

# The Lorentz Group and Singular Lorentz Transformations

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*abstract*

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## I. THE LORENTZ TRANSFORMATION

In this section the two null directions inherent in all Lorentz transformations, except the singular Lorentz transformations, are illustrated. The Lorentz Transform is defined by

$$(x, y, z, t) \rightarrow (x', y', z', t') \text{ such that}$$

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

If the transformation preserves the orientation of the spatial axes then it is called a proper Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also If  $t \geq 0 \Rightarrow t' \geq 0$  then it is called an orthochronous Lorentz transformation. This ensures that the time direction is preserved. In this project the “Lorentz transformation” will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the  $x$  direction at the speed of light,  $c = 1$ , and starting at  $x = 0$ . The space-time for such a photon can be illustrated as follows (FIGURE). It is clear that there are two null directions in this space-time,  $x = \pm t$ . To see this use the standard Lorentz transformation:

$$x' = \gamma(x - vt) \text{ , where } \gamma = (1 - v^2)^{-1/2}$$

$$t' = \gamma(t - vx)$$

Rearranging:

$$x' - t' = \gamma(1 + v)(x - t)$$

$$x' + t' = \gamma(1 - v)(x + t)$$

It is clear that:

$$x = \pm t \leftrightarrow x' = \pm t'$$

Thus there are two null directions (are null directions by definition invariant??? yes) in this space-time at  $x = \pm t$ . It can be shown that all Lorentz transformations have two invariant null directions except the singular Lorentz transformation which has only one fixed null direction.

## II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

In this section the Kasner solution of the vacuum field equations is derived from the Schwarzschild solution by taking the limit as the mass goes to infinity. It is then shown that the special case of the Kasner solution with no mass is equivalent to a novel form of Minkowskian space-time. (what do we want the Kasner vacuum solution for???). Starting with the Schwarzschild Solution of the vacuum field equations:

$$\epsilon ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right) dt^2 \quad (1)$$

Now make the Eddington-Finkelstein coordinate transformation:

$$u = t - r - 2m \ln(r - 2m) \quad (2)$$

Working out the differential:

$$du = dt - dr - \frac{2m dr}{r - 2m}$$

SEE CALCULATIONS PG1

Then write Eqn.(1) in terms of  $u$  to obtain:

$$\epsilon ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2dudr - du^2 + \frac{2m}{r}du^2$$

Note that if  $m = 0$  the space-time becomes Minkowskian, as expected. Setting  $r = 0$  in this Minkowskian space-time the line element becomes:

$$\epsilon ds^2 = -du^2 \Rightarrow t = -1$$

Which implies that  $r = 0$  is a time-like world-line in Minkowskian space-time, with proper time  $u$ .(CHECK ITS A GEODESIC???)

We want to find the limit of the Schwarzschild solution as  $m \rightarrow \infty$ . In its current form, the limit cannot be calculated so first a suitable coordinate transformation must be made.

$$\begin{aligned} u &= \mu u' , \text{ where } \mu = \text{const} \\ r &= \mu^{-1} r' \end{aligned}$$

So that

$$\begin{aligned} du &= \mu du' \\ dr &= \mu^{-1} dr' \\ \Rightarrow dudr &= du' dr' \end{aligned}$$

To obtain (SEE CALCS PG3):

$$\epsilon ds^2 = r'^2 \sin^2 \theta \left\{ \frac{d\theta^2}{\mu^2 \sin^2 \theta} + \mu^{-2} d\phi^2 \right\} - 2du' dr' - \left( \mu^2 - \frac{2m\mu^3}{r'} du' \right)$$

Now set  $m\mu^3 = k = \text{const} \Rightarrow m = k\mu^{-3}$  and make another transformation first done by Ivor Robinson(check this???) given by:

$$\sin \theta = \frac{1}{\cosh(\mu\xi)} , \quad \mu^{-1}\phi = \eta \quad (3)$$

Which results in (SEE CALCS PG 3):

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 \mu\xi} (d\xi^2 + d\eta^2) - 2dudr - \left( \mu^2 - \frac{2k}{r} \right) du^2$$

Where the primes have been dropped for notational simplicity. This is now in an appropriate form to take the limit  $m \rightarrow \infty$  which is equivalent to  $\mu \rightarrow 0$ . This limit gives:

$$\epsilon ds^2 = r^2(d\xi^2 + d\eta^2) - 2dudr - \frac{2k}{r} du^2 \quad (4)$$

This is still a solution of the field equations, but it is no longer the Schwarzschild solution. It is found that this is the Kasner solution. To see this (SEE CALCS PG ???):

$$\epsilon ds^2 = T^{-2/3} dX^2 + T^{4/3} (dY^2 + dZ^2) - dT^2 \quad (5)$$

By definition the Kasner solution is given by:

$$\epsilon ds^2 = T^{2p} dX^2 + T^{2q} dY^2 + T^{2r} dZ^2 - dT^2$$

With:

$$p + q + r = 1 = p^2 + q^2 + r^2$$

So it is clear that Eqn.(5) is the Kasner solution with  $p = -1/3$  and  $q = r = 2/3$ .

### A. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting  $k = 0$  in Eqn.(4), which is equivalent to  $m = 0$ .

$$\epsilon ds^2 = r^2(d\xi^2 + d\eta^2) - 2dudr \quad (6)$$

Setting  $r = 0$  it can be shown that  $\epsilon ds^2 = 0$  in this case. Then  $r = 0$  is a null geodesic with  $u$  an affine parameter along it. To demonstrate these properties first let  $x^i = (x, y, z, t)$  be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element:

$$\epsilon ds_0^2 = dx^2 + dy^2 + dz^2 - dt^2$$

We note that  $C : x = 0, y = 0, z = t$  is a null geodesic as it will lie in the light cone of Minkowskian space-time. it we write it parametrically as  $x^i = w^i(u)$  such that  $w^i = (0, 0, u, u)$  then  $u$  is an affine parameter along  $C$ . The tangent to  $C$  is then computed as:

$$v^i(u) = \frac{dw^i}{du} = (0, 0, 1, 1)$$

As  $C$  is a null geodesic the first integral will be  $v_i v^i = 0$  (IS THIS CALLED THE FIRST INTEGRAL?) and thus  $v_i = (0, 0, 1, -1)$  where we have chosen the convention  $(+, +, +, -)$ .

The position vector of a point in Minkowskian space time can be written in the form(DO THE PICTURE FROM 2:1):

$$\begin{aligned} x^i &= w^i(u) = rk^i \\ \text{or } x^i &= w^i(u) + rk^i \end{aligned}$$

Thus  $r$  is a new parameter which tell us the shortest distance between  $C$  and some point  $x^i$ , and  $k^i$  is the unit vector in that direction. As  $k^i$  is a unit vector it satisfies the realtions:

$$k^i k_i = 0 \quad (7)$$

$$k^i v_i = -1 \quad (8)$$

Thus  $k^i$  is normalized so that  $k^i$  and  $v^i$  are both future pointing (HOW DOES THIS MAKE THEM BOTH FUTURE POINTING?). Making the parameterisation:

$$\begin{aligned} k^i &= (\xi, \eta, A, B) \\ \Rightarrow k_i &= (\xi, \eta, A, -B) \end{aligned}$$

We can choose any variable for the first two slots of  $k^i$  so we choose  $\xi$  and  $\eta$  from before for convenience. Using the relation (7) it is clear that:

$$\xi^2 + \eta^2 + A^2 - B^2 = 0$$

and using the relation (8) it is found that:

$$A - B = -1 \quad (9)$$

$$\Rightarrow A^2 - B^2 = (A + B)(A - B) = -(A + B) \quad (10)$$

Which implies:

$$\xi^2 + \eta^2 = A + B \quad (11)$$

So expressions for  $A$  and  $B$  are found using Eqn.(9) and Eqn.(11):

$$A = \frac{1}{2}(-1 + \xi^2 + \eta^2)$$

$$B = \frac{1}{2}(1 + \xi^2 + \eta^2)$$

In summary so far we have:

$$x^i = w^i(u) + rk^i \quad (12)$$

$$w^i = (0, 0, u, u) \quad (13)$$

$$k^i = (\xi, \eta, \frac{1}{2}(-1 + \xi^2 + \eta^2), \frac{1}{2}(1 + \xi^2 + \eta^2)) \quad (14)$$

$$x^i = (x, y, z, t) \quad (15)$$

Consider Eqn.(12) as a coordinate transformation from  $(x, y, z, t)$  to  $(\xi, \eta, r, u)$  such that:

$$x = r\xi$$

$$y = r\eta$$

$$z = u + \frac{r}{2}(-1 + \xi^2 + \eta^2)$$

$$t = u + \frac{r}{2}(1 + \xi^2 + \eta^2) \quad (16)$$

as is clear from Eqn.(12) - Eqn.(15). Now this is applied to the Minkowskian line element Eqn.(6). First, the  $x$  and  $y$  differentials are:

$$dx = rd\xi + \xi dr$$

$$dy = r d\eta + \eta dr$$

Which gives:

$$dx^2 + dy^2 = r^2(d\xi^2 + d\eta^2) + 2r\xi d\xi dr + 2r\eta d\eta dr + (\xi^2 + \eta^2)dr^2 \quad (17)$$

Next, the  $z$  and  $t$  differentials:

$$z + t = 2u + r(\xi^2 + \eta^2)$$

$$z - t = -r$$

$$dz + dt = 2du + (\xi^2 + \eta^2)dr + 2r\xi d\xi + 2r\eta d\eta$$

$$dz - dt = -dr$$

Using difference of two squares to obtain:

$$dz^2 - dt^2 = -2dudr - (\xi^2 + \eta^2)dr^2 - 2r\xi d\xi dr - 2r\eta d\eta dr \quad (18)$$

Combining Eqn.(17) and (18) to get:

$$dx^2 + dy^2 + dz^2 - dt^2 = r^2(d\xi^2 + d\eta^2) - 2dudr$$

and from this it is clear that Eqn.(6) is the line element of Minkowskian space-time with  $r = 0$  a null geodesic with affine parameter  $u$  along it as before.

### III. THE SINGULAR LORENTZ TRANSFORMATION

In this section a Lorentz transformation that leaves our line element Eqn.(6) invariant is constructed. This transformation is then expressed in terms of  $(x, y, z, t)$  and examined to see what for it has. First we define an arbitrary complex parameter by  $\zeta = \xi + i\eta$  so that the differentials are given by:

$$\begin{aligned} d\zeta &= d\xi + i d\eta \\ d\bar{\zeta} &= d\bar{\xi} + i d\bar{\eta} \end{aligned}$$

and the line element can be rewritten as:

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2du dr$$

In this form the transformation  $\zeta \rightarrow \zeta + w$ , where  $w \in \mathbb{C}$ , is trivial. It leaves the line element unchanged and the null geodesic  $r = 0$  trivially invariant. This is a Lorentz transformation which leaves one null direction invariant. Therefore it is a two real parameter, singular Lorentz transformation, where the two parameters come from the complex variable  $w$ . With this form of the line element the transformation is obviously trivial, we now want to see what this transformation looks like in terms of the usual variables  $(x, y, z, t)$ .

First we invert the transformation (??) and use the new variable  $\zeta$ :

$$\begin{aligned} x + iy &= r(\xi + i\eta) = r\zeta \\ z &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta}) \\ t &= u + \frac{r}{2}(1 + \zeta\bar{\zeta}) \end{aligned}$$

From this it is clear that:

$$\begin{aligned} t - z &= r, \text{ and} \\ t + z &= 2u + r\zeta\bar{\zeta} \end{aligned}$$

So finally:

$$\begin{aligned} \zeta &= \frac{x + iy}{t - z} \\ r &= t - z \\ u &= \frac{1}{2}(t + z) - \frac{(x^2 + y^2)}{2(t - z)} \end{aligned}$$

Now make the desired transformation:

$$\begin{aligned} \zeta' &\rightarrow \zeta + w \\ \bar{\zeta}' &\rightarrow \bar{\zeta} + \bar{w} \\ r' &= r \\ u' &= u \end{aligned}$$

(SEE NOTES PG 2:6 FOR CALC)

To obtain:

$$x' + iy' = x + iy + w(t - z) \tag{19}$$

$$z' - t' = -r = z - t \tag{20}$$

$$z' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z) \tag{21}$$

Next, it is necessary to show that this is indeed a Lorentz transformation by verifying the usual Lorentz invariant quantity. First from Eqn.(19) implies:

$$\begin{aligned} x'^2 + y'^2 &= (x + iy + w(t - z))(x - iy + \bar{w}(t - z)) \\ &= x^2 + y^2 + \bar{w}(t - z)(x + iy) + w(t - z)(x - iy) + w\bar{w}(t - z)^2 \end{aligned}$$

Then Eqn.(20) and Eqn.(21) imply:

$$\begin{aligned} (z' + t')(z' - t') &= z'^2 - t'^2 \\ &= z^2 - t^2 + (z - t)w(x - iy) + (z - t)\bar{w}(x + iy) + (z - t)w\bar{w}(t - z) \end{aligned}$$

Thus the Lorentz invariant quantity in the primed frame is the same as that of the unprimed frame:

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

It is also clear from Eqn.(20) that the null direction  $z = t$  is invariant under this Lorentz transformation.

In conclusion, this transformation involves one complex parameter and thus two real parameters. In the usual Cartesian coordinates it is described by Eqns.(19) - (21) and in the coordinates  $(\xi, \eta, r, u)$  (SHOULD THIS NOT BE WITH A  $\zeta$  ???) derived in previous sections, it is expressed simply as:

$$\begin{aligned} \zeta' &= \zeta + w \\ r' &= r \\ u' &= u \end{aligned}$$

Note that the operation of addition of complex numbers is commutative so that:

$$\begin{aligned} \zeta' &= \zeta + w_1, \text{ and } \zeta'' = \zeta' + w_2 \\ \Rightarrow \zeta'' &= \zeta + w_1 + w_2 \end{aligned}$$

and thus these transformations form a 2-parameter abelian subgroup of the Lorentz group with the binary operation of addition of complex numbers. (SHOW ALL 2 PARAM ABELIAN SUBGROUPS OF THE LORENTZ GROUP ARE SINGULAR TRANSFORMATIONS???)

#### IV. ACKNOWLEDGEMENTS

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