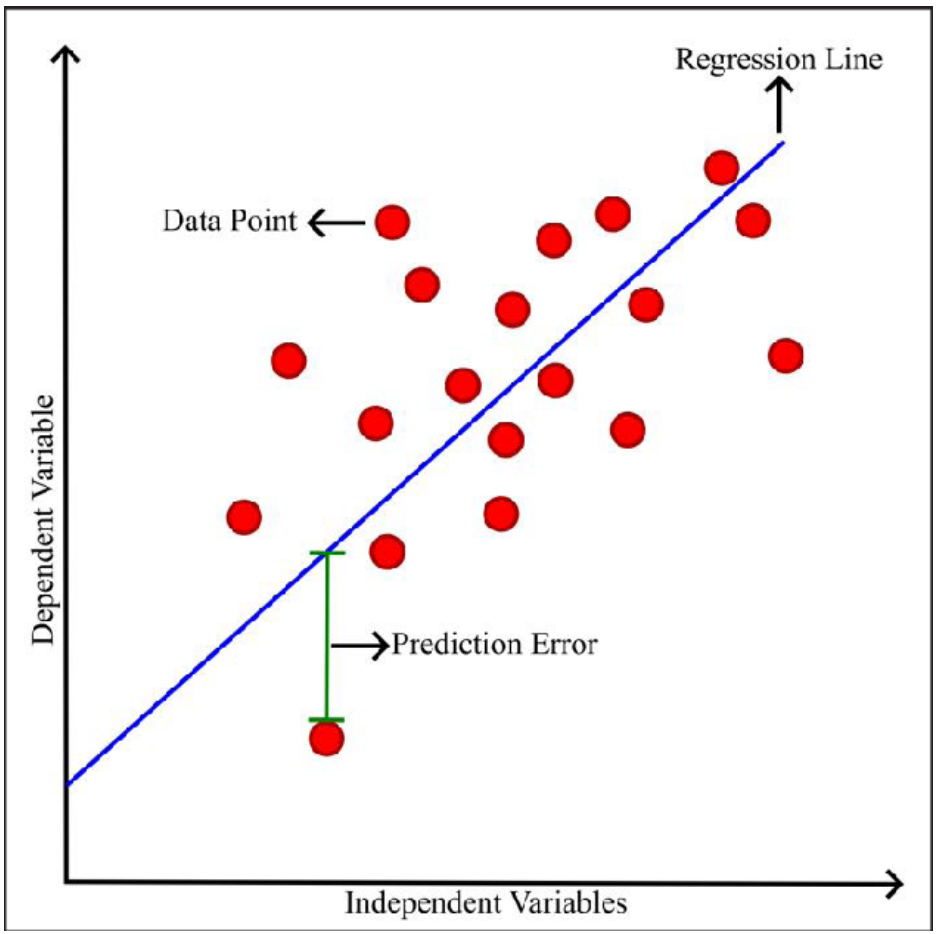
Regression Review

Review/Background - Introduction to Causal Inference

Kevin Li

Linear Regression

$$Y_i = \underbrace{\beta_0 + \beta_1 X_i}_{\text{best-fit line}} + \varepsilon_i$$



β_0 is the **intercept** of the line: expected value of Y when $X = 0$.

β_1 is the slope/**coefficient**: for a one unit increase in X, there is an expected β_1 change in Y.

ε_i is the **error**: not all data-points will be exactly on the line, and this represents how far individual i's Y value is from the line.

Multiple Linear Regression

We can have multiple independent variables $X_1, X_2, \ldots, X_p$:

$$Y_i = \underbrace{\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_p X_{ip}}_{\text{regression best-fit hyperplane (p-dimensional)}} + \varepsilon_i$$

- For any $\beta_j = \beta_1, \ldots, \beta_p$: for a one unit increase in X_j, there is an expected β_j change in Y, *holding all other independent variables constant*.

We can also write the same regression in matrix form in two ways:

$$Y_i = \mathbf{X}_i^\top \boldsymbol{\beta} + \varepsilon_i \iff \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

- The first form is used often to shorten the long multiple regression equation. The second form is used for proofs.

Fitted Values

The previous two slides introduced the **population model of regression** - i.e. the true relationship between independent and dependent variables in the population.

We generally do not know the true population relationships, so we have to use our **sample** to create estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$ and $\hat{\varepsilon}_i$.

- Estimation procedure on the next page.

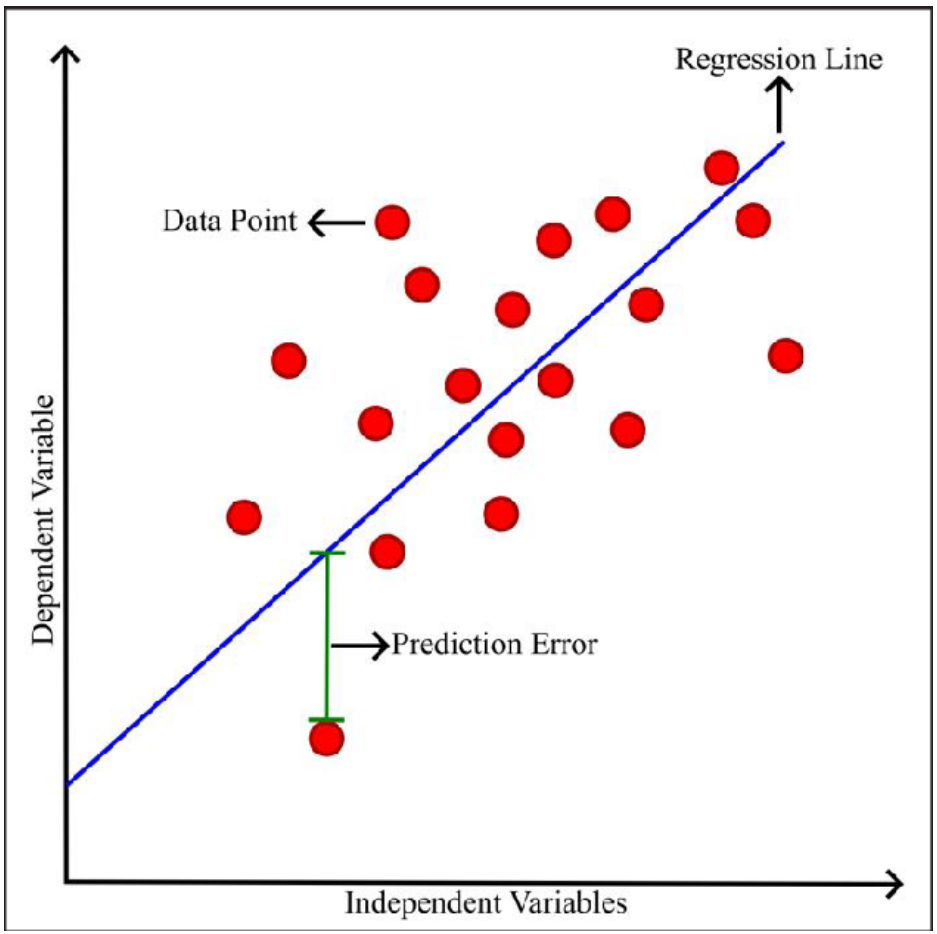
Once we estimate, we can generate our "fitted" model (fitted values), which is our sample-estimated best-fit prediction line.

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{i1} + \hat{\beta}_2 X_{i2} + \cdots + \hat{\beta}_p X_{ip}$$

- No ε_i in our prediction - why? On average, the prediction error should be 0, so we do not need to include it.

Estimating $\hat{\beta}_0, \ldots, \hat{\beta}_p$

Sum of Squared Errors $= \sum (Y_i - \hat{Y}_i)^2$



We want to find values for $\hat{\beta}_0, \ldots, \hat{\beta}_p$ that minimise the **sum of squared** (prediction) **errors**.

Why squared? We don't care about positive or negative errors, just the size of errors. Squaring gets rid of sign.

Makes sense: we want our fitted line to have as little prediction error as possible to capture the true relationships.

OLS Estimator

Estimating $\hat{\beta}_0, \ldots \hat{\beta}_p$ by finding $\hat{\beta}_0, \ldots, \hat{\beta}_p$ by minimising the sum of squared errors is called the **OLS** estimator:

$$\text{minimise:} \quad SSE(\hat{\boldsymbol{\beta}}) = \sum (Y_i - \hat{Y}_i)^2 = \underbrace{(\mathbf{y} - \hat{\mathbf{y}})^\top (\mathbf{y} - \hat{\mathbf{y}})}_{\text{linear algebra form}}$$

1. Multiple out the linear algebra equation above.
2. Take the partial derivative in respect to vector $\hat{\boldsymbol{\beta}}$.
3. Solve for $\hat{\boldsymbol{\beta}}$ to get the OLS estimates that minimise the SSE:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

Vector $\hat{\boldsymbol{\beta}}$ will contain estimated values for $\hat{\beta}_0, \ldots, \hat{\beta}_p$.

Estimators and Uncertainty

We usually only have a sample of individuals from the population, when we estimate $\widehat{\beta}$.

- What if we had a different sample with different individuals? We would get different $\widehat{\beta}$ estimate.

So we have to account for **sampling uncertainty**. This is done with a sampling distribution.

- Sampling distribution is basically - imagine you take a sample, estimate $\widehat{\beta}$. then take another hypothetical sample, and another. The distribution of estimates is the sampling distribution.

Standard deviation of a sampling distribution is the **standard error** of the estimate.

Unbiasedness and Variance

Our goal in statistics is to use our sample estimator $\hat{\beta}$ to estimate the true population β (we do not know the value of).

If the expected value of the sampling distribution $\mathbb{E}[\hat{\beta}]$ is equal to the true population value β (that we do not know), then the estimator is considered **unbiased**.

- That means on average, any estimate we run will have an expected value of the true population value.
- Thus, we want an unbiased estimator, since any specific estimate with any specific sample will be on average, correct.

We generally prefer unbiased estimators that have low variance.

- Low variance means estimates are less spread apart. If our estimator is unbiased, and the variance is low, that means any individual estimate is close to the true population value.

Gauss-Markov: Unbiasedness

Gauss-Markov theorem (at least part of it) states that OLS is an unbiased estimator of the true $\widehat{\beta}$ under the following conditions.

1. **Linearity in parameters**: The true relationship between X and Y can be represented by some form of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.
2. **Random Sampling**: Random sample from a population.
3. **No Perfect Multicollineraity**: No 100% (exact) linear correlations between explanatory variables.
4. **Strict Exogeneity**: Formally defined as $\mathbb{E}[\boldsymbol{\varepsilon}|\mathbf{X}] = 0$. This implies that $Cov(\varepsilon, X_j) = 0$ for any explanatory variable X_j.

Violations to exogeneity are often caused by omitted confounders. Thus, ommitted confounders = biased (bad) estimates (important!)

Variance (Heteroscedasticity)

The estimated variance of the OLS estimator (under heteroscedasticity):

$$\widehat{Var}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top \underbrace{\begin{pmatrix} \hat{\varepsilon}_1^2 & 0 & \dots & 0 \\ 0 & \hat{\varepsilon}_2^2 & \dots & 0 \\ \vdots & \dots & \ddots & 0 \\ 0 & 0 & \dots & \hat{\varepsilon}_i^2 \end{pmatrix}}_{\varepsilon \text{ variance matrix}} \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}$$

The estimate of the standard error of $\hat{\beta}$, $\widehat{se}(\hat{\beta})$, is just the square root of our estimated variance.

Sometimes, we will use other standard errors, like clustered standard errors (or rarely, normal homoscedastic errors). These simply just modify the ε variance matrix.

Hypothesis Testing

There is uncertainty in our estimates of any coefficient $\hat{\beta}_j$. How do we know the true β is **not** 0 with just our estimate (our null: $H_0 : \beta_j = 0$)?

We calculate a t-test statistic:

$$t = \frac{\hat{\beta}_j}{\widehat{se}(\hat{\beta}_j)}$$

Then, we use the t-test statistic and a t-distribution with $n - p - 1$ degrees of freedom to calculate the **p-value**.

- p-value is the probability the null is true ($\beta = 0$), given our estimate $\hat{\beta}$. If this is lower than 5%, the null is unlikely, so we reject the null and conclude there is significant relationship between X_j and Y.