The Lorentz Group and Singular Lorentz Transformations

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abstract

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I. SPECIAL LINEAR MATRICES OF THE LORENTZ TRANSFORMATION

Let $\vec{x} = (x, y, z, t)$ be the position vector of a point in minkowskian space-time. Knowing $\vec{x}$ we can construct the following 2×2 hermitian matrix:

$$A = \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix} \tag{1}$$

with $A^\dagger(\vec{x}) = A(\vec{x})$. This is useful as its determinant is the same as the usual lorentz invariant quantity(WHAT DO WE CALL THIS???):

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2$$

Consider any 2×2 hermitian matrix H.

$$H = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \ , H^\dagger = \begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix}$$

It is know that $H^\dagger(\vec{x}) = H(\vec{x})$ so it is clear that $p = \bar{p}$ and $s = \bar{s}$ and thus $p, s \in \mathbb{R}$. Also $q = \bar{r}$ and then of course $\bar{q} = r$. Hence knowing p, q, r and s is equivalent to knowing 4 real numbers, two from p and two from q. From these parameters the coordinates (x, y, z, t) of a point in Minkowskian space-time can be constructed as:

$$x + iy = q = \bar{r} \ , t - z = p \ , t + z = s$$

by comparing with matrix A above. Hence it is true that there is a one to one correspondence between points in Minkowskian space-time and 2×2 hermitian matrices.

Construct the following matrix:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with α,β,γ,$\delta \in \mathbb{C}$. With the condition that $\det(U) = 1$. Such matrices U form a group called the special linear group, which is denoted by $SL(2, \mathbb{C})$. Given $A(\vec{x})$ consider $UA(\vec{x})U^\dagger$. This is a 2×2 hermitian matrix since:

$$\begin{aligned} cl(UA(\vec{x})U^\dagger)^\dagger &= (U^\dagger)^\dagger A^\dagger(\vec{x}) U^\dagger \\ &= UA(\vec{(x)})U^\dagger \end{aligned}$$

since $(U^\dagger)^\dagger = U$ and $A^\dagger = A$. Hence there exists a point $\vec{x'} = (x', y', z', t')$ in minkowskian space-time for which:

$$A(\vec{x'}) = UA(\vec{x})U^\dagger \tag{2}$$

Any U involves 6 real parameters, 2 each from the four complex components, with the condition $\det(U) = 1$ supplying two constraints. One on the real parts and one on the imaginary parts of the components. Now calculate the determinant of the matrix in the primed frame

$$\begin{aligned} \det(A(\vec{x'})) &= \det(UA(\vec{x})U^\dagger) \\ &= (\det(U))(\det(A(\vec{x}))(\det(U^\dagger)) \\ &= (\det(U))(\det(A(\vec{x}))(\det\bar{(U)}) \\ &= \det(A(\vec{x})) \end{aligned}$$

Thus we have the relation:

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$$

Hence the transformation $\vec{x} \rightarrow \vec{x'}$ implicit in Eqn.(2) is a Lorentz transformation. Eqn.(2) describes the most general proper, orthochronous Lorentz transformation.

It is useful to calculate the matrix U for some examples of Lorentz transformations. First, write Eqn.(2) in terms of its components:

$$\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha} + (x+iy)\bar{\beta} & (t-z)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)\bar{alpha} + (t+z)\bar{\beta} & (x-iy)\bar{\gamma} + (t+z)\bar{\delta} \end{pmatrix}$$

Thus we have the relations:

$$t'-z' = (t-z)\alpha\bar{\alpha} + (x+iy)\alpha\bar{\beta} + (x-iy)\beta\bar{alpha} + (t+z)\beta\bar{\beta} \tag{2a}$$

$$x'+iy' = (t-z)\alpha\bar{\gamma} + (x+iy)\alpha\bar{\delta} + (x-iy)\beta\bar{\gamma} + (t+z)\beta\bar{\delta} \tag{2b}$$

$$t'+z' = (t-z)\gamma\bar{\gamma} + (x+iy)\gamma\bar{\delta} + (x-iy)\delta\bar{\gamma} + (t+z)\delta\bar{\delta} \tag{2c}$$

A. Example 1: Rotational Transformation

Find U 2a2b2c corresponding to the one parameter Lorentz transformation:

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \\ t' &= t \end{aligned}$$

This implies that:

$$\begin{aligned} t'-z' &= t-z \\ x'+iy' &= (x+iy)e^{-i\theta} \\ t'+z' &= t+z \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(2a) to obtain: (SEE CALS)

B. Example 2: Special Relativity Lorentz Transfomation

(CALCS)

C. Example 3: Singular Lorentz Transformation

(CALCS)

It is clear that there will always be two matrices $\pm U$ corresponding to every Lorentz transformation, since if U satisfies $A(\vec{x'}) = UA(\vec{x})U^\dagger$ then so does $-U$. Hence there is a 2 to 1 correspondece between the elements of $SL(2,\mathbb{C})$ and the proper orthochronous Lorentz transofrmation.

II. INFINITESIMAL LORENTZ TRANSFORMATIONS

There are Lorentz transformations that are small perturbations of the identity transformation and so $U \in SL(2,\mathbb{C})$ has the form

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix},$$

where $a, b, c, f \in \mathbb{C}$ and ϵ is a small real parameter. Here terms of order ϵ^2 will be neglected. As $U \in SL(2,\mathbb{C})$ its determinant is calculated as

$$\det(U) = 1 + O(\epsilon^2).$$

Using this it is possible to obtain a relation between f and a

$$\begin{aligned}(1+\epsilon a)(1+\epsilon f) - \epsilon^2 bc &= 1 + O(\epsilon^2), \\ 1 + \epsilon(a+f) &= 1 + O(\epsilon^2), \\ \Rightarrow f &= -a + O(\epsilon).\end{aligned}$$

Hence

$$U = \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix},$$

Now the explicit infinitesimal Lorentz transformations are calculated as in section I, by substituting U into

$$A(\vec{x}') = UA(\vec{x})U^{\dagger}.$$

Now writing this out in component form to obtain

$$\begin{aligned}\begin{pmatrix} t'-z' & x'+iy' \\ x'+iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} 1+\epsilon\bar{a} & \epsilon\bar{c} \\ \epsilon\bar{b} & 1-\epsilon\bar{a} \end{pmatrix}, \\ &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} (t-z)(1+\epsilon\bar{a}) + \epsilon\bar{b}(x+iy) & (t-z)\epsilon\bar{c} + (1-\epsilon\bar{a})(x+iy) \\ (x-iy)(1+\epsilon\bar{a}) + \epsilon\bar{b}(t+z) & (x-iy)\epsilon\bar{c} + (1-\epsilon\bar{a})(t+z) \end{pmatrix}.\end{aligned}$$

This then implies the three relations

$$\begin{aligned}t'-z' &= t-z+\epsilon(a+\bar{a})(t-z)+\epsilon(b+\bar{b})x+i\epsilon(\bar{b}-b)y+O(\epsilon^2) \\ t'+z' &= t+z-\epsilon(a+\bar{a})(t+z)+\epsilon(c+\bar{c})x+i\epsilon(c-\bar{c})y+O(\epsilon^2) \\ x'+iy' &= x+iy+\epsilon(a-\bar{a})(x+iy)+\epsilon(b+\bar{c})t+\epsilon(b-\bar{c})z+O(\epsilon^2)\end{aligned}$$

III. ACKNOWLEDGEMENTS