The Lorentz Group and Singular Lorentz Transformations

Kevin Maguire (10318135)
(Dated: March 2, 2014)

abstract

Contents

I. INFINITESIMAL LORENTZ TRANSFORMATIONS

There are Lorentz transformations that are small perturbations of the identity transformation and so $U \in SL(2,\mathbb{C})$ has the form

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix}, \tag{1}$$

where $a, b, c, f \in \mathbb{C}$ and ϵ is a small real parameter. Here terms of order ϵ^2 will be neglected. As $U \in SL(2,\mathbb{C})$ its determinant is calculated as

$$\det(U) = 1 + O(\epsilon^2).$$

Using this it is possible to obtain a relation between f and a

$$\begin{aligned} (1+\epsilon a)(1+\epsilon f) - \epsilon^2 bc &= 1 + O(\epsilon^2), \\ 1 + \epsilon(a+f) &= 1 + O(\epsilon^2), \\ \Rightarrow f &= -a + O(\epsilon). \end{aligned}$$

Hence

$$U = \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix},$$

Now the explicit infinitesimal Lorentz transformations are calculated as in section **??**, by substituting U into

$$A(\vec{x}') = UA(\vec{x})U^{\dagger}.$$

Now writing this out in component form to obtain

$$\begin{aligned} \begin{pmatrix} t'-z' & x'+iy' \\ x'+iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} 1+\epsilon\bar{a} & \epsilon\bar{c} \\ \epsilon\bar{b} & 1-\epsilon\bar{a} \end{pmatrix}, \\ &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} (t-z)(1+\epsilon\bar{a}) + \epsilon\bar{b}(x+iy) & (t-z)\epsilon\bar{c} + (1-\epsilon\bar{a})(x+iy) \\ (x-iy)(1+\epsilon\bar{a}) + \epsilon\bar{b}(t+z) & (x-iy)\epsilon\bar{c} + (1-\epsilon\bar{a})(t+z) \end{pmatrix}. \end{aligned}$$

This then implies the three relations

$$t'-z' = t-z+\epsilon(a+\bar{a})(t-z)+\epsilon(b+\bar{b})x+i\epsilon(\bar{b}-b)y+O(\epsilon^2), \tag{2}$$

$$t'+z' = t+z-\epsilon(a+\bar{a})(t+z)+\epsilon(c+\bar{c})x+i\epsilon(c-\bar{c})y+O(\epsilon^2), \tag{3}$$

$$x'+iy' = x+iy+\epsilon(a-\bar{a})(x+iy)+\epsilon(b+\bar{c})t+\epsilon(b-\bar{c})z+O(\epsilon^2). \tag{4}$$

As $a, b, c \in \mathbb{C}$, set

$$a = a_1 + ia_2,\ b = b_1 + ib_2,\ c = c_1 + ic_2.$$

Then subbing these into the above equations, eliminating t and z respectively from Eqn.(2) and (3) and taking real and imaginary parts of Eqn.(4) to obtain

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -2a_2 & (b_1-c_1) & (b_1+c_1) \\ 2a_2 & 0 & (b_2+c_2) & (b_2-c_2) \\ -(b_1-c_1) & -(b_2-c_2) & 0 & -2a_1 \\ (b_1+c_1) & (b_2-c_2) & -2a_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + O(\epsilon^2). \tag{5}$$

The above 4×4 matrix will be denoted as $L^i{}_j$, so that Eqn.(5) can be written simply as

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2). \tag{6}$$

Where $\bar{x}^i = (x', y', z', t')$. It is also necessary to check that the Lorentz invariance of the quadratic form still holds.

$$\begin{aligned} x'^2 + y'^2 + z'^2 - t'^2 &= x^2 + y^2 + z^2 - t^2 - 4\epsilon a_2 xy + 2\epsilon(b_1 + c_1)xt \\ &\quad +2\epsilon(b_1 - c_1)xz + 4\epsilon a_2 yx + 2\epsilon(b_2 - c_2)yt \\ &\quad +2\epsilon(b_2 + c_2)yz - 4\epsilon a_1 zt + 2\epsilon(c_1 - b_1)zx \\ &\quad -2\epsilon(c_2 + b_2)zy + 4\epsilon a_1 tz - 2\epsilon(c_1 + b_1)tx \\ &\quad -2\epsilon(b_2 - c_2)ty + O(\epsilon^2) \\ &= x^2 + y^2 + z^2 - t^2 + O(\epsilon^2) \end{aligned}$$

Hence this transforamtion is still a Lorentz Transformation if we neglect terms of order ϵ^2.

Consider the time-like world line (SEE FIG pg 5:3)of a particle in Minkowkian space-time $x^i = x^i(s)$. If s is arc length or proper time then $v^i(s) = \frac{dx^i}{ds}$ is the unit tangent(NOT SURE WHY???) vector field. It is clear that $v^i(s)$ must be time-like as $x^i(s)$ is time-like, thus

$$\eta_{ij} v^i v^j = -1.$$

Where $\eta_{ij} - \text{diag}(1, 1, 1, -1)$ is the metric of minkowskiam space-time. This implies that

$$(v^1)^2 + (v^2)^2 + (v^3)^2 - (v^4)^2 = -1.$$

Now consider taking a step along the world line of the particle. Define $\bar{s} = s + \alpha$, where α is some real parameter, so that $v^i(s + \alpha) := \bar{v}^i(\bar{s})$. Hence we also have

$$(\bar{v}^1)^2 + (\bar{v}^2)^2 + (\bar{v}^3)^2 - (\bar{v}^4)^2 = -1,$$

and so $v^i(s)$ and $\bar{v}^i(\bar{s})$ are related by a Lorentz transformation. In particlular $v^i(s + \epsilon)$ and $v^i(s)$ are related by an infinitesimal Lorentz Transformation given by Eqn.(6),

$$v^i(s + \epsilon) = v^i(s) = \epsilon L^i{}_j(s) v^j(s) + O(\epsilon^2). \tag{7}$$

Rearranging to obtain

$$\frac{v^i(s + \epsilon) - v^i(s)}{\epsilon} = L^i{}_j(s) v^j(s) + O(\epsilon). \tag{8}$$

Now taking the limit as the infinitesimal step, ϵ goes to zero to obtain a continuous differentiable equation,

$$\frac{dv^i}{ds} = L^i{}_j(s) v^j(s). \tag{9}$$

This equation determines the trajectory of the particle through Minkowskian space-time. In terms of x this is equivalent to

$$\frac{d^2 x^i}{ds^2} = L^i{}_j(s) \frac{dx^j}{ds}.$$

It is interesting to write these equations in terms of the particles 3-velocity given by

$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right).$$

Start by using the chain rule on v^i,

$$v^i = \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1 \right) \frac{dt}{ds}.$$

Now determine the first integral of v^i, which is equal to -1 as v^i is time-like,

$$-1 = \eta_{ij} v^i v^j = \left\{ \left(\frac{d}{dt}\right)^2 + \left(\frac{d}{dt}\right)^2 + \left(\frac{d}{dt}\right)^2 - 1 \right\} \left(\frac{dt}{ds}\right)^2,$$

as this is just the scalar product in Minkowskian space-time. Therefore (NOT SURE WHERE THIS COMES FROM)

$$\frac{dt}{ds} = \gamma(s) := (1 - u^2)^{-1/2},$$

where $u = |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$. Thus from Eqn.(I)

$$v^i = \gamma(u)(\vec{u}, 1) \tag{10}$$

It is now conveinient to display Eqn.(9) as two equations denoting the spacial part and the temproal part, in terms of γ and u. Again using the chain rule to obtain

$$\frac{dt}{ds}\frac{dv^i}{dt} = L^i{}_j v^j.$$

This then implies that

$$\begin{aligned} \gamma(u)\frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_j v^j, \\ \gamma(u)\frac{d}{dt}\gamma(u) &= L^4{}_j v^j, \end{aligned} \tag{11}$$

as $v^i = \gamma(u)(\vec{u}, 1)$. Here we have used the usual convention that greek indices denote the sum over the spacial indices only, thus $\alpha = 1, 2, 3$. Now Eqn.(10) can be used to rewrite the $L^i{}_j$ coefficients to get

$$\begin{aligned} L^\alpha{}_j v^j &= \gamma(u)(L^\alpha{}_\beta u^\beta + L^\alpha{}_4) \\ L^4{}_j v^j &= \gamma(u)(L^4{}_\alpha u^{\alpha 0}) \end{aligned} \tag{12}$$

where $L^4{}_4 = 0$ from Eqn.(5). Putting together Eqns.(11) and (12) to obtain differential equations for the spacial and temporal coordinates in terms of the particles 3-velocity,

$$\begin{aligned} \frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_\beta u^\beta + L^\alpha{}_4, \\ \frac{d\gamma(u)}{dt} &= L^4{}_\alpha u^\alpha. \end{aligned}$$

These can be written explicitely as four equations

$$\frac{d}{dt}(\gamma(u)u^{(1)}) = -2a_2u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1, \tag{13}$$
$$\frac{d}{dt}(\gamma(u)u^{(2)}) = 2a_2u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2, \tag{14}$$
$$\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1, \tag{15}$$
$$\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1u^{(3)}. \tag{16}$$

Now define the 3-vectors $\vec{P}$ and $\vec{Q}$ such that

$$\vec{P} = (b_1 + c_1, b_2 - c_2, -2a_1), \vec{Q} = (b_2 + c_2, -(b_1 - c_1), -2a_2).$$

It is clear that Eqns.(13)-(16) can be written in terms of $\vec{P}$ and $\vec{Q}$ as follows,

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q}, \tag{17}$$
$$\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}. \tag{18}$$

Note that these expressions look remarkably like the Lorentz force in electromagnetism. It is easily shown that Eqn.(17) implies Eqn.(18). To see this, first take the scalar product of Eqn.(17) with $\vec{U}$.

$$\vec{u} \cdot \frac{d}{dt}(\gamma(u)\vec{u}) = \vec{u} \cdot \vec{P} + \vec{u} \cdot (\vec{u} \times \vec{Q}), \tag{19}$$
$$\gamma\vec{u}\frac{d\vec{u}}{dt} + \vec{u} \cdot \vec{u}\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}, \tag{20}$$

by using the product rule and as the scalar product of the cross product with a repeated vector is zero in the third term. The quantity γ is known in terms of u, so it is possible to write the derivative in the first term as a derivative of γ as follows,

$$\gamma^{-2} = 1 - u^2 = 1 - \vec{u} \cdot \vec{u},$$
$$\Rightarrow -2\gamma^{-3}\frac{d\gamma}{dt} = -2\vec{u} \cdot \frac{d\vec{u}}{dt}.$$

Subbing this result back into Eqn.(20) to obtain

$$\gamma\gamma^{-3}\frac{d\gamma}{dt} + u^2\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P},$$
$$(\gamma^{-2} + u^2)\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}.$$

Therefore

$$\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P},$$

and so it is shown that Eqn.(18) is a generalization of Eqn.(17) and contains no new information.

The dependance of the 3-force acting on a particle as shown by Eqn.(17), depends in general on the particles 3-velocity $\vec{u}$ in a special way, in order to be compatible with Special Relativity. Thus in particular *the Lorentz 3-force acting on a particle of rest mass m and charge q must depend upon $\vec{u}$ as in Eqn.(17) to be compatible with special Relativity.* So the Lorentz force of electromagnetism is a special case of the a charged particle moving through Minkowskian space-time along a world-line of infiniteimal Lorentz transformations. In this case, make the identifications

$$\vec{P} = \frac{q}{m}\vec{E}, \text{ and } \vec{Q} = \frac{q}{m}\vec{B}, \tag{21}$$

where $\vec{E}$ is the external electric field and $\vec{B}$ is the external magnetic field in which the particle is moving. Then Eqn.(17) takes the familiar form

$$m\frac{d}{dt}(\gamma(u)\vec{u}) = q(\vec{E} + \vec{u} \times \vec{B}).$$

Or in the case of a slow moving particle $\gamma \approx 1$ and

$$m\vec{a} = q(\vec{E} + \vec{u} \times \vec{B}).$$

A. Fractional Linear Transformations of the Infinitesimal Linear Transformation

Recall that the fractional linear transformation constructed in section (**??**) had a one to one correspondence with proper orthochronous Lorentz transforamtions, and the fixed points of the fractional transformation corresponded to null directions of the Lorentz transformation. As in Eqn.(**??**), section (**??**) construct the fractional linear transformation of the special linear ($SL(2,(C))$) matrix U for the infinitesimal Lorentz transformation given in Eqn.(1). It is found to be

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

Then the fixed points are given when $\zeta' = \zeta$, which implies,

$$\begin{aligned} \epsilon\bar{b}\zeta^2 + (\epsilon\bar{a} + \epsilon\bar{a})\zeta - \epsilon\bar{c} &= O(\epsilon^2), \\ \Rightarrow \bar{b}\zeta^2 + 2\bar{a}\zeta - \bar{c} &= O(\epsilon). \end{aligned} \tag{22}$$

Of interest here are the singular Lorentx transforamtions, so it is required that the roots of this quadratic are the same. Thus the usual discriminant is set to zero,

$$4\bar{a}^2 + 4\bar{b}\bar{c} = 0.$$

Therefore,

$$\bar{a}^2 + \bar{b}\bar{c} \Leftrightarrow a^2 + bc = 0.$$

Write these equations out explicly and equate real and imaginary coefficients to obtain

$$a_1^2 = a_2^2 + b_1c_1 - b_2c_2 = 0, \tag{23}$$

$$2a_1a_2 + b_2c_1 + b_1c_2 = 0. \tag{24}$$

It is interesting to write these equations in terms of the electric and magnetic vectors, namely $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$. The relation between the a,b and c and the $\vec{B}$ and $\vec{E}$ coefficients comes from Eqn.(21), where the factor q/m has been suppressed for conveinience.

$$a_1 = -\frac{1}{2}E^3,\ b_2 = \frac{1}{2}(E^2 + B^1),\ c_1 = \frac{1}{2}(E^1 + B^2), \tag{25}$$

$$a_2 = -\frac{1}{2}B^3,\ b_1 = \frac{1}{2}(E^1 - B^2),\ c_2 = \frac{1}{2}(B^1 - E^2). \tag{26}$$

So Eqn.(23) implies

$$\frac{1}{4}(E^3)^2 - \frac{1}{4}(B^3)^2 + \frac{1}{4}((E^1)^2 - (B^2)^2) + \frac{1}{4}((E^2)^2 - (B^1)^2) = 0, \Rightarrow (E^1)^2 + (E^2)^2 + (E^3)^2 = (B^1)^2 + (B^2)^2 + (B^3)^2.$$

Thus it is clear that

$$|\vec{E}|^2 = |\vec{B}|^2 \tag{27}$$

Similarly, Eqn.(24) implies

$$\begin{aligned} \frac{1}{2}E^3B^3 + \frac{1}{4}(E^1E^2 + E^1B^1 + E^2B^2 + B^1B^2) \\ + \frac{1}{4}(-E^1E^2 + E^1B^1 + E^2B^2 - B^1B^2) = 0, \\ \Rightarrow E^3B^3 + E^1B^1 + E^2B^2 = 0. \end{aligned}$$

So it is shown that

$$\vec{E} \cdot \vec{B} = 0. \tag{28}$$

The above Eqns.(27) and (28) are the (Lorentz invariant) conditions that the electromagnetic field in which the charged particle is moving is a pure radiation field. Thus in conclusion, *if the world line of the charged particle is generated by infinitesimal singular Lorentz transformations then the particles moving in a pure radiation electromagnetic field.* (CASE WHERE ITS A PLAVE WAVE WORTH DOING???)

B. Pure Radiation Field Conditions in Minkowskian Space-Time

Eqns.(27) and (28) are the pure radiation field conditions in physical space, $\mathbb{R}^2$. It is also interesting to see what form these equations take in Minkowskian space-time. To do this, solve the quadratic equation in Eqn.(22) for the case where the roots coincide, to find the single fixed point of the system. It is clear that

$$\zeta = -\frac{\bar{a}}{\bar{b}},$$

is the fixed point. Now determine the corresponding null direction k^i. (HOW DID WE DO THIS???)

$$k^i = (\bar{\zeta} + \zeta, i(\bar{\zeta} - \zeta), \bar{\zeta}\zeta - 1, \bar{\zeta}\zeta + 1).$$

Now relate k^i to $L_{ij} = \eta_{ij}L^k{}_j$. From the relations in (25) and (26) it is clear that

$$L_{ij} = \frac{q}{m}\begin{pmatrix} 0 & B^3 & -B^2 & E^1 \\ -B^3 & 0 & B^1 & E^2 \\ B^2 & -B^1 & 0 & E^3 \\ -E^1 & -E^2 & -E^3 & 0 \end{pmatrix} = -(L_{ij}).$$

The dual of this quantity is defined by

$$^*L_{ij} = \frac{1}{2}\epsilon_{ijkl}L^{kl}$$

II. ACKNOWLEDGEMENTS