Microeconomic Theory for Political Analysis

Book 1 of the Political Economics Sequence in An Introduction to Political Economics

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Part I

Introductory Microeconomic Theory

Chapter 1

Preferences and Utility Theory

Note: This course uses single variable calculus, set theory, and probability theory extensively. See the Quantitative Methods Sequence for help with mathematical concepts.

1.1 Agents and Preferences

Agents are anything that makes decisions. They can be people, firms, political parties, candidates, etc., anything that can make decisions.

Let us define X as a set of **alternative outcomes** that agents can achieve. Alternatives are both mutually exclusive and exhaustive:

- Mutually exclusive means that ending up with one outcome, means you cannot end up with any outcome. For example, on a math test, you have the potential outcomes of any score between 0 and 100. If you end up with a 95, you cannot at the same time, end up with an 84.
- Exhaustive means that every possible outcome of a situation is included in set X. For example, on a math test (assuming no extra credit), the set of all possible outcomes is $X = \{x : x \in \mathbb{R}; 0 \leq x \leq 100\}$. Or in more intuitive terms, set X contains all real numbers between 0 and 100 (we are assuming you can get decimal percentages).

$\succsim$ is a **preference relation** over the set of alternatives X. It indicates the relationship between two elements within X.

If $x, y \in X$, and when $x \succsim y$, that means the agent **weakly prefers** alternative x to y. Weakly preferences means that an agent either prefers alternative x to y, or is indifferent between x and y. However, they never prefer y to x.

Other preference relations include:

- **Strict Preference**: When $x \succ y$, that means an agent always prefers x to y. i.e. if they are given opportunities to choose between outcomes x and y, they will always, 100% of the time, select x over y.
- **Indifference**: When $x \sim y$, that means an agent is indifferent between x and y - they do not have a preference over which outcome they get.

1.2 Preferences and Rationality

We assume that agents have **rational** preferences. Preferences are considered rational if they are both complete and transitive.

Complete preferences means that for every two elements $x, y \in X$, either $x \succsim y$ or $y \succsim x$, or both.

- In more intuitive terms, for every two possible alternatives, agents must either prefer one alternative over another, or be indifferent between the two. They cannot be "unsure" of their preferences (note: unsure is not the same as indifferent).
- For example, if set X is all the possible outcomes of my math test, I must be able to select any two scores that are possible, and be able to compare them. I prefer outcome 100 to outcome 95 on my test. Maybe I am indifferent between 92 and 91. However, when asked, I am never "not sure" between any two outcomes.

Transitive preferences means that for three elements $x, y, z \in X$, if $x \succsim y$, and $y \succsim z$, then $x \succsim z$.

- A simple example is that - if I prefer apples to oranges, and I prefer oranges to watermelon, then, I should prefer apples to watermelon.
- This might seem arbitary with fruits, however, with numbers, it makes much more sense. If I prefer a score of 100 compared to 95, and I prefer a score of 95 to 90, then I should prefer 100 to 90.

If individual preferences of an agent are both complete and transitive, then, they are rational.

- Note the word "individual". As we will cover later, group preferences are often not rational in this sense.

1.3 Utility Function and Representation Theorem

Any rational set of preferences can be represented by a utility function.

A **utility function** is a way to map preferences over X into numerical values. Or more mathematically, $u : X \to \mathbb{R}$.

The utility function $u : X \to \mathbb{R}$ should be such that, for any $x, y \in X$, $u(x) \geq u(y)$ if and only if $x \succsim y$.

- More intuitively, this says that the utility function should match our preference relations. If we weakly prefer x to y, then our utility function should mirror this, so $U(x) \geq U(y)$.
- The same is true for $u(x) > u(y) \Rightarrow x \succ y$ and $u(x) = u(y) \Rightarrow x \sim y$.

The utility function allows us to represent the binary preferences (comparisons between two alternatives) in a numeric form, which allows us to compare all outcomes, and is much easier to manipulate mathematically.

One important thing to note is that utility functions are **ordinal**: they can be used to rank alternatives, such as if $x \succ y$. However, they cannot be used to tell <u>how much</u> one prefers x to y. In other words, the value $u(x) - u(y)$ has no inherent meaning.

A preference relation can be represented by a utility function only if it is rational (i.e. complete and transitive).

- If it is not complete, then for some outputs we would not be able to assign utilities, which would break the utility function.
- If it is not transitive, it cannot be represented in numbers, as numbers are transitive. For example, $5 > 4$, and $4 > 3$, so $5 > 3$ must be true for numbers. If preferences are not transitive, they cannot be represented by numbers which are bound by transitivity.

1.4 Agent Rationality

Given a set of choices X, what is the "best choice" of an individual. This is an important concept, since without considering an agent's best choice, we cannot make any predictions on the likely outcomes of scenarios involving decision making.

We can define the best choice as the alternative outcomes included in the **maximal set** of X, notated $M(\succsim, X)$, which is defined as:

$$M(\succsim, X) = \{x \in X : x \succsim y \forall y \in X\}$$

Or in intuitive terms, the maximal set is the set of all alternatives x in X, such that x is weakly preferred over alternative y for every alternative y in X.

- Or even more simply, basically, when comparing one outcome to all other outcomes, not a single other outcome is preferred over this outcome. If that is the case, add that outcome to the maximal set.

All outcome alternatives included in the maximal set $M(\succsim, X)$ are considered the "best choices" of an individual, since they are always weakly preferred to any other alternative $y \in X$.

If the utility function $u : X \to \mathbb{R}$ is a utility representation of the preference relations $\succsim$ of an agent on X, then the elements of the maximal set are:

$$M(\succsim, X) = \arg\max_{x \in X}\{u(x)\}$$

Or in more intuitive terms, the inputs $x \in X$ that make the utility function $u : X \to \mathbb{R}$ achieve its maximum value, are the elements of x in the maximal set $M(\succsim, X)$.

- Intuitively, since the maximal set contains the alternatives that are weakly preferred to all other alternative outcomes, the utility function output of that weakly preferred outcome should be equal or higher than all other utility function outputs for other alternatives.

This course on the microeconomic theory and its applications to political situations is focused on how agents, such as candidates, political parties, government officials, and others, make decisions according to their maximal set, and how these decisions impact political outcomes.

Chapter 2

Introduction to Game Theory

2.1 Modelling and Game Theory

A model is a simplified version of reality to help us understand the complex world around us. This is especially useful in the complex world of the social sciences.

To simplify reality, models make **assumptions**. These assumptions should be similar to reality, so we can draw conclusions to the real world. However, these assumptions should also make the model simple, and only focus on the relevant features and details.

For example, a driving map is a model. It does not contain all the information in the world - it does not record the height of buildings, the colour of the terrain, and so on. It even simplifies the spherical earth into a 2-dimensional flat map.

However, despite simplifications of reality, a driving map is amazing for navigation. If you are trying to get from point A to point B by car, you do not care about the spherical nature of the globe. The simplifying assumptions on a driving map allow us to navigate the world around us, without being a perfect copy of reality.

There are different forms of modelling in the social sciences, including decision theory, competitive equilibrium, and game theory. This book will mainly focus on game theory.

Game Theory is the study of mathematical models of conflict and cooperation between rational players. The key property of game theory is that <u>one player's actions, affect another player's outcomes/gains/payoffs</u>.

Other key properties of Game Theory include:

- There is **common knowledge** of the rules of the game. This basically means that every player knows that every other player knows the rules of the game.
- A player will predict what their opponents will do, and will react to their opponents.
- The optimal decision of a player, depends on their beliefs on what the other players will do.

2.2 Properties of Games and Preferences

Games are the environment in which strategic interaction between players occurs. For each game, it must consist of:

1. A set of **players**
2. For each player, a set of **actions** and **strategies**
3. For each player, **preferences** over the outcomes associated with the actions, such that they have an opinion on every possible outcome of the game

Players will take actions, and they yield some outcomes. The pathway between actions and outcomes is the **environment** - essentially the function that takes your action input, and outputs your outcome.

A player's belief on the environment - which includes the rules of the game, the other players' strategies, and the other players' beliefs, influence a player's choice of actions.

A player's preferences over the outcomes of games, in game theory, must be both **complete** and **transitive** (see chapter 1 for more detail).

- Complete preferences means that given any 2 outcomes, you can always compare them. Or in more intuitive terms, the player always has an opinion on the outcome - they are never "not sure" about how they feel about one outcome compared to another (however, they can be indifferent).
- Transtive preferences means that if you prefer a over b, and b over c, then you prefer a over c. It is basically that your preferences make sense and are consistent.

Preferences over outcomes can either be **ordinal** or **cardinal**

- Ordinal preferences means that you can rank your preferences of the potential outcomes, but you cannot give a specific value to that preference. For example, you can say you like a more than b, but you cannot say exactly how much you like a over b
- Cardinal preferences means that not only can you rank your preferences, you can also describe the magnitude of your preferences. For example, if you say you like a 3 times better than b, then you have cardinal preferences

We can assign numbers to both cardinal and ordinal preferences. Just be aware, if preferences are ordinal (and they often are), the numbers represent relative rankings, not specific magnitudes of preference.

- For example, if preferences are ordinal, the "distance" between preference values 1 and 2, may not be equal to the distance between 2 and 3

2.3 Types of Games

Games in Game Theory can be categorised according to two key properties: the **timing** of a game, and the **availability of information** in the game.

The timing of the game determines how players move relative to each other:

- A **static game** is when all players move "simultaneously". This does not actually mean they have to move at the exact same moment - it simply means that when one player moves, they cannot yet observe how the other player has moved. Thus, the players must anticipate the other players moves.
- A **dynamic game** is when players move sequentially, in an order. These are also called extensive form games. This means that players can see how their opponents moved, before they choose their move. For example, Chess is a dynamic game.

The availability of information is another distinction between types of games:

- A game of **complete information** is one where all payoffs are known to all players.
- A game of **incomplete information** is when some payoffs or some actions are not common knowledge. These games are quite a bit more complex.

We will start with static games of complete information - the simplest form and the best type of game to introduce core concepts.

2.4 Game Theory Notation

Before we start discussing games, we need to discuss notation. Notation is key, because it allows us to be very precise and concise in defining our games.

Players are denoted $i = 1, 2, ...N$.

- When we say player i, we are referring to any specific player in the game.
- When we say player 1, or player 2..., we are referring to a specific player in the game.
- N represents the total number of players in a game.

The actions of a player i are denoted a_i

- So, the actions of player 1 are denoted a_1, and so on

The strategies of a player i are denoted s_i

- A player's selected strategy, if applicable, is denoted with a star: s_i^*
- All other strategies are denoted with an apostrophe (more specifically, a set complement sign): s_i'

The action profile (of all players' actions) is a vector $a = (a_1, a_2, ..., a_N)$

- We can split this vector into 2 different parts. a_i denotes player i's actions, while a_{-i} denotes the action profiles of all other players not player i

Preferences are represented by a payoff function $u_i(a_i, a_{-i})$

- Essentially, u_i is a function with the inputs of player i's actions a_i, and the actions of everyone else a_{-i}

Part II

Static Games of Complete Information

Chapter 3

Dominant Strategy Equilibrium

3.1 Dominant Strategies

As we have established, game theory is when one person's actions affect another's payoffs. Thus, players have to anticipate what other players will do, and act accordingly.

However, sometimes, one of the actions available to a player i, is always better than any of the other actions available to them, no matter what their opponent decides to do. This strategy that is always better than the alternatives, no matter what the other player does, is called a **dominant strategy**.

Formally, a dominant strategy, notated s_i^D, is a dominant strategy, if it is player i's best strategy, no matter the strategies the other players pick. No matter what the opponent does, the payoff is highest for player i when they play s_i^D

Mathematically, we can represent that statement as following:

$$u_i(s_i^D, s_{-i}) > u_i(s_i', s_{-i})$$

Which translates into words as - the utility of player i choosing their dominant strategy s_i^D, is always greater than the utility of player i choosing another strategy s_i', no matter the strategies chosen by the opponents s_{-i}

If player i has a dominant strategy s_i^D, then their other strategies s_i' are considered to be **dominated**.

3.2 Dominant Strategies in Prisoner's Dilemma

An example of a dominant strategy is the famous game: prisoner's dilemma. The story is as follows:

Two suspects are arrested. The police holds the suspects in separate cells and explains the consequences of their actions. If they both remain quiet (cooperate with each other), then both will be convicted for minor offences and sentenced to 1 month in jail. If both rat each other out (defect), *then both will be sentenced to jail for 8 months. Finally, if one defects but the other does not, then the confessor is immediately released and the other is sentenced to 10 months, 8 for the crime and 2 for obstructing justice.*

We can represent this game in a matrix, which is often called the normal form of the game. Note, the first value in each cell is the payoff of player 1 (p1), and the second value in each cell is the payoff of player 2 (p2).

	Cooperate (p2)	Defect (p2)
Cooperate (p1)	-1, -1	-10, 0
Defect (p1)	0, -10	-8, -8

Let us look at player 1. What is the best move for player 1?

Pretend player 2 plays cooperate. Thus, we can hide the defect column:

	Cooperate (p2)
Cooperate (p1)	-1, -1
Defect (p1)	0, -10

What is player 1's best strategy? Well, it is to defect, since a payoff of 0 is greater than a payoff of 1. <u>So, when player 2 picks cooperate, player 1 should pick defect</u>.

Now, pretend player 2 plays defect. Thus, we can hide the cooperate column:

	Defect (p2)
Cooperate (p1)	-10, 0
Defect (p1)	-8, -8

What is player 1's best strategy? Well, it is to defect, since a payoff of -8 is greater than a payoff of -10. <u>So, when player 2 picks defect, player 1 should pick defect</u>.

So, we have established that player 1 will pick strategy defect, for both the scenarios that player 2 picks cooperate or player 2 picks defect. So, player 1 should always pick defect.

Thus, player 1's strategy of defect is the strategy that maximises their payoff, regardless of the strategy of player 2. Thus, <u>player 1's strategy of defect is a dominant strategy.</u>

We can do the same thing for player 2. We first pretend player 1 will pick cooperate, and we find player 2 should pick defect as their best strategy. Then we pretend player 1 will play defect, and we find player 2 should also pick defect as their best strategy. <u>Thus, player 2 also has a dominant strategy of defect.</u>

The Prisoner's Dilemma thus has a dominant strategy for both players - defect.

3.3 Solution Concepts

One of the key things about modelling situations is that, well, we want to find a likely outcome. An equilibrium solution concept is a concept that shows a likely solution to a game, by restricting potential strategies being played to the most reasonable one.

Any equilibrium solution concept has a few assumptions:

1. Players are rational - they choose actions that maximise their own payoffs. Without this assumption, it is very difficult to find a likely outcome
2. Structure of the game and player's rationality is common knowledge. If players do not know about the structure of the game, they well, cannot make best response decisions to their opponents
3. Any predictions must be self-enforcing - essentially, if we end up in a solution outcome, the players should not have any reason to want to leave that outcome, or else it would not be a final solution.

3.4 Dominant Strategy Equilibrium

A dominant strategy equilibrium is a solution concept where all players play dominant strategies.

- After all, if everyone has a best response to all other strategies of other players, and everyone is playing their best response, that is a very likely outcome of the game.

Formally, a strategy profile of all players $S^D = (S_1^D, S_2^D, ..., S_N^D)$ is a dominant strategy equilibrium, if for all players i, $u_i(S_i^D, s_{-i}) > u_i(s_i', s_{-i})$ for all s_{-i}

- Thie second part is the definition of dominant strategy. So, this basically says that a a set of strategies (strategy profile) is a dominant strategy equilibrium, given every player is playing a dominant strategy.

For example, take the **prisoner's dilemma**. We have already established that for both players, defect is a dominant strategy. Thus, if both players play defect, that is a dominant strategy equilibrium.

There, at most, can only be one dominant strategy equilibrium in a game. This makes dominant strategy equilibrium a great solution concept for predicting - as there will always only be one solution.

However, many games do not have any dominant strategies, and thus, many games do not have a dominant strategy equilibrium. As a result, this solution concept is not possible on a vast majority of games.

Chapter 4

Nash Equilibrium

4.1 Best Response

We previously discussed that dominant strategies do not appear in most games, and thus, we need an alternate way of determining a solution.

Thus, we need to introduce the concept of a **best response**. We already introduced the intuition of a best response in the section on the prisoner's dilemma.

A strategy for player i is a best response to another strategy played by the other players, if that chosen strategy for player i gives more payoff than any other strategy player i could choose.

- Essentially, we hold the other player's strategies constant. Then, we find what player i should play. We do this for every combination of other player's strategies.

More formally, the strategy s_i is a best response to the opponents specific strategy s_{-i}, if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, where s_i' are the other strategy profiles of player i that are not the chosen s_i

A strategy profile cannot be a best response, if there exists a **profitable deviation**.

- Essentially, given the opponent plays a certain strategy, if you can switch to a more profitable strategy, then the original strategy is not a best response.

4.2 Finding Best Responses

To find the best responses for player 1, do the following:

1. Assume player 2 plays a specific strategy
2. Find which player 1 strategy maximises player 1's payoff, given player 2 plays that specific strategy
3. Now assume player 2 plays another strategy
4. Now, find which player 1 strategy maximises player 1's payoff, given player 2 plays that specific strategy
5. Continue until all you have a best response to ever potential strategy of player 2

For example, let us find the best responses for player 1 in the following game (called the stag-hunt game)

	Stag (p2)	Hunt (p2)
Stag (p1)	2, 2	0, 1
Hunt (p1)	1, 0	1, 1

First, let us assume player 2 plays stag. Thus, let us hide the hunt (p2) column.

	Stag (p2)
Stag (p1)	2, 2
Hunt (p1)	1, 0

What is player 1's best response to player 2 playing stag? Well, since $2 > 1$, stag is the best response of player 1.

Now, let us assume player 2 plays hunt. Thus, let us hide the stag (p2) column.

	Hunt (p2)
Stag (p1)	0, 1
Hunt (p1)	1, 1

What is player 1's best response to player 2 playing hunt? Well, since $1 > 0$, hunt is the best response of player 1.

Thus, we have found all the best responses for player 1. You would do the same for player 2, and for any other players in any other game.

4.3 Nash Equilibrium

Nash Equilibrium is another solution concept. This solution concept is obtained when both players are playing a best response to the other player's chosen strategy.

As we discussed previously, a best response means a player has no profitable deviation. Thus, when both players are playing a best response to each other in a Nash Equilibrium, neither player has a profitable deviation.

- Thus, no player can do better by choosing a different strategy. That means the Nash Equilibria is stable: once we enter the equilibria, no one wants to leave it.

More formally, strategy profile s^* is a Nash Equilibrium, when $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$ for all s_i' and for all players i

- Or in more intuitive terms, a strategy profile is a Nash Equilibrium, when the chosen strategy of player i, given an opponents strategies yields a higher payoff than any other strategy that player i could choose.

All dominant strategy equilibria are Nash Equilibria, but not all Nash Equilibria are dominant strategy equilibria.

However, there are a few important caveats of Nash Equilibria:

- Nash Equilibria do not always make sense as sensible outcomes - we will see this throughout the book
- Often, there is more than one Nash Equilibria, and we are not sure which solution is the likely outcome of the game
- Sometimes, Nash Equilibria do not exist in a game (although this is less frequent than Dominant Strategy Equilibria)
- Finally, Nash Equilibria have many issues when it comes to dynamic/sequential games, which we will cover later.

4.4 Finding Nash Equilibrium

Finding Nash Equilibria is quite simple: we find the best responses for both players, just like we did in the "finding best responses" section previously.

For example, take this following game (game of chicken):

	Swerve (p2)	Straight (p2)
Swerve (p1)	0, 0	-1, 1
Straight (p1)	1, -1	-9, -9

Let us first find the best responses of player 1, while fixing player 2's strategies. The best responses are underlined/bolded

	Swerve (p2)	Straight (p2)
Swerve (p1)	0, 0	**-1**, 1
Straight (p1)	**1**, -1	-9, -9

Next, let us find the best responses of player 2, while fixing player 1's strategies. The best responses are underlined/bolded

	Swerve (p2)	Straight (p2)
Swerve (p1)	0, 0	**-1**, **1**
Straight (p1)	**1**, **-1**	-9, -9

We can see that the strategy profiles, where both players are playing best responses, are (Swerve, Straight), and (Straight, Swerve). Thus, these are the two Nash Equilibria.

4.5 Public Goods Contribution Games

So far, we have focused on relatively simple 2-player games. However, let us introduce a slightly more complex game: the public goods contribution game. The game takes the following form:

Each of n people chooses whether to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows:

- *(i) any outcome in which the good is provided and she does not contribute*
- *(ii) any outcome in which the good is provided and she contributes*
- *(iii) any outcome in which the good is not provided and she does not contribute*
- *(iv) and outcome in which the good is not provided and she contributes.*

What are the Nash Equilibria of this game? This game is more complex than the previous examples, since it involves both more players, and also, unclear amounts (such as k).

To approach this problem, we can do the following:

1. Divide the problem into different "scenarios", based on number of players that contribute relative to k, the threshold of providing the public good. For example, we could split this game into a scenario where less that k people contribute, where exactly k people contribute, and where more than k people contribute.
2. Find if there is any Nash Equilibria. While we cannot find the "best responses" as easily, recall that Nash Equilibria also is defined as no player having a profitable deviation. We can look at each player, and ask - *do they have a profitable deviation?* - and if they do not, we can conclude that is a Nash Equilibrium.

First, let us look at the scenario that *less than k people contribute.* Is there a Nash Equilibrium?

- Let us consider that 0 people contribute to the public good. Does any player have any incentive to deviate?
  - All players are playing "not contribute". Let us assume player i is one of them. Player i does not have a profitable deviation to "contribute", since that k, the threshold of the public good, cannot be lower than 2. Thus, deviating to "contribute" would lower player i's payoff from payoff number iii to payoff number iv (see question above)
  - Thus, 0 people contributing is a Nash Equilibrium, as no player has an incentive to deviate.
- Let us now consider that 1 person is contributing to the public good. Does any player have an incentive to deviate?
  - Let us assume player i is the only player that is contributing so far. Player i does have a profitable deviation given his opponents current strategies, since if he changes to "not contribute", he will increase his payoff from outcome iv to outcome iii.
  - This will be the case for all cases where $k-2$ or less people contribute, since no player's individual action will be able to reach the threshold of the public good, so all players

contributing would have an incentive to not contribute.

  - Thus, 1 person, and anything between and including 1 and $k-2$ people contributing, is not a nash equilibrium, since a profitable deviation exists.

- What if there are $k-1$ people contributing though?
  - Well, then if player i is not contributing, he has an incentive to deviate to "contribute", as that will reach the threshold k for the provision of the public good, thus boosting their payoff from outcome iii to outcome ii.
  - Thus, $k-1$ players contributing is also not a Nash Equilibrium, since a profitable deviation exists.

Now, let us consider that k people contribute to the public good. Does any player have any incentive to deviate?

- Let us say player i is already contributing. If they were to deviate to "not contribute", the total number of contributors would fall below k, thus moving player i from outcome ii to outcome iii, lowering his payoffs.
- Let us say player i is not-contributing. if they were to deviate to "contribute", the good is already and still being provided, so they would go from outcome i to outcome ii, lowering their payoffs.
- Thus, <u>k players contributing is a Nash Equilibrium</u>, since no player has a profitable deviation.

Finally, there is the scenario that more than k people contribute. Does any player have an incentive to deviate?

- Let us say player i is already contributing. If they were to switch to "not contribute", the public good would still be provided. They would move from outcome ii to outcome i, increasing their payoffs.
- Thus, there is no Nash Equilibrium when more than k people contribute, since there is a profitable deviation.

<u>Thus, there are two Nash Equilibria in this game: at 0 people contributing, and at k people contributing</u>. This game shows how you can analyse more complex games and find Nash Equilibria - just divide the game into scenarios, and look for players with profitable deviations.

Chapter 5

Pareto Efficiency and Extensions of Nash Equilibrium

5.1 Pareto Efficiency

Pareto efficient outcomes are outcomes that may or may not be feasible from a solution concept wise, but in theory, offer the best case for both players.

The definition of a **Pareto Efficient** outcome is that, given that outcome, it is impossible to move to any other outcome that makes at least one player better off, without making anyone else worse off.

- The key point here is **without making anyone worse off**. For example, if a game had two outcomes (2, 0) and (1, 10000000000), both would be pareto efficient, including (2, 0), since player 1 is made worse off when moving to outcome 2.

For example, let us take the classic prisoner's dilemma game:

	Cooperate (p2)	Defect (p2)
Cooperate (p1)	-1, -1	-10, 0
Defect (p1)	0, -10	-8, -8

The outcomes that are pareto efficient are (Cooperate, Cooperate), (Cooperate, Defect), and (Defect, Cooperate). All three are outcomes where it is impossible to move to another potential outcome, without making someone worse off.

The opposite of pareto efficient is **pareto inefficient** - an outcome where it is possible to make at least one player better off, without making anyone worse off.

- In the prisoner's dilemma, (Defect, Defect) is pareto inefficient, since we can move to (Cooperate, Cooperate), which makes both players better off, and no one worse off.

5.2 Prisoner's Dilemma as a Category of Games

A **Prisoner's dilemma** is not just one specific game with the storyline of the prisoners. It refers to all games that meet the following criteria:

- Each player has 2 strategies: cooperate or defect
- Each player has a **dominant strategy of defecting**
- The Nash Equilibrium that arises is thus (Defect, Defect), and that is the outcome of the game
- However, while (Defect, Defect) is the outcome of the game, it is pareto inefficient - both players in theory could be better off with (Cooperate, Cooperate), but because of the structure of the game, they both have the incentive to defect.

Prisoner's Dilemma games are used to show how individual rationality and group rationality differ.

- The group can get the best outcome (the pareto efficient ones) if they all cooperate.
- However, if you choose to cooperate, you run the risk that someone else is going to defect, undercutting you and giving you the worst payoff.
- Thus, it is in individual's incentives to always defect.

The Prisoner's Dilemma is frequently used to explain issues of **Public Goods**.

- Public goods are non-excludable - you cannot exclude people from using the good. An example of a Public Good is National Defence, or the air around us.
- Since these goods are non-excludable, people cannot be prevented from using the good, even if they do not contribute to the upkeep. Thus, there is a incentive for individuals to under-invest in the upkeep of public goods, reducing the quality for everyone.
- Under-invest in this case is considered the "defect" strategy, while invest is the "cooperate" strategy. Individually, you are better off if you defect, as you can still enjoy the benefits of this non-excludable good, without any of the costs. However, when everyone does this rational action, it results in a poor condition of the public good, thus hurting everyone (a non-pareto efficient outcome)
- An example is Air Quality - individuals and companies poluate and refuse to contribute to the upkeep of good air quality, despite the fact that poor air quality is worse for everyone (pareto inefficient)

Other examples of the application of Prisoner's Dilemmas include:

- Overuse of common resources (like fisheries), due to individual incentives, resulting in a pareto-inefficient outcome
- Training of workers - because firms are afraid that workers will leave and join a rival, firms do not train their workers properly, resulting in a less-than-optimal economy for everyone.
- Cartels - cartels can make the most money when everyone makes very little of a product (raising prices). However, if one person defects and makes a lot, they get a huge increase in profits, while every one else loses. Thus, fearful of others defecting, everyone defects, making production very high, and making everyone worse off (pareto inefficient)

5.3 Multiple Nash Equilibrium and Focal Points

We discussed how one of the weaknesses of Nash Equilibrium is the fact that multiple can exist in a single game (as we have shown with both the chicken, and stag-hunt game).

In theory, all Nash Equilibria are supposed to be equally likely to occur - there is no mathematical proof that one would occur more frequently than others.

However, there are two ways in which one Nash Equilibria may stand above the rest, and be the most likely outcome of the game.

First, if one Nash Equilibria is Pareto efficient, when the others are not, then that one is likely to be the outcome.

- This is because all players know the payoffs, since that is common knowledge, and all players are able to calculate Nash Equilibria.
- Since Pareto Efficient outcomes mean no player can do better without making anyone worse, and pareto-inefficient outcomes mean that at least one player can do better without making anyone worse, it generally makes sense for players to choose the Nash Equilibria that is Pareto Efficient.

Second, is the concept proposed by Schelling - Focal Points

- Sometimes, some Nash Equilibria are culturally more "notable" and likely than the other Nash Equilibria.
- For example, take the following game: You want to meet a friend in Paris, but your phones are both dead. If you both choose the same place, then you get a full payoff. if you two choose different places to meet, then neither get any payoff.
- In theory, any location in Paris is a Nash Equilibria - since as long as both of you are at the same place, you are getting the optimal outcome.
- However, culturally, there are some locations in Paris that are more "obvious". For example, many people would probably meet at the Eiffel Tower. Thus, the Eiffel Tower is a focal point - a likely solution to the game, due to cultural reasons.

5.4 Trembling Hand Perfect Equilibrium

Game theory, as we know, assumes players are rational, and play optimal outcomes. Empirical evidence has shown that this is sort-of true: on average, a large amount of people, when their actions are aggregated and averaged, do tend to make rational decisions.

- Of course, there are key exceptions to this (which is the topic of study for the fields of Behvioural Economics and Behvaioural Political Economy).

However, in no world is every single individual a rational one. Since Game Theory and models is often about predicting likely outcomes, we want to make sure our solution concepts, like Nash Equilibria, will survive even in the real world, where not everyone is rational.

- Essentially, we want our predictions to have some robustness. We want to make sure our model was not "cherry picked" just to make a certain prediction come true.

The **Trembling Hand Perfect Equilibrium**, proposed by economist Reinhard Selten, is a way to test the robustness of Nash Equilibria.

Essentially, we assume that one player of a game plays the "suboptimal" strategy at a probability of ϵ. Or in other words, they make a "mistake" at probability ϵ, and play the "correct" strategy at probability $1-\epsilon$.

Given one player does this, we see how the other player reacts, and if they still decide to go with the Nash Equilibrium strategy. If the other player decides to still go with the Nash Equilibrium strategy despite the suboptimal strategy of one player, the Nash Equilibrium is robust.

For example, let us take the following game:

	Left (p2)	Right (p2)
Up (p1)	1, 1	2, 0
Down (p1)	0, 2	2, 2

The two Nash Equilibria of the game are (Up, Left), and (Down, Right).

Let us test the robustness of the (Up, Left) equilibrium.

Assume that player 1 might be irrational at probability ϵ and select "Down". That means they will be rational at probability $1-\epsilon$, and select "Up". Let us call this probability of playing up at $1-\epsilon$ and probability of playing down at ϵ as the strategy profile s_1^{ϵ}

Let us assume player 2 is still rational. Does (Up, Left) remain a Nash Equilibria? To test this, let us find player 2's utilities for playing "left" and "right", given player 1 does what was listed above. If player 2 still prefers "left", then it is a robust Nash Equilibria.

The expected utility of player 2, when they play left, is defined as $u_2(L, s_1^{\epsilon}) = 1 \times Pr(Up) + 2 \times Pr(Down)$

- Where $Pr(Up) = 1-\epsilon$ is the probability player 1 plays up, and $Pr(Down) = \epsilon$ is the probability that player 1 plays down.
- Plugging in the probabilities, we get: $u_2(L, s_1^{\epsilon}) = 1(1-\epsilon) + 2\epsilon = 1+\epsilon$

The expected utility of player 2, when they play right, is defined as $u_2(R, s_1^{\epsilon}) = 0 \times Pr(Up) + 2 \times Pr(Down)$

- Where $Pr(Up) = 1-\epsilon$ is the probability player 1 plays up, and $Pr(Down) = \epsilon$ is the probability that player 1 plays down.
- Plugging in the probabilities, we get $u_2(R, s_1^{\epsilon}) = 0(1-\epsilon) + 2\epsilon = 2\epsilon$

Thus, when player 2 plays left, they expect a utility of $1-\epsilon$, and when the play right, they expect a utility of 2ϵ. The Nash Equilibrium (Up, Left) is only true of player 2 plays left, it is only true of $1-\epsilon > 2\epsilon$. Let us solve for ϵ

$$1 > 3\epsilon \rightarrow \epsilon < 1/3$$

Thus, the Nash Equilibrium (Up, Left) remains true as long as player 1 is irrational less than 1/3 of the time.

- This is usually considered quite robust. We usually assume ϵ is quite a small value, such as 0.01, or perhaps at most, 0.1.
- Thus, the (Up, Left) Nash Equilibria is quite robust against irrational play.

We can test the other Nash Equilibrium (Down, Right) with the same parameters:

- When player 2 plays left, $u_2(L, s_1^\epsilon) = 1\epsilon + 2(1-\epsilon) = 2 - \epsilon$
- When player 2 plays right, $u_2(R, s_1^\epsilon) = 0\epsilon + 2(1-\epsilon) = 2 - 2\epsilon$

(Down, Right) is only a Nash Equilibrium when player 2 plays right. Thus, $2 - 2\epsilon > 2 - \epsilon$. Let us solve for ϵ:

$$2 - 2 > -\epsilon + 2\epsilon \rightarrow \epsilon < 0$$

So, (Down, Right) is only a Nash Equilibrium when player 2 is irrational less than 0% of the time, which is impossible. Thus, (Down, Right) is not a robust Nash Equilibrium

Chapter 6

Mixed Strategy Nash Equilibrium

Chapter 7

Downsian Model of Electoral Competition

7.1 Downsian Model of Electoral Competition

Have you ever noticed that many businesses, that are competitors, are often located in the same place? For example, there are often competing petrol stations all around the same intersection. Or multiple fast food or coffee chains in the same place? Or a bunch of Chinese stores concentrated in one small area?

Hotelling argued this is no accident - these businesses have incentives to locate here. These locations are Nash Equilibriums of the competitive game between businesses.

The **Downsian Model**, also called the **Median Voter Theory**, applies this same logic to political parties. The model takes the following characteristics:

There are two political parties, who both want to win an election. The only "tool", or more formally, action, the two parties can take is to choose where to position their party's policy platform on a left-right spectrum (between -1 and 1). Citizens also have preferences along this left-right spectrum between -1 and 1, with the median voter's position being defined as m. Citizens will always vote for the party that is positioned closest to their preference on the left-right spectrum. The party that obtains the most citizens' votes wins. Indifferent citizens (that are equidistant to both party's positions) will have a 50/50 chance to vote for either party. If the parties tie in terms of votes, they each win the election with a 50/50 chance.

We can formalise this game into a strategic game:

- There are two players - political parties. Let us label them A and B
- The players actions are to simultaneously choose a policy position between -1 and 1: $p_i \in [-1, 1]$
- The objective is to attract the most votes (so 50% or above, since there are 2 parties).

What is the Nash Equilibrium of this game?

First, let us think about how any party would win the election: they need 50% of the votes. By definition, the median voter is in the 50th percentile. Thus, to win at least 50 percent of the votes, you must win the median voter. To win the median voter, you must be closer to the median voter's position m, than the other party.

Let us look at a few scenarios:

- What if party A is positioned to the right of the median, so $p_A > m$? Party B can best respond by playing anything closer to the median voter, which are $-p_A < p_B < p_A$. Then, party A can respond to that, and move even closer to the median voter, and so on...
- The same is the case if party A is positioned to the left of the median. Party B will best respond by playing anything closer to the median voter, and party A will do the same, and so on...

Eventually, the two parties will converge at the median voter's preference m. Since you always want to be closer to the median in order to win 50% of the vote, both parties will coverge at m.

Thus, the Nash Equilibrium of the Downsian Model is strategy profile $(p_A = m, p_B = m)$.

- We can check this by checking profitable deviations. Here, with both players playing m, they get a payoff of 0.5 chance of winning the election.
- If either player i moves away from m, they automatically lose the election since they are further from the median voter, thus they get payoff 0. Thus, there is no profitable deviation.

Is the Downsian Model correct? Do parties actually converge at the median? There is some mixed evidence.

An important thing to understand when evaluating models is that we do not evaluate models based on their conclusions. This is because the results of a formal mathematical model come directly from the assumptions of the model. To assess any model, we must critically assess the assumptions on which the model is built on.

7.2 Unbounded and Multi-Dimensional Models

In the Downsian Model, we assumed policy preferences are distributed in an interval $[-1, 1]$. However, even without this assumption, the Downsian Model still has a Nash Equilibrium at the median voter position m

- This is because the median voter is by definition, the 50th percentile. To win a majority, you need to win more than 50%. Thus, any bounds of is policy preferences will still have a Nash Equilibrium at the median voter position m

In the Downsian Model, we also assumed policy preferences are only in a one-dimensional plain R^1 (the real number line). However, in the real world, policy is often made on many different dimensions, such as a spectrum for environmental policy, immigration policy, social policy, and so on.

When we extend the model beyond one dimension, there is no Nash Equilibrium at all for the parties. The reason for this is that it imitates another game, called the *Budget Allocatio*n or *Divide the Dollar* game.

In this game, there are 3 "regions" $1, 2, 3$ that want budget from the central government, which has only a limited amount of funds. From each region, a politician is elected to represent them. To approve any budget, 2/3 of the politicians need to agree.

Let us represent the total central government budget as 1 (100%). Thus, the amount of money distributed to each region q_i, must meet the criteria: $q_1 + q_2 + q_3 = 1$. Each politician wants to get the maximum amount of budget for their region.

At first, we might think that a fair distribution between the 3 regions is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. However, there is a profitable deviation to this outcome. For example, take the budget $(\frac{1}{2}, \frac{1}{2}, 0)$ - remember, only 2/3 politicians need to agree to approve a budget. In this new budget, politician 1 increases the budget for their region from 1/3 to 1/2,, and politician 2 also increases the budget for their region. Thus, 2/3 politicians agree to the new budget, passing it.

However, there is also a profitable deviation to the budget $(\frac{1}{2}, \frac{1}{2}, 0)$ - $(0, \frac{3}{5}, \frac{2}{5})$. Here, politicians 2 and 3 are increasing their allocation of the budget, and thus, will vote in favor of the new budget.

In fact, for every single budget proposal, there is always an incentive for 2 players to deviate. We can actually demonstrate this with a cyclical example:

1. Let us start with distribution $(\frac{1}{2}, \frac{1}{2}, 0)$
2. We can deviate to distribution $(0, \frac{3}{5}, \frac{2}{5})$, as player 2 and 3 improve their payoffs.
3. Then, we deviate to $(\frac{2}{5}, 0, \frac{3}{5})$, as player 1 and 3 improve their payoffs.
4. Then, we can deviate back to our first distribution $(\frac{1}{2}, \frac{1}{2}, 0)$, and restart the cycle.

Essentially, there is never a Nash Equilibrium, because there is always an incentive for 2 players to deviate.

We can model a multidimensional Downsian Model in the same way. Each "dimension" of policy, like environmental, defense, social, etc., is a different budget allocation. Each party will always be able to "outflank" the other party by diverting budget away from one issue and giving it to another, making some citizens happier. Thus, there is no Nash Equilibria is the Downsian Model has multiple policy dimensions.

7.3 Other Distributions of Citizens

In the Downsian Model, we assumed that citizen/voter preferences are uniformly distributed. However, with any distribution of voter preferences, the median voter theory still holds. This does not matter if the voters are polarised with very few voters in the middle, if they are normally distributed, if they all lean left or right, etc.

The reason for this is that the median voter position, m, by definition, is the 50th percentile - 50% of the citizens are to the right of them, and 50% are to the left of them. Thus, to win any 2-party election, you need to win more than 50% of the votes, and by definition, the 50th percentile must be included in that winning majority.

Note that the median value can change. For example, in a country with very left-leaning citizens, the median position m might not be at 0, it could be $m = -0.5$. However, the optimal positioning of political parties will still be at the median voter m, no matter what value m takes.

7.4 Asymmetrical Voter Preferences

In the Downsian Model, we assumed that citizens will vote for the party whose preferences are closest to them. For example, if the citizen had an option of a party to the left within distance 0.2 and a party to the right within distance 0.3, they would prefer the party to the left.

However, not all real world preferences of voters are symmetrical. For example, a left-wing voter could be uncomfortable with radical-left, so they would actual prefer a candidate 0.2 to the right over a further left candidate 0.1 to the left. This preference would be non-symmetric - i.e. the citizen favours deviations in one direction over another, from their preferred position.

Even with asymmetric voter preferences, the median voter is still the Nash Equilibrium, as long as the preferences of citizens are single-peaked. Single peaked preference means that they only have one most-preferred policy position, and moving away from this most-preferred policy position from both sides will reduce utility.

7.5 Costs of Voting and Abstaining Voters

In the Downsian Model, we assumed that there is no cost to voting for citizens, and all citizens will vote (rather than abstain).

In this scenario, there is no longer a Nash Equilibrium at the median voter. If both parties are situated at the median, no rational voter will vote if voting is costly - after all, no matter how they vote, the winning party will have the policy platform of the median voter. Thus, their vote is not pivotal to affect the policy actually implemented, and voting (with cost) would simply not be worth it.

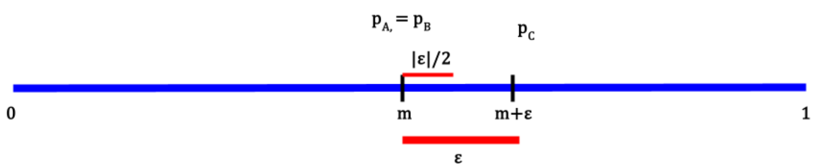
Only if parties are somewhat different, will voters be willing to bear the cost of voting to cast a vote. This way, their vote will actually matter, and they will be willing to bear the cost of voting. Thus, there is still a converging force towards the median in this scenario (since more towards the median means a higher chance of winning), but the two party's positions would stay far enough apart to incentivise voters to vote.

7.6 Expanding Downsian Competition to 3 Parties

In the Downsian Model, we assumed that there are only 2 parties and candidates. What if there were more than 2 parties (as is frequently the case)?

In a 3 party election, there would no longer be a Nash Equilibrium at the median voter position. This is because any party i has a profitable deviation in this scenario.

If party i were to slightly deviate by ϵ from the median voter, their new position on the spectrum would be $m + \epsilon$. The figure below shows this scenario:



In the figure above, we can see party i that deviates ϵ from m will win a vote share of $\frac{1}{2} - \frac{1}{2}\epsilon$, while the other two parties combine for $\frac{1}{2} + \frac{1}{2}\epsilon$, which is split between the two parties, so each party has a vote share $\frac{1}{4} + \frac{1}{4}\epsilon$.

Thus, we can write an inequality to see for what deviation ϵ for player i, where player i's vote share will be higher than the other two party's vote share.

$$\frac{1}{2} - \frac{1}{2}\epsilon > \frac{1}{4} + \frac{1}{4}\epsilon$$

$$\frac{1}{4} > \frac{3}{4}\epsilon$$

$$\epsilon < \frac{1}{3}$$

Thus, when $\epsilon < \frac{1}{3}$, then party i gets the most votes of any party. Since if they did not deviate by ϵ and stayed at the median, they would have tied, that means party i has a profitable deviation when the deviation $\epsilon < \frac{1}{3}$. Since party i has a profitable deviation, we can conclude that <u>there is no Nash Equilibrium at the median voter in a 3 party election</u>.

Chapter 8

Further Spatial Models

8.1 Policy-Seeking Models

In the Downsian Model, we assumed that parties only care about winning elections (called rent-seeking behaviour). Thus, they are willing to adopt any policy that will help them win elections.

However, many politicians actually hold strong beliefs over what policies are right and wrong. Thus, it is reasonable to expect that some parties may derive utility from the specific policies that can implement (policy-seeking behaviour). For example, while a socialist candidate may more easily win elections acting as a moderate at the median voter, they might also lose utility because they personally believe strongly in socialist policies.

We can create a model for this situation. Let us pretend there are two parties, the *right* party R and the *left* party L, who are on some political spectrum in $[-1, 1]$. Let us assume party R prefers the policy position $p_R = 1$ the most, while party L prefers the policy $p_L = -1$ the most.

- We can represent this in a utility function - where parties get more utility when the policy implemented by the winning party is closer to their preferred position, and get less utility when the policy implemented by the winning party is further from their preferred position.
- Mathematically, utility of party R is $u_R(p) = -(p-1)^2$, where p is the implemented policy position. This is because as the implemented policy p goes further from 1, the utility falls in the utility function. The peak utility of this function is at policy position 1.
- Similarly, utility of party L is $u_L(p) = -(p+1)^2$, where p is the implemented policy position. This function decreases as you go away from policy position -1, and has a maximum utility at policy position -1.

We can also define $\pi(p_L, p_R)$ as the probability party L wins the election. Thus, we can write out the utility function of party L:

$$u_L(p_L, p_R) = \pi(p_L, p_R) \times -(p_L+1)^2 + [1 - \pi(p_L, p_R)] \times -(p_R+1)^2$$

Let us break this utility function down into its constituent parts to better understand it:

- The left hand side of the equation, $u_L(p_L, p_R)$, is the utility function of party L, given party L plays some position p_L, and party R plays some position p_R.
- The first part of the right hand side, $\pi(p_L, p_R) \times -(p_L + 1)^2$, consists of two parts. $\pi(p_L, p_R)$ is the probability party L wins the election. $-(p_L + 1)^2$ is the utility function of party L we created previously, but since party L wins the election, we replace p with p_L. These two parts essentially state: given the probability party L wins the election, how does it feel about the own policies it is implementing?
- The second part of the right hand side, $[1 - \pi(p_L, p_R)] \times -(p_R + 1)^2$, consists of two parts. $[1 - \pi(p_L, p_R)]$ represents the probability that party R wins the election, since $\pi(p_L, p_R)$ is the probability that party L wins the election, and in a two party election, the probability that party R wins the election is simply $1-$ the probability that party L wins. $-(p_R + 1)^2$ is the utility function of party L we created previously, but since party R wins the election, we replace p with p_R. These two parts essentially state: given the probability party R wins the election, how does party L feel about the policies that party R is implementing.

So essentially, the utility function of player L is a expected utility of the probability of party L winning the election and implementing its own policies, and the probability of party R winning the election and implementing their probabilities.

Despite this additional complexity with the policy-seeking behaviour of policies, the median voter is still the Nash Equilibrium of this game. This is because while both parties would like to implement non-median policies, if they deviate away from the median, they will let the other party win, and thus have to deal with the other party's policies, which they find even worse. If you win the election, you implement your own policy, and deny the opponent their chance of implementing their policy that you disagree with.

8.2 Citizen-Candidate Model

The Downsian Model assumes that parties can select any policy position on the spectrum that will help them win the election. However, in reality, this is not the case. Let us take Bernie Sanders - pretend he was in a presidential election. Bernie decides that in order to maximise his chances of winning, he is going to move to the very centre of the political spectrum where the median voter is. Do you think the voters will believe Bernie has suddenly became a centrist? Likely not - they will assume that Bernie is still a far-left candidate, and many will still vote as if Bernie was a far-left candidate, despite his changed position.

Now, it is not about the politician's policy promises on the political spectrum. Instead, voters are voting on the politician's personal identity and personal preferences on the political spectrum.

The **Citizen-Candidate Game** is a new strategic game to represent this situation.

- The players of the game are a continuum of citizens along a spectrum, that are both potential voters and candidates.
- Each citizen has two actions: either remain a voter, or decide to run for office as a candidate. Citizens who remain voters will vote for the candidate closest to their personal preferences.

- The utility of a player is determined by how close the personal preferences of the candidate is from their own personal preferences.
- When no candidate enters the race, some status-quo policy $\hat{p}$ is implemented.
- There is a cost to choosing the action of becoming a candidate, the cost being some number δ.
- For simplicity, let us assume the preferred policy of the median citizen is $m = 0$, on a spectrum between $[-1, 1]$

First, let us test if there is a Nash Equilibrium at the median voter $m = 0$, where only the median voter decides to become a candidate:

- The median citizen will only enter as a candidate, if the cost of voting δ is less than their dislike of the status-quo policy $\hat{p}$. After all, if they do not dislike the status-quo policy, why bear the cost of becoming a candidate?
- Let us say that indeed, $\delta \leq |\hat{p}|$, and the median voter enters the election. If the median voter enters the election, we know that no other voters will enter the race. This is because the median voter will always beat anyone who is not at the median, so other citizens will have no incentive to bear the cost of entering as a candidate δ if they know they will lose for sure.
- Thus, there is a Nash Equilibrium where the median voter runs, given $\delta \leq |\hat{p}|$, as no other citizen has an incentive to join the election.

Let us now test if there are any equilibrium with 2 candidates deciding to enter the race:

- First, we know that any pair of candidates running must be equidistant from the median. After all, if one candidate is closer to the median, they will for sure win the election, and thus, the other candidate will have no incentive to join.
- Thus, we will have two players (let us call them R and L), where their positions are $p_R = \Delta$ and $p_L = -\Delta$ (you could more accurately write them as $m \pm \Delta$, but since we assume $m = 0$, we do not need to include the m).
- If L does not enter the race, then player R will win the race and implement their policy. Same with the reverse.
- However, L and R will have no incentive to enter the election, if they do not hate each other's policies that much. More specifically, they will not enter the race if their dislike of each other's policy does not exceed the cost of entry δ
- Thus, when $\delta \leq 2\Delta$ (where 2Δ is the distance between Δ and $-\Delta$), both player L and R will join the election. No other candidates will join - citizens closer to the median do not dislike either L or R policies enough to endure the cost of joining δ, and any other citizen further from the median will for sure lose to player L and R, thus unwilling to bear the cost δ if they are guaranteed to lose.
- Thus, there is an equilibrium with 2 players becoming candidates, where the two players are equidistant distance Δ from the median m, and when the inequality $\delta \leq 2\Delta$ is met.

8.3 Public Finance Model

Parameters of the Model

When political parties and candidates run, they often talk about how much taxes they will implement in order to provide government services and public goods. How should political parties decide what tax policy to implement, and how much public goods to provide?

We can represent this situation more formally:

- Citizens are different from each other, with varying incomes y_i. This is become some are "talented" and earn a lot, while others make less income.
- Citizens care about how much they consume privately c_i, and also how much they can enjoy the public good g
- The government provides the public good g, but can only pay for this public good through taxes t (we are assuming that there is no borrowing for funding public goods).
- However, increasing taxes t will be taken from the citizen's income y_i, which will reduce the amount they will be able to consume privately c_i

The game that follows has the following rules:

- The government provides a public good $g \geq 0$, financed by taxes $t \in [0, 1]$
- The government can only spend money on the public good, and cannot raise money in any other ways than taxes. Thus, $ty_1 + ty_2 + ... + ty_n = nt\bar{y} = g$. More simply, taxes raised is the tax rate times one's individual income, and the aggregate tax collected from everyone is the tax rate, times the average income, times the number of individuals. That is equal to the amount spent on the public good g.
- Finally, we assume the mean of income $E[y_i] = \bar{y}$ is more than the median of income y_m, since almost every distribution of income in the world is right-skewed.

The utility of any citizen i is $u_i = c_i + H(g)$, where c_i is the private consumption of the individual (income after taxes), and $H(g)$ is the utility derived from the public good.

- Since c_i is private consumption, which is income after taxes, we can define it as a function of taxes and individual income: $c_i = (1 - t)y_i$, or essentially, all the income left over after taxes.
- We will assume $H(g)$, the utility derived from the public good, has decreasing marginal utility. This means that more g will always yield a higher utility, however, with each additional g, the gain in utility is smaller.

Individual Citizen Preferences

Now the question is, what is each individual citizen's preferred amount of public good g? Well, we can solve for this.

Recall that utility of any citizen is $u_i = c_i + H(g)$. We also know that $c_i = (1 - t)y_i$. Let us substitute c_i into the utility function:

$$u_i = c - I + H(g)$$

$$u_i = (1-t)y_i + H(g)$$

Next, recall that earlier, we determined that $nt\bar{y} = g$, or essentially, aggregate tax income equals public good provision. We can solve for t, to get $t = \frac{g}{n\bar{y}}$. We can plug in that t into our utility function:

$$u_i = (1-t)y_i + H(g)$$

$$u_i = \left(1 - \frac{g}{n\bar{y}}\right) y_i + H(g)$$

Now, notice how we only have one variable left in this equation, g, which is good, since we are interested in the citizen's preferred amount of g. Let us rewrite the function as a function of g:

$$u_i(g) = \left(1 - \frac{g}{n\bar{y}}\right) y_i + H(g)$$

Now, what is each citizen's preferred g? Well naturally, it is the amount of g that maximises their utility function. How do we find maximums? By taking the derivative, and setting the derivative equal to 0:

$$u_i(g) = \left(1 - \frac{g}{n\bar{y}}\right) y_i + H(g)$$

$$u_i(g) = y_i - \frac{y_i}{n\bar{y}} g + H(g)$$

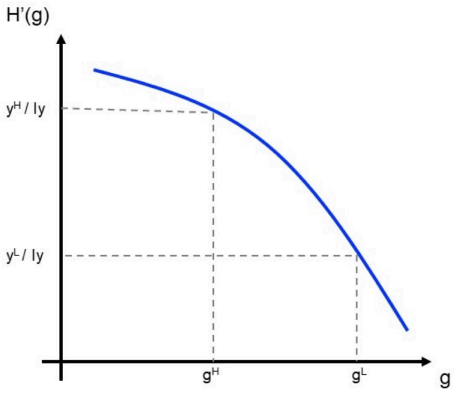
$$u_i'(g) = -\frac{y_i}{n\bar{y}} + H'(g)$$

Set the derivative equal to 0:

$$0 = -\frac{y_i}{n\bar{y}} + H'(g)$$

$$H'(g) = \frac{y_i}{n\bar{y}}$$

What does this equation tell us? First, we know that $H(g)$ had decreasing marginal utility, which means $H'(g)$ must be decreasing as g increases (see blue line below). We also know that when income y_i is high, $H'(g)$ is high as well. When y_i is low, $H'(g)$ is low.



Thus, we can see, when y_i is high y^H (which means $H'(g)$ is high), the corresponding g is quite small. When y_i is low y^L (which means $H'(g)$ is low), the corresponding g is quite large.

This tells us that richer individuals prefer smaller g (less public good, which also implies lower taxes), which poorer individuals prefer larger g (more public good, which also implies higher taxes). This makes sense based on common intuition.

Spatial Model

Now, we can model these preferences in a Downsian Game. Two parties A and B compete on the policy position of size of government, choosing positions p_A and p_B on the amount of public good provision (and rate of tax). These parties can either be rent-seeking or policy seeking, this does not matter. The party who wins a majority of the votes implements the public good policy that have chosen.

As we have established before in the Downsian Model, the median voter is the Nash Equilibrium where both parties will converge to. Thus, the median voter's preferences, given their income y_m, is the outcome of the game:

$$H'(g) = \frac{y_i}{n\bar{y}} = \frac{y_m}{n\bar{y}}$$

This model thus makes a few predictions:

- A more skewed income distribution with a few very very rich people, which would mean a smaller y_m, should mean that in a 2-party democracy, we should see a larger government that provides more public goods.
- A richer country, thus with a higher median income y_m, should also have a larger government.

These predictions can be empirically tested. This shows how microeconomic models can be used to create hypotheses and theories regarding political phenomena, which we can then test with econometric methods and data.

8.4 Valence Politics Model

Part III

Dynamic Games

Chapter 9

Dynamic Games of Complete Information

Chapter 10

Elections as Incentive Devices

Part IV

Bayesian Games

Chapter 11

Dynamic Games of Incomplete Information

Chapter 12

Perfect Bayesian Equilibrium

Chapter 13

Signalling Games

Part V

Advanced Games