Thus the relations

$$t' - z' = (t-z)\alpha\bar{\alpha} + (x+iy)\alpha\bar{\beta} + (x-iy)\beta \bar{alpha} + (t+z)\beta\bar{\beta}, \tag{28a}$$
$$x' + iy' = (t-z)\alpha\bar{\gamma} + (x+iy)\alpha\bar{\delta} + (x-iy)\beta\bar{\gamma} + (t+z)\beta\bar{\delta}, \tag{28b}$$
$$t' + z' = (t-z)\gamma\bar{\gamma} + (x+iy)\gamma\bar{\delta} + (x-iy)\delta\bar{\gamma} + (t+z)\delta\bar{\delta}. \tag{28c}$$

are obtained. Now these equations are used on some specific cases.

A. Example 1: Rotational Transformation

Find U corresponding to the one parameter Lorentz transformation,

$$\begin{aligned}x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta, \\ z' &= z, \\ t' &= t.\end{aligned}$$

This implies that

$$t' - z' = t - z$$
$$x' + iy' = (x + iy)e^{-i\theta}$$
$$t' + z' = t + z$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta} + \bar{\alpha}\beta = 0, \tag{29a}$$
$$i(\alpha\bar{\beta} - \bar{\alpha}\beta) = 0, \tag{29b}$$
$$-\alpha\bar{\alpha} + \beta\bar{\beta} = -1, \tag{29c}$$
$$\alpha\bar{\alpha} + \beta\bar{\beta} = 1. \tag{29d}$$

Then Eqn.(29a) and (29b) imply $\alpha\bar{\beta} = 0$ so $\alpha = 0$ or $\beta = 0$. Also Eqn.(29c) and (29d) imply $2\beta\bar{\beta} = 0$ so $\beta = 0$ and $2\alpha\bar{\alpha} = 0$ so $\alpha \neq 0$ and $\alpha\bar{\alpha} = 1$. Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$e^{-i\theta} = \alpha\bar{\delta} + \beta\bar{\gamma}, \tag{30a}$$
$$e^{-i\theta} = \alpha\bar{\delta} - \beta\bar{\gamma}, \tag{30b}$$
$$0 = -\alpha\bar{\gamma} + \beta\bar{\delta}, \tag{30c}$$
$$0 = \alpha\bar{\gamma} + \beta\bar{\delta}. \tag{30d}$$

With $\beta = 0$, Eqn.(30a) and (30b) imply $\alpha\bar{\delta} = e^{-i\theta}$. Also Eqn.(30c) and (30d) imply $\alpha\bar{\gamma} = 0$ so $\gamma = 0$ since $\alpha \neq 0$. Then using $\alpha\bar{\alpha} = 1$

$$\alpha\bar{\delta} = e^{-i\theta},$$
$$\bar{\alpha}\alpha\bar{\delta} = \bar{\alpha}e^{-i\theta},$$
$$\bar{\delta} = \bar{\alpha}e^{-i\theta}.$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28c) to obtain

$$\gamma\bar{\delta} + \delta\bar{\gamma} = 0, \tag{31a}$$
$$\gamma\bar{\delta} - \delta\bar{\gamma} = 0, \tag{31b}$$
$$-\gamma\bar{\gamma} + \delta\bar{\delta} = 1, \tag{31c}$$
$$\gamma\bar{\gamma} + \delta\bar{\delta} = 1. \tag{31d}$$

Eqn(31a) and (31b) are satisfied since $\gamma = 0$, this also implies that $\delta\bar{\delta} = 1$ from Eqn(31c). Now use the fact that $\det(U) = 1$, which implies

$$\alpha\delta - \beta\gamma = 1,$$

thus $\alpha\delta = 1$ as $\beta = 0$. Then using $\alpha\bar{\alpha} = 1$ again and $\delta\bar{\delta} = 1$

$$\alpha^2 e^{-\theta} = 1,$$
$$\alpha^2 = e^{-i\theta},$$
$$\alpha = \pm e^{-i\theta/2},$$

which finally implies that $\delta = \pm e^{i\theta/2}$. Hence there are 2 matrices U corresponding to the spacial rotation, namely

$$U = \pm \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

The next two examples are then very similar.

B. Example 2: Special Relativity Lorentz Transformation

Find U corresponding to the one parameter Lorentz transformation,

$$\begin{aligned} x' &= \gamma_0(x - vt), \\ t' &= \gamma_0(t - vx), \\ y' &= y, \\ z' &= z, \end{aligned}$$

Where $\gamma_0 = (1 - v^2)^{-1/2}$. This implies that

$$\begin{aligned} t' - z' &= -\gamma_0 vx + \gamma_0 t - z \\ x' + iy' &= \gamma_0 x - v\gamma_0 t + iy \\ t' + z' &= -v\gamma_0 x + \gamma_0 t + z \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta} + \bar{\alpha}\beta = -\gamma_0 v, \tag{32a}$$

$$i(\alpha\bar{\beta} - \bar{\alpha}\beta) = 0, \tag{32b}$$

$$-\alpha\bar{\alpha} + \beta\bar{\beta} = -1, \tag{32c}$$

$$\alpha\bar{\alpha} + \beta\bar{\beta} = \gamma_0. \tag{32d}$$

Then Eqn.(32a) and (32b) imply that $\alpha\bar{\beta} = \bar{\alpha}\beta$. Also Eqn.(32c) and (32d) imply that

$$\beta\bar{\beta} = \frac{\gamma_0 - 1}{2}, \tag{33}$$

$$\alpha\bar{\alpha} = \frac{\gamma_0 + 1}{2}. \tag{34}$$

Thus β can be written in terms of α using Eqn.(32a)

$$\begin{aligned} \alpha\bar{\beta} &= -\frac{\gamma_0 v}{2}, \\ \alpha\bar{\alpha}\bar{\beta} &= -\bar{\alpha}\frac{\gamma_0 v}{2}, \\ \beta &= -\alpha\frac{\gamma_0 v}{\gamma_0 + 1}. \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$\gamma_0 = \alpha\bar{\delta} + \beta\bar{\gamma}, \tag{35a}$$

$$1 = \alpha\bar{\delta} - \beta\bar{\gamma}, \tag{35b}$$

$$0 = -\alpha\bar{\gamma} + \beta\bar{\delta}, \tag{35c}$$

$$-v\gamma_0 = \alpha\bar{\gamma} + \beta\bar{\delta}. \tag{35d}$$

Eqn.(35a) and (35b) imply that

$$\begin{aligned} \beta\bar{\gamma} &= \frac{\gamma_0 - 1}{2}, \\ \alpha\bar{\delta} &= \frac{\gamma_0 + 1}{2}. \end{aligned}$$

Thus δ can be written in terms of α by using Eqn.(34)

$$\begin{aligned} \alpha\bar{\alpha}\bar{\delta} &= \bar{\alpha}\frac{\gamma_0 + 1}{2}, \\ \bar{\delta} &= \bar{\alpha}, \\ \delta &= \alpha. \end{aligned}$$

Also, using Eqn.(33) γ can be written in terms of β

$$\begin{aligned} \beta\bar{\gamma} &= \frac{\gamma_0 - 1}{2}, \\ \bar{\beta}\beta\bar{\gamma} &= \bar{\beta}\frac{\gamma_0 - 1}{2}, \\ \bar{\gamma} &= \bar{\beta}, \\ \gamma &= \beta. \end{aligned}$$

Now use the fact that $\det(U) = 1$, which implies

$$\alpha\delta - \beta\gamma = 1,$$

thus $\alpha^2 - \beta^2 = 1$ as $\delta = \alpha$ and $\gamma = \beta$. Now replace β

$$\begin{aligned} \alpha^2 \left(\frac{(\gamma_0 + 1)^2 - (\gamma_0 v)^2}{(\gamma_0 + 1)^2} \right) &= 1 \\ \alpha &= \pm \frac{\gamma_0 + 1}{\sqrt{(\gamma_0 + 1)^2 - (\gamma_0 v)^2}}. \end{aligned}$$

Now rewrite the denominator of α using $\gamma_0 v = \sqrt{{\gamma_0}^2 - 1}$

$$\begin{aligned} (\gamma_0 + 1)^2 - (\gamma_0 v)^2 &= {\gamma_0}^2 + 1 + 2\gamma_0 - {\gamma_0}^2 + 1, \\ &= 2(\gamma_0 + 1). \end{aligned}$$

So finally

$$\alpha = \pm\frac{\sqrt{\gamma_0 + 1}}{2}.$$

Hence there are 2 matrices U corresponding to this Lorentz transformation given by

$$U = \pm \begin{pmatrix} \frac{\sqrt{\gamma_0+1}}{2} & -\frac{\sqrt{\gamma_0-1}}{2} \\ -\frac{\sqrt{\gamma_0-1}}{2} & \frac{\sqrt{\gamma_0+1}}{2} \end{pmatrix}$$

C. Example 3: Singular Lorentz Transformation

(CALCS)

It is clear that there will always be two matrices $\pm U$ corresponding to every Lorentz transformation, since if U satisfies $A(\vec{x}') = UA(\vec{x})U^\dagger$ then so does $-U$. Hence there is a 2 to 1 correspondence between the elements of $SL(2, \mathbb{C})$ and the proper orthochronous Lorentz transformation.

V. STEREOGRAPHIC PROJECTION AND THE EXTENDED COMPLEX PLANE

Stereographic projection is the mapping of points on a sphere to points on a plane. In $\mathbb{R}^3$ with rectangular Cartesian coordinates, x, y, z, consider the unit sphere with centre $(0,0,0,0)$, defined by

$$\mathbb{S}^2 \subset \mathbb{R}^3 : x^2 + y^2 + x^2 = 1.$$

(SEE FIG PG 4:1) $Q = Q(X, Y, 0)$
The projection $P \to Q$ is a stereographic projection. A relationship between X,Y and (x, y, z) is constructed as follows. P is subdivided into the line segment NQ in some ratio, $l : m$ say. By coordinate geometry

$$\begin{aligned} x &= \frac{lX + mO}{l+m} = \frac{lX}{l+m}, \\ y &= \frac{lY + mO}{l+m} = \frac{lY}{l+m}, \\ z &= \frac{l.0 + m.1}{l+m} = \frac{m}{l+m}. \end{aligned}$$

This implies that

$$\begin{aligned} 1 - z &= \frac{l}{l+m}, \\ x &= (1-z)X, \\ y &= (1-z)Y. \end{aligned}$$

It is also known that $x^2 + y^2 + z^2 = 1$, so using this relation it is clear that

$$\begin{aligned} x^2 + y^2 &= 1 - z^2 = (1-z)(1+z) \\ \Rightarrow (1-z^2)(X^2+Y^2) &= (1-z)(1+z) \end{aligned}$$

If the point N is excluded, i.e. $z \neq 1$ then dividing by $(1-z)^2$ to obtain

$$X^2 + Y^2 = \frac{1+z}{1-z}.$$

Rearranging to find that

$$z = \frac{X^2 + y^2 - 1}{X^2 + y^2 + 1} \tag{36}$$

Define $\zeta = X + iY$ and rewrite Eqn.(36) to see that

$$z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1},$$

Which implies that

$$1 - z = \frac{2}{\zeta\bar{\zeta} + 1}.$$

So relations for $(x, y, z) \in \mathbb{S}^2 \backslash \{N\}$ have been obtained in terms of ζ.

$$x + iy = \frac{2\zeta}{\zeta\bar{\zeta}}, \tag{37}$$
$$zz = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}. \tag{38}$$

Hence the points on $\mathbb{S}^2\backslash\{N\}$ are labelled by complex numbers $\zeta \in \mathbb{C}$. If a point $\zeta = \infty$, called the point at infinity of $\mathbb{C}$, is allowed then the following limits hold:

$$\begin{aligned} x + iy &= \frac{2/\bar{\zeta}}{1 + 1/\zeta\bar{\zeta}} \to 0, \text{ as } \zeta \to \infty \\ z &= \frac{1 - 1/\zeta\bar{\zeta}}{1 + 1/\zeta\bar{\zeta}} \to 1, \text{ as } \zeta \to \infty \end{aligned}$$

Then $N = (0, 0, 1)$ corresponds to $\zeta = \infty$. Thus in this way there is a one to one correspondence between the points of $\mathbb{S}^2$ and the points of the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, which is the usual complex plane with the point at infinity added. Since $\mathbb{S}^2$ has finite surface area, and is therefore called a *compact manifold*, the identification of the points of $\hat{\mathbb{C}}$ with the points of $\mathbb{S}^2$ is called the *compactification* of $\hat{\mathbb{C}}$.

$(\zeta, \bar{\zeta})$ are called the *stereographic coordinates* on $\mathbb{S}^2\backslash\{N\}$. How are they related to the polar angles θ and ϕ? To investigate this write the usual spherical polar coordinates in terms of ζ. First it is known that

$$\begin{aligned} x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{aligned}$$

So using Eqn.(37) it is easy to show that

$$\begin{aligned} \cos\theta &= \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1} \\ \Rightarrow \zeta\bar{\zeta}\cos\theta + \cos\theta &= \zeta\bar{\zeta} - 1 \\ \Rightarrow \zeta\bar{\zeta} = \frac{1 + \cos\theta}{1 - \cos\theta} &= \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} \\ \Rightarrow \zeta\bar{\zeta} &= \cot^2(\theta/2). \end{aligned}$$

Now using Eqn.(38) to obtain:

$$\begin{aligned} \sin\theta(\cos\phi + i\sin\phi) &= \frac{2\zeta}{\cot^2(\theta/2)}, \\ 2\sin(\theta/2)\cos(\theta/2)e^{i\phi} &= 2\zeta\sin^2(\theta/2), \\ \Rightarrow \zeta &= e^{i\phi}\cot(\theta/2). \end{aligned}$$

This makes sense as it is clear that if $\zeta = \infty$ then $\theta = 0$ as one would expect. In summary the following coordinate transformations have been constructed

$$\vec{n} = (x, y, z) \;= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{39}$$
$$= \left(\tfrac{\bar{\zeta}+\zeta}{\bar{\zeta}\zeta+1}, i\tfrac{\bar{\zeta}-\zeta}{\bar{\zeta}\zeta+1}, \tfrac{\bar{\zeta}\zeta-1}{\bar{\zeta}\zeta+1}\right). \tag{40}$$

Where here $\vec{n}$ is a unit vector in $\mathbb{R}^3$ such that $\vec{n} \cdot \vec{n} = 1$.

Let $\vec{x} = (x, y, z, t)$ be a point on the future null cone with vector $(0, 0, 0, 0)$. Denote the future null cone as N^+ (not sure???), so that

$$N^+ : x^2 + y^2 + z^2 - t^2 = 0, \text{ for } t > 0$$

as all the vectors in the null cone have a Lorentz quadratic form equal to zero by definition. (SEE FIG PG 4:5). The intersection of the space-like hypersurface $t = \text{const} > 0$ is a 2-sphere denoted by

$$\mathbb{S}^2(t) : x^2 + y^2 + z^2 - t^2 = \text{const}.$$

There is a generator of N^+ passing through each point of $\mathbb{S}^2(t)$. These generators are the null geodesics tangent to N^+ and passing through the point $(0,0,0,0)$. Hence the points of $\mathbb{S}^2(t)$, denoted by (θ, ϕ) or ζ, label the *generators* of N^+.

For any $t > 0$ it is clear that

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = 1.$$

Hence we can write

$$\begin{aligned} \vec{x} &= t(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 1), \\ &= t\left(\tfrac{\bar{\zeta}+\zeta}{\zeta\bar{\zeta}+1}, i\tfrac{\bar{\zeta}-\zeta}{\zeta\bar{\zeta}+1}, \tfrac{\bar{\zeta}\zeta-1}{\zeta\bar{\zeta}+1}, 1\right). \end{aligned} \quad (41)$$

It is shown explicitly that the direction of $\vec{x}$ is determined by (θ, ϕ) or ζ by comparing this to Eqn.(40). All possible directions of $\vec{x}$ on N^+ are covered if $\zeta \in \hat{\mathbb{C}}$. Now the Lorentz transformation $\vec{x} \to \vec{x'}$ is investigated. This transformation takes the form

$$\vec{x} \to \vec{x'} = t'\left(\frac{\bar{\zeta}' + \zeta'}{\bar{\zeta}'\zeta' + 1}, i\frac{\bar{\zeta}' - \zeta'}{\bar{\zeta}'\zeta' + 1}, \frac{\bar{\zeta}'\zeta' - 1}{\bar{\zeta}'\zeta' + 1}, 1\right).$$

We say that the null direction ζ is transformed to the null direction ζ'. The relation between ζ and ζ' must also be determined. Construct the matrix $A(\vec{x})$ as in Eqn.(26).

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix}.$$

Where $c_0 = 2t/\zeta\bar{\zeta} + 1 \in \mathbb{R}^2$. Note that as $\vec{x}$ is a null vector $\det(A(\vec{x})) = 0$. Thus the transformed matrix $A(\vec{x'})$ is given similarly as

$$A(\vec{x'}) = c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix}.$$

Now, as in the examples in section (IV) we determine the special linear matrix U such that

$$A(\vec{x'}) = UA(\vec{x})U^\dagger \quad (42)$$

As before, this matrix equation is written component wise as

$$c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

Then three separate relations between ζ and ζ' are obtained

$$
\begin{aligned}
{c_0}' &= c_0(\alpha\bar{\alpha} + \alpha\bar{\beta}\zeta + \bar{\alpha}\beta\bar{\zeta} + \beta\bar{\beta}\zeta\bar{\zeta}), & (43)\\
{c_0}'\zeta' &= c_0(\alpha\bar{\gamma} + \alpha\bar{\delta}\zeta + \bar{\gamma}\beta\bar{\zeta} + \beta\bar{\delta}\zeta\bar{\zeta}), & (44)\\
{c_0}'\zeta'\bar{\zeta}' &= c_0(\gamma\bar{\gamma} + \gamma\bar{\delta}\zeta + \bar{\gamma}\delta\bar{\zeta} + \delta\bar{\delta}\zeta\bar{\zeta}). & (45)\\
& & (46)
\end{aligned}
$$

Using Eqns.(43) and (44) to obtain

$$
\zeta' = \frac{{c_0}'\zeta'}{{c_0}'} = \frac{\alpha(\bar{\gamma} + \bar{\delta}\zeta) + \beta\bar{\zeta}(\bar{\gamma} + \bar{\delta}\zeta)}{\alpha(\bar{\alpha} + \bar{\beta}\zeta) + \beta\bar{\zeta}(\bar{\alpha} + \bar{\beta}\zeta)}
$$

Thus

$$
\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)}, \tag{47}
$$

with $\alpha\delta - \beta\gamma = 1$ as before. This is a fractional linear transformation of the extended complex plane $\hat{\mathbb{C}}$. There is a one to one correspondence here between proper, orthochronous Lorentz transformations and fractional linear transformations of the extended complex plane. This is because the matrices $\pm U$ both satisfy Eqn.(42) as in the previous section., but now both matrices give the same transformation as the signs will cancel in the fractional transformation.

A. Singular and Non-Singular Lorentz Transformations

A given Lorentz transformation is equivalent to known α, β , γ and δ parameters module a sign and therefore gives an explicit fractional linear transformation. For a given Lorentz transformation a *fixed point* of the corresponding fractional linear transformation corresponds to an invariant null direction. The fixed points ζ satisfy the relation $\zeta' = \zeta$. Thus

$$
\begin{aligned}
\frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)} &= \zeta, \\
\Rightarrow \bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} &= 0. & (48)
\end{aligned}
$$

Clearly this is a quadratic equation over the field $\mathbb{C}$, thus it has two roots in general. Hence a Lorentz transformation does indeed leave two null directions invariant in general. The non-singular case is when these roots do not coincide. If Eqn.(48) has only one root then the corresponding Lorentz transformation leaves one null direction invariant, this is the singular case.

Consider Eqn.(48) again. Divide by ζ^2 to obtain

$$
\bar{\beta} + (\bar{\alpha} - \bar{\delta})\zeta^{-1} - \bar{\gamma}\zeta^{-2} = 0.
$$

Hence $\zeta = \infty$ is a solution of this equation if $\beta = 0$. If $zeta = \infty$ then $\vec{x}$ is given by $\vec{x} = t(0,0,1,1)$ by Eqn.(41). Thus it is clear that this corresponds to the null direction $z = t$. Compare this to example 3,section (IV C). Here β is zero AND the null direction is $z = t$ as expected. If $\zeta = 0$ is a solution to Eqn.(48) it is required that $\gamma = 0$, thus $\vec{x} = (0,0,-1,1)$. So it is predicted that a Lorentz transformation with a special linear matrix of the form

$$
U = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},
$$

will leave the $Z = -t$ null direction invariant.

B. Example: Standard Lorentz Transformation

Continuing on from Example 2, section (IV C), where α, β, γ and δ were determined. ζ' can now be expressed as

$$\zeta' = \frac{-\sqrt{\gamma_0 - 1} + \sqrt{\gamma_0 + 1}\zeta}{\sqrt{\gamma_0 + 1} - \sqrt{\gamma_0 - 1}\zeta}.$$

If the condition $\zeta' = \zeta$ is imposed then

$$\sqrt{\gamma_0 - 1}(\zeta^2 - 1) = 0$$
$$\Rightarrow \zeta = \pm 1$$

In the $\zeta = +1$ case $\vec{x} = t(1, 0, 0, 1)$ and the invariant direction is $x = t$. Similarly in the $\zeta = -1$ case $\vec{x} = t(-1, 0, 0, 1)$ and the invariant direction is $x = -t$.

(ADD SECTION 4b IN NOTES, MAYBE IN THE APPENDIX?)

VI. INFINITESIMAL LORENTZ TRANSFORMATIONS

There are Lorentz transformations that are small perturbations of the identity transformation and so $U \in SL(2, \mathbb{C})$ has the form

$$U = \pm \begin{pmatrix} 1 + \epsilon a & \epsilon b \\ \epsilon c & 1 + \epsilon f \end{pmatrix}, \tag{49}$$

where $a, b, c, f \in \mathbb{C}$ and ϵ is a small real parameter. Here terms of order ϵ^2 will be neglected. As $U \in SL(2, \mathbb{C})$ its determinant is calculated as

$$\det(U) = 1 + O(\epsilon^2).$$

Using this it is possible to obtain a relation between f and a

$$(1 + \epsilon a)(1 + \epsilon f) - \epsilon^2 bc = 1 + O(\epsilon^2),$$
$$1 + \epsilon(a + f) = 1 + O(\epsilon^2),$$
$$\Rightarrow f = -a + O(\epsilon).$$

Hence

$$U = \begin{pmatrix} 1 + \epsilon a & \epsilon b \\ \epsilon c & 1 - \epsilon a \end{pmatrix},$$

Now the explicit infinitesimal Lorentz transformations are calculated as in section IV, by substituting U into

$$A(\vec{x}') = U A(\vec{x}) U^\dagger.$$

Now writing this out in component form to obtain

$$\begin{pmatrix} t' - z' & x' + iy' \\ x' + iy' & t' + z' \end{pmatrix} = \begin{pmatrix} 1 + \epsilon a & \epsilon b \\ \epsilon c & 1 - \epsilon a \end{pmatrix} \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix} \begin{pmatrix} 1 + \epsilon \bar{a} & \epsilon \bar{c} \\ \epsilon \bar{b} & 1 - \epsilon \bar{a} \end{pmatrix},$$
$$= \begin{pmatrix} 1 + \epsilon a & \epsilon b \\ \epsilon c & 1 - \epsilon a \end{pmatrix} \begin{pmatrix} (t - z)(1 + \epsilon \bar{a}) + \epsilon \bar{b}(x + iy) & (t - z)\epsilon \bar{c} + (1 - \epsilon \bar{a})(x + iy) \\ (x - iy)(1 + \epsilon \bar{a}) + \epsilon \bar{b}(t + z) & (x - iy)\epsilon \bar{c} + (1 - \epsilon \bar{a})(t + z) \end{pmatrix}.$$

This then implies the three relations

$$t' - z' = t - z + \epsilon(a + \bar{a})(t - z) + \epsilon(b + \bar{b})x + i\epsilon(\bar{b} - b)y + O(\epsilon^2), \quad (50)$$
$$t' + z' = t + z - \epsilon(a + \bar{a})(t + z) + \epsilon(c + \bar{c})x + i\epsilon(c - \bar{c})y + O(\epsilon^2), \quad (51)$$
$$x' + iy' = x + iy + \epsilon(a - \bar{a})(x + iy) + \epsilon(b + \bar{c})t + \epsilon(b - \bar{c})z + O(\epsilon^2). \quad (52)$$

As $a, b, c \in \mathbb{C}$, set

$$a = a_1 + ia_2,\ b = b_1 + ib_2,\ c = c_1 + ic_2.$$

Then subbing these into the above equations, eliminating t and z respectively from Eqn.(50) and (51) and taking real and imaginary parts of Eqn.(52) to obtain

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -2a_2 & (b_1 - c_1) & (b_1 + c_1) \\ 2a_2 & 0 & (b_2 + c_2) & (b_2 - c_2) \\ -(b_1 - c_1) & -(b_2 - c_2) & 0 & -2a_1 \\ (b_1 + c_1) & (b_2 - c_2) & -2a_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + O(\epsilon^2). \quad (53)$$

The above 4×4 matrix will be denoted as ${L^i}_j$, so that Eqn.(53) can be written simply as

$$\bar{x}^i = x^i + \epsilon {L^i}_j x^j + O(\epsilon^2). \quad (54)$$

Where $\bar{x}^i = (x', y', z', t')$. It is also necessary to check that the Lorentz invariance of the quadratic form still holds.

$$\begin{aligned} {x'}^2 + {y'}^2 + {z'}^2 - {t'}^2 &= x^2 + y^2 + z^2 - t^2 - 4\epsilon a_2 xy + 2\epsilon(b_1 + c_1)xt \\ &\quad +2\epsilon(b_1 - c_1)xz + 4\epsilon a_2 yx + 2\epsilon(b_2 - c_2)yt \\ &\quad +2\epsilon(b_2 + c_2)yz - 4\epsilon a_1 zt + 2\epsilon(c_1 - b_1)zx \\ &\quad -2\epsilon(c_2 + b_2)zy + 4\epsilon a_1 tz - 2\epsilon(c_1 + b_1)tx \\ &\quad -2\epsilon(b_2 - c_2)ty + O(\epsilon^2) \\ &= x^2 + y^2 + z^2 - t^2 + O(\epsilon^2) \end{aligned}$$

Hence this transformation is still a Lorentz Transformation if we neglect terms of order ϵ^2.

Consider the time-like world line (SEE FIG pg 5:3)of a particle in Minkowskian space-time $x^i = x^i(s)$. If s is arc length or proper time then $v^i(s) = \frac{dx^i}{ds}$ is the unit tangent(NOT SURE WHY???) vector field. It is clear that $v^i(s)$ must be time-like as $x^i(s)$ is time-like, thus

$$\eta_{ij} v^i v^j = -1.$$

Where $\eta_{ij} - \mathrm{diag}(1, 1, 1, -1)$ is the metric of Minkowskian space-time. This implies that

$$(v^1)^2 + (v^2)^2 + (v^3)^2 - (v^4)^2 = -1.$$

Now consider taking a step along the world line of the particle. Define $\bar{s} = s + \alpha$, where α is some real parameter, so that $v^i(s + \alpha) := \bar{v}^i(\bar{s})$. Hence we also have

$$(\bar{v}^1)^2 + (\bar{v}^2)^2 + (\bar{v}^3)^2 - (\bar{v}^4)^2 = -1,$$

and so $v^i(s)$ and $\bar{v}^i(\bar{s})$ are related by a Lorentz transformation. In particular $v^i(s + \epsilon)$ and $v^i(s)$ are related by an infinitesimal Lorentz Transformation given by Eqn.(54),

$$v^i(s + \epsilon) = v^i(s) = \epsilon {L^i}_j(s) v^j(s) + O(\epsilon^2). \quad (55)$$

Rearranging to obtain

$$\frac{v^i(s+\epsilon)-v^i(s)}{\epsilon} = L^i{}_j(s)v^j(s) + O(\epsilon). \tag{56}$$

Now taking the limit as the infinitesimal step, ϵ goes to zero to obtain a continuous differentiable equation,

$$\frac{dv^i}{ds} = L^i{}_j(s)v^j(s). \tag{57}$$

This equation determines the trajectory of the particle through Minkowskian space-time. In terms of x this is equivalent to

$$\frac{d^2x^i}{ds^2} = L^i{}_j(s)\frac{dx^j}{ds}.$$

It is interesting to write these equations in terms of the particles 3-velocity given by

$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

Start by using the chain rule on v^i,

$$v^i = \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right)\frac{dt}{ds}.$$

Now determine the first integral of v^i, which is equal to -1 as v^i is time-like,

$$-1 = \eta_{ij}v^iv^j = \left\{\left(\frac{d}{dt}\right)^2 + \left(\frac{d}{dt}\right)^2 + \left(\frac{d}{dt}\right)^2 - 1\right\}\left(\frac{dt}{ds}\right)^2,$$

as this is just the scalar product in Minkowskian space-time. Therefore (NOT SURE WHERE THIS COMES FROM)

$$\frac{dt}{ds} = \gamma(s) := (1-u^2)^{-1/2},$$

where $u = |\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$. Thus from Eqn.(VI)

$$v^i = \gamma(u)(\vec{u}, 1) \tag{58}$$

It is now convenient to display Eqn.(57) as two equations denoting the spacial part and the temporal part, in terms of γ and u. Again using the chain rule to obtain

$$\frac{dt}{ds}\frac{dv^i}{dt} = L^i{}_jv^j.$$

This then implies that

$$\begin{aligned} \gamma(u)\frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_jv^j, \\ \gamma(u)\frac{d}{dt}\gamma(u) &= L^4{}_jv^j, \end{aligned} \tag{59}$$

as $v^i = \gamma(u)(\vec{u}, 1)$. Here we have used the usual convention that Greek indices denote the sum over the spacial indices only, thus $\alpha = 1, 2, 3$. Now Eqn.(58) can be used to rewrite the $L^i{}_j$ coefficients to get

$$\begin{aligned} L^\alpha{}_j v^j &= \gamma(u)(L^\alpha{}_\beta u^\beta + L^\alpha{}_4) \\ L^4{}_j v^j &= \gamma(u)(L^4{}_\alpha u^{\alpha 0}) \end{aligned} \tag{60}$$

where $L^4{}_4 = 0$ from Eqn.(53). Putting together Eqns.(59) and (60) to obtain differential equations for the spacial and temporal coordinates in terms of the particles 3-velocity,

$$\begin{aligned} \frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_\beta u^\beta + L^\alpha{}_4, \\ \frac{d\gamma(u)}{dt} &= L^4{}_\alpha u^\alpha. \end{aligned}$$

These can be written explicitly as four equations

$$\frac{d}{dt}(\gamma(u)u^{(1)}) = -2a_2 u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1, \tag{61}$$

$$\frac{d}{dt}(\gamma(u)u^{(2)}) = 2a_2 u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2, \tag{62}$$

$$\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1, \tag{63}$$

$$\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1 u^{(3)}. \tag{64}$$

Now define the 3-vectors $\vec{P}$ and $\vec{Q}$ such that

$$\vec{P} = (b_1 + c_1, b_2 - c_2, -2a_1), \vec{Q} = (b_2 + c_2, -(b_1 - c_1), -2a_2).$$

It is clear that Eqns.(61)-(64) can be written in terms of $\vec{P}$ and $\vec{Q}$ as follows,

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q}, \tag{65}$$

$$\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}. \tag{66}$$

Note that these expressions look remarkably like the Lorentz force in electromagnetism. It is easily shown that Eqn.(65) implies Eqn.(66). To see this, first take the scalar product of Eqn.(65) with $\vec{U}$.

$$\vec{u} \cdot \frac{d}{dt}(\gamma(u)\vec{u}) = \vec{u} \cdot \vec{P} + \vec{u} \cdot (\vec{u} \times \vec{Q}), \tag{67}$$

$$\gamma \vec{u} \frac{d\vec{u}}{dt} + \vec{u} \cdot \vec{u} \frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}, \tag{68}$$

by using the product rule and as the scalar product of the cross product with a repeated vector is zero in the third term. The quantity γ is known in terms of u, so it is possible to write the derivative in the first term as a derivative of γ as follows,

$$\begin{aligned} \gamma^{-2} &= 1 - u^2 = 1 - \vec{u} \cdot \vec{u}, \\ \Rightarrow -2\gamma^{-3}\frac{d\gamma}{dt} &= -2\vec{u} \cdot \frac{d\vec{u}}{dt}. \end{aligned}$$

Subbing this result back into Eqn.(68) to obtain

$$\gamma\gamma^{-3}\frac{d\gamma}{dt} + u^2\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P},$$
$$(\gamma^{-2}+u^2)\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P}.$$

Therefore

$$\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P},$$

and so it is shown that Eqn.(66) is a generalization of Eqn.(65) and contains no new information.

The dependence of the 3-force acting on a particle as shown by Eqn.(65), depends in general on the particles 3-velocity $\vec{u}$ in a special way, in order to be compatible with Special Relativity. Thus in particular *the Lorentz 3-force acting on a particle of rest mass m and charge q must depend upon $\vec{u}$ as in Eqn.(65) to be compatible with special Relativity.* So the Lorentz force of electromagnetism is a special case of the a charged particle moving through Minkowskian space-time along a world-line of infinitesimal Lorentz transformations. In this case, make the identifications

$$\vec{P} = \frac{q}{m}\vec{E}, \text{ and } \vec{Q} = \frac{q}{m}\vec{B}, \tag{69}$$

where $\vec{E}$ is the external electric field and $\vec{B}$ is the external magnetic field in which the particle is moving. Then Eqn.(65) takes the familiar form

$$m\frac{d}{dt}(\gamma(u)\vec{u}) = q(\vec{E}+\vec{u}\times\vec{B}).$$

Or in the case of a slow moving particle $\gamma \approx 1$ and

$$m\vec{a} = q(\vec{E}+\vec{u}\times\vec{B}).$$

A. Fractional Linear Transformations of the Infinitesimal Linear Transformation

Recall that the fractional linear transformation constructed in section (V) had a one to one correspondence with proper orthochronous Lorentz transformations, and the fixed points of the fractional transformation corresponded to null directions of the Lorentz transformation. As in Eqn.(47), section (V) construct the fractional linear transformation of the special linear $(SL(2,(C)))$ matrix U for the infinitesimal Lorentz transformation given in Eqn.(49). It is found to be

$$\zeta' = \frac{\zeta + \epsilon(\bar{c}-\bar{a}\zeta) + O(\epsilon^2)}{1+\epsilon(\bar{a}+\bar{b}\zeta)+O(\epsilon^2)}.$$

Then the fixed points are given when $\zeta' = \zeta$, which implies,

$$\epsilon\bar{b}\zeta^2 + (\epsilon\bar{a}+\epsilon\bar{a})\zeta - \epsilon\bar{c} = O(\epsilon^2),$$
$$\Rightarrow \bar{b}\zeta^2 + 2\bar{a}\zeta - \bar{c} = O(\epsilon). \tag{70}$$

Of interest here are the singular Lorentz transformations, so it is required that the roots of this quadratic are the same. Thus the usual discriminant is set to zero,

$$4\bar{a}^2 + 4\bar{b}\bar{c} = 0.$$

Therefore,

$$\bar{a}^2 + \bar{b}\bar{c} \Leftrightarrow a^2 + bc = 0. \tag{71}$$

Write these equations out explicitly and equate real and imaginary coefficients to obtain

$$a_1^2 = a_2^2 + b_1c_1 - b_2c_2 = 0, \tag{72}$$
$$2a_1a_2 + b_2c_1 + b_1c_2 = 0. \tag{73}$$

It is interesting to write these equations in terms of the electric and magnetic vectors, namely $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$. The relation between the a,b and c and the $\vec{B}$ and $\vec{E}$ coefficients comes from Eqn.(69), where the factor q/m has been suppressed for convenience.

$$a_1 = -\frac{1}{2}E^3,\ b_2 = \frac{1}{2}(E^2 + B^1),\ c_1 = \frac{1}{2}(E^1 + B^2), \tag{74}$$
$$a_2 = -\frac{1}{2}B^3,\ b_1 = \frac{1}{2}(E^1 - B^2),\ c_2 = \frac{1}{2}(B^1 - E^2). \tag{75}$$

So Eqn.(72) implies

$$\frac{1}{4}{(E^3)}^2 - \frac{1}{4}{(B^3)}^2 + \frac{1}{4}({(E^1)}^2 - {(B^2)}^2) + \frac{1}{4}({(E^2)}^2 - {(B^1)}^2) = 0, \Rightarrow {(E^1)}^2 + {(E^2)}^2 + {(E^3)}^2 = {(B^1)}^2 + {(B^2)}^2 + {(B^3)}^2.$$

Thus it is clear that

$$|\vec{E}|^2 = |\vec{B}|^2 \tag{76}$$

Similarly, Eqn.(73) implies

$$\begin{aligned} &\frac{1}{2}E^3B^3 + \frac{1}{4}(E^1E^2 + E^1B^1 + E^2B^2 + B^1B^2) \\ &+\frac{1}{4}(-E^1E^2 + E^1B^1 + E^2B^2 - B^1B^2) = 0, \\ &\Rightarrow E^3B^3 + E^1B^1 + E^2B^2 = 0. \end{aligned}$$

So it is shown that

$$\vec{E} \cdot \vec{B} = 0. \tag{77}$$

The above Eqns.(76) and (77) are the (Lorentz invariant) conditions that the electromagnetic field in which the charged particle is moving is a pure radiation field. Thus in conclusion, *if the world line of the charged particle is generated by infinitesimal singular Lorentz transformations then the particles moving in a pure radiation electromagnetic field.* (CASE WHERE ITS A PLANE WAVE WORTH DOING???)

B. Pure Radiation Field Conditions in Minkowskian Space-Time

Eqns.(76) and (77) are the pure radiation field conditions in physical space, $\mathbb{R}^2$. It is also interesting to see what form these equations take in Minkowskian space-time. To do this, solve the quadratic equation in Eqn.(70) for the case where the roots coincide, to find the single fixed point of the system. It is clear that

$$\zeta = -\frac{\bar{a}}{\bar{b}}, \tag{78}$$

is the fixed point. Now determine the corresponding null direction k^i. (HOW DID WE DO THIS???)

$$k^i = (\bar{\zeta} + \zeta, i(\bar{\zeta} - \zeta), \bar{\zeta}\zeta - 1, \bar{\zeta}\zeta + 1).$$

Now relate k^i to $L_{ij} = \eta_{ij} L^k{}_j$. From the relations in (74) and (75) it is clear that

$$L_{ij} = \frac{q}{m} \begin{pmatrix} 0 & B^3 & -B^2 & E^1 \\ -B^3 & 0 & B^1 & E^2 \\ B^2 & -B^1 & 0 & E^3 \\ -E^1 & -E^2 & -E^3 & 0 \end{pmatrix} = -(L_{ij}).$$

The dual of this quantity is defined by

$$^*L_{ij} = \frac{1}{2}\epsilon_{ijkl} L^{kl},$$

where ϵ_{ijkl} is the Levi-Civita permutation symbol in 4 dimensions. To work out the components of $^*L_{ij}$, first use the raising and lowering of operators with the metric $\eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ to show that

$$L_{\alpha\beta} = L^\alpha{}_\beta, L_{4j} = -L^4{}_j.$$

Where $\alpha, \beta = 1, 2, 3$ and $j = 1, 2, 3, 4$. Thus the components of $^*L_{ij}$ are calculated as follows

$$\begin{aligned}
^*L_{12} &= \epsilon_{1234} L^{34} == L^{34} = -L^3{}_4 = -E^3, \\
^*L_{13} &= \epsilon_{1324} L^{24} = -L^{24} = L^2{}_4 = E^2, \\
^*L_{14} &= \epsilon_{1423} L^{23} = L^{23} = L^2{}_3 = B^1, \\
^*L_{23} &= \epsilon_{2314} L^{14} = L^{14} = -L^1{}_4 = -E^1, \\
^*L_{24} &= \epsilon_{2413} L^{13} = -L^{13} = -L^1{}_3 = B^2, \\
^*L_{34} &= \epsilon_{3412} L^{12} = L^{12} = L^1{}_2 = B^3.
\end{aligned}$$

Now construct the matrix (WHATS THE PHYSICAL INTERPRETATION OF THIS MATRIX???)

$$\begin{aligned}
(L_{ij} + i^* L_{ij}) = & \frac{q}{m} \begin{pmatrix} 0 & B^3 - iE^3 & -B^2 + iE^2 & E^1 + iB^1 \\ -B^3 + iE^3 & 0 & B^1 - iE^1 & E^2 + iB^2 \\ B^2 - iE^2 & -B^1 + iE^1 & 0 & E^3 + iB^3 \\ -E^1 - iB^1 & -E^2 - iB^2 & -E^3 - iB^3 & 0 \end{pmatrix} \\
= & \frac{q}{m} \begin{pmatrix} 0 & 2ia & (b-c) & (b+c) \\ -2ia & 0 & -i(b+c) & -i(b-c) \\ -(b-c) & i(b+c) & 0 & -2a \\ -(b+c) & i(b-c) & 2a & 0 \end{pmatrix}
\end{aligned}$$

Now calculate the various components of the product $(L_{ij} + i^* Lij)k^j$, the first is done as an example.

$$\begin{aligned}
(L_{1j} + i^* L1j)k^j =& \frac{q}{m}(2iak^2 + (b-c)k^3 + (b+c)k^4), \\
=& \frac{q}{m}(-2a(\zeta - \bar{\zeta}) + 2b\zeta\bar{\zeta} + 2c), \\
=& \frac{2q}{m}(-a\bar{\zeta} + a\zeta + b\zeta\bar{\zeta} + c), \\
=& \frac{2q}{mb}(a^2 + bc),
\end{aligned}$$

calculated at the fixed point $\zeta = -\bar{\alpha}\ \bar{\beta}$ from Eqn.(78). Recall from the start of section (VI A) that the discriminant of the quadratic in Eqn.(70) was set to zero to obtain a single root and thus a singular Lorentz transformation. This resulted in the condition given also in Eqn.(71), namely

$$a^2 + bc = 0.$$

Considering this it is clear that

$$(L_{1j} + i^* L1j)k^j = \frac{2q}{mb}(a^2 + bc) = 0$$

.
Indeed it is easy to show that every component of the product is zero.

$$\begin{aligned}
(L_{1j} + i^* L1j)k^j &= \frac{2q}{mb}(a^2 + bc) = 0, \\
(L_{2j} + i^* L2j)k^j &= \frac{2iq}{m}(a^2 + bc) = 0, \\
(L_{3j} + i^* L3j)k^j &= -\frac{2q\bar{a}}{mb\bar{b}}(a^2 + bc) = 0, \\
(L_{4j} + i^* L4j)k^j &= \frac{2q\bar{a}}{mb\bar{b}}(a^2 + bc) = 0.
\end{aligned}$$

In conclusion, it is shown that $(L_{ij} + i^* Lij)k^j = 0$ if and only if $a^2 + bc = 0$ which in turn implies that the field generated is a pure radiation field as in Eqns.(76) and (77). Notice that $(L_{ij} + i^* Lij)k^j$ is nothing but the scalar product of (WHAT SHOULD WE CALL IT???) and k^j. Thus (WHAT DO WE CALL IT??) and k^j are orthogonal in Minkowskian space-time, which implies that k^j *is the propagation direction in Minkowskian space-time of the electromagnetic radiation.*

As stared at the beginning of the section, Eqns.(76) and (77) which describe a pure radiation field in physical space $\mathbb{R}^3$, have been written as $(L_{ij} + i^* L_{ij})k^j = 0$ which describes the radiation field in Minkowskian space-time.

VII. ACKNOWLEDGEMENTS
