Singular Lorentz Transformations and Pure Radiation Fields

Kevin Maguire

April 21, 2014

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1. Introduction: Lorentz Transformations
2. Strange Minkowskian Line Element
3. Singular Lorentz transformation
4. $SL(2,\mathbb{C})$ Matrices of the Lorentz Transformation

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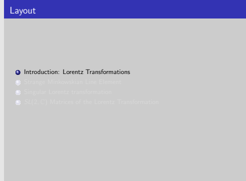

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Introduction: Lorentz Transformations

- A Lorentz transformation is defined by the preservation of the quadratic form

$$x'^2+y'^2+z'^2-t'^2 = x^2+y^2+z^2-t^2,$$

 in the transformation $(x, y, z, t) \to (x', y', z', t')$
- Take the Proper Orthochronous Lorentz Transformations(POLTs) which form the restricted Lorentz group $SO^{+}(1,3)$
- In general lorentz transformations have two invariant null directions

- –Proper is det 1 . preserves the orientation of spacial axes, preserves handedness
- –orthochronous means time is always positive and the direction of time is preserve
- –Think of the standard Lornetz transformation, always two null directions at $x \pm$

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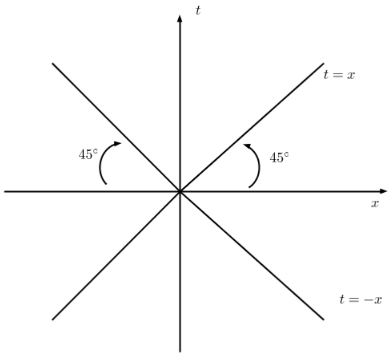

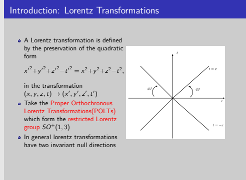

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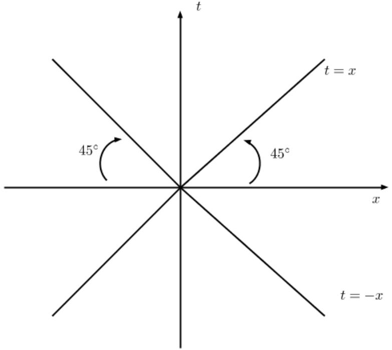

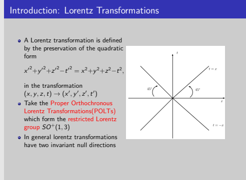

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- derive a strange minkowskian line element
- making a complicated transformation that keeps a single null geodesic fixed look trivial

Strange Minkowskian Line Element

- Start with the Schwarzschild solution

$$\epsilon \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) - \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2.$$

- Make the Eddington-Finkelstein coordinate transformation [2]

$$u = t - r - 2m\ln(r - 2m).$$

- Make further coordinate transformations to obtain

$$\epsilon \mathrm{d}s^2 = \frac{r^2}{\cosh^2 \mu\xi}(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) - 2\mathrm{d}u\mathrm{d}r - \left(\mu^2 - \frac{2k}{r}\right) \mathrm{d}u^2.$$

- Taking the limit as the energy, $\mu \to 0$ gives The Kasner Solution

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LTs that leave one null invariant direction are constructed

Singular Lorentz Transformation

- Define an arbitrary complex parameter $\zeta := \xi + i\eta$, to get the new line element[3]
$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr.$$
- The transformation $\zeta \to \zeta + w$, where $w \in \mathbb{C}$ is then trivial and leaves the single null geodesic $r = 0$ invariant.
- In Cartesian coordinates this transformation becomes

$$\begin{aligned} x' + iy' &= x + iy + w(t - z), \\ z' - t' &= -r = z - t, \\ z' + t' &= z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z). \end{aligned}$$

- Addition of complex numbers is commutative, and w has two parameters, so the singular Lorentz transformations form a 2-parameter abelian subgroup of the Lorentz group

- This is what we want, An LT which leaves one null invariant.
- The use in the previous coord transforms was to make this transformation look trivial
- So this is what the seemingly trivial transformation looks like in cartesians
- Again its clear that $r = 0$ keeps one direction fixed, as then z=t
- but it doesn't work both ways, not all 2 parameter abelian subgroups are singular lorentz transformations

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- shown here that there is a 2 to 1 correspondence between SL(2,C) and POLTs
- first show there 1 to 1 correspondence between points in Minkowskian space time and Hermitian matrices

$SL(2,\mathbb{C})$ Matrices of the POLT

- There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices
- Contruct the following matrix

$$A = \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix},$$

- This is useful as its determinant is the Lorentz quadratic form modulo a sign

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2.$$

- It is also closely related to spinors

$$A = [t\mathbb{I}_2 - \vec{x}\cdot\vec{\sigma}].$$

- Construct the transformation $A(\vec{x}') = UA(\vec{x})U^\dagger$, where

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

is an element of $SL(2,\mathbb{C})$

$SL(2,\mathbb{C})$ Matrices of the POLT
There is a one to one correspondence between points in Minkowskian space-time and Hermitian matrices

- Complex Hermitian matrices have 4 independant components, so the element of such a matrix can be used to represent points in Minkowskian space-time.
- Where σ are the pauli matirces which form a basis for the Lie algebra of $SU(2)$
- where $\alpha,\beta,\gamma,\delta$ are complex its an element of the special linear group. This mean it has determinant 1. **write it on the board**

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- It is also closely related to spinors

$$A = [t\mathbb{I}_2 - \vec{x}\cdot\vec{\sigma}].$$

- Construct the transformation $A(\vec{x}') = UA(\vec{x})U^{\dagger}$, where

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

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- $A(\vec{x}')$ and $A(\vec{x})$ have the same determinant so the above transformation preserves the Lorentz quadratic form, thus is is a Lorentz transformation.
- Write this transformation component wise

$$\begin{pmatrix} t'-z' & x'+iy' \\ x'-iy' & t'+z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix},$$
$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha}+(x+iy)\bar{\beta} & (t-z)\bar{\gamma}+(x+iy)\bar{\delta} \\ (x-iy)\bar{\alpha}+(t+z)\bar{\beta} & (x-iy)\bar{\gamma}+(t+z)\bar{\delta} \end{pmatrix}.$$

- Thus the general relations

$$t'-z' = (t-z)\alpha\bar{\alpha}+(x+iy)\alpha\bar{\beta}+(x-iy)\beta\bar{\alpha}+(t+z)\beta\bar{\beta},$$
$$x'+iy' = (t-z)\alpha\bar{\gamma}+(x+iy)\alpha\bar{\delta}+(x-iy)\beta\bar{\gamma}+(t+z)\beta\bar{\delta},$$
$$t'+z' = (t-z)\gamma\bar{\gamma}+(x+iy)\gamma\bar{\delta}+(x-iy)\delta\bar{\gamma}+(t+z)\delta\bar{\delta}.$$

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Example: Singular Lorentz transformation

- Take the singular Lorentz transformation from earlier

$$
\begin{aligned}
t' - z' &= t - z, \\
x' + iy' &= x + iy + w(t - z), \\
t' + z' &= t + z + w(x - iy) + \bar{w}(x + iy) + w\bar{w}(t - z).
\end{aligned}
$$

- Equate coefficients on the RHS of this equation with the RHS of the general relations on the previous slide to obtain

$$
\begin{aligned}
\alpha &= \pm 1, \qquad & \beta &= 0, \\
\gamma &= \bar{w}a, \qquad & \delta &= \alpha.
\end{aligned}
$$

- So there are always two possible choices of U

$$U = \pm \begin{pmatrix} 1 & 0 \\ \bar{w} & 1 \end{pmatrix}$$

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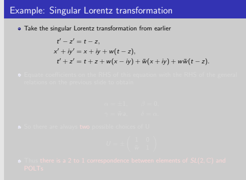

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