The Lorentz Group and Singular Lorentz Transformations

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abstract

Contents

I. THE LORENTZ TRANSFORMATION

In this section the two null directions inherent in all Lorentz transformations, except the singular Lorentz transformations, are illustrated. The Lorentz Transform is defined by

$$(x, y, z, t) \rightarrow (x', y', z', t') \text{ such that}$$
$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

If the transformation preserves the orientation of the spatial axes then is it called a proper Lorentz transformation. This is equivalent to saying the transformation does not change the handedness of the axes. Also If $t \geq 0 \Rightarrow t' \geq 0$ then it is called an orthochronous Lorentz transforamtion. This ensures that the time direction is preserved. In this project the "Lorentz transforamation" will refer to the proper, orthochronous Lorentz transformation.

Consider a photon moving in the x direction at the speed of light, $c = 1$, and starting at $x = 0$. The space-time for such a photon can be illustrated as follows (FIGURE). It is clear that there are two null directions in this space-time, $x = \pm t$. To see this use the standard Lorentz transformation:

$$x' = \gamma(x - vt) \quad , \text{ where } \gamma = (1 - v^2)^{-1/2}$$
$$t' = \gamma(t - vx)$$

Rearranging:

$$x' - t' = \gamma(1 + v)(x - t)$$
$$x' + t' = \gamma(1 - v)(x + t)$$

It is clear that:

$$x = \pm t \leftrightarrow x' = \pm t'$$

Thus there are two null directions (are null directions by definition invarient??? yes)in this space-time at $x = \pm t$. It can be shown that all Lorentz transformations have two invarient null directions except the singular Lorentz transformation which has only one fixed null direcion.

II. REPARAMETERISATION OF THE SCHWARZSCHILD SOLUTION

In this section the Kasner solution of the vacuum field equations is derived from the Schwarschild solution by taking the limit as the mass goes to infinity. It is then shown that the special case of the Kasner solution with no mass is equivalent to a novel form of Minkowskian space-time. (what do we want the Kasner vacuum solution for???). Starting with the Schwarschild Solution of the vacuum field equations:

$$\epsilon ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2m}{r}\right)dt^2 \tag{1}$$

Now make the Eddington-Finkelstein coordinate transformation:

$$u = t - r - 2m\ln(r - 2m) \tag{2}$$

Working out the differential:

$$du = dt - dr - \frac{2mdr}{r - 2m}$$

SEE CALCULATIONS PG1

Then write Eqn.(1) in terms of u to obtain:

$$\epsilon ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2dudr - du^2 + \frac{2m}{r}du^2$$

Note that if $m = 0$ the space-time becomes Minkowskian, as expected. Setting $r = 0$ in this Minkowskian space-time the line element becomes:

$$\epsilon ds^2 = -du^2 \Rightarrow t = -1$$

Which implies that $r = 0$ is a time-like world-line in Minkowskian space-time, with proper time u.(CHECK ITS A GEODESIC???)

We want to find the limit of the Schwarzschild solution as $m \to \infty$. In its current form, the limit cannot be calculated so first a suitable coordinate transformation must be made.

$$u = \mu u' \quad \text{, where } \mu = \text{ const}$$
$$r = \mu^{-1} r'$$

So that

$$du = \mu du'$$
$$dr = \mu^{-1} dr'$$
$$\Rightarrow dudr = du'dr'$$

To obtain (SEE CALCS PG3):

$$\epsilon ds^2 = r'^2 \sin^2\theta \left\{ \frac{d\theta^2}{\mu^2 \sin^2\theta} + \mu^{-2} d\phi^2 \right\} - 2du'dr' - \left(\mu^2 - \frac{2m\mu^3}{r'} du' \right)$$

Now set $m\mu^3 = k =$ const $\Rightarrow m = k\mu^{-3}$ and make another transformation first done by Ivor Robinson(check this???) given by:

$$\sin\theta = \frac{1}{\cosh(\mu\xi)} \quad , \mu^{-1}\phi = \eta \tag{3}$$

Which results in (SEE CALCS PG 3):

$$\epsilon ds^2 = \frac{r^2}{\cosh^2 \mu\xi}(d\xi^2 + d\eta^2) - 2dudr - \left(\mu^2 - \frac{2k}{r} \right) du^2$$

Where the primes have been dropped for notational simplicity. This is now in an appropriate form to take the limit $m \to \infty$ which is equivalent to $\mu \to 0$. This limit gives:

$$\epsilon ds^2 = r^2(d\xi^2 + d\eta^2) - 2dudr - \frac{2k}{r}du^2 \tag{4}$$

This is still a solution of the field equations, but it is no longer the Schwarzschild solution. It is found that this is the Kasner solution. To see this (SEE CALCS PG ???):

$$\epsilon ds^2 = T^{-2/3}dX^2 + T^{4/3}\left(dY^2 + dZ^2\right) - dT^2 \tag{5}$$

By definition the Kasner solution is given by:

$$\epsilon ds^2 = T^{2p}dX^2 + T^{2q}dY^2 + T^{2r}dZ^2 - dT^2$$

With:

$$p + q + r = 1 = p^2 + q^2 + r^2$$

So it is clear that Eqn.(5) is the Kasner solution with $p = -1/3$ and $q = r = 2/3$.

A. Line Element of Minkowskian Space-Time

Minkowskian space-time reemerges again by setting $k = 0$ in Eqn.(4), which is equivalent to $m = 0$.

$$\epsilon ds^2 = r^2(d\xi^2 + d\eta^2) - 2dudr \tag{6}$$

Setting $r = 0$ it can be shown that $\epsilon ds^2 = 0$ in this case. Then $r = 0$ is a null geodesic with u an affine parameter along it. To demonstrate these properties first let $x^i = (x, y, z, t)$ be rectangular Cartesian coordinates with time in Minkowskian space-time with the usual line element:

$$\epsilon {ds_0}^2 = dx^2 + dy^2 + dz^2 - dt^2$$

We note that $C : x = 0, y = 0, z = t$ is a null geodesic as it will lie in the light cone of Minkowskian space-time. it we write it parametrically as $x^i = w^i(u)$ such that $w^i = (0, 0, u, u)$ then u is an affine parameter along C. The tangent to C is then computed as:

$$v^i(u) = \frac{dw^i}{du} = (0, 0, 1, 1)$$

As C is a null geodesic the first integral will be $v_i v^i = 0$ (IS THIS CALLED THE FIRST INTEGRAL?) and thus $v_i = (0, 0, 1, -1)$ where we have chosen the convention $(+, +, +, -)$.

The position vector of a point in Minkowskian space time can be written in the form(DO THE PICTURE FROM 2:1):

$$\begin{aligned} x^i &= w^i(u) = rk^i \\ \text{or } x^i &= w^i(u) + rk^i \end{aligned}$$

Thus r is a new parameter which tell us the shortest distance between C and some point x^i, and k^i is the unit vector in that direction. As k^i is a unit vector it satisfies the realtions:

$$k^i k_i = 0 \tag{7}$$
$$k^i v_i = -1 \tag{8}$$

Thus k^i is normalized so that k^i and v^i are both future pointing (HOW DOES THIS MAKE THEM BOTH FUTURE POINTING?). Making the parameterisation:

$$\begin{aligned} k^i &= (\xi, \eta, A, B) \\ \Rightarrow k_i &= (\xi, \eta, A, -B) \end{aligned}$$

We can choose any variable for the first two slots of k^i so we choose ξ and η from before for conveinience. Using the relation (7) it is clear that:

$$\xi^2 + \eta^2 + A^2 - B^2 = 0$$

and using the relation (8) it is found that:

$$A - B = -1 \tag{9}$$
$$\Rightarrow A^2 - B^2 = (A + B)(A - B) = -(A + B) \tag{10}$$

Which implies:

$$\xi^2 + \eta^2 = A + B \tag{11}$$

So expressions for A and B are found using Eqn.(9) and Eqn.(11):

$$A = \frac{1}{2}(-1+\xi^2+\eta^2)$$
$$B = \frac{1}{2}(1+\xi^2+\eta^2)$$

In summary so far we have:

$$x^i = w^i(u) + rk^i \quad (12)$$
$$w^i = (0,0,u,u) \quad (13)$$
$$k^i = (\xi, \eta, \frac{1}{2}(-1+\xi^2+\eta^2), \frac{1}{2}(1+\xi^2+\eta^2)) \quad (14)$$
$$x^i = (x,y,z,t) \quad (15)$$

Consider Eqn.(12) as a coordinate transformation from (x,y,z,t) to (ξ,η,r,u) such that:

$$x = r\xi$$
$$y = r\eta$$
$$z = u + \frac{r}{2}(-1+\xi^2+\eta^2)$$
$$t = u + \frac{r}{2}(1+\xi^2+\eta^2) \quad (16)$$

as is clear from Eqn.(12) - Eqn.(15). Now this is applied to the Minkowskian line element Eqn.(6). First, the x and y differentials are:

$$dx = rd\xi + \xi dr$$
$$dy = rd\eta + \eta dr$$

Which gives:

$$dx^2 + dy^2 = r^2(d\xi^2 + d\eta^2) + 2r\xi d\xi dr + 2r\eta d\eta dr + (\xi^2+\eta^2)dr^2 \quad (17)$$

Next, the z and t differentials:

$$z + t = 2u + r(\xi^2+\eta^2)$$
$$z - t = -r$$
$$dz + dt = 2du + (\xi^2+\eta^2)dr + 2r\xi d\xi + 2r\eta d\eta$$
$$dz - dt = -dr$$

Using difference of two squares to obtain:

$$dz^2 - dt^2 = -2dudr - (\xi^2+\eta^2)dr^2 - 2r\xi d\xi dr - 2r\eta d\eta dr \quad (18)$$

Combining Eqn.(17) and (18) to get:

$$dx^2 + dy^2 + dz^2 - dt^2 = r^2(d\xi^2 + d\eta^2) - 2dudr$$

and from this it is clear that Eqn.(6) is the line element of Minkowskian space-time with $r = 0$ a null geodesic with affine parameter u along it as before.

III. THE SINGULAR LORENTZ TRANSFORMATION

In this section a Lorentz transformation that leaves our line element Eqn.(6) invarient is constructed. This transformation is then expressed in terms of (x, y, z, t) and examined to see what for it has. First we define an arbitrary complex parameter by $\zeta = \xi + i\eta$ so that the differentials are given by:

$$\begin{aligned} d\zeta &= d\xi + id\eta \\ d\bar{\zeta} &= d\bar{\xi} + id\bar{\eta} \end{aligned}$$

and the line element can be rewritten as:

$$\epsilon ds^2 = r^2 d\zeta d\bar{\zeta} - 2dudr$$

In this form the transforamtion $\zeta \to \zeta + w$, where $w \in \mathbb{C}$, is trivial. It leaves the line element unchanged and the null geodesic $r = 0$ trivially invariant. This is a lorentz transformation which leaves one null direction invarient. Therefore it is a two real parameter, singular lorentz transformation, where the two parameters come from the complex variable w. With this form of the line element the transformation is obviously trivial, we now want to see what this transformation looks like in terms of the usual variables (x, y, z, t).

First we invert the transformation (**??**) and use the new variable ζ:

$$\begin{aligned} x + iy &= r(\xi + i\eta) = r\zeta \\ z &= u + \frac{r}{2}(-1 + \zeta\bar{\zeta}) \\ t &= u + \frac{r}{2}(1 + \zeta\bar{\zeta}) \end{aligned}$$

From this it is clear that:

$$\begin{aligned} t - z &= r, \text{ and} \\ t + z &= 2u + r\zeta\bar{\zeta} \end{aligned}$$

So finally:

$$\begin{aligned} \zeta &= \frac{x + iy}{t - z} \\ r &= t - z \\ u &= \frac{1}{2}(t + z) - \frac{(x^2 + y^2)}{2(t - z)} \end{aligned}$$

Now make the desired transformation:

$$\begin{aligned} \zeta' &\to \zeta + w \\ \bar{\zeta}' &\to \bar{\zeta} + \bar{w} \\ r' &= r \\ u' &= u \end{aligned}$$

(SEE NOTES PG 2:6 FOR CALC)
To obtain:

$$x' + iy' = x + iy + w(t - z) \tag{19}$$
$$z' - t' = -r = z - t \tag{20}$$
$$z' + t' = z + t + w(x - iy) + w(x + iy) + w\bar{w}(t - z) \tag{21}$$

Next, it is necessary to show that this is indeed a Lorentx transformation by verifying the usual Lorentz invarient quantity. First from Eqn.(19) implies:

$$x'^2 + y'^2 = (x + iy + w(t - z))(x - iy + \bar{w}(t - z))$$
$$= x^2 + y^2 + \bar{w}(t - z)(x + iy) + w(t - z)(x - iy) + w\bar{w}(t - z)^2$$

Then Eqn.(20) and Eqn.(21) imply:

$$(z' + t')(z' - t') = z'^2 - t'^2$$
$$= z^2 - t^2 + (z - t)w(x - iy) + (z - t)\bar{w}(x + iy) + (z - t)w\bar{w}(t - z)$$

Thus the Lorentz invarient quantity in the primed frame is the same as that of the unprimed frame:

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2$$

It is also clear from Eqn.(20) that the null direction $z = t$ is invarient under this Lorentz transformation.

In conclusion, this transformation involves one complex parameter and thus two real parameters. In the usual Cartesian coordinates it is described by Eqns.(19) - (19) and in the coordinates (ξ, η, r, u) (SHOULD THIS NOT BE WITH A ζ ???) derived in previous sections, it is expressed simply as:

$$\zeta' = \zeta + w$$
$$r' = r$$
$$u' = u$$

Note that the operation of addition of complex numbers is commutative so that:

$$\zeta' = \zeta + w_1, \text{ and } \zeta'' = \zeta' + w_2$$
$$\Rightarrow \zeta'' = \zeta + w_1 + w_2$$

and thus these transformations form a 2-parameter abelian subgroup of the Lorentz group with the binary operation of addition of complex numbers. (SHOW ALL 2 PARAM ABELIAN SUBGROUPS OF THE LORENTZ GROUP ARE SINGULAR TRANSFORMATIONS???)

IV. SPECIAL LINEAR MATRICES OF THE LORENTZ TRANSFORMATION

Let $\vec{x} = (x, y, z, t)$ be the position vector of a point in minkowskian space-time. Knowing $\vec{x}$ we can construct the following 2×2 hermitian matrix:

$$A = \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix} \tag{22}$$

with $A^\dagger(\vec{x}) = A(\vec{x})$. This is useful as its determinant is the same as the usual lorentz invariant quantity(WHAT DO WE CALL THIS???):

$$\det(A(\vec{x})) = t^2 - x^2 - y^2 - z^2$$

Consider any 2×2 hermitian matrix H.

$$H = \begin{pmatrix} p & q \\ r & s \end{pmatrix} , H^\dagger = \begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix}$$

It is know that $H^{\dagger}(\vec{x}) = H(\vec{x})$ so it is clear that $p = \bar{p}$ and $s = \bar{s}$ and thus $p, s \in \mathbb{R}$. Also $q = \bar{r}$ and then of course $\bar{q} = r$. Hence knowing p, q, r and s is equivalent to knowing 4 real numbers, two from p and two from q. From these parameters the coordinates (x, y, z, t) of a point in Minkowskian space-time can be constructed as:

$$x + iy = q = \bar{r}\ ,\ t - z = p\ ,\ t + z = s$$

by comparing with matrix A above. Hence it is true that there is a one to one correspondence between points in Minkowskian space-time and 2×2 hermitian matrices.

Construct the following matrix:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with α,β,γ,$\delta \in \mathbb{C}$. With the condition that $\det(U) = 1$. Such matrices U form a group called the special linear group, which is denoted by $SL(2, \mathbb{C})$. Given $A(\vec{x})$ consider $UA(\vec{x})U^{\dagger}$. This is a 2×2 hermitian matrix since:

$$\begin{aligned} cl(UA(\vec{x})U^{\dagger})^{\dagger} &= (U^{\dagger})^{\dagger} A^{\dagger}(\vec{x}) U^{\dagger} \\ &= UA(\vec{(x)})U^{\dagger} \end{aligned}$$

since $(U^{\dagger})^{\dagger} = U$ and $A^{\dagger} = A$. Hence there exists a point $\vec{x'} = (x', y', z', t')$ in minkowskian space-time for which:

$$A(\vec{x'}) = UA(\vec{x})U^{\dagger} \tag{23}$$

Any U involves 6 real parameters, 2 each from the four complex components, with the condition $\det(U) = 1$ supplying two constraints. One on the real parts and one on the imaginary parts of the components. Now calculate the determinant of the matrix in the primed frame

$$\begin{aligned} \det(A(\vec{x'})) &= \det(UA(\vec{x})U^{\dagger}) \\ &= (\det(U))(\det(A(\vec{x}))(\det(U^{\dagger})) \\ &= (\det(U))(\det(A(\vec{x}))(\det\bar{(}U)) \\ &= \det(A(\vec{x})) \end{aligned}$$

Thus we have the relation:

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$$

Hence the transformation $\vec{x} \to \vec{x'}$ implicit in Eqn.(23) is a Lorentz transformation. Eqn.(23) describes the most general proper, orthochronous Lorentz transformation.

It is useful to calculate the matrix U for some examples of Lorentz transformations. First, write Eqn.(23) in terms of its components:

$$\begin{aligned} \begin{pmatrix} t' - z' & x' + iy' \\ x' - iy' & t' + z' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t - z & x + iy \\ x - iy & t + z \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (t-z)\bar{\alpha} + (x+iy)\bar{\beta} & (t-z)\bar{\gamma} + (x+iy)\bar{\delta} \\ (x-iy)al\bar{p}ha + (t+z)\bar{\beta} & (x-iy)\bar{\gamma} + (t+z)\bar{\delta} \end{pmatrix} \end{aligned}$$

Thus we have the relations:

$$t' - z' = (t-z)\alpha\bar{\alpha} + (x+iy)\alpha\bar{\beta} + (x-iy)\beta al\bar{p}ha + (t+z)\beta\bar{\beta} \tag{23a}$$

$$x' + iy' = (t-z)\alpha\bar{\gamma} + (x+iy)\alpha\bar{\delta} + (x-iy)\beta\bar{\gamma} + (t+z)\beta\bar{\delta} \tag{23b}$$

$$t' + z' = (t-z)\gamma\bar{\gamma} + (x+iy)\gamma\bar{\delta} + (x-iy)\delta\bar{\gamma} + (t+z)\delta\bar{\delta} \tag{23c}$$

A. Example 1: Rotational Transformation

Find U 23a23b23c corresponding to the one parameter Lorentz transformation:

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \\ t' &= t \end{aligned}$$

This implies that:

$$\begin{aligned} t' - z' &= t - z \\ x' + iy' &= (x+iy)e^{-i\theta} \\ t' + z' &= t + z \end{aligned}$$

Equating coefficients of x, y, z, t on both sides of Eqn.(23a) to obtain: (SEE CALS)

B. Example 2: Special Relativity Lorentz Transfomation

(CALCS)

C. Example 3: Singular Lorentz Transformation

(CALCS)

It is clear that there will always be two matrices $\pm U$ corresponding to every Lorentz transformation, since if U satisfies $A(\vec{x'}) = UA(\vec{x})U^\dagger$ then so does $-U$. Hence there is a 2 to 1 correspondece between the elements of $SL(2,\mathbb{C})$ and the proper orthochronous Lorentz transofrmation.

V. STEREOGRAPHIC PROJECTTION AND THE EXTENDED COMPLEX PLANE

Stereographic projection is the mapping of points on a sphere to points on a plane. In $\mathbb{R}^3$ with rectangular cartesian coordinates, x, y, z, consider the unit sphere with centre $(0,0,0,0)$, defined by

$$\mathbb{S}^2 \subset \mathbb{R}^3 : x^2 + y^2 + x^2 = 1.$$

(SEE FIG PG 4:1) $Q = Q(X,Y,0)$

The projection $P \to Q$ is a stereographic projection. A relationship between X,Y and (x,y,z) is constructed as follows. P is subdivided into the line segment NQ in some ratio, $l : m$ say. By coordinate geometry

$$\begin{aligned} x &= \frac{lX + mO}{l+m} = \frac{lX}{l+m}, \\ y &= \frac{lY + mO}{l+m} = \frac{lY}{l+m}, \\ z &= \frac{l.0 + m.1}{l+m} = \frac{m}{l+m}. \end{aligned}$$

This implies that

$$\begin{aligned} 1 - z &= \frac{l}{l+m}, \\ x &= (1-z)X, \\ y &= (1-z)Y. \end{aligned}$$

It is also known that $x^2+y^2+z^2=1$, so using this relation it is clear that

$$\begin{aligned} x^2+y^2 &= 1-z^2 = (1-z)(1+z) \\ \Rightarrow (1-z^2)(X^2+Y^2) &= (1-z)(1+z) \end{aligned}$$

If the point N is excluded, i.e. $z \neq 1$ then dividing by $(1-z)^2$ to obtain

$$X^2+Y^2 = \frac{1+z}{1-z}.$$

Rearranging to find that

$$z = \frac{X^2+y^2-1}{X^2+y^2+1} \tag{24}$$

Define $\zeta = X+iY$ and rewrite Eqn.(24) to see that

$$z = \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1},$$

Which implies that

$$1-z = \frac{2}{\zeta\bar{\zeta}+1}.$$

So relations for $(x,y,z) \in \mathbb{S}^2 \backslash \{N\}$ have been obtained in terms of ζ.

$$x+iy = \frac{2\zeta}{\zeta\bar{\zeta}}, \tag{25}$$

$$zz = \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1}. \tag{26}$$

Hence the points on $\mathbb{S}^2 \backslash \{N\}$ are labelled by complex numbers $\zeta \in \mathbb{C}$. If a point $\zeta = \infty$, called the point at infinity of $\mathbb{C}$, is allowed then the following limits hold:

$$\begin{aligned} x+iy &= \frac{2/\bar{\zeta}}{1+1/\zeta\bar{\zeta}} \to 0, \text{ as } \zeta \to \infty \\ z &= \frac{1-1/\zeta\bar{\zeta}}{1+1/\zeta\bar{\zeta}} \to 1, \text{ as } \zeta \to \infty \end{aligned}$$

Then $N=(0,0,1)$ corresponds to $\zeta=\infty$. Thus in this way there is a one to one correspondence between the points of $\mathbb{S}^2$ and the points of the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$, which is the usual complex plane with the point at infinity added. Since $\mathbb{S}^2$ has finite surface area, and is therefore called a *compact manifold*, the identification of the points of $\hat{\mathbb{C}}$ with the points of $\mathbb{S}^2$ is called the *compactification* of $\hat{\mathbb{C}}$.

$(\zeta, \bar{\zeta})$ are called the *stereographic coordinates* on $\mathbb{S}^2 \backslash \{N\}$. How are they related to the polar angles θ and ϕ? To investigate this write the usual spherical polar coordinates in terms of ζ. First it is known that

$$\begin{aligned} x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{aligned}$$

So using Eqn.(25) it is easy to show that

$$
\begin{aligned}
\cos\theta &= \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1} \\
\Rightarrow \zeta\bar{\zeta}\cos\theta + \cos\theta &= \zeta\bar{\zeta}-1 \\
\Rightarrow \zeta\bar{\zeta} = \frac{1+\cos\theta}{1-\cos\theta} &= \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} \\
\Rightarrow \zeta\bar{\zeta} &= \cot^2(\theta/2).
\end{aligned}
$$

Now using Eqn.(26) to obtain:

$$
\begin{aligned}
\sin\theta(\cos\phi + i\sin\phi) &= \frac{2\zeta}{\cot^2(\theta/2)}, \\
2\sin(\theta/2)\cos(\theta/2)e^{i\phi} &= 2\zeta\sin^2(\theta/2), \\
\Rightarrow \zeta &= e^{i\phi}\cot(\theta/2).
\end{aligned}
$$

This makes sense as it is clear that if $\zeta = \infty$ then $\theta = 0$ as one would expect. In summary the following coordinate transformations have been constructed

$$
\vec{n} = (x,y,z) \; = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{27}
$$

$$
= \left(\tfrac{\bar{\zeta}+\zeta}{\zeta\zeta+1}, i\tfrac{\bar{\zeta}-\zeta}{\zeta\zeta+1}, \tfrac{\bar{\zeta}\zeta-1}{\zeta\zeta+1}\right). \tag{28}
$$

Where here $\vec{n}$ is a unit vector in $\mathbb{R}^3$ such that $\vec{n}\cdot\vec{n} = 1$.

Let $\vec{x} = (x,y,z,t)$ be a point on the future null cone with vector $(0,0,0,0)$. Denote the future null cone as N^+ (not sure???), so that

$$
N^+ : x^2 + y^2 + z^2 - t^2 = 0, \text{ for } t > 0
$$

as all the vectors in the null cone have a Loretz quadratic form equal to zero by definition. (SEE FIG PG 4:5). The intersection of the spave-like hypersurface $t = \text{const} > 0$ is a 2-sphere denoted by

$$
\mathbb{S}^2(t) : x^2 + y^2 + z^2 - t^2 = \text{const}.
$$

There is a generator of N^+ passing through each point of $\mathbb{S}^2(t)$. These generators are the null geodesics tangent to N^+ and passing through the point $(0,0,0,0)$. Hence the points of $\mathbb{S}^2(t)$, denoted by (θ,ϕ) or ζ, label the *generators* of N^+.

For any $t > 0$ it is clear that

$$
\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t}\right)^2 = 1.
$$

Hence we can write

$$
\vec{x} \; = t(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 1),
$$

$$
= t\left(\tfrac{\bar{\zeta}+\zeta}{\zeta\zeta+1}, i\tfrac{\bar{\zeta}-\zeta}{\zeta\zeta+1}, \tfrac{\bar{\zeta}\zeta-1}{\zeta\zeta+1}, 1\right). \tag{29}
$$

It is shown explicitely that the direction of $\vec{x}$ is determined by (θ,ϕ) or ζ by comparing this to Eqn.(28). All possible directions of $\vec{x}$ on N^+ are covered if $\zeta \in \hat{\mathbb{C}}$. Now the Lorentz transformation $\vec{x} \to \vec{x'}$ is investigated. This tranformation takes the form

$$
\vec{x} \to \vec{x'} = t'\left(\frac{\bar{\zeta}'+\zeta'}{\bar{\zeta}'\zeta'+1}, i\frac{\bar{\zeta}'-\zeta'}{\bar{\zeta}'\zeta'+1}, \frac{\bar{\zeta}'\zeta'-1}{\bar{\zeta}'\zeta'+1}, 1\right).
$$

We say that the null direction ζ is trandformed to the null direction ζ'. The realtion between ζ and ζ' must also be determined. Construct the matix $A(\vec{x})$ as in Eqn.(22).

$$A(\vec{x}) = \begin{pmatrix} \frac{2t}{\zeta\bar{\zeta}+1} & \frac{2t\zeta}{\zeta\bar{\zeta}+1} \\ \frac{2t\bar{\zeta}}{\zeta\bar{\zeta}+1} & \frac{2t\zeta\bar{\zeta}}{\zeta\bar{\zeta}+1} \end{pmatrix} = c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix}.$$

Where $c_0 = 2t/\zeta\bar{\zeta} + 1 \in \mathbb{R}^2$. Note that as $\vec{x}$ is a null vector $\det(A(\vec{x})) = 0$. Thus the transformed matrix $A(\vec{x'})$ is given similarly as

$$A(\vec{x'}) = c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta'} & \bar{\zeta}'\zeta' \end{pmatrix}.$$

Now, as in the examples in section (IV) we determine the special linear matrix U such that

$$A(\vec{x'}) = UA(\vec{x})U^\dagger \tag{30}$$

As before, this matrix equation is written component wise as

$$c_0{}' \begin{pmatrix} 1 & \zeta' \\ \bar{\zeta'} & \bar{\zeta}'\zeta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} c_0 \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \bar{\zeta}\zeta \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}.$$

Then three separate relations between ζ and ζ' are obtained

$$c_0{}' = c_0(\alpha\bar{\alpha} + \alpha\bar{\beta}\zeta + \bar{\alpha}\beta\bar{\zeta} + \beta\bar{\beta}\zeta\bar{\zeta}), \tag{31}$$
$$c_0{}'\zeta' = c_0(\alpha\bar{\gamma} + \alpha\bar{\delta}\zeta + \bar{\gamma}\beta\bar{\zeta} + \beta\bar{\delta}\zeta\bar{\zeta}), \tag{32}$$
$$c_0{}'\zeta'\bar{\zeta}' = c_0(\gamma\bar{\gamma} + \gamma\bar{\delta}\zeta + \bar{\gamma}\delta\bar{\zeta} + \delta\bar{\delta}\zeta\bar{\zeta}). \tag{33}$$
$$\tag{34}$$

Using Eqns.(31) and (32) to obtain

$$\zeta' = \frac{c_0{}'\zeta'}{c_0{}'} = \frac{\alpha(\bar{\gamma} + \bar{\delta}\zeta) + \beta\bar{\zeta}(\bar{\gamma} + \bar{\delta}\zeta)}{\alpha(\bar{\alpha} + \bar{\beta}\zeta) + \beta\bar{\zeta}(\bar{\alpha} + \bar{\beta}\zeta)}$$

Thus

$$\zeta' = \frac{(\bar{\gamma} + \bar{\delta}\zeta)}{(\bar{\alpha} + \bar{\beta}\zeta)},$$

with $\alpha\delta - \beta\gamma = 1$ as before. This is a fractional linear transformation of the extended complex plane $\hat{\mathbb{C}}$. There is a one to one correspondence here between proper, orthochronous Lorentz transformations and fracational linear transformations of the extended complex plane. This is because the matrices $\pm U$ both satisfy Eqn.(30) as in the previous section., but now both matricex give the same transformation as the signs will cancel in the fractional transformation.

A. Singular and Non-Singular Lorentz Transformations

A given Lorentz transformation is equivalent to known α, β , γ and δ parameters module a sign and therefore gives an explicit fractional linear transformation. For a given Lorentz transformation a *fixed point* of the correspondong fractional linear transformation corresponds to an invarient null direction. The fixed points ζ satisfy the relation $\zeta' = \zeta$. Thus

$$\frac{(\bar{\gamma}+\bar{\delta}\zeta)}{(\bar{\alpha}+\bar{\beta}\zeta)} = \zeta,$$
$$\Rightarrow \bar{\beta}\zeta^2 + (\bar{\alpha}-\bar{\delta})\zeta - \bar{\gamma} = 0. \tag{35}$$

Clearly this is a quadratic equation over the field $\mathbb{C}$, thus it has two roots in general. Hence a Lorentz transformation does indeed leave two null directions invarient in general. The non-sigular case is when these roots do not coincide. If Eqn.(35) has only one root then the corresponding Lorentz transformation leaves one null direction invarient, this is the singular case.

Consider Eqn.(35) again. Divide by ζ^2 to obtain

$$\bar{\beta} + (\bar{\alpha}-\bar{\delta})\zeta^{-1} - \bar{\gamma}\zeta^{-2} = 0.$$

Hence $\zeta = \infty$ is a solution os this eqution if $\beta = 0$. If $zeta = \infty$ then $\vec{x}$ is given by $\vec{x} = t(0,0,1,1)$ by Eqn.(29). Thus it is clear that this corresponds to the null direction $z = t$. Compare this to example 3,section (IV C). Here β is zero AND the null direction is $z = t$ as expected. If $\zeta = 0$ is a solution to Eqn.(35) it is required that $\gamma = 0$, thus $\vec{x} = (0,0,-1,1)$. So it is predicted that a Lorentz transformation with a special linear matrix of the form

$$U = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},$$

will leave the $Z = -t$ null direction invariant.

B. Example: Standard Lorentz Transformation

Continuing on from Example 2, section (IV C), where α, β, γ and δ were determined. ζ' can now be expressed as

$$\zeta' = \frac{-\sqrt{\gamma_0 - 1} + \sqrt{\gamma_0 + 1}\zeta}{\sqrt{\gamma_0 + 1} - \sqrt{\gamma_0 - 1}\zeta}.$$

If th condition $\zeta' = \zeta$ is imposed then

$$\sqrt{\gamma_0 - 1}(\zeta^2 - 1) = 0$$
$$\Rightarrow \zeta = \pm 1$$

In the $\zeta = +1$ case $\vec{x} = t(1,0,0,1)$ and the invariant direction is $x = t$. Similarly in the $\zeta = -1$ case $\vec{x} = t(-1,0,0,1)$ and the invariant direction is $x = -t$.

(ADD SECTION 4b IN NOTES, MAYBE IN THE APPENDIX?)

VI. INFINITESIMAL LORENTZ TRANSFORMATIONS

There are Lorentz transformations that are small perturbations of the identity transformation and so $U \in SL(2,\mathbb{C})$ has the form

$$U = \pm \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon f \end{pmatrix},$$

where $a, b, c, f \in \mathbb{C}$ and ϵ is a small real parameter. Here terms of order ϵ^2 will be neglected. As $U \in SL(2,\mathbb{C})$ its determinant is calculated as

$$\det(U) = 1 + O(\epsilon^2).$$

Using this it is possible to obtain a relation between f and a

$$\begin{aligned}(1+\epsilon a)(1+\epsilon f)-\epsilon^2 bc &= 1+O(\epsilon^2),\\ 1+\epsilon(a+f) &= 1+O(\epsilon^2),\\ \Rightarrow f &= -a+O(\epsilon).\end{aligned}$$

Hence

$$U = \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix},$$

Now the explicit infinitesimal Lorentz transformations are calculated as in section IV, by substituting U into

$$A(\vec{x}') = UA(\vec{x})U^\dagger.$$

Now writing this out in component form to obtain

$$\begin{aligned}\begin{pmatrix} t'-z' & x'+iy' \\ x'+iy' & t'+z' \end{pmatrix} &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix}\begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix}\begin{pmatrix} 1+\epsilon\bar{a} & \epsilon\bar{c} \\ \epsilon\bar{b} & 1-\epsilon\bar{a} \end{pmatrix},\\ &= \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1-\epsilon a \end{pmatrix}\begin{pmatrix} (t-z)(1+\epsilon\bar{a})+\epsilon\bar{b}(x+iy) & (t-z)\epsilon\bar{c}+(1-\epsilon\bar{a})(x+iy) \\ (x-iy)(1+\epsilon\bar{a})+\epsilon\bar{b}(t+z) & (x-iy)\epsilon\bar{c}+(1-\epsilon\bar{a})(t+z) \end{pmatrix}.\end{aligned}$$

This then implies the three relations

$$t'-z' = t-z+\epsilon(a+\bar{a})(t-z)+\epsilon(b+\bar{b})x+i\epsilon(\bar{b}-b)y+O(\epsilon^2), \tag{36}$$

$$t'+z' = t+z-\epsilon(a+\bar{a})(t+z)+\epsilon(c+\bar{c})x+i\epsilon(c-\bar{c})y+O(\epsilon^2), \tag{37}$$

$$x'+iy' = x+iy+\epsilon(a-\bar{a})(x+iy)+\epsilon(b+\bar{c})t+\epsilon(b-\bar{c})z+O(\epsilon^2). \tag{38}$$

As $a, b, c \in \mathbb{C}$, set

$$a = a_1+ia_2,\ b = b_1+ib_2,\ c = c_1+ic_2.$$

Then subbing these into the above equations, eliminating t and z respectively from Eqn.(36) and (37) and taking real and imaginary parts of Eqn.(38) to obtain

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \epsilon\begin{pmatrix} 0 & -2a_2 & (b_1-c_1) & (b_1+c_1) \\ 2a_2 & 0 & (b_2+c_2) & (b_2-c_2) \\ -(b_1-c_1) & -(b_2-c_2) & 0 & -2a_1 \\ (b_1+c_1) & (b_2-c_2) & -2a_1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + O(\epsilon^2). \tag{39}$$

The above 4×4 matrix will be denoted as $L^i{}_j$, so that Eqn.(39) can be written simply as

$$\bar{x}^i = x^i + \epsilon L^i{}_j x^j + O(\epsilon^2). \tag{40}$$

Where $\bar{x}^i = (x', y', z', t')$. It is also necessary to check that the Lorentz invariance of the quadratic form still holds.

$$\begin{aligned}x'^2+y'^2+z'^2-t'^2 &= x^2+y^2+z^2-t^2-4\epsilon a_2 xy+2\epsilon(b_1+c_1)xt\\ &\quad +2\epsilon(b_1-c_1)xz+4\epsilon a_2 yx+2\epsilon(b_2-c_2)yt\\ &\quad +2\epsilon(b_2+c_2)yz-4\epsilon a_1 zt+2\epsilon(c_1-b_1)zx\\ &\quad -2\epsilon(c_2+b_2)zy+4\epsilon a_1 tz-2\epsilon(c_1+b_1)tx\\ &\quad -2\epsilon(b_2-c_2)ty+O(\epsilon^2)\\ &= x^2+y^2+z^2-t^2+O(\epsilon^2)\end{aligned}$$

Hence this transforamtion is still a Lorentz Transformation if we neglect terms of order ϵ^2.

Consider the time-like world line (SEE FIG pg 5:3)of a particle in Minkowkian space-time $x^i = x^i(s)$. If s is arc length or proper time then $v^i(s) = \frac{dx^i}{ds}$ is the unit tangent(NOT SURE WHY???) vector field. It is clear that $v^i(s)$ must be time-like as $x^i(s)$ is time-like, thus

$$\eta_{ij} v^i v^j = -1.$$

Where $\eta_{ij} - \text{diag}(1,1,1,-1)$ is the metric of minkowskiam space-time. This implies that

$$(v^1)^2 + (v^2)^2 + (v^3)^2 - (v^4)^2 = -1.$$

Now consider taking a step along the world line of the particle. Define $\bar{s} = s + \alpha$, where α is some real parameter, so that $v^i(s+\alpha) := \bar{v}^i(\bar{s})$. Hence we also have

$$(\bar{v}^1)^2 + (\bar{v}^2)^2 + (\bar{v}^3)^2 - (\bar{v}^4)^2 = -1,$$

and so $v^i(s)$ and $\bar{v}^i(\bar{s})$ are related by a Lorentz transformation. In particlular $v^i(s+\epsilon)$ and $v^i(s)$ are related by an infinitesimal Lorentz Transformation given by Eqn.(40),

$$v^i(s+\epsilon) = v^i(s) = \epsilon L^i{}_j(s) v^j(s) + O(\epsilon^2). \tag{41}$$

Rearranging to obtain

$$\frac{v^i(s+\epsilon) - v^i(s)}{\epsilon} = L^i{}_j(s) v^j(s) + O(\epsilon). \tag{42}$$

Now taking the limit as the infinitesimal step, ϵ goes to zero to obtain a continuous differentiable equation,

$$\frac{dv^i}{ds} = L^i{}_j(s) v^j(s). \tag{43}$$

This equation determines the trajectory of the particle through Minkowskian space-time. In terms of x this is equivalent to

$$\frac{d^2x^i}{ds^2} = L^i{}_j(s) \frac{dx^j}{ds}.$$

It is interesting to write these equations in terms of the particles 3-velocity given by

$$\vec{u} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right).$$

Start by using the chain rule on v^i,

$$v^i = \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, 1 \right) \frac{dt}{ds}.$$

Now determine the first integral of v^i, which is equal to -1 as v^i is time-like,

$$-1 = \eta_{ij} v^i v^j = \left\{ \left(\frac{d}{dt} \right)^2 + \left(\frac{d}{dt} \right)^2 + \left(\frac{d}{dt} \right)^2 - 1 \right\} \left(\frac{dt}{ds} \right)^2,$$

as this is just the scalar product in Minkowskian space-time. Therefore (NOT SURE WHERE THIS COMES FROM)

$$\frac{dt}{ds} = \gamma(s) := (1 - u^2)^{-1/2},$$

where $u = |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$. Thus from Eqn.(VI)

$$v^i = \gamma(u)(\vec{u}, 1) \tag{44}$$

It is now conveinient to display Eqn.(43) as two equations denoting the spacial part and the temproal part, in terms of γ and u. Again using the chain rule to obtain

$$\frac{dt}{ds}\frac{dv^i}{dt} = L^i{}_j v^j.$$

This then implies that

$$\begin{aligned} \gamma(u)\frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_j v^j, \\ \gamma(u)\frac{d}{dt}\gamma(u) &= L^4{}_j v^j, \end{aligned} \tag{45}$$

as $v^i = \gamma(u)(\vec{u}, 1)$. Here we have used the usual convention that greek indices denote the sum over the spacial indices only, thus $\alpha = 1, 2, 3$. Now Eqn.(44) can be used to rewrite the $L^i{}_j$ coefficients to get

$$\begin{aligned} L^\alpha{}_j v^j &= \gamma(u)(L^\alpha{}_\beta u^\beta + L^\alpha{}_4) \\ L^4{}_j v^j &= \gamma(u)(L^4{}_\alpha u^{\alpha 0}) \end{aligned} \tag{46}$$

where $L^4{}_4 = 0$ from Eqn.(39). Putting together Eqns.(45) and (46) to obtain differential equations for the spacial and temporal coordinates in terms of the particles 3-velocity,

$$\begin{aligned} \frac{d}{dt}(\gamma(u)u^\alpha) &= L^\alpha{}_\beta u^\beta + L^\alpha{}_4 \\ \frac{d\gamma(u)}{dt} &= L^4{}_\alpha u^\alpha \end{aligned}$$

(ON TOP OF pg 5:6)

VII. ACKNOWLEDGEMENTS