Worksheet 2 – Accelerated Motion

A Preview of the Derivative:

Some of you are taking calculus for the first time this semester and you will be learning how to take the derivative and the integral of a function. The derivative of the function represents the *rate of change* of that function. We know that we can find the average velocity from change in position and change in time; the derivative of the position is simply the average velocity, but for a super tiny amount of time called infinitesimal:

$$\vec{v}_{avg} = \langle \vec{v} \rangle = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Just like velocity is the rate of change of position, acceleration is the rate of change of velocity:

$$\vec{a}_{avg} = \langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

This means that the acceleration is the derivative of the derivative of the position, and we call this the second derivative and write it like this:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

You are going to learn why this is true and what the mathematical methods of evaluating the derivative are in your calculus class. In this class, we will simply use the tools that you learn about in calculus to solve some problems. Here is a short list of things you will learn in calculus this semester that we are going to use in this class:

1. Notation:

   The derivative may be written as either $\frac{d}{dx} f(x)$ or as $f'(x)$.

2. Powers:

   The derivative of x to a power is given by:

   $$\frac{d}{dx} ax^n = nax^{n-1}$$

3. The Sum of Two Functions:

   $$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

4. Polynomials:

   Putting 2 and 3 together yields:

$$\frac{d}{dx}\left[ax^n + bx^m + cx^\ell\right] = nax^{n-1} + mbx^{m-1} + \ell cx^{\ell-1}$$

5. The Product Rule:

   The derivative of the product of two functions is given:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

6. Trig Functions:

   The derivatives of sine and cosine are given:

$$\frac{d}{dx}A\sin x = A\cos x$$

$$\frac{d}{dx}A\cos x = -A\sin x$$

Developing the Equations for Constantly Accelerated Motion:

We have learned that when the derivative of a vector is taken, you simply take the derivative of each component separately. When we apply this idea to the position vector we get:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt} \quad \text{and} \quad v_z = \frac{dz}{dt}$$

When we apply the same principle to take the derivative of the velocity vector we have:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k})$$

$$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

Spe

$$a_x = \frac{dv_x}{dt} \quad \text{and} \quad a_y = \frac{dv_y}{dt} \quad \text{and} \quad a_z = \frac{dv_z}{dt}$$

What happens if the acceleration vector is a constant? It turns out that the motion lies within a plane. Because it lies within a plane, we can choose the z axis to be perpendicular to the plane so that $z = 0$ for the entire path of the object, and this means we have two-dimensional

motion. Finally, we will choose to make the y axis parallel to the direction of the constant acceleration vector. If we do these two things we get:

$$z = 0 \qquad v_z = 0 \qquad a_z = 0$$

$$a_x = 0 \qquad \text{and} \qquad a_y = a$$

Putting this into the equations for the velocity vector and the acceleration vector gives us:

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \qquad \text{Eq. (1)}$$

$$\vec{a} = a\hat{\jmath} \qquad \text{Eq. (2)}$$

For this type of motion, the general equation that results for the position and velocity vectors are:

$$\vec{r} = (x_0 + v_{0x}t)\hat{\imath} + \left(y_0 + v_{0y}t + \tfrac{1}{2}at^2\right)\hat{\jmath} \qquad \text{Eq. (3)}$$

$$\vec{v} = v_{0x}\hat{\imath} + (v_{0y} + at)\hat{\jmath} \qquad \text{Eq. (4)}$$

Note that in these equations all of the symbols on the right side of the equals signs *except time* are constants; they do not change as time progresses.

Problem 1 – Verify the equations for constantly accelerated motion by taking the derivative.

(a) Take the derivative of the position vector in equation 3, above. Does it equal the expression for the velocity vector in equation 4?

$$\frac{d}{dt}\vec{r} = \frac{d}{dt}(x_0 + V_{0x}t)\hat{\imath} + \frac{d}{dt}\left(y_0 + V_{0y}t + \tfrac{1}{2}at^2\right)\hat{\jmath}$$

$$\vec{V} = V_{0x}\hat{\imath} + (V_{0y} + at)\hat{\jmath} = \text{Eq } 4$$

yes

(b) Take the derivative of the velocity vector in equation 4, above. Does it equal the equation for the acceleration vector in equation 2?

$$\frac{d}{dt}\vec{v} = \frac{d}{dt}V_{0x}\hat{\imath} + \frac{d}{dt}(V_{0y} + at)\hat{\jmath}$$

$$\vec{a} = a\hat{\jmath} = \text{Eq } 2$$

yes

If you answer yes to (a) and (b), then congratulations!!! You have just derived the time dependent equations for constantly accelerated motion. These equations are typically written in their component form, written on the next page.

Two Dimensional Constantly Accelerated Motion Equations:

When acceleration is constant and is in the $+y$-axis direction:

$$x = x_0 + v_{0x}t \quad \text{where} \quad v_x = v_{0x} = \text{a constant} \qquad \text{Eq. (5)}$$

$$y = y_0 + v_{0y}t + \frac{1}{2}at^2 \quad \text{and} \quad v_y = v_{0y} + at \qquad \text{Eq. (6)}$$

Constant Acceleration in only One Dimension:

When the initial velocity vector is parallel or antiparallel to the constant acceleration vector, the resulting motion occurs along a line (the x-axis). This means that:

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \quad \text{and} \quad v_x = v_{0x} + a_x t \qquad \text{Eq. (6) again}$$

Problem Solving Strategies:

When you solve problems with constantly accelerated motion, you should always apply the following problem-solving method:

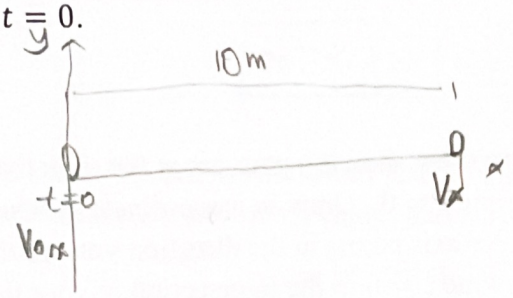
1. Read the problem statement slowly and carefully. Now read it again, but pay attention to the numbers that are specified.
2. Draw a cross section figure (from the side view) of the physical system at time $t = 0$. Then draw in a dashed line showing the path of the object as time progresses. Also draw in the object for which you wish to find something, call this time t.
3. Draw in a coordinate system showing the origin. You should always choose either the x or y axis parallel or antiparallel to the direction of acceleration.
4. Now you will apply the equations for constantly accelerated motion to solve the problem. This means use equations 5 and 6. Start by writing down the constant values for the equation as well as the final values of x, v and t:

$x_0 =$	$x =$	$v_{0x} =$	$v_x =$	$a_x =$
$y_0 =$	$y =$	$v_{0y} =$	$v_y =$	$a_y =$
$t =$				

5. Now read through the problem statement again and write in any known values of these constants.
6. Identify a strategy for applying the equations 5 and 6 to solve the problem. Do it.

Problem 2 - You are crossing a 10 m wide intersection, but you are speeding up as you go because you notice the countdown light telling you how soon the light will change. Assume that you start at rest but that you accelerate at a constant rate of 0.45 m/s^2. Find how much time it takes you to cross the street as well as the speed you have as you finish crossing the street.

(a) Draw a figure from the side view showing the street and you walking at time $t = 0$. Label your coordinate system, choosing the origin of the x axis to be where you are at time $t = 0$.



(b) The list of variables for this problem are shown below. Fill in any known values based on the information in the problem statement and your coordinate system.

$x_0 =$ 0m $x =$ 10m $v_{0x} =$ 0m/s $v_x =$

$a_x =$ 0.45m/s^2 $t =$

(c) Use equation 6 from page 4 to find the time it takes you to cross the street as well as your speed as you reach the other side of the street.

$x = x_0 + V_{0x}t + \frac{1}{2}at^2$, $V_x = V_{0x} + at$

$V_x = 0 + 0.45t$

$10 = 0 + 0 + \frac{1}{2}(0.45) \times t^2$

$\frac{391}{40} = t^2$

$\boxed{t = 3.13s}$ $\rightarrow V_x = 0.45t$

$\boxed{V_x = 1.41 m/s}$

$\boxed{\therefore t = 3.13s \quad V_x = 1.41 m/s}$

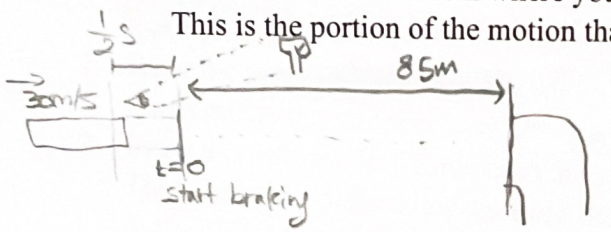
Worksheet 2 – Accelerated Motion

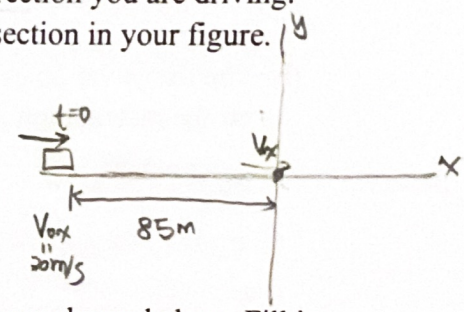
Problem 3 - You are driving at a constant speed of 30 m/s when you see a sign that indicates that you need to turn at the next block which is 100 m from you. Because of your reaction time, you do not start to brake the car until *half a second after* you see the sign for your turn. When you are braking your car experiences a constant acceleration of 1 m/s^2 that slows you down. Your goal in this problem is to find out how fast you are going as you reach the turn.

(a) How far from the intersection are you the moment that you begin to slow down?

$$100 - \left(30\,m/s \times \tfrac{1}{2}\,s\right) = \boxed{85\,m}$$

(b) Draw a top view of a street and an intersection showing your car at the time that you begin to slow down. We will call this time $t = 0$. Draw in a coordinate system where the origin is at the intersection and the $+x$-axis points in the direction you are driving. Draw a dashed line from where you are time $t = 0$ to the intersection in your figure. This is the portion of the motion that we will analyze.





(c) The variables for this portion of the motion that we will analyze are shown below. Fill in any known values based on the problem statement, what you found out in part (a) and the coordinate system you are using. In particular, pay attention to the sign of your values. Using this coordinate system has some consequences on signs.

Constants for 2nd Part:

$x_0 = 0$ $\quad x = 85$ $\quad v_{0x} = 30\,m/s$

$v_x =$ $\quad a_x = -1\,m/s^2$ $\quad t =$

(d) Now apply equation 6 from page 4 to solve for how fast you are going as you reach the intersection. Note that you will need to either first solve for t, or symbolically eliminate it.

$$\boxed{\therefore\ t = 2.98\,s \quad V_x = 27.02\,m/s}$$

$$x = x_0 + V_{0x}t + \tfrac{1}{2}at^2$$
$$85 = 0 + 30t + \tfrac{1}{2}(-1)t^2$$
$$85 = 30t - \tfrac{1}{2}t^2$$
$$t^2 - 60t + 170 = 0$$

quadratic formula $\quad t = \dfrac{30 \pm \sqrt{900 - 170}}{1}$

$$= 57.02\,s,\ 2.98\,s$$

57.02s doesnot make sense with the problem

So $\boxed{t = 2.98\,s}$

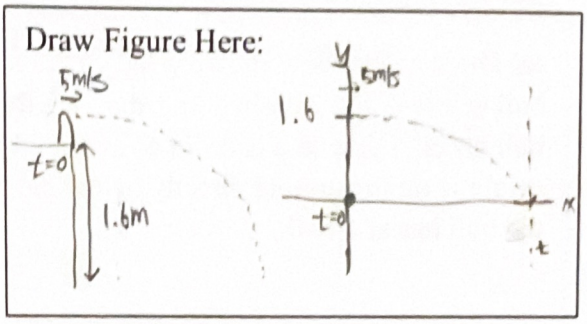
$$V_x = V_{0x} + at$$
$$V_x = 30\,m/s + (-1\,m/s^2)t$$
$$= (30 - t)\,m/s$$
$$V_x = (30 - 2.98)\,m/s$$
$$\boxed{V_x = 27.02\,m/s}$$

Worksheet 2 – Accelerated Motion

Problem 4 - You throw a ball horizontally with a speed of 5 m/s from a height of 1.6 m.

(a) Draw a side view showing the position of the ball at $t = 0$ and a dashed line showing the path the ball takes. Draw in a coordinate system where the origin is at the position on the ground directly below the ball at $t = 0$.

Draw Figure Here:



(b) Consider the motion of the ball from just after you throw it to just before it hits the ground. For this motion, identify all known values and enter them below.

$x_0 = 0$ $\quad x =$ $\quad v_{0x} = 5\,m/s$ $\quad v_x = 5\,m/s$

$y_0 = 1.6$ $\quad y = 0$ $\quad v_{0y} = 0$ $\quad v_y =$

$a_x = 0$ $\quad a_y = -9.8\,m/s^2$ $\quad t =$

(c) Find out how far from your feet the ball lands.

$x = x_0 + V_{0x}t + \frac{1}{2}a_x t^2$

$R = 0 + 5t + \frac{1}{2}(0)t^2$

$R = 0 + 5\left(\frac{4}{7}\right) + 0$

$\boxed{R = \frac{20}{7}\,m}$

$y = y_0 + V_{0y}t + \frac{1}{2}a_y t^2$

$0 = 1.6 + (0)t + \frac{1}{2}(-9.8)t^2$

$0 = 1.6 - 4.9t^2$

$4.9t^2 = 1.6$

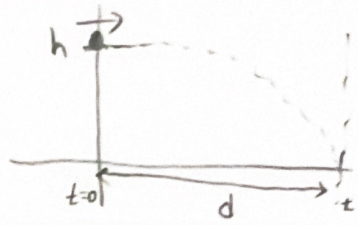
$t = \frac{4}{7}$

Distance from feet = _____

Problem 5 - You throw a ball horizontally from a height h and it lands a distance d from your feet.

(a) Draw a side view showing the position of the ball at $t = 0$ and a dashed line showing the path the ball takes. Draw in a coordinate system where the origin is on the ground directly below the position the ball has at $t = 0$.

Draw Figure Here:



(b) Consider the motion of the ball from just after you throw it to just before it hits the ground. For this motion, identify all known values and enter them below.

$x_0 =$ 0 $\quad x =$ d $\quad v_{0x} =$ $\quad v_x =$

$y_0 =$ h $\quad y =$ 0 $\quad v_{0y} =$ 0 $\quad v_y =$

$a_x =$ 0 $\quad a_y =$ $-g$ $\quad t =$

(c) Find the speed with which you threw the ball in terms of h, d and g.

$x = x_0 + V_{0x}t + \frac{1}{2}at^2$

$d = 0 + V_{0x}t + 0$

$d = V_{0x}\left(\sqrt{\frac{2h}{g}}\right)$

$\therefore V_{0x} = \dfrac{\sqrt{\frac{2h}{g}}}{d}$

$y = y_0 + V_{0x}t + \frac{1}{2}at^2$

$0 = h + 0 + \frac{1}{2}(-g)t^2$

$h = \frac{1}{2}gt^2$

$\sqrt{\frac{2h}{g}} = t$

Speed you threw ball = _____