The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

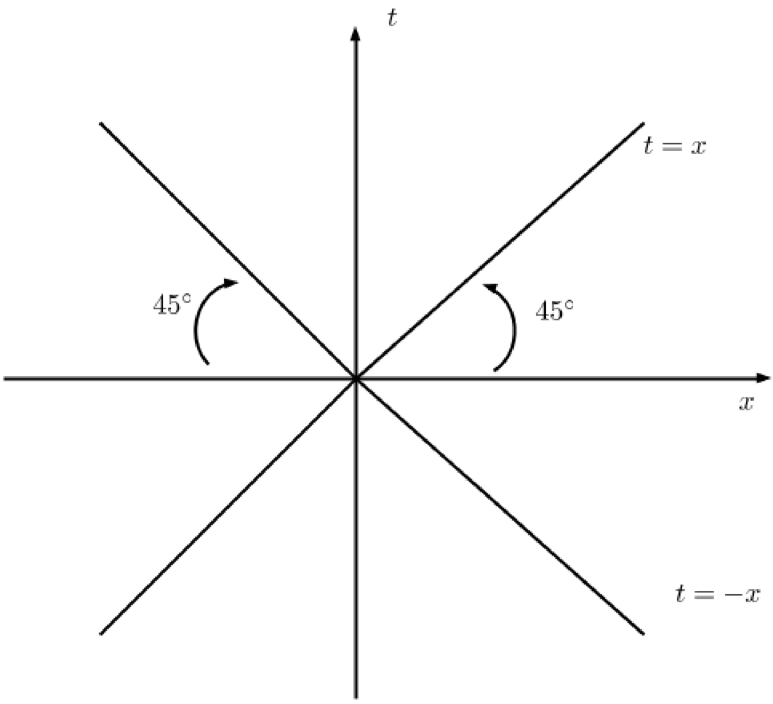
I. THE LORENTZ TRANSFORMATION

In this section the two null directions inherent in all Lorentz transformations, except the singular Lorentz transformations, are illustrated. The Lorentz Transform is defined by $(x, y, z, t) \to (x', y', z', t')$ such that

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2.$$

If the transformation preserves the orientation of the spatial axes then is it called a *proper* Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also if $t \geq 0$ implies that time is always positive then it is called an *orthochronous* Lorentz transformation, which ensures that the time direction is preserved. In this project the "Lorentz transformation" will refer to the proper, orthochronous Lorentz transformation.

FIG. 1: *Space-Time Diagram for a photon moving at the speed of light, $c = 1$ and starting from $x = 0$. Two null directions can be seen.*



Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as in Fig.(1). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx), \end{aligned}$$

where $\gamma = (1 - v^2)^{-1/2}$. Rearrange to obtain

$$\begin{aligned} x' - t' &= \gamma(1 + v)(x - t), \\ x' + t' &= \gamma(1 - v)(x + t). \end{aligned}$$

It is clear that $x = \pm t$ implies $x' = \pm t'$. Thus there are two null directions in this space-time at $x = \pm t$, as null directions are by definition invariant under a Lorentz transformation. It can be shown that all Lorentz transformations have two invariant null directions except the singular Lorentz transformation which has only one fixed null direction.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

In this section the Kasner solution of the vacuum field equations is derived from the Schwarzschild solution by taking the limit as the mass goes to infinity. It is then shown that the special case of the Kasner solution with no mass is equivalent to a novel form of Minkowskian space-time. Start with the Schwarzschild Solution of the vacuum field equations given by

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2. \tag{1}$$

A. Eddington-Finkelstein Coordinate Transformation

First, make the Eddington-Finkelstein coordinate transformation

$$u = t - r - 2m \ln(r - 2m). \tag{2}$$

Calculate the differentials

$$\begin{aligned}
\mathrm{d}u &= \mathrm{d}t - \mathrm{d}r - \frac{2m\mathrm{d}r}{r - 2m}, \\
&= \mathrm{d}t - \mathrm{d}r\left(1 - \frac{2m}{r}\right)^{-1}, \\
\mathrm{d}t &= \mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r,
\end{aligned}$$

and sub them into Eqn.(1).

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right)\left(\mathrm{d}u + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r\right)^2,$$

which gives the result,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - 2\mathrm{d}u\mathrm{d}r - \mathrm{d}u^2 + \frac{2m}{r}\mathrm{d}u^2. \tag{3}$$

Note that if $m = 0$ the space-time becomes Minkowskian, as expected. If $r = 0$ in this Minkowskian space-time the line element becomes

$$\epsilon \mathrm{d}s^2 = -\mathrm{d}u^2,$$

which implies that $\epsilon = -1$ and the first integral of this trajectory is also equal to -1. Thus $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u. It can also be shown that $r = 0$ is a geodesic with proper time, u an affine parameter along it, this is done here in two ways. The first method is to write the line element in cartesian coordinates using the transformation

$$\begin{aligned}
x &= r \sin\theta \cos\phi, \\
y &= r \sin\theta \sin\phi, \\
z &= r \cos\theta, \\
t &= u + r.
\end{aligned}$$

It is clear that setting $r = 0$ will give $x = y = z = 0$ and $t = u$. So the world-line is the t-axis with u a parameter along it an is of course a time-like geodesic. The second method is much longer and involves calculaing the metric and proving the geodesic equations, it is done here as it will be familiar to many readers. The Langrangian method described by the following equations is used

$$L = g_{ij}\dot{x}^i\dot{x}^j, \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 2w_i,$$
$$w^j = g^{ji}w_i,$$

with a geodesic defined by $w^j = 0$. The metric of this space-time in coordinates (r, θ, ϕ, u) is determined from Eqn.(3) with $m = 0$

$$g_{ij}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin\theta & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

The inverse of the metric is calculated to be

$$g^{ij}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

w^j will be calculated for r and u and simply stated for θ and ϕ. The Lagrangian for r is

$$w_r = -\ddot{u} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2),$$

and the Lagrangian for u is

$$w_u = -\ddot{u} - \ddot{r}$$

Now w^r is given by

$$w^r = g^{rj}w_j$$
$$= w_r - w_u$$

and w^u is

$$w^u = g^{uj}w_j$$
$$= -w_r$$

So putting these together to obtain $w^r = 2w_r$ and $w^u = -w_r$ which implies

$$w^r = -2\ddot{u} - 2r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2),$$
$$w^u = \ddot{u} + r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2),$$

and also for θ and ϕ

$$w^\theta = r\ddot{\theta} + 2\dot{\theta}\dot{r} - r\sin\theta\cos\theta\dot{\phi}^2,$$
$$w^\phi = \sin\theta(\dot{\phi}\dot{r} + r\ddot{\phi}) + 2r\dot{\phi}\cos\theta\dot{\theta}.$$

Then set $r=0$ to obtain the equations

$$\begin{aligned} w^r &= -2\ddot{u}, \\ w^u &= \ddot{u}, \\ w^\theta &= 0, \\ w^\phi &= 0, . \end{aligned}$$

Now $s=u$ as u is proper time then $w^r = w^u = 0$, which implies that the trajectory $r=0$ is a geodesic with u an affine parameter along it.

The limit of the Schwarzschild solution as $m \to \infty$ must be calculated to find the Kasner solution. In its current form the limit cannot be taken, so two suitable coordinate transformations must be made to get it in a more useful form. First Set

$$\begin{aligned} u &= \mu u', \\ r &= \mu^{-1} r', \end{aligned}$$

where $\mu =$ const so that

$$\begin{aligned} \mathrm{d}u &= \mu \mathrm{d}u', \\ \mathrm{d}r &= \mu^{-1} \mathrm{d}r'. \end{aligned}$$

The product of the differentials is then invariant

$$\mathrm{d}u\mathrm{d}r = \mathrm{d}u'\mathrm{d}r'.$$

When the new coordinates are subbed into Eqn.(3) the Schwarzschild solution becomes

$$\epsilon \mathrm{d}s^2 = r'^2 \sin^2\theta \left\{ \frac{\mathrm{d}\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2}\mathrm{d}\phi^2 \right\} - 2\mathrm{d}u'\mathrm{d}r' - \left(\mu^2 - \frac{2m\mu^3}{r'} \right) \mathrm{d}u'^2. \tag{4}$$

Now set $m\mu^3 = k$ or $m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by

$$\sin\theta = \frac{1}{\cosh(\mu\xi)}, \mu^{-1}\phi = \eta.$$

The second of these transformations gives simply $\mu^{-2}\mathrm{d}\phi^2 = \mathrm{d}\eta^2$. To rewrite the first coordinate transformation, first differentiate.

$$\begin{aligned} \cos\theta \mathrm{d}\theta &= \frac{-1}{(\cosh(\mu\xi))^2} \sinh(\mu\xi)\mu \mathrm{d}\xi, \\ &= -\sin^2\theta \sinh(\mu\xi)\mu \mathrm{d}\xi. \end{aligned}$$

Use the forumla $\cosh^2 A - \sinh^2 A = 1$, divide by $\mathrm{d}\xi$ and simplfy using trigonometric identities

$$\begin{aligned} \cos\theta \frac{\mathrm{d}\theta}{\mathrm{d}\xi} &= -\mu \sin^2\theta \sqrt{\frac{1}{\sin^2\theta} - 1}, \\ &= -\mu \sin\theta \cos\theta. \end{aligned}$$

Finally rewrite in terms of $\mathrm{d}\xi$

$$\mathrm{d}\xi^2 = \left(\frac{\mathrm{d}\theta}{\mu \sin\theta}\right)^2.$$

Subbing these transformations into Eqn.(4) gives

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right)\mathrm{d}u^2,$$

where the primes have been dropped for convenience.

B. The Kasner Solution

This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$, to obtain

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \frac{2k}{r}\mathrm{d}u^2 \tag{5}$$

This is still a solution of the field equations but it is no longer the Schwarzschild solution. In this section it is shown to be the Kasner Solution(READ UP ABOUT THIS), which by definition is given by

$$\epsilon \mathrm{d}s^2 = T^{2p}\mathrm{d}X^2 + T^{2q}\mathrm{d}Y^2 + T^{2r}\mathrm{d}Z^2 - \mathrm{d}T^2, \tag{6}$$

such that

$$p + q + r = 1 = p^2 + q^2 + r^2.$$

To write Eqn.(5) in this form first make the transformation

$$\xi' = \lambda^{-1}\xi,\ \ \eta' = \lambda^{-1}\eta,$$
$$r' = \lambda r,\ \ u' = \lambda^{-1}u,$$

with $\lambda := k^{-1/3}$. Subbing in these new coordinates gives

$$\epsilon \mathrm{d}s^2 = r'^2(\mathrm{d}\xi'^2 + \mathrm{d}\eta'^2) - 2\mathrm{d}u'\mathrm{d}r' + \frac{2}{r'}\mathrm{d}u'^2.$$

Now add and subtract $(r'/2)\mathrm{d}r'^2$ to complete the square as follows

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\left(\mathrm{d}u - \frac{r}{2}\mathrm{d}r\right)^2 - \frac{r}{2}\mathrm{d}r^2,$$

where the primes have again been dropped for convenience. Now set

$$\bar{X} = u - \frac{r^2}{4},$$

so the differential of $\bar{X}$ is

$$\mathrm{d}\bar{X} = \mathrm{d}u - \frac{r}{2}\mathrm{d}r,$$

and the line element can be rewritten in terms of $\bar{X}$,

$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{2}{r}\mathrm{d}\bar{X}^2 - \left(\frac{r^{1/2}}{\sqrt{2}}\mathrm{d}r\right)^2.$$

Now define T such that

$$T = \frac{\sqrt{2}}{3}r^{3/2}, \text{ and } r = \left(\frac{3}{\sqrt{2}}\right)^{2/3} T^{2/3},$$

which results in the new line element

$$\epsilon \mathrm{d}s^2 = \left(\frac{3}{\sqrt{2}}\right)^{4/3} T^{4/3}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2\left(\frac{\sqrt{2}}{3}\right)^{2/3} T^{-2/3}\mathrm{d}\bar{X}^2 - \mathrm{d}T^2.$$

Then a final coordinate transformation can be made to remove the unwanted constants, given by

$$\begin{aligned} X &= \left[2\left(\frac{\sqrt{2}}{3}\right)\right]^{1/2} \bar{X}, \\ Y &= \left(\frac{3}{\sqrt{2}}\right)^{2/3} \xi, \\ Z &= \left(\frac{3}{\sqrt{2}}\right)^{2/3} \eta, \end{aligned}$$

to obtain

$$\epsilon \mathrm{d}s^2 = T^{-2/3}\mathrm{d}X^2 + T^{4/3}\left(\mathrm{d}Y^2 + \mathrm{d}Z^2\right) - \mathrm{d}T^2. \tag{7}$$

Comparing this result to the general form of the Kasner solution in Eqn.(6) it is clear that they have the same form with $p = -1/3$ and $q = r = 2/3$. Thus the solution obtained by taking the limit of the Schwarzschild solution as $m \to \infty$ is indeed the Kasner Solution.

C. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting $k = 0$ in Eqn.(5), which is equivalent to $m = 0$.

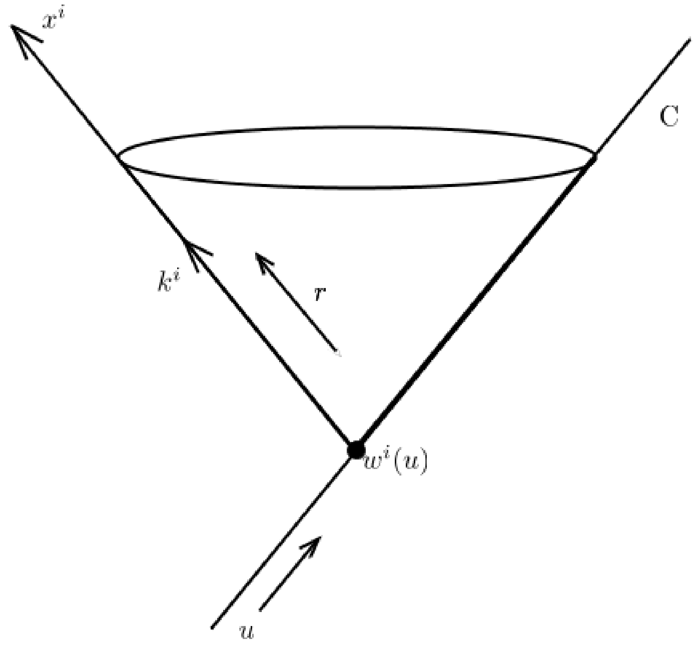
$$\epsilon \mathrm{d}s^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r \tag{8}$$

Setting $r = 0$ gives $\epsilon \mathrm{d}s^2 = 0$. In this section it is demonstrated that $r = 0$ is a null geodesic with u an affine parameter along it and that Eqn.(8) is indeed the Minkowskian space-time line element. To verify these properties first let $x^i = (x, y, z, t)$ be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element

$$\epsilon \mathrm{d}{s_0}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2.$$

We note that the trajectory C defined by $x = 0,\ y = 0,\ z = t$ is a null geodesic as it will lie on the light cone of Minkowskian space-time. If C is writen parametrically as $x^i = w^i(u)$ such that $w^i = (0, 0, u, u)$ then u is an affine parameter along it. The tangent to C is then computed as

FIG. 2: *Minkowskian space-time illustrating the new parameter r which is the shortest distance between some point x^i and the trajectory C along the k^i direction. The parameter u which determines the distance travelled along C is also shown*



$$v^i(u) = \frac{\mathrm{d}w^i}{\mathrm{d}u} = (0, 0, 1, 1).$$

As C is a null geodesic the first integral will be $v_i v^i = 0$ and thus $v_i = (0, 0, 1, -1)$ where we have chosen the convention $(+, +, +, -)$.

The position vector of a point in Minkowskian space time can be written in the form

$$x^i - w^i(u) = rk^i,$$
$$\text{or } x^i = w^i(u) + rk^i.$$

Thus r is a new parameter which tell us the shortest distance between C and some point x^i, and k^i is the unit vector in that direction, see Fig.(2). As k^i is a unit vector it satisfies the relations

$$k^i k_i = 0, \tag{9}$$
$$k^i v_i = -1. \tag{10}$$

Thus k^i is normalized so that k^i and v^i are both future pointing. Making the parameterisation

$$k^i = (\xi, \eta, A, B),$$
$$k_i = (\xi, \eta, A, -B).$$

We can choose any variable for the first two slots of k^i so we choose ξ and η from before for convenience. Using the relation (9) it is clear that

$$\xi^2 + \eta^2 + A^2 - B^2 = 0,$$

and using the relation (10) it is found that

$$A - B = -1, \tag{11}$$
$$\Rightarrow A^2 - B^2 = (A+B)(A-B) = -(A+B). \tag{12}$$

Which implies

$$\xi^2 + \eta^2 = A + B. \tag{13}$$

So expressions for A and B are found using Eqn.(11) and Eqn.(13):

$$A = \frac{1}{2}(-1 + \xi^2 + \eta^2)$$
$$B = \frac{1}{2}(1 + \xi^2 + \eta^2)$$

In summary so far we have

$$x^i = w^i(u) + rk^i, \tag{14}$$
$$w^i = (0, 0, u, u), \tag{15}$$
$$k^i = (\xi, \eta, \frac{1}{2}(-1 + \xi^2 + \eta^2), \frac{1}{2}(1 + \xi^2 + \eta^2)), \tag{16}$$
$$x^i = (x, y, z, t). \tag{17}$$

Consider Eqn.(14) as a coordinate transformation from (x, y, z, t) to (ξ, η, r, u) such that

$$\begin{aligned} x &= r\xi, \\ y &= r\eta, \\ z &= u + \frac{r}{2}(-1 + \xi^2 + \eta^2), \\ t &= u + \frac{r}{2}(1 + \xi^2 + \eta^2), \end{aligned} \tag{18}$$

which is clear from Eqns.(14) - (17). Now this is applied to the Minkowskian line element of Eqn.(8). First, the x and y differentials are

$$dx = r\mathrm{d}\xi + \xi\mathrm{d}r,$$
$$\mathrm{d}y = r\mathrm{d}\eta + \eta\mathrm{d}r.$$

Which gives

$$\mathrm{d}x^2 + \mathrm{d}y^2 = r^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + 2r\xi\mathrm{d}\xi\mathrm{d}r + 2r\eta\mathrm{d}\eta\mathrm{d}r + (\xi^2 + \eta^2)\mathrm{d}r^2. \tag{19}$$

Next, the z and t differentials

$$\begin{aligned} z + t &= 2u + r(\xi^2 + \eta^2), \\ z - t &= -r, \\ \mathrm{d}z + \mathrm{d}t &= 2\mathrm{d}u + (\xi^2 + \eta^2)\mathrm{d}r + 2r\xi\mathrm{d}\xi + 2r\eta\mathrm{d}\eta, \\ \mathrm{d}z - \mathrm{d}t &= -\mathrm{d}r. \end{aligned}$$

Then using difference of two squares to obtain

$$dz^2 - dt^2 = -2dudr - (\xi^2 + \eta^2)dr^2 - 2r\xi d\xi dr - 2r\eta d\eta dr. \tag{20}$$

Combining Eqn.(19) and (20) to get:

$$dx^2 + dy^2 + dz^2 - dt^2 = r^2(d\xi^2 + d\eta^2) - 2dudr$$

and from this it is clear that Eqn.(8) is the line element of Minkowskian space-time with $r = 0$ a null geodesic with affine parameter u along it as stated at the beginning of the section.(ERROR. REFER TO GEODESIC CALC FROM EARLIER)

Thus it has been shown that the Kasner solution to the vacuum filed equations can be obtained from the Schwarzschild solution of the vacuum field equations by taking the special case where the mass goes to infinity. Then the Minkowskian space-time line element in a particular set of coordinates (ξ, η, r, u), can be derived from the Kasner solution by setting the energy, k to zero. It has been shown that the special case where $r = 0$ in this Minkowskian space-time is then a null geodesic with u an affine parameter along it.

III. THE SINGULAR LORENTZ TRANSFORMATION

In this section a Lorentz transformation that leaves our line element Eqn.(8) invariant is constructed. This transformation is then expressed in terms of (x, y, z, t) and examined to see what form it has. The subgroup of the Lorentz group that it makes is then examined. First we define an arbitrary complex parameter by $\zeta = \xi + i\eta$ so that the differentials are given by

$$\begin{aligned} d\zeta &= d\xi + id\eta, \\ d\bar{\zeta} &= d\bar{\xi} - id\bar{\eta}, \end{aligned}$$

and the line element can be rewritten as

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$

In this form the transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$, is trivial. It leaves the line element unchanged and the null geodesic $r = 0$ trivially invariant. This is a Lorentz transformation which leaves one null direction invariant. Therefore it is a two real parameter, singular Lorentz transformation, where the two parameters come from the complex variable w. With this form of the line element the transformation is obviously trivially invariant, we now want to see what this transformation looks like in terms of the usual coordinates (x, y, z, t).

First invert the transformation (18) and use the new variable ζ

$$\begin{aligned} x + iy &= r(\xi + i\eta) = r\zeta \\ z &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta}) \\ t &= u + \frac{r}{2}(1 + \zeta\bar{\zeta}) \end{aligned} \tag{21}$$

From this it is clear that

$$\begin{aligned} t - z &= r, \\ t + z &= 2u + r\zeta\bar{\zeta}. \end{aligned}$$

So finally

$$r = t - z,$$
$$\zeta = \frac{x+iy}{t-z},$$
$$u = \frac{1}{2}(t+z) - \frac{(x^2+y^2)}{2(t-z)}. \tag{22}$$

Now make the desired transformation $(\zeta', \bar{\zeta}', r', u') \to (\zeta + w, \bar{\zeta} + \bar{w}, r, u)$ by first replacing these new quantities into transformation (21)

$$x' + iy' = r'\zeta' = r(\zeta + w),$$
$$z' = u + \frac{r}{2}(-1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}),$$
$$t' = u + \frac{r}{2}(1 + \zeta\bar{\zeta} + \zeta\bar{w} + \bar{\zeta}w + w\bar{w}).$$

So the transformed Cartesian coordinates have been written in terms of the untransformed particular coordinates, (ζ, r, u). Next, using the relations (22), write the transformed Cartesian coordinates in terms of the untransformed Cartesian coordinates.

$$x' + iy' = x + iy + w(t-z), \tag{23}$$
$$z' - t' = -r = z - t, \tag{24}$$
$$z' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t-z). \tag{25}$$

It is necessary to show that this is indeed a Lorentz transformation by verifying the usual Lorentz invariant quadratic form. First Eqn.(23) implies

$$\begin{aligned} x'^2 + y'^2 &= (x + iy + w(t-z))(x - iy + \bar{w}(t-z)) \\ &= x^2 + y^2 + \bar{w}(t-z)(x+iy) + w(t-z)(x-iy) + w\bar{w}(t-z)^2. \end{aligned}$$

Then Eqn.(24) and Eqn.(25) imply

$$\begin{aligned} z'^2 - t'^2 &= (z' + t')(z' - t') \\ &= z^2 - t^2 + (z-t)w(x-iy) + (z-t)\bar{w}(x+iy) + (z-t)w\bar{w}(t-z) \end{aligned}$$

Thus the extra terms cancel and Lorentz invariant quadratic form in the primed frame is the same as that of the unprimed frame,

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2.$$

It is also clear from Eqn.(24) that the null direction $z = t$ is invariant under this Lorentz transformation.

In conclusion, this transformation involves one complex parameter and thus two real parameters. In the usual Cartesian coordinates it is described by Eqns.(23) - (25) and in the coordinates (ξ, η, r, u), also denoted by (ζ, r, u) derived in previous sections, it is expressed simply as

$$\zeta' = \zeta + w,$$
$$r' = r,$$
$$u' = u.$$

Note that the operation of addition of complex numbers is commutative so that if $\zeta' = \zeta + w_1$ and $\zeta'' = \zeta' + w_2$ then

$$\zeta'' = \zeta + w_1 + w_2 = \zeta + w_3.$$

Thus these transformations form a 2-parameter abelian subgroup of the Lorentz group with the binary operation of addition of complex numbers.

So a Lorentz transformation that preserves the line element Eqn.(8) has been constructed. It is found that this transformation is a singular Lorentz transformation as it keeps the null direction $r = 0$ fixed. Thus it is shown that all singular Lorentz transformations are two parameter abelian subgroups of the Lorentz group. Note that not all two parameter abelian subgroups of the Lorentz group are singular Lorentz transformations.

IV. SPECIAL LINEAR MATRICES OF THE LORENTZ TRANSFORMATION

In this section it is shown that there is a 2 to 1 correspondence between the matrices $SL(2, \mathbb{C})$ and the proper orthochronous Lorentz transformations. First it is demonstrated that there is a one to one correspondence between 2×2 Hermitian matrices and Minkowskian space-time, this is then applied to Lorentz transformations. Let $\vec{x} = (x, y, z, t)$ be the position vector of a point in Minkowskian space-time. Knowing $\vec{x}$ we can construct the following 2×2 Hermitian matrix

$$A = \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix}, \tag{26}$$

with $A^\dagger(\vec{x}) = A(\vec{x})$. This is useful as its determinant is the same as the Lorentz invariant quadratic form, up to an arbitrary sign.

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

Consider any 2×2 Hermitian matrix H, thus

$$H = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \;,\; H^\dagger = \begin{pmatrix} \bar{p} & \bar{r} \\ \bar{q} & \bar{s} \end{pmatrix}$$

It is know that $H^\dagger(\vec{x}) = H(\vec{x})$ so it is clear that $p = \bar{p}$ and $s = \bar{s}$ and thus p and s are real numbers. Also $q = \bar{r}$ and then of course $\bar{q} = r$. Hence knowing p, q, r and s is equivalent to knowing 4 real numbers, one from p, one from s and two from q. From these parameters the coordinates (x, y, z, t) of a point in Minkowskian space-time can be constructed as

$$\begin{aligned} x + iy &= q = \bar{r}, \\ t - z &= p, \\ t + z &= s. \end{aligned}$$

by comparing with matrix A above. Hence it is true that there is a one to one correspondence between points in Minkowskian space-time and 2×2 Hermitian matrices.

Construct the following matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with α, β, γ, $\delta \in \mathbb{C}$, and the condition that $\det(U) = 1$. Such matrices U form a group called the special linear group, which is denoted by $SL(2, \mathbb{C})$. Given $A(\vec{x})$ consider $UA(\vec{x})U^\dagger$. This is a 2×2 Hermitian matrix since

$$\begin{aligned} (UA(\vec{x})U^\dagger)^\dagger &= (U^\dagger)^\dagger A^\dagger(\vec{x})U^\dagger \\ &= UA(\vec{x})U^\dagger, \end{aligned}$$

as $(U^\dagger)^\dagger = U$ and $A^\dagger = A$. Hence there exists a point $\vec{x}' = (x', y', z', t')$ in Minkowskian space-time for which

$$A(\vec{x}') = UA(\vec{x})U^\dagger. \tag{27}$$

Any U involves 6 real parameters, 2 each from the four complex components, with the condition $\det(U) = 1$ supplying two constraints, one on the real parts and one on the imaginary parts of the components. Now calculate the determinant of the matrix in the primed frame

$$\begin{aligned}
\det(A(\vec{x}')) &= \det(UA(\vec{x})U^\dagger), \\
&= (\det(U))(\det(A(\vec{x}))(\det(U^\dagger)), \\
&= (\det(U))(\det(A(\vec{x}))(\det\bar{(}U)), \\
&= \det(A(\vec{x})).
\end{aligned}$$

Thus we have the relation

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2.$$

Hence the transformation $\vec{x} \to \vec{x}'$ implicit in Eqn.(27) is a Lorentz transformation. This equation describes the most general proper, orthochronous Lorentz transformation.

It is useful to calculate the matrix U for some examples of Lorentz transformations. First, write Eqn.(27) in terms of its components

$$\begin{aligned}
\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \\
&= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha} + (x+iy)\bar{\beta} & (t-z)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha} + (t+z)\bar{\beta} & (x-iy)\bar{\gamma} + (t+z)\bar{\delta} \end{pmatrix}.
\end{aligned}$$

Thus the relations

$$t' - z' = (t-z)\alpha\bar{\alpha} + (x+iy)\alpha\bar{\beta} + (x-iy)\beta \bar{alpha} + (t+z)\beta\bar{\beta}, \tag{28a}$$

$$x' + iy' = (t-z)\alpha\bar{\gamma} + (x+iy)\alpha\bar{\delta} + (x-iy)\beta\bar{\gamma} + (t+z)\beta\bar{\delta}, \tag{28b}$$

$$t' + z' = (t-z)\gamma\bar{\gamma} + (x+iy)\gamma\bar{\delta} + (x-iy)\delta\bar{\gamma} + (t+z)\delta\bar{\delta}. \tag{28c}$$

are obtained. Now these equations are used on some specific cases.

A. Example 1: Rotational Transformation

Find U corresponding to the one parameter Lorentz transformation,

$$\begin{aligned}
x' &= x\cos\theta + y\sin\theta, \\
y' &= -x\sin\theta + y\cos\theta, \\
z' &= z, \\
t' &= t.
\end{aligned}$$

This implies that

$$\begin{aligned}
t' - z' &= t - z \\
x' + iy' &= (x+iy)e^{-i\theta} \\
t' + z' &= t + z
\end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta} + \bar{\alpha}\beta = 0, \tag{29a}$$
$$i(\alpha\bar{\beta} - \bar{\alpha}\beta) = 0, \tag{29b}$$
$$-\alpha\bar{\alpha} + \beta\bar{\beta} = -1, \tag{29c}$$
$$\alpha\bar{\alpha} + \beta\bar{\beta} = 1. \tag{29d}$$

Then Eqn.(29a) and (29b) imply $\alpha\bar{\beta} = 0$ so $\alpha = 0$ or $\beta = 0$. Also Eqn.(29c) and (29d) imply $2\beta\bar{\beta} = 0$ so $\beta = 0$ and $\alpha\bar{\alpha} = 1$ so $\alpha \neq 0$. Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$e^{-i\theta} = \alpha\bar{\delta} + \beta\bar{\gamma}, \tag{30a}$$
$$e^{-i\theta} = \alpha\bar{\delta} - \beta\bar{\gamma}, \tag{30b}$$
$$0 = -\alpha\bar{\gamma} + \beta\bar{\delta}, \tag{30c}$$
$$0 = \alpha\bar{\gamma} + \beta\bar{\delta}. \tag{30d}$$

With $\beta = 0$, Eqn.(30a) and (30b) imply $\alpha\bar{\delta} = e^{-i\theta}$. Also Eqn.(30c) and (30d) imply $\alpha\bar{\gamma} = 0$ so $\gamma = 0$ since $\alpha \neq 0$. Then using $\alpha\bar{\alpha} = 1$

$$\begin{aligned} \alpha\bar{\delta} &= e^{-i\theta}, \\ \bar{\alpha}\alpha\bar{\delta} &= \bar{\alpha}e^{-i\theta}, \\ \bar{\delta} &= \bar{\alpha}e^{-i\theta}. \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28c) to obtain

$$\gamma\bar{\delta} + \delta\bar{\gamma} = 0, \tag{31a}$$
$$\gamma\bar{\delta} - \delta\bar{\gamma} = 0, \tag{31b}$$
$$-\gamma\bar{\gamma} + \delta\bar{\delta} = 1, \tag{31c}$$
$$\gamma\bar{\gamma} + \delta\bar{\delta} = 1. \tag{31d}$$

Eqn(31a) and (31b) are satisfied since $\gamma = 0$, this also implies that $\delta\bar{\delta} = 1$ from Eqn(31c). Now use the fact that $\det(U) = 1$, which implies

$$\alpha\delta - \beta\gamma = 1,$$

thus $\alpha\delta = 1$ as $\beta = 0$. Then using $\alpha\bar{\alpha} = 1$ again and $\delta\bar{\delta} = 1$

$$\begin{aligned} \alpha^2 e^{-\theta} &= 1, \\ \alpha^2 &= e^{-i\theta}, \\ \alpha &= \pm e^{-i\theta/2}, \end{aligned}$$

which finally implies that $\delta = \pm e^{i\theta/2}$. Hence there are 2 matrices U corresponding to the spacial rotation, namely

$$U = \pm \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

The next two examples are then very similar.

B. Example 2: Standard Lorentz Transformation

Find U corresponding to the one parameter Lorentz transformation,

$$\begin{aligned} x' &= \gamma_0(x - vt), \\ t' &= \gamma_0(t - vx), \\ y' &= y, \\ z' &= z, \end{aligned} \tag{32}$$

Where $\gamma_0 = (1 - v^2)^{-1/2}$. This implies that

$$\begin{aligned} t' - z' &= -\gamma_0 vx + \gamma_0 t - z \\ x' + iy' &= \gamma_0 x - v\gamma_0 t + iy \\ t' + z' &= -v\gamma_0 x + \gamma_0 t + z \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta} + \bar{\alpha}\beta = -\gamma_0 v, \tag{33a}$$

$$i(\alpha\bar{\beta} - \bar{\alpha}\beta) = 0, \tag{33b}$$

$$-\alpha\bar{\alpha} + \beta\bar{\beta} = -1, \tag{33c}$$

$$\alpha\bar{\alpha} + \beta\bar{\beta} = \gamma_0. \tag{33d}$$

Then Eqn.(33a) and (33b) imply that $\alpha\bar{\beta} = \bar{\alpha}\beta$. Also Eqn.(33c) and (33d) imply that

$$\beta\bar{\beta} = \frac{\gamma_0 - 1}{2}, \tag{34}$$

$$\alpha\bar{\alpha} = \frac{\gamma_0 + 1}{2}. \tag{35}$$

Thus β can be written in terms of α using Eqn.(33a)

$$\begin{aligned} \alpha\bar{\beta} &= -\frac{\gamma_0 v}{2}, \\ \alpha\bar{\alpha}\bar{\beta} &= -\bar{\alpha}\frac{\gamma_0 v}{2}, \\ \beta &= -\alpha\frac{\gamma_0 v}{\gamma_0 + 1}. \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$\gamma_0 = \alpha\bar{\delta} + \beta\bar{\gamma}, \tag{36a}$$

$$1 = \alpha\bar{\delta} - \beta\bar{\gamma}, \tag{36b}$$

$$0 = -\alpha\bar{\gamma} + \beta\bar{\delta}, \tag{36c}$$

$$-v\gamma_0 = \alpha\bar{\gamma} + \beta\bar{\delta}. \tag{36d}$$

Eqn.(36a) and (36b) imply that

$$\begin{aligned} \beta\bar{\gamma} &= \frac{\gamma_0 - 1}{2}, \\ \alpha\bar{\delta} &= \frac{\gamma_0 + 1}{2}. \end{aligned}$$

Thus δ can be written in terms of α by using Eqn.(35)

$$\begin{aligned}\alpha\bar{\alpha}\bar{\delta} &= \bar{\alpha}\frac{\gamma_0+1}{2},\\ \bar{\delta} &= \bar{\alpha},\\ \delta &= \alpha.\end{aligned}$$

Also, using Eqn.(34) γ can be written in terms of β

$$\begin{aligned}\beta\bar{\gamma} &= \frac{\gamma_0-1}{2},\\ \bar{\beta}\beta\bar{\gamma} &= \bar{\beta}\frac{\gamma_0-1}{2},\\ \bar{\gamma} &= \bar{\beta},\\ \gamma &= \beta.\end{aligned}$$

Now use the fact that $\det(U) = 1$, which implies

$$\alpha\delta - \beta\gamma = 1,$$

thus $\alpha^2 - \beta^2 = 1$ as $\delta = \alpha$ and $\gamma = \beta$. Replace β

$$\begin{aligned}\alpha^2\left(\frac{(\gamma_0+1)^2-(\gamma_0 v)^2}{(\gamma_0+1)^2}\right) &= 1\\ \alpha &= \pm\frac{\gamma_0+1}{\sqrt{(\gamma_0+1)^2-(\gamma_0 v)^2}}.\end{aligned}$$

Rewrite the denominator of α using $\gamma_0 v = \sqrt{\gamma_0{}^2-1}$

$$\begin{aligned}(\gamma_0+1)^2-(\gamma_0 v)^2 &= \gamma_0{}^2+1+2\gamma_0-\gamma_0{}^2+1,\\ &= 2(\gamma_0+1).\end{aligned}$$

So finally

$$\alpha = \pm\frac{\sqrt{\gamma_0+1}}{2}.$$

Hence there are 2 matrices U corresponding to this Lorentz transformation given by

$$U = \pm\begin{pmatrix}\frac{\sqrt{\gamma_0+1}}{2} & -\frac{\sqrt{\gamma_0-1}}{2}\\ -\frac{\sqrt{\gamma_0-1}}{2} & \frac{\sqrt{\gamma_0+1}}{2}\end{pmatrix}$$

C. Example 3: Singular Lorentz Transformation

Find U corresponding to the two parameter Lorentz transformation,

$$\begin{aligned}t'-z' &= t-z,\\ x'+iy' &= x+iy+w(t-z),\\ t'+z' &= t+z+w(x-iy)+\bar{w}(x+iy)+w\bar{w}(t-z).\end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta}+\bar{\alpha}\beta=0, \tag{37a}$$
$$i(\alpha\bar{\beta}-\bar{\alpha}\beta)=0, \tag{37b}$$
$$-\alpha\bar{\alpha}+\beta\bar{\beta}=-1, \tag{37c}$$
$$\alpha\bar{\alpha}+\beta\bar{\beta}=1. \tag{37d}$$

Then Eqn.(37a) and (37b) imply $\alpha\bar{\beta}=0$ so $\alpha=0$ or $\beta=0$. Also Eqn.(37c) and (37d) imply $2\beta\bar{\beta}=0$ so $\beta=0$ and $\alpha\bar{\alpha}=1$ so $\alpha\neq 0$. Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$1=\alpha\bar{\delta}+\beta\bar{\gamma}, \tag{38a}$$
$$1=\alpha\bar{\delta}-\beta\bar{\gamma}, \tag{38b}$$
$$-w=-\alpha\bar{\gamma}+\beta\bar{\delta}, \tag{38c}$$
$$w=\alpha\bar{\gamma}+\beta\bar{\delta}. \tag{38d}$$

With $\beta=0$, Eqn.(38a) implies $\alpha\bar{\delta}=1$, so multiply by $\bar{\alpha}$ and use $\alpha\bar{\alpha}=1$ to obtain $\delta=\alpha$. Also Eqn.(38c) and (38d) imply $\alpha\bar{\gamma}=w$ so $\gamma=\bar{w}\alpha$ since $\alpha\bar{\alpha}=1$. Now use the fact that $\det(U)=1$, which implies

$$\alpha\delta-\beta\gamma=1,$$

thus $\alpha^2=1$ as $\beta=0$ and $\gamma=\alpha$. So $\alpha=\pm 1$, thus $\gamma=\bar{w}$. Hence there are 2 matrices U corresponding to this two parameter transformation, namely

$$U=\pm\begin{pmatrix}1 & 0\\ \bar{w} & 1\end{pmatrix}$$

It is clear that there will always be two matrices, $\pm U$ corresponding to every Lorentz transformation, since if U satisfies $A(\vec{x}')=UA(\vec{x})U^{\dagger}$ then so does $-U$. Hence there is a 2 to 1 correspondence between the elements of $SL(2,\mathbb{C})$ and the proper orthochronous Lorentz transformation, with A and A' corresponding to differenent coordinates of Minkowskian space-time and $\pm U$ corresponding to proper orthochronous Lorentz transformations.

V. THE FRACTIONAL LINEAR TRANSFORMATION

In this section a fractional linear transformation is derived from extending the stereographic projection between a 2-sphere and the complex plane to Minkowskian space-time. The fixed points of this transformation are determined and found to be directly related to the null directions of a Lorentz transformation. Some examples of fractional linear transformations and their associated null directions are then given.

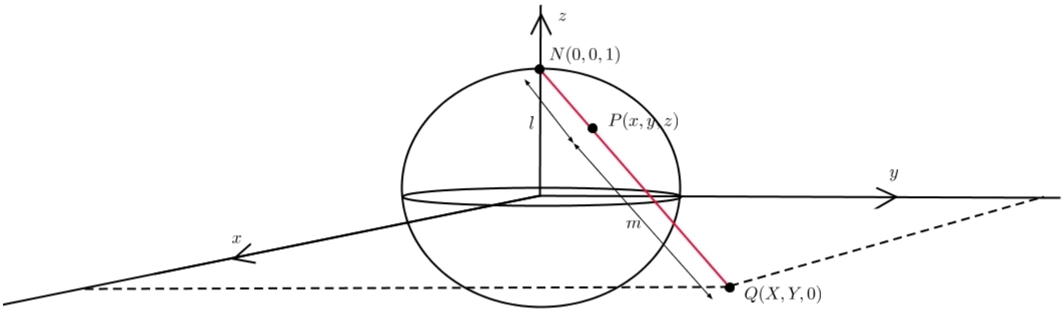
A. Stereographic Projection and the Extended Complex Plane

First, a one to one mapping between the 2-sphere, $\mathbb{S}^2$ and the extended complex plane $\hat{\mathbb{C}}$ is constructed. *Stereographic projection* is the mapping of points on a sphere to points on a plane. In $\mathbb{R}^3$ with rectangular Cartesian coordinates (x,y,z), consider the unit sphere with centre $(0,0,0)$, defined by

$$\mathbb{S}^2\subset\mathbb{R}^3 : x^2+y^2+x^2=1.$$

The projection $P\to Q$ is a stereographic projection, see Fig.(3). A relationship between $(X,Y,0)$ and (x,y,z) is constructed as follows. P is subdivided into the line segment NQ in some ratio, $l:m$ say.
By coordinate geometry

FIG. 3: *The projection from the unit 2-sphere to the x,y-plane is called a Stereographic projection. The point P on the shpere is mapped to the point Q on the plane by translation along the line joining the points N and P*



$$
\begin{aligned}
x &= \frac{lX + mO}{l+m} = \frac{lX}{l+m}, \\
y &= \frac{lY + mO}{l+m} = \frac{lY}{l+m}, \\
z &= \frac{l.0 + m.1}{l+m} = \frac{m}{l+m}.
\end{aligned}
$$

Where O is the position of the origin. This implies that

$$
\begin{aligned}
1 - z &= \frac{l}{l+m}, \\
x &= (1-z)X, \\
y &= (1-z)Y.
\end{aligned}
$$

It is also known that $x^2 + y^2 + z^2 = 1$, so using this relation it is clear that

$$
\begin{aligned}
x^2 + y^2 = 1 - z^2 &= (1-z)(1+z), \\
(1-z)^2(X^2+Y^2) &= (1-z)(1+z).
\end{aligned}
$$

If the point N is excluded, i.e. $z \neq 1$ then divide by $(1-z)^2$ to obtain

$$X^2 + Y^2 = \frac{1+z}{1-z},$$

and Rearrange to find

$$z = \frac{X^2 + y^2 - 1}{X^2 + y^2 + 1}. \tag{39}$$

Define $\zeta = X + iY$ and rewrite Eqn.(39) to see that

$$z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1},$$

Which implies

$$1 - z = \frac{2}{\zeta\bar{\zeta} + 1}.$$

So relations for $(x, y, z) \in \mathbb{S}^2 \backslash \{N\}$ have been obtained in terms of ζ, such that

$$x + iy = \zeta(1 - z) = \frac{2\zeta}{\zeta\bar{\zeta} + 1}, \tag{40}$$

$$z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}. \tag{41}$$

Hence the points on $\mathbb{S}^2 \backslash \{N\}$ are labelled by complex numbers $\zeta \in \mathbb{C}$. If a point $\zeta = \infty$, called the point at infinity of $\mathbb{C}$, is allowed then the following limits hold

$$\begin{aligned} x + iy &= \frac{2/\bar{\zeta}}{1 + 1/\zeta\bar{\zeta}} \to 0, \text{ as } \zeta \to \infty, \\ z &= \frac{1 - 1/\zeta\bar{\zeta}}{1 + 1/\zeta\bar{\zeta}} \to 1, \text{ as } \zeta \to \infty. \end{aligned}$$

Then $N = (0, 0, 1)$ corresponds to $\zeta = \infty$. Thus in this way there is a one to one correspondence between the points of $\mathbb{S}^2$ and the points of the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, which is the usual complex plane with the point at infinity added. Since $\mathbb{S}^2$ has finite surface area, and is therefore called a *compact manifold*, the identification of the points of $\hat{\mathbb{C}}$ with the points of $\mathbb{S}^2$ is called the *compactification* of $\hat{\mathbb{C}}$.

The pair $(\zeta, \bar{\zeta})$ are called the *stereographic coordinates* on $\mathbb{S}^2 \backslash \{N\}$. How are they related to the polar angles θ and ϕ? To investigate this write the usual spherical polar coordinates in terms of ζ. First it is known that

$$\begin{aligned} x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{aligned}$$

So by Eqn.(41) it is clear that

$$\begin{aligned} z = \cos\theta &= \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1} \\ \zeta\bar{\zeta}\cos\theta + \cos\theta &= \zeta\bar{\zeta} - 1 \\ \zeta\bar{\zeta} &= \frac{1 + \cos\theta}{1 - \cos\theta} = \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} \\ \zeta\bar{\zeta} &= \cot^2(\theta/2). \end{aligned}$$

Now use Eqn.(40) to obtain

$$\begin{aligned} \sin\theta(\cos\phi + i\sin\phi) &= \frac{2\zeta}{\cot^2(\theta/2) + 1}, \\ 2\sin(\theta/2)\cos(\theta/2)e^{i\phi} &= 2\zeta\sin^2(\theta/2), \\ \zeta &= e^{i\phi}\cot(\theta/2). \end{aligned}$$

This makes sense as if $\zeta = \infty$ then $\theta = 0$ as one would expect. In summary the following coordinate transformations have been constructed

$$\vec{n} = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{42}$$

$$= \left(\frac{\bar{\zeta} + \zeta}{\bar{\zeta}\zeta + 1}, i\frac{\bar{\zeta} - \zeta}{\bar{\zeta}\zeta + 1}, \frac{\bar{\zeta}\zeta - 1}{\bar{\zeta}\zeta + 1}\right), \tag{43}$$

where here $\vec{n}$ is a unit vector in $\mathbb{R}^3$ such that $\vec{n} \cdot \vec{n} = 1$.

B. Extension to Minkowskian Sapce-Time

These results are now extended to Minkowskian space-time to derive an expression for the fractional Linear transformation. Let $\vec{x} = (x, y, z, t)$ be a point on the future null cone with origin $(0, 0, 0, 0)$. Denote the future null cone as N^+, so that

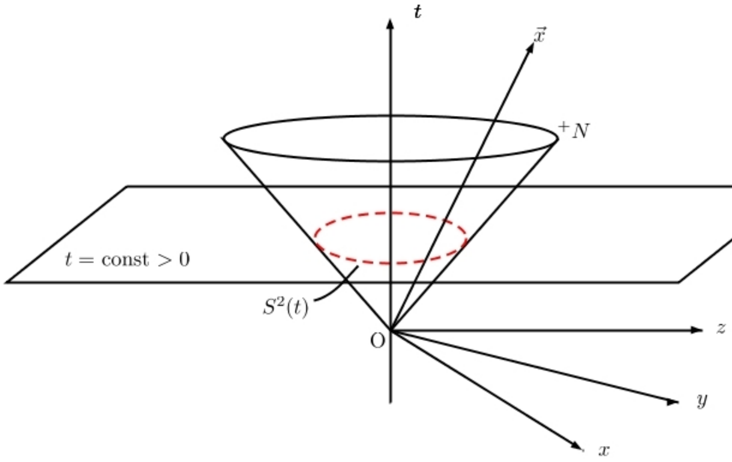
$$N^+ : x^2 + y^2 + z^2 - t^2 = 0, \text{ for } t > 0,$$

as all the vectors in the null cone have a Lorentz quadratic form equal to zero by definition. The intersection with the space-like hypersurface $t = \text{const} > 0$ is a 2-sphere denoted by

$$\mathbb{S}^2(t) : x^2 + y^2 + z^2 = t^2 = \text{const}, \tag{44}$$

see Fig.(4). There is a generator of N^+ passing through each point of $\mathbb{S}^2(t)$, they are the null geodesics tangent to N^+ and passing through the point $(0, 0, 0, 0)$. Hence the points of $\mathbb{S}^2(t)$, denoted by (θ, ϕ) or ζ, label the *generators* of N^+.

FIG. 4: *Future null cone of Minkowskian space-time. Shown in red is the intersection between the future null cone and the hyperplane t = const. This surface is the sphere in three dimensions from which a stereographic projection was derived earlier.*



For every generator and any $t > 0$ it is clear that

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = 1.$$

This is just the definition of a 2-sphere, as in Eqn.(44), normalised with respect to a constant t. Hence we can write $\vec{x}$ in terms of the coordinate transformation of (43)

$$\begin{aligned}\vec{x} &= t(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 1),\\ &= t\left(\frac{\bar{\zeta}+\zeta}{\bar{\zeta}\zeta+1}, i\frac{\bar{\zeta}-\zeta}{\bar{\zeta}\zeta+1}, \frac{\bar{\zeta}\zeta-1}{\bar{\zeta}\zeta+1}, 1\right).\end{aligned} \tag{45}$$

Where a 4^{th} component is included as $\vec{x}$ is an element of Minkowskian space-time. Thus the direction of $\vec{x}$ is determined explicitly by (θ, ϕ) or ζ as $\vec{x}$ has the same form as the unit vector of Eqn.(43). Note that all possible directions of $\vec{x}$ on N^+ are covered if $\zeta \in \hat{\mathbb{C}}$. Now the Lorentz transformation $\vec{x} \to \vec{x}'$ is investigated, it takes the form

$$\vec{x} \to \vec{x}' = t'\left(\frac{\bar{\zeta}'+\zeta'}{\bar{\zeta}'\zeta'+1}, i\frac{\bar{\zeta}'-\zeta'}{\bar{\zeta}'\zeta'+1}, \frac{\bar{\zeta}'\zeta'-1}{\bar{\zeta}'\zeta'+1}, 1\right).$$

The null direction ζ is transformed to the null direction ζ', where the relation between them must be determined. Construct the matrix $A(\vec{x})$ as in Eqn.(26),

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix},$$

where $c_0 = 2t/(\zeta\bar{\zeta}+1) \in \mathbb{R}^2$. Note that as $\vec{x}$ is a null vector $\det(A(\vec{x})) = 0$. Thus the transformed matrix $A(\vec{x}')$ is given similarly as

$$A(\vec{x}') = c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix}.$$

Now, as in the examples in section (IV) we determine the special linear matrix U such that

$$A(\vec{x}') = UA(\vec{x})U^{\dagger} \tag{46}$$

As before, this matrix equation is written component wise as

$$c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta}' & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

Then three separate relations between ζ and ζ' are obtained

$$c_0{}' = c_0(\alpha\bar{\alpha} + \alpha\bar{\beta}\zeta + \bar{\alpha}\beta\bar{\zeta} + \beta\bar{\beta}\zeta\bar{\zeta}), \tag{47}$$
$$c_0{}'\zeta' = c_0(\alpha\bar{\gamma} + \alpha\bar{\delta}\zeta + \bar{\gamma}\beta\bar{\zeta} + \beta\bar{\delta}\zeta\bar{\zeta}), \tag{48}$$
$$c_0{}'\zeta'\bar{\zeta}' = c_0(\gamma\bar{\gamma} + \gamma\bar{\delta}\zeta + \bar{\gamma}\delta\bar{\zeta} + \delta\bar{\delta}\zeta\bar{\zeta}). \tag{49}$$

Using Eqns.(47) and (48) and factorising to obtain

$$\zeta' = \frac{c_0{}'\zeta'}{c_0{}'} = \frac{\alpha(\bar{\gamma} + \bar{\delta}\zeta) + \beta\bar{\zeta}(\bar{\gamma} + \bar{\delta}\zeta)}{\alpha(\bar{\alpha} + \bar{\beta}\zeta) + \beta\bar{\zeta}(\bar{\alpha} + \bar{\beta}\zeta)}$$

Thus

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)}, \tag{50}$$

with $\alpha\delta - \beta\gamma = 1$ as before. This is a *fractional linear transformation* of the extended complex plane $\hat{\mathbb{C}}$. There is a one to one correspondence here between proper, orthochronous Lorentz transformations and fractional linear transformations of the extended complex plane. This is because the matrices $\pm U$ both satisfy Eqn.(46) as in the previous section, but now both matrices give the same transformation as the signs will cancel in the fractional transformation.

C. Fixed points and Their Associated Null Directions

A given Lorentz transformation is equivalent to known α, β , γ and δ parameters module a sign and therefore gives an explicit fractional linear transformation. For a given Lorentz transformation a *fixed point* of the corresponding fractional linear transformation corresponds to an invariant null direction. The fixed points ζ satisfy the relation $\zeta' = \zeta$. Thus from Eq.(50)

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)} = \zeta,$$
$$\bar{\beta}\zeta^2 + (\bar{\alpha} - \bar{\delta})\zeta - \bar{\gamma} = 0. \tag{51}$$

Clearly this is a quadratic equation over the field $\mathbb{C}$, thus it has two roots in general. The non-singular case is when these roots do not coincide, hence a Lorentz transformation does indeed leave two null directions invariant in general. If Eqn.(51) has only one root then the corresponding Lorentz transformation leaves one null direction invariant, this is the singular case.

Consider Eqn.(51) again. Divide by ζ^2 to obtain

$$\bar{\beta} + (\bar{\alpha} - \bar{\delta})\zeta^{-1} - \bar{\gamma}\zeta^{-2} = 0.$$

Hence $\zeta = \infty$ is a solution of this equation if $\beta = 0$. If $\zeta = \infty$ then $\vec{x}$ is given by $\vec{x} = t(0,0,1,1)$ by Eqn.(45). Thus it is clear that this corresponds to the null direction $z = t$. Compare this to Example 3, Section (IV C). Here β is zero AND the null direction is $z = t$ as expected. If $\zeta = 0$ is a solution to Eqn.(51) it is required that $\gamma = 0$, thus $\vec{x} = (0,0,-1,1)$. So it is predicted that a Lorentz transformation with a special linear matrix of the form

$$U = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},$$

will leave the $z = -t$ null direction invariant. The example in section (V D) illustrates a case where the null direction is found to be $x = \pm t$, due to the form of the matrix U. The choice of the components of the matrix A in Eqn.(26) determine which direction is the null direction. The initial choice of A gives special significance to the z component, this of course is arbitrary. In Appendix (A), the example in Section (IV C) is repeated for the case when special significance is given to x.

D. Example: Standard Lorentz Transformation

Continuing on from Example 2, section (IV B), where α, β, γ and δ were determined. ζ' can now be expressed as a fractional linear transformation by Eqn.(50)

$$\zeta' = \frac{-\sqrt{\gamma_0 - 1} + \sqrt{\gamma_0 + 1}\zeta}{\sqrt{\gamma_0 + 1} - \sqrt{\gamma_0 - 1}\zeta}.$$

If the condition $\zeta' = \zeta$ is imposed then

$$\sqrt{\gamma_0 - 1}(\zeta^2 - 1) = 0,$$
$$\zeta = \pm 1.$$

In the $\zeta = +1$ case, $\vec{x} = t(1,0,0,1)$ and the invariant direction is $x = t$. Similarly in the $\zeta = -1$ case $\vec{x} = t(-1,0,0,1)$ and the invariant direction is $x = -t$.

VI. INFINITESIMAL LORENTZ TRANSFORMATIONS

In this section the infinitesimal Lorentz transformations are derived. These transformations can be used to determine the equations of motion of a particle as it travels through Minkowskian space-time. It is interesting to write these equations in terms of the particles 3-velocity as the form of the Lorentz force emerges. It is shown that the Lorentz force must depend on the particle's 3-velocity in a special way in order to be compatible with special relativty. Then the fractional linear transformation associated with the infinitesimal transformation is derived to find that the resulting radiation field must be a pure radiation field. Finally the conditions for a pure radiation field are extended to Minkowskian space-time.

The Infinitesimal transformations are Lorentz transformations that are small perturbations of the identity transformation and so $U \in SL(2,\mathbb{C})$ has the form

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix}, \tag{52}$$

where $a, b, c, f \in \mathbb{C}$ and ϵ is a small real parameter. Here terms of order ϵ^2 will be neglected. As $U \in SL(2,\mathbb{C})$ its determinant is calculated as

$$\det(U) = 1 + O(\epsilon^2).$$

Using this it is possible to obtain a relation between f and a

$$\begin{aligned} (1+\epsilon a)(1+\epsilon f) - \epsilon^2 bc &= 1 + O(\epsilon^2), \\ 1 + \epsilon(a+f) &= 1 + O(\epsilon^2), \\ f &= -a + O(\epsilon). \end{aligned}$$

Hence

$$U = \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix}. \tag{53}$$

Now the explicit infinitesimal Lorentz transformations are calculated as in section IV, by substituting U into

$$A(\vec{x}') = UA(\vec{x})U^{\dagger}.$$

Writing this out in component form gives

$$\begin{aligned} \begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} 1+\epsilon\bar{a} & \epsilon\bar{c} \\ \epsilon\bar{b} & 1-\epsilon\bar{a} \end{pmatrix}, \\ &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix} \begin{pmatrix} (t-z)(1+\epsilon\bar{a}) + \epsilon\bar{b}(x+iy) & (t-z)\epsilon\bar{c} + (1-\epsilon\bar{a})(x+iy) \\ (x-iy)(1+\epsilon\bar{a}) + \epsilon\bar{b}(t+z) & (x-iy)\epsilon\bar{c} + (1-\epsilon\bar{a})(t+z) \end{pmatrix}. \end{aligned}$$

This then implies the three relations

$$t'-z' = t-z+\epsilon(a+\bar{a})(t-z)+\epsilon(b+\bar{b})x+i\epsilon(\bar{b}-b)y+O(\epsilon^2), \tag{54}$$

$$t'+z' = t+z-\epsilon(a+\bar{a})(t+z)+\epsilon(c+\bar{c})x+i\epsilon(c-\bar{c})y+O(\epsilon^2), \tag{55}$$

$$x'+iy' = x+iy+\epsilon(a-\bar{a})(x+iy)+\epsilon(b+\bar{c})t+\epsilon(b-\bar{c})z+O(\epsilon^2). \tag{56}$$

As $a, b, c \in \mathbb{C}$, set

$$a = a_1 + ia_2,\ b = b_1 + ib_2,\ c = c_1 + ic_2.$$

Then subbing these into the above equations, eliminating t and z respectively from Eqn.(54) and (55) and taking real and imaginary parts of Eqn.(56) to obtain

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -2a_2 & (b_1-c_1) & (b_1+c_1) \\ 2a_2 & 0 & (b_2+c_2) & (b_2-c_2) \\ -(b_1-c_1) & -(b_2+c_2) & 0 & -2a_1 \\ (b_1+c_1) & (b_2-c_2) & -2a_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + O(\epsilon^2). \tag{57}$$

The above 4×4 matrix will be denoted as $L^i{}_j$, so that Eqn.(57) can be written simply as

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2), \tag{58}$$

where $\bar{x}^i = (x', y', z', t')$. It will become clear that L_{ij} is the *electromagnetic tensor* which is usually denoted by F_{ij}. It is necessary to check that the Lorentz invariance of the quadratic form still holds.

$$\begin{aligned} {x'}^2 + {y'}^2 + {z'}^2 - {t'}^2 &= x^2 + y^2 + z^2 - t^2 - 4\epsilon a_2 xy + 2\epsilon(b_1 + c_1)xt \\ &+ 2\epsilon(b_1 - c_1)xz + 4\epsilon a_2 yx + 2\epsilon(b_2 - c_2)yt \\ &+ 2\epsilon(b_2 + c_2)yz - 4\epsilon a_1 zt + 2\epsilon(c_1 - b_1)zx \\ &- 2\epsilon(c_2 + b_2)zy + 4\epsilon a_1 tz - 2\epsilon(c_1 + b_1)tx \\ &- 2\epsilon(b_2 - c_2)ty + O(\epsilon^2) \\ &= x^2 + y^2 + z^2 - t^2 + O(\epsilon^2) \end{aligned}$$

Hence this transformation is still a Lorentz Transformation if we neglect terms of order ϵ^2.

Consider the time-like world line of a particle in Minkowskian space-time $x^i = x^i(s)$. If s is arc length or proper time then $v^i(s) = \frac{dx^i}{ds}$ is the unit tangent vector field. It is clear that $v^i(s)$ must be time-like as $x^i(s)$ is time-like, thus

$$\eta_{ij} v^i v^j = -1,$$

where $\eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ is the metric of Minkowskian space-time. This implies that

$$(v^1)^2 + (v^2)^2 + (v^3)^2 - (v^4)^2 = -1.$$

Now consider taking a step along the world line of the particle. Define $\bar{s} = s + \alpha$, where α is some real parameter, so that $v^i(s + \alpha) := \bar{v}^i(\bar{s})$, see Fig.(5).

Hence we also have

$$(\bar{v}^1)^2 + (\bar{v}^2)^2 + (\bar{v}^3)^2 - (\bar{v}^4)^2 = -1,$$

and so $v^i(s)$ and $\bar{v}^i(\bar{s})$ are related by a Lorentz transformation. In particular $v^i(s + \epsilon)$ and $v^i(s)$ are related by an infinitesimal Lorentz Transformation given by Eqn.(58),

$$v^i(s + \epsilon) = v^i(s) + \epsilon L^i{}_j(s) v^j(s) + O(\epsilon^2). \tag{59}$$

Rearranging to obtain

$$\frac{v^i(s + \epsilon) - v^i(s)}{\epsilon} = L^i{}_j(s) v^j(s) + O(\epsilon). \tag{60}$$

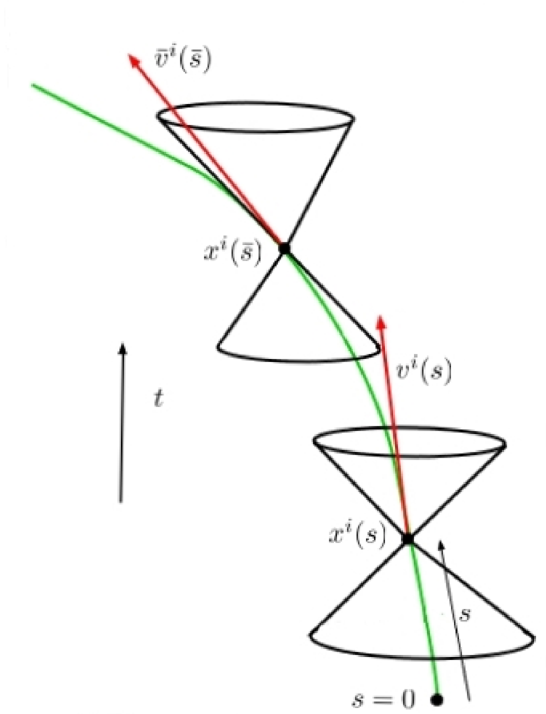
Now taking the limit as the infinitesimal step, ϵ goes to zero to obtain a continuous differential equation,

$$\frac{dv^i}{ds} = L^i{}_j(s) v^j(s). \tag{61}$$

This equation determines the trajectory of the particle through Minkowskian space-time. In terms of x this is equivalent to

$$\frac{d^2 x^i}{ds^2} = L^i{}_j(s) \frac{dx^j}{ds}.$$

FIG. 5: *The trajectory of a particle moving through Minkowskian space time. The position along the path is parameterised by* s*. A step along the curve from* s *to* $s+\alpha$ *is shown. This step is found to be a Lorentz transformation as both* $v^i(s)$ *and* $\bar{v}^i(\bar{s})$ *have equal Lorentz quadratic forms.*



A. The Lorentz Force in Special Relativity

Now the form of the Lorentz force can be derived by writing these equations in terms of the particles 3-velocity given by

$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

Start by using the chain rule on v^i,

$$v^i = \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1\right)\frac{dt}{ds}. \tag{62}$$

Now determine the first integral of v^i, which is equal to -1 as v^i is time-like,

$$\begin{aligned} -1 = \eta_{ij}v^iv^j &= \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - 1\right\}\left(\frac{dt}{ds}\right)^2, \\ &= (u^2-1)\left(\frac{dt}{ds}\right)^2 \end{aligned}$$

as the first three terms are just the scalar product of the three velocity, where $u = |\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$. Rearrage this and define

$$\frac{dt}{ds} = (1-u^2)^{-1/2} := \gamma(s).$$

Thus from Eqn.(62)

$$v^i = \gamma(u)(\vec{u}, 1) \tag{63}$$

It is now convenient to display Eqn.(61) as two equations denoting the spacial part and the temporal part, in terms of γ and u. Again using the chain rule to obtain

$$\frac{dt}{ds}\frac{dv^i}{dt} = L^i{}_j v^j.$$

This then implies that

$$\begin{aligned} \gamma(u)\frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_j v^j, \\ \gamma(u)\frac{d}{dt}\gamma(u) &= L^4{}_j v^j, \end{aligned} \tag{64}$$

as $v^i = \gamma(u)(\vec{u}, 1)$. Here we have used the usual convention that Greek indices denote the sum over the spacial indices only, thus $\alpha = 1, 2, 3$. Now Eqn.(63) can be used to rewrite the $L^i{}_j$ coefficients to get

$$\begin{aligned} L^\alpha{}_j v^j &= \gamma(u)(L^\alpha{}_\beta u^\beta + L^\alpha{}_4) \\ L^4{}_j v^j &= \gamma(u)(L^4{}_\alpha u^\alpha) \end{aligned} \tag{65}$$

where $L^4{}_4 = 0$ from Eqn.(57). Putting together Eqns.(64) and (65) to obtain differential equations for the spacial and temporal coordinates in terms of the particles 3-velocity,

$$\begin{aligned} \frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_\beta u^\beta + L^\alpha{}_4, \\ \frac{d\gamma(u)}{dt} &= L^4{}_\alpha u^\alpha. \end{aligned}$$

These can be written explicitly as four equations

$$\frac{d}{dt}(\gamma(u)u^{(1)}) = -2a_2u^{(2)} + (b_1 - c_1)u^{(3)} + b_1 + c_1, \tag{66}$$

$$\frac{d}{dt}(\gamma(u)u^{(2)}) = 2a_2u^{(1)} + (b_2 + c_2)u^{(3)} + b_2 - c_2, \tag{67}$$

$$\frac{d}{dt}(\gamma(u)u^{(3)}) = -(b_1 - c_1)u^{(1)} - (b_2 + c_2)u^{(2)} - 2a_1, \tag{68}$$

$$\frac{d\gamma(u)}{dt} = (b_1 + c_1)u^{(1)} + (b_2 - c_2)u^{(2)} - 2a_1u^{(3)}. \tag{69}$$

Now define the 3-vectors $\vec{P}$ and $\vec{Q}$ such that

$$\begin{aligned} \vec{P} &= (b_1 + c_1, b_2 - c_2, -2a_1), \\ \vec{Q} &= (b_2 + c_2, -(b_1 - c_1), -2a_2). \end{aligned}$$

It is clear that Eqns.(66)-(69) can be written in terms of $\vec{P}$ and $\vec{Q}$ as follows,

$$\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{P} + \vec{u} \times \vec{Q}, \tag{70}$$

$$\frac{d\gamma}{dt} = \vec{u} \cdot \vec{P}. \tag{71}$$

Note that these expressions look remarkably like the Lorentz force in electromagnetism. It is easily shown that Eqn.(70) implies Eqn.(71). To see this, first take the scalar product of Eqn.(70) with $\vec{u}$.

$$\vec{u}\cdot\frac{d}{dt}(\gamma(u)\vec{u}) = \vec{u}\cdot\vec{P} + \vec{u}\cdot(\vec{u}\times\vec{Q}), \tag{72}$$

$$\gamma\vec{u}\frac{d\vec{u}}{dt} + \vec{u}\cdot\vec{u}\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P}, \tag{73}$$

by using the product rule on the left hand side and as the scalar product of the cross product with a repeated vector is zero in the third term. The quantity γ is known in terms of u, so it is possible to write the derivative in the first term as a derivative of γ as follows,

$$\gamma^{-2} = 1 - u^2 = 1 - \vec{u}\cdot\vec{u},$$

$$-2\gamma^{-3}\frac{d\gamma}{dt} = -2\vec{u}\cdot\frac{d\vec{u}}{dt}.$$

Subbing this result back into Eqn.(73) to obtain

$$\gamma\gamma^{-3}\frac{d\gamma}{dt} + u^2\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P},$$

$$(\gamma^{-2} + u^2)\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P}.$$

Therefore

$$\frac{d\gamma}{dt} = \vec{u}\cdot\vec{P},$$

and so it is shown that Eqn.(71) is a generalization of Eqn.(70) and contains no new information.

The dependence of the 3-force acting on a particle as shown by Eqn.(70), depends in general on the particles 3-velocity $\vec{u}$ in a special way, in order to be compatible with Special Relativity. Thus in particular *the Lorentz 3-force acting on a particle of rest mass m and charge q must depend upon $\vec{u}$ as in Eqn.(70) to be compatible with special Relativity.* So the Lorentz force of electromagnetism is a special case of a charged particle moving through Minkowskian space-time along a world-line of infinitesimal Lorentz transformations. In this case, make the identifications

$$\vec{P} = \frac{q}{m}\vec{E}, \qquad \vec{Q} = \frac{q}{m}\vec{B}, \tag{74}$$

where $\vec{E}$ is the external electric field and $\vec{B}$ is the external magnetic field in which the particle is moving. Then Eqn.(70) takes the familiar form

$$m\frac{d}{dt}(\gamma(u)\vec{u}) = q(\vec{E} + \vec{u}\times\vec{B}).$$

Or in the case of a slow moving particle with $\gamma \approx 1$

$$m\vec{a} = q(\vec{E} + \vec{u}\times\vec{B}).$$

B. Fractional Linear Transformations of the Infinitesimal Linear Transformation

Recall that the fractional linear transformation constructed in section (V) had a one to one correspondence with proper orthochronous Lorentz transformations, and the fixed points of the fractional transformation corresponded to null directions of the Lorentz transformation. As in Eqn.(50), section (V B) construct the fractional linear transformation of the special linear $SL(2,\mathbb{C})$ matrix U for the infinitesimal Lorentz transformation given in Eqn.(53). It is found to be

$$\zeta' = \frac{\zeta + \epsilon(\bar{c} - \bar{a}\zeta) + O(\epsilon^2)}{1 + \epsilon(\bar{a} + \bar{b}\zeta) + O(\epsilon^2)}.$$

Then the fixed points are given when $\zeta' = \zeta$, which implies,

$$\begin{aligned} \epsilon\bar{b}\zeta^2 + (\epsilon\bar{a} + \epsilon\bar{a})\zeta - \epsilon\bar{c} &= O(\epsilon^2), \\ \bar{b}\zeta^2 + 2\bar{a}\zeta - \bar{c} &= O(\epsilon). \end{aligned} \tag{75}$$

Of interest here are the singular Lorentz transformations, so it is required that the roots of this quadratic are the same. Thus the usual discriminant is set to zero,

$$4\bar{a}^2 + 4\bar{b}\bar{c} = 0.$$

Therefore,

$$a^2 + bc = 0. \tag{76}$$

Write these equations out explicitly and equate real and imaginary coefficients to obtain

$$a_1^2 - a_2^2 + b_1c_1 - b_2c_2 = 0, \tag{77}$$

$$2a_1a_2 + b_2c_1 + b_1c_2 = 0. \tag{78}$$

It is interesting to write these equations in terms of the electric and magnetic vectors, namely $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$. The relation between the a,b and c and the $\vec{B}$ and $\vec{E}$ coefficients comes from Eqn.(74), where the factor q/m has been suppressed for convenience.

$$a_1 = -\frac{1}{2}E^3, \qquad b_2 = \frac{1}{2}(E^2 + B^1), \qquad c_1 = \frac{1}{2}(E^1 + B^2), \tag{79}$$

$$a_2 = -\frac{1}{2}B^3, \qquad b_1 = \frac{1}{2}(E^1 - B^2), \qquad c_2 = \frac{1}{2}(B^1 - E^2). \tag{80}$$

So Eqn.(77) implies

$$\frac{1}{4}(E^3)^2 - \frac{1}{4}(B^3)^2 + \frac{1}{4}((E^1)^2 - (B^2)^2) + \frac{1}{4}((E^2)^2 - (B^1)^2) = 0,$$
$$(E^1)^2 + (E^2)^2 + (E^3)^2 = (B^1)^2 + (B^2)^2 + (B^3)^2.$$

Thus it is clear that

$$|\vec{E}|^2 = |\vec{B}|^2 \tag{81}$$

Similarly, Eqn.(78) implies

$$\frac{1}{2}E^3B^3 + \frac{1}{4}(E^1E^2 + E^1B^1 + E^2B^2 + B^1B^2) + \frac{1}{4}(-E^1E^2 + E^1B^1 + E^2B^2 - B^1B^2) = 0,$$
$$E^3B^3 + E^1B^1 + E^2B^2 = 0.$$

So it is shown that

$$\vec{E} \cdot \vec{B} = 0. \tag{82}$$

The above Eqns.(81) and (82) are the (Lorentz invariant) conditions that the electromagnetic field in which the charged particle is moving is a pure radiation field. Thus in conclusion, *if the world line of the charged particle is generated by infinitesimal singular Lorentz transformations then the particle is moving in a pure radiation electromagnetic field.* An example of such a field would be monochromatic plane electromagnetic waves, or the formulas above would pick up a factor of ω if the waves were not monochromatic.

C. Pure Radiation Field Conditions in Minkowskian Space-Time

Eqns.(81) and (82) are the pure radiation field conditions in physical space, $\mathbb{R}^3$. It is interesting to see what form these equations take in Minkowskian space-time. To do this, solve the quadratic equation in Eqn.(75) for the case where the roots coincide, to find the single fixed point of the system. It is clear that

$$\zeta = -\frac{\bar{a}}{\bar{b}}, \tag{83}$$

is the fixed point. Then determine the corresponding null direction k^i as was done in Eqn.(43), Section (V A)

$$k^i = \left(\bar{\zeta} + \zeta, i(\bar{\zeta} - \zeta), \bar{\zeta}\zeta - 1, \bar{\zeta}\zeta + 1\right). \tag{84}$$

Now relate k^i to $L_{ij} = \eta_{ij} L^k{}_j$. From the relations in (79) and (80) it is clear that

$$L_{ij} = \frac{q}{m} \begin{pmatrix} 0 & B^3 & -B^2 & E^1 \\ -B^3 & 0 & B^1 & E^2 \\ B^2 & -B^1 & 0 & E^3 \\ -E^1 & -E^2 & -E^3 & 0 \end{pmatrix} = -(L_{ji}).$$

Using this matrix and Eqn.(61) for an arbitrary 4-vector it is possible to derive the transformation relations for the electromagnetic fields under the standard Lorentz transformation, see Appendix (B). The dual of this quantity is defined by

$$^*L_{ij} = \frac{1}{2}\epsilon_{ijkl} L^{kl},$$

where ϵ_{ijkl} is the Levi-Civita permutation symbol in 4 dimensions. To work out the components of $^*L_{ij}$, first use the raising and lowering of operators with the metric $\eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ to show that

$$L_{\alpha\beta} = L^\alpha{}_\beta, \qquad L_{4j} = -L^4{}_j.$$

Where $\alpha, \beta = 1, 2, 3$ and $j = 1, 2, 3, 4$. Thus the components of $^*L_{ij}$ are calculated as follows

$$\begin{aligned} ^*L_{12} &= \epsilon_{1234} L^{34} = L^{34} = -L^3{}_4 = -E^3, \\ ^*L_{13} &= \epsilon_{1324} L^{24} = -L^{24} = L^2{}_4 = E^2, \\ ^*L_{14} &= \epsilon_{1423} L^{23} = L^{23} = L^2{}_3 = B^1, \\ ^*L_{23} &= \epsilon_{2314} L^{14} = L^{14} = -L^1{}_4 = -E^1, \\ ^*L_{24} &= \epsilon_{2413} L^{13} = -L^{13} = -L^1{}_3 = B^2, \\ ^*L_{34} &= \epsilon_{3412} L^{12} = L^{12} = L^1{}_2 = B^3. \end{aligned}$$

Now construct the matrix

$$\begin{aligned} \mathcal{L}_{ij} := (L_{ij} + i^*L_{ij}) =& \frac{q}{m} \begin{pmatrix} 0 & B^3 - iE^3 & -B^2 + iE^2 & E^1 + iB^1 \\ -B^3 + iE^3 & 0 & B^1 - iE^1 & E^2 + iB^2 \\ B^2 - iE^2 & -B^1 + iE^1 & 0 & E^3 + iB^3 \\ -E^1 - iB^1 & -E^2 - iB^2 & -E^3 - iB^3 & 0 \end{pmatrix} \\ =& \frac{q}{m} \begin{pmatrix} 0 & 2ia & (b-c) & (b+c) \\ -2ia & 0 & -i(b+c) & -i(b-c) \\ -(b-c) & i(b+c) & 0 & -2a \\ -(b+c) & i(b-c) & 2a & 0 \end{pmatrix} \end{aligned}$$

Most of what follows could be obtained by just using L_{ij} instead of $(L_{ij} + i^* L_{ij})$, but by using this quantity the information contained in L_{ij} is more readily available. This is a standard mathematical trick to simplify later calculations. Now calculate the various components of the product $(L_{ij} + i^* Lij)k^j$, the first is done as an example.

$$\begin{aligned}(L_{1j} + i^* L_{1j})k^j =& \frac{q}{m}(2iak^2 + (b-c)k^3 + (b+c)k^4),\\ =& \frac{q}{m}(-2a(\zeta - \bar{\zeta}) + 2b\zeta\bar{\zeta} + 2c),\\ =& \frac{2q}{m}(-a\bar{\zeta} + a\zeta + b\zeta\bar{\zeta} + c),\\ =& \frac{2q}{mb}(a^2 + bc),\end{aligned}$$

calculated at the fixed point $\zeta = -\bar{\alpha}/\bar{\beta}$ from Eqn.(83). Recall from the start of section (VI B) that the discriminant of the quadratic in Eqn.(75) was set to zero to obtain a single root and thus a singular Lorentz transformation. This resulted in the condition given also in Eqn.(76), namely

$$a^2 + bc = 0.$$

With this it is clear that

$$(L_{1j} + i^* L_{1j})k^j = \frac{2q}{mb}(a^2 + bc) = 0.$$

Indeed it is easy to show that every component of the product is zero.

$$\begin{aligned}(L_{1j} + i^* L_{1j})k^j &= \frac{2q}{mb}(a^2 + bc) = 0,\\ (L_{2j} + i^* L_{2j})k^j &= \frac{2iq}{m}(a^2 + bc) = 0,\\ (L_{3j} + i^* L_{3j})k^j &= -\frac{2q\bar{a}}{mb\bar{b}}(a^2 + bc) = 0,\\ (L_{4j} + i^* L_{4j})k^j &= \frac{2q\bar{a}}{mb\bar{b}}(a^2 + bc) = 0.\end{aligned}$$

In conclusion, it is shown that $\mathcal{L}_{ij}k^j = 0$ if and only if $a^2 + bc = 0$ which in turn implies that the field generated is a pure radiation field by Eqns.(81) and (82) and that the infinitesimal Lorentz transformation is singular. Notice that $\mathcal{L}_{ij}k^j$ is nothing but the scalar product of $\mathcal{L}_{ij}$ and k^j. Thus $\mathcal{L}_{ij}$ and k^j are orthogonal in Minkowskian space-time if the Lorentz transfomation is singular, which implies that *k^j is the propagation direction in Minkowskian space-time of the electromagnetic radiation.* It is shown briefly in Appendix (C) that this also implies $\vec{E}$, $\vec{B}$ and $\vec{n}$ form a right-handed triad. Also in Appendix (D) some familiar electromagnetism equations are derived from the pure radiation conditions.

As stared at the beginning of the section, Eqns.(81) and (82) which describe a pure radiation field in physical space $\mathbb{R}^3$, have been written as $\mathcal{L}_{ij}k^j = 0$ which describes the radiation field in Minkowskian space-time.

Appendix A: Singular Lorentz Transformation with Special Significance Given to x

The choice of components of the matrix A in Eqn.(26) gives an arbitrary but special significance to the corrdinate z. In this calculation the matrix U of Eqn.(27) is determined for a singular Lorentz transformation, in which special significance has been given to the coordinate x. This transformation is given by

$$\begin{aligned} t'-x' &= t-x, \\ z'+iy' &= z+iy+w(t-x), \\ t'+x' &= t+x+w(z-iy)+\bar{w}(z+iy)+w\bar{w}(t-x). \end{aligned}$$

Notice that this is the same tranformation as in Example 3, Section (IV C), with x and z swapped. This is equivalent to swapping these two coordinates in the matrix A only. Note that if x and z were exchanged in both A and the transformation then the U obtained would be exactly the same as the example done previously.

First, the components of the matrix A must be construced. Thus the transformation must be written out explicitly as

$$\begin{aligned} x' &= x+\frac{1}{2}(w+\bar{w})z+\frac{i}{2}(\bar{w}-w)y+\frac{1}{2}w\bar{w}(t-x), \\ y' &= y+\frac{i}{2}(\bar{w}-w)(t-x), \\ z' &= z+\frac{1}{2}(w+\bar{w})(t-x), \\ t' &= t+\frac{1}{2}(w+\bar{w})z+\frac{i}{2}(\bar{w}-w)y+\frac{1}{2}w\bar{w}(t-x). \end{aligned}$$

Now construct the components of A

$$\begin{aligned} t'-z' &= t-z+\frac{1}{2}(w+\bar{w})z+\frac{i}{2}(\bar{w}-w)y+\frac{1}{2}w\bar{w}(t-x)-\frac{1}{2}(\bar{w}+w)(t-x), \\ t'+z' &= t+z+\frac{1}{2}(w+\bar{w})z+\frac{i}{2}(\bar{w}-w)y+\frac{1}{2}w\bar{w}(t-x)+-\frac{1}{2}(\bar{w}+w)(t-x), \\ x'+iy' &= x+\frac{1}{2}(w+\bar{w})z+\frac{i}{2}(\bar{w}-w)y+\frac{1}{2}w\bar{w}(t-x)-\frac{1}{2}(\bar{w}+w)(t-x)+iy. \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28a) to obtain

$$\alpha\bar{\beta}+\bar{\alpha}\beta = \frac{1}{2}(\bar{w}+w)-\frac{1}{2}w\bar{w}, \tag{A1a}$$

$$\alpha\bar{\beta}-\bar{\alpha}\beta = \frac{1}{2}(\bar{w}-w), \tag{A1b}$$

$$-\alpha\bar{\alpha}+\beta\bar{\beta} = -1+\frac{1}{2}(\bar{w}+w), \tag{A1c}$$

$$\alpha\bar{\alpha}+\beta\bar{\beta} = 1+\frac{1}{2}w\bar{w}-\frac{1}{2}(\bar{w}+w). \tag{A1d}$$

Then Eqn.(A1a) and (A1b) imply $\alpha\bar{\beta} = \frac{1}{2}\bar{w}-\frac{1}{4}w\bar{w}$. Also Eqn.(A1c) and (A1d) imply $\beta\bar{\beta} = \frac{1}{4}w\bar{w}$ and $\alpha\bar{\alpha} = 1+\frac{1}{4}w\bar{w}-\frac{1}{2}\bar{w}-\frac{1}{2}w$. Hence α can be written in terms of β

$$\begin{aligned} \alpha\bar{\beta} &= (\frac{1}{2}\bar{w}-\frac{1}{4}w\bar{w}), \\ \frac{1}{4}w\bar{w}\alpha &= \frac{1}{2}\bar{w}(1-\frac{1}{2}w)\beta, \\ \alpha &= \frac{(2-w)}{w}\beta. \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28b) to obtain

$$\alpha\bar{\delta}+\beta\bar{\gamma}=1-\frac{1}{2}w\bar{w}+\frac{1}{2}(\bar{w}-w), \tag{A2a}$$

$$\alpha\bar{\delta}-\beta\bar{\gamma}=\frac{1}{2}(\bar{w}-w)+1, \tag{A2b}$$

$$-\alpha\bar{\gamma}+\beta\bar{\delta}=\frac{1}{2}(\bar{w}+w), \tag{A2c}$$

$$\alpha\bar{\gamma}+\beta\bar{\delta}=\frac{1}{2}w\bar{w}-\frac{1}{2}(\bar{w}-w). \tag{A2d}$$

Now Eqn.(A2a) and (A2b) imply $\beta\bar{\gamma}=-\frac{1}{4}w\bar{w}$. So using $\bar{\beta}\beta=\frac{1}{4}w\bar{w}$ again to obtain

$$\begin{aligned}\frac{1}{4}w\bar{w}\bar{\gamma}&=\bar{\beta}\beta\bar{\gamma},\\ \frac{1}{4}w\bar{w}\bar{\gamma}&=-\frac{1}{4}w\bar{w}\bar{\beta},\\ \gamma&=-\beta.\end{aligned}$$

Also Eqn.(A2c) and (A2d) imply $\beta\bar{\delta}=\frac{1}{4}w\bar{w}+\frac{1}{2}w$. Thus

$$\begin{aligned}\frac{1}{4}w\bar{w}\bar{\delta}&=\left(\frac{1}{4}w\bar{w}+\frac{1}{2}w\right)\bar{\beta},\\ \delta&=\left(\frac{w\bar{w}+2\bar{w}}{w\bar{w}}\right)\beta.\end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(28c) to obtain

$$\gamma\bar{\delta}+\delta\bar{\gamma}=-\frac{1}{2}w\bar{w}-\frac{1}{2}(w+\bar{w}), \tag{A3a}$$

$$\gamma\bar{\delta}-\delta\bar{\gamma}=\frac{1}{2}(\bar{w}-w), \tag{A3b}$$

$$-\gamma\bar{\gamma}+\delta\bar{\delta}=1+\frac{1}{2}(w+\bar{w}), \tag{A3c}$$

$$\gamma\bar{\gamma}+\delta\bar{\delta}=1+\frac{1}{2}w\bar{w}+\frac{1}{2}(\bar{w}+w). \tag{A3d}$$

Here Eqn(A3a) and (A3b) imply $\gamma\bar{\delta}=-\frac{w}{4}(2+\bar{w})$. Also, Eqn(A3c) and Eqn(A3d) imply $\gamma\bar{\gamma}=\frac{1}{4}w\bar{w}$. So using these relations δ can be written in terms of γ

$$\begin{aligned}\gamma\bar{\gamma}\bar{\delta}&=\frac{w}{4}(2+\bar{w})\bar{\gamma},\\ \frac{1}{4}w\bar{w}\bar{\delta}&=\frac{w}{4}(2+\bar{w})\bar{\gamma},\\ \delta&=-\frac{(2+w)}{w}\gamma.\end{aligned}$$

At this point β and δ have been written in terms of γ and α is written in terms of β. Write α in terms of γ

$$\alpha=\frac{(2-w)}{w}\beta=\frac{(w-2)}{w}\gamma.$$

Now use the condition that $\det(U)=1$ and replace everything in favour of γ

$$\begin{aligned}
\alpha\delta - \beta\gamma &= 1, \\
-\frac{(w^2-4)}{w^2}\gamma^2 + \gamma^2 &= 1, \\
\frac{4}{w^2}\gamma^2 &= 1, \\
\gamma &= \pm\frac{w}{2}.
\end{aligned}$$

Replace γ and write all the components of U as functions of w only

$$\begin{aligned}
\alpha &= \pm\frac{1}{2}(w-2), \\
\beta &= \mp\frac{w}{2}, \\
\gamma &= \pm\frac{w}{2}, \\
\delta &= \mp\frac{1}{2}(w+2).
\end{aligned}$$

Finally it is found that

$$U = \pm\begin{pmatrix} \frac{w-2}{2} & -\frac{w}{2} \\ \frac{w}{2} & -\frac{(w+2)}{2} \end{pmatrix}.$$

Now the fractional linear trasnformation associated with U must be determined. Using Eqn.(50) to find that

$$\zeta' = \frac{\bar{w} - (\bar{w}+2)\zeta}{\bar{w} - 2 - \bar{w}\zeta}.$$

The fixed points of the system and thus the null directions are found by setting $\zeta' = \zeta$ to obtain

$$\begin{aligned}
\bar{w}\zeta^2 - 2\bar{w}\zeta + \bar{w} &= 0, \\
\bar{w}(\zeta - 1)^2 &= 0.
\end{aligned}$$

Thus there is a fixed point at $\zeta = 1$. The corresponding null direction is then given by Eqn.(45) such that $\vec{x} = t(1, 0, 0, 1)$. Hence finally the null direction is the generator of N^+, $x = t$. So in this case where x was given special significance instead of z with the given singular Lorentz transformation it is found that the null direction is $x = t$ instead of $z = t$. This is as expected as by exchanging z with x the coordinate system has been rotated.

Appendix B: Standard Lorentz Transformation of the Electromagnetic Field Vectors

In this appendix the transformation laws for $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$ under the standard Lorentz transformation of Eqn.(32) are derived. If p^i is any vector transported along $x^i = x^i(s)$ then, in Eqn.(61), v^i can be replaced with p^i, such that

$$\frac{\mathrm{d}p^i}{\mathrm{d}s} = L^i{}_j p^j.$$

Here $s =$ arc length so the usual line element

$$\epsilon \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2,$$

is formed. p^i is a 4-vector so it transforms like $x^i = (x, y, z, t)$ under the standard Lorentz transformation, such that

$$\begin{aligned}
\bar{p}^1 &= \gamma(p^1 - vp^4),\\
\bar{p}^2 &= p^2,\\
\bar{p}^3 &= p^3,\\
\bar{p}^4 &= \gamma(p^4 - vp^1).
\end{aligned}$$

Which implies that

$$\mathrm{d}p^i = (\mathrm{d}p^1, \mathrm{d}p^2, \mathrm{d}p^3, \mathrm{d}p^4),$$

is a 4-vector and ds is invariant, thus $\frac{\mathrm{d}p^i}{\mathrm{d}s}$ is also a 4-vector. Hence $L^i{}_j p^j$ is a 4-vector for any 4-vector p^i, where $L^i{}_j$ is given by

$$L^i{}_j = \begin{pmatrix} 0 & B^3 & -B^2 & E^1 \\ -B^3 & 0 & B^1 & E^2 \\ B^2 & -B^1 & 0 & E^3 \\ E^1 & E^2 & E^3 & 0 \end{pmatrix}$$

and a factor of q/m has been left out as it will play no role. Write out $L^i{}_j p^j$ explicitly

$$\begin{aligned}
L^1{}_j p^j &= B^3p^2 - B^2p^3 + E^1p^4,\\
L^2{}_j p^j &= -B^3p^1 + B^1p^3 + E^2p^4,\\
L^3{}_j p^j &= B^2p^1 - B^1p^2 + E^3p^4,\\
L^4{}_j p^j &= E^1p^1 + E^2p^2 + E^3p^3.
\end{aligned}$$

Since $L^i{}_j p^j$ is a 4-vector it transforms under the standard Lorentz transformation, such that

$$\bar{L}^1{}_j \bar{p}^j = \gamma(L^1{}_j p^j - vL^4{}_j p^j), \tag{B1}$$
$$\bar{L}^2{}_j \bar{p}^j = L^2{}_j p^j, \tag{B2}$$
$$\bar{L}^3{}_j \bar{p}^j = L^3{}_j p^j, \tag{B3}$$
$$\bar{L}^4{}_j \bar{p}^j = \gamma(L^4{}_j p^j - vL^1{}_j p^j). \tag{B4}$$

Write out Eqn.(B1) explicitly

$$\begin{aligned}
\bar{B}^3\bar{p}^2 - \bar{B}^2\bar{p}^3 + \bar{E}^1\bar{p}^4 &= \gamma(B^3p^2 - B^2p^3 + E^1p^4 - v(E^1p^1 + E^2p^2 + E^3p^3)),\\
\bar{B}^3\bar{p}^2 - \bar{B}^2\bar{p}^3 + \bar{E}^1\gamma(p^4 - vp^1) &= \gamma(B^3p^2 - B^2p^3 + E^1p^4 - vE^1p^1 - vE^2p^2 - vE^3p^3).
\end{aligned}$$

Now equate the coefficients of p^1, p^2, p^3 and p^4, which gives

$$\bar{E}^1 = E^1, \tag{B5a}$$
$$\bar{B}^3 = \gamma(B^3 - vE^2), \tag{B5b}$$
$$\bar{B}^2 = \gamma(B^2 + vE^3). \tag{B5c}$$

Where generally one of the relations given is trivial. Continue by writting Eqn.(B2) explicitly and again equate coefficients to get

$$B^3 = \gamma(\bar{B}^3 + v\bar{E}^2), \tag{B6}$$
$$\bar{B}^1 = B^1, \tag{B7}$$
$$E^2 = \gamma(v\bar{B}^3 + \bar{E}^2). \tag{B8}$$

Eqn(B6) and (B8) now need to be inverted using $\gamma(1 - v^2) = \gamma^{-1}$, to obtain

$$\bar{B}^3 = \gamma(B^3 - vE^2), \tag{B9a}$$
$$\bar{E}^2 = \gamma(E^2 - vB^3) \tag{B9b}$$

It is also necessary to write out Eqn.(B3) explicitly and equate the coefficients of p to get

$$B^2 = \gamma(\bar{B}^2 - v\bar{E}^3), \tag{B10}$$
$$\bar{B}^1 = B^1, \tag{B11}$$
$$E^3 = \gamma(\bar{E}^3 - v\bar{B}^2). \tag{B12}$$

Again, Eqn(B10) and (B12) need to be inverted to give

$$\bar{E}^3 = \gamma(E^3 + vB^2), \tag{B13a}$$
$$\bar{B}^2 = \gamma(B^2 + vE^3) \tag{B13b}$$

Following a similar procedure write Eqn.(B4) out explicitly, equate coefficients to get

$$\begin{aligned}
\bar{E}^1 &= E^1, \\
\bar{E}^2 &= \gamma(E^2 - vB^3), \\
\bar{E}^3 &= \gamma(E^3 + vB^2),
\end{aligned}$$

as obtained already. Thus in summary the final form of the standard Lorentz transformation of the electromagnetic vectors E and B is given by

$$\begin{aligned}
\bar{E}^1 &= E^1, \quad \bar{E}^2 = \gamma(E^2 - vB^3), \quad \bar{E}^3 = \gamma(E^3 + vB^2), \\
\bar{B}^1 &= B^1, \quad \bar{B}^2 = \gamma(B^2 + vE^3), \quad \bar{B}^3 = \gamma(B^3 - vE^2).
\end{aligned}$$

Using these transformations it is possible to verify that the quantities in Eqn.(81) and (82 are invariant, by direct computation. Start with

$$\begin{aligned}
|\bar{E}|^2 - |\bar{B}|^2 &= (\bar{E}^1)^2 + (\bar{E}^2)^2 + (\bar{E}^3)^2 - (\bar{B}^1)^2 - (\bar{B}^2)^2 - (\bar{B}^3)^2, \\
&= (E^1)^2 + \gamma^2((E^2)^2 - 2vE^2B^3 + v^2(B^3)^2) \\
&\quad + \gamma^2((E^3)^2 + 2vE^3B^2 + v^2(B^2)^2) - (B^1)^2 \\
&\quad - \gamma^2((B^2)^2 + 2vB^2E^3 + v^2(B^3)^2) \\
&\quad - \gamma^2((B^3)^2 - 2vB^3E^2 + v^2(E^2)^2),
\end{aligned}$$

which gives after some cancellations

$$\begin{aligned}
|\bar{E}|^2 - |\bar{B}|^2 &= (E^1)^2 + \gamma^2(1 - v^2)(E^2)^2 + \gamma^2(1 - v^2)(E^3)^2, \\
&\quad - (B^1)^2 - \gamma^2(1 - v^2)(B^2)^2 - \gamma^2(1 - v^2)(B^3)^2, \\
&= (E^1)^2 + (E^2)^2 + (E^3)^2 - (B^1)^2 - (B^2)^2 - (B^3)^2,
\end{aligned}$$

as $\gamma^2(1-v^2)=1$. Finaly the next equation is verified as follows

$$\begin{aligned}\bar{E}\cdot\bar{B} &= \bar{E}^1\bar{B}^1+\bar{E}^2\bar{B}^2+\bar{E}^3\bar{B}^3,\\ &= E^1B^1\\ &+\gamma^2(E^2B^2+vE^2E^3-vB^2B^3-v^2E^3B^3)\\ &+\gamma^2(E^3B^3-vE^2E^3+vB^2B^3-v^2E^2B^2),\\ &= E^1B^1+\gamma^2(1-v^2)E^2B^2\gamma^2(1-v^2)E^3B^3,\\ &= E^1B^1+E^2B^2+E^3B^3.\end{aligned}$$

Appendix C: The Triad of Electromagnetic Radiation

In this appendix it is shown that the vectors $(\vec{n},\vec{B},\vec{E})$ form a right-handed triad.
From Eqn.(84), k^i can be rewritten as

$$k^i=(\zeta\bar{\zeta}+1)(\vec{n},1),$$

where $\vec{n}=(n^1,n^2,n^3)$ is a unit vector such that $\vec{n}\cdot\vec{n}=1$. Expanding the relation $\mathcal{L}_{ij}k^i=0$ in terms of $\vec{n}$ gives

$$(B^3-iE^3)n^2-(B^2-iE^2)n^3+E^1+iB^1=0, \tag{C1a}$$
$$-(B^3-iE^3)n^1+(B^1-iE^1)n^3+E^2+iB^2=0, \tag{C1b}$$
$$(B^2-iE^2)n^1-(B^1-iE^1)n^2+E^3+iB^3=0, \tag{C1c}$$
$$-(E^1+iB^1)n^1-(E^2+iB^2)n^2-(E^3+iB^3)=0. \tag{C1d}$$

Then by equating real and imaginary parts of Eqn.(C1d) it is clear that $\vec{E}\cdot\vec{n}=\vec{B}\cdot\vec{n}$. Also, from Eqns.(C1a) - (C1c) the following series of equations are seen, first Eqn.(C1a) implies

$$\begin{aligned}E^1&=B^2n^3-B^3n^2,\\ B^1&=E^3n^2-E^2n^3,\end{aligned}$$

while Eqn.(C1b) gives

$$\begin{aligned}E^2&=B^3n^1-B^1n^3,\\ B^2&=E^1n^3-E^3n^1.\end{aligned}$$

Then Eqn.(C1c) implies

$$\begin{aligned}E^3&=B^1n^2-B^2n^1,\\ B^3&=E^2n^1-E^1n^2.\end{aligned}$$

Thus is it clear that these are the components of the curl $\vec{E}=\vec{B}\times\vec{n}$ and $\vec{B}=\vec{n}\times\vec{E}$. This curl is proof that the vectors $(\vec{n},\vec{E},\vec{B})$ form a right-handed triad.

Appendix D: Derivation of Familiar Electromagnetism Equations from Pure Radiation Conditions

In this appendix the familiar equations

$$L_{ij}L^{ij} = 0 = L^*_{ij}L_{ij},$$

are shown to be equivalent to the pure radiation conditions of Eqn.(81) and (82). First consider the scalar product

$$\begin{aligned}\mathcal{L}_{ij}\mathcal{L}^{ij} = 2(&\mathcal{L}_{12}\mathcal{L}_{12} + \mathcal{L}_{13}\mathcal{L}_{13} + \mathcal{L}_{23}\mathcal{L}_{23} \\ &- \mathcal{L}_{14}\mathcal{L}_{14} - \mathcal{L}_{24}\mathcal{L}_{24} - \mathcal{L}_{34}\mathcal{L}_{34})\end{aligned}$$

where $\mathcal{L}_{ij} = L_{ij} + i^*L^{ij}$ as before. Writting this in terms of the components of L gives

$$\begin{aligned}\frac{1}{2}\mathcal{L}_{ij}\mathcal{L}^{ij} &= \frac{q^2}{m^2}(-4a^2 + b^2 - 2bc + c^2 - b^2 - 2bc - c^2 \\ &- b^2 - 2bc - c^2 + b^2 - 2bc + c^2 - 4a^2) \\ &= -\frac{8q^2}{m^2}(a^2 + bc) = 0,\end{aligned}$$

as $a^2 + bc = 0$ for a singular Lorentz transformation and this is equivalent to the pure radiation field conditions as shown in Section (VI B). This scalar product can also to written in terms of L_{ij} such that

$$\mathcal{L}_{ij}\mathcal{L}^{ij} = L_{ij}L^{ij} -^* L^*_{ij}L^{ij} + 2iL^*_{ij}L^{ij}.$$

Rewritting the middle terms component-wise gives the result

$$\begin{aligned}{}^*L^*_{ij}L^{ij} = 2(&{}^*L^*_{12}L_{12} +^* L^*_{13}L_{13} +^* L^*_{23}L_{23} \\ &-^* L^*_{14}L_{14} -^* L^*_{24}L_{24} -^* L^*_{34}L_{34}) \\ 2(&L_{34}L_{34} + L_{24}L_{24} + L_{14}L_{14} \\ &- L_{23}L_{23} - L_{13}L_{13} - L_{12}L_{12}) \qquad = -L_{ij}L^{ij},\end{aligned}$$

where ${}^*L_{ij} = \frac{1/2}{\epsilon}{}_{ijkl}L^{kl}$ has been used. So the scalar product becomes

$$\mathcal{L}_{ij}\mathcal{L}^{ij} = 2L_{ij}L^{ij} + 2iL^*_{ij}L^{ij}.$$

So the condition that the Lorentz transformation is singular gives $a^2 + bc = 0$ which implies first that the pure radiation conditions of Eqn.(81) and (82) hold and second that the scalar product $\mathcal{L}_{ij}\mathcal{L}^{ij}$ vanishes, which is equivalent to

$$L_{ij}L^{ij} = 0 = L^*_{ij}L_{ij},$$