Econometrics for Political Analysis

Book 1 of the Econometrics Sequence in An Introduction to Political Economics

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Part I

Econometrics and Causal Inference

Chapter 1

Goals of Econometrics

This book is designed to be both an approachable, but also rigorous, introduction to Econometrics, specifically for the analysis of political institutions and actors. I highly recommend that anyone who is interested in studying these concepts review the mathematical/statistical theory introduced in the Quantitative Methods sequence (especially the first book).

This book will also use the R language to implement some methods, so a basic understanding of the language is useful.

1.1 Econometrics

The goal of econometrics is to use data to answer economic questions. These economic questions often revolved around human decision making and social settings, in which traditional statistical methods did not work well for.

Political Science has increasing adopted these econometric approaches, as just like economics, political science is about how individuals and institutions behave in social settings. The similarities between the two fields allows econometric techniques to work quite well in studying political questions.

Econometrics studies relationships between features, so we can make inferences: using facts that we do know (data) to learn facts that we do not know. There are 3 types of inference that are common in econometrics:

1. **Descriptive Inference**: learning about the "true value" of a relationship between features in the world. For example, what is the actual relationship between economic development and democratisation?
2. **Predictive Inference**: learning about new observations from previous observations. For example, can we predict how some person will vote, based on how others vote? Can we predict the winner of a political election, given past historical electoral trends?
3. **Causal Inference**: how a certain feature (treatment) directly causes some outcome. This is very important for understanding the world. For example, if we are making a policy to fix a problem, we want to be sure that the policy causes a certain outcome that fixes that problem.

1.2 Descriptive Inference

Descriptive inference is about learning about some relationship in the world. This might seem very simple: just measure the relationship! However, there are many issues in reality when trying to do this.

This biggest issue is with how large populations are. For example, if we want to see the relationship between individual income and likelihood to vote in the UK, we would have to ask every single voter in the UK, likely over 50 million individuals. Researchers do not have the time or money to do this!

Thus, we need some way to use a smaller number of individuals, to learn about a larger population. This is the goal of descriptive inference.

Generally, researchers will get a **sample** of the population - which is a small subset of the population. If this sample is representative of the population, we might be able to study the sample, then infer our conclusions back to the population.

This is generally done with techniques that we will learn about throughout this book, such as hypothesis testing, confidence intervals, and sampling distributions.

Before we begin discussing this topic, it is important you already understand what correlations are, and what explanatory/independent and dependent/outcome variables are. If this confuses you, consult the Quantitative Methods sequence.

1.3 Predictive Inference

Predictive inference is also of major importance in the social sciences. After all, being able to make good predictions can help us identify potential effects of policies, what areas need help, and how to best allocate resources. For example, economists often use predictive inference to predict future economic conditions, which helps governments plan how to respond to potential challenges.

In politics, prediction is also important. You might want to use predictive inference to predict potential political disasters, such as Civil Wars, in order to better prepare humanitarian aid. You might want to use predictive inference to determine which citizens in what neighbourhoods are most at risk, so you can help them out. You might want to use predictive inference to predict who will vote, and who will win elections.

The goal in predictive inference is to create a model with the data we have, that will perform well in predicting observations that we do not have.

In this book, we will not focus too much on predictive inference. We will build models that can make predictions, but the focus of these models will be other types of inference. However, this book builds the foundational methods and techniques for predictive inference, which are then built on in the next two books in this sequence: *Further Econometrics for Political Analysis* and *Econometrics and Big Data*.

Chapter 2

Causal Inference

2.1 Introduction to Causal Inference

Causal inference is the primary focus of econometrics and the social sciences. This is because causal effects are what drive things in the world. Knowing how things cause other things is critical for understanding the world around us, and designing better policies.

An important thing to know is that correlation does not equal causation. Descriptive inference is concerned about how to find correlations/relationships in the population. That is not the goal of Causal Inference: the goal of causal inference is to find how things cause each other.

- For example, ice cream sales are strongly correlated with shark attack frequency. Does that mean ice cream sales directly cause shark attacks? No - they happen to be correlated, because there is another variable, the warm weather and people going to the beach, that causes both to rise at the same time. Ice cream sales itself does not have any independent affect on shark attacks, they just happen to be correlated.

In the first section of this book, we will establish what a causal effect even is, and discuss how randomised experiments are the "gold standard" of establishing causal relationships. However, we will explore why randomised experiments are often impossible in the social sciences. The rest of the book will focus on techniques that try to find causal effects without randomised experiments.

2.2 Potential Outcomes Framework

A **causal effect** is a change in some feature of the world Y, that would directly result from a change in some other feature D. Essentially, change in D causes change in Y.

This causal effect implies some **counterfactual**. A counterfactual is a comparison between the outcome in a real world, and the outcome in a hypothetical world, where both worlds are identical up to the point where feature D is implemented.

- Imagine that there are 2 worlds, that are exactly the same until treatment D occurs. One world gets the treatment D, and the other world does not get this treatment.

- Since these 2 worlds are identical besides the treatment D, the difference between the world's Y outcomes are the effect of our treatment D.

Causal States are these hypothetical states of the world. The control state is the world where a unit does not receive the treatment D. The treatment state is identical to the control state, with the only exception that a unit receives the treatment D.

- We can define the control state as when $D = 0$, and the treatment state where $D = 1$

For each unit of observation i, we can define two potential outcomes to the control state:

- Y_{1i} is the potential outcome for unit i, given it is in the treatment state $D_i = 1$.
- Y_{0i} is the potential outcome for unit i, given it is in the control state $D_i = 0$.

Thus, the **causal effect** τ of the treatment D for any unit i must be $\tau_i = Y_{1i} - Y_{0i}$. This is because since the two states of the world are identical except for treatment D, the resulting difference must be as a result of treatment D

However, in the real world, we do not have parallel worlds (unfortunately). However, we do know which units have undergone treatment in the real world, and the units who have not undergone the treatment in the real world. - We can label those who have undergone the treatment in the real world as $D_i = 1$, and those who have not undergone the treatment as $D_i = 0$

Thus, the observed Y outcome of any unit i is given by the equation:

$$Y_i = D_i \times Y_{1i} + (1 - D_i) \times Y_{0i}$$

This equation might be a little abstract, however, it is easy to understand by plugging numbers in. When a unit does undergo treatment, plug in $D_i = 1$, and you will get outcome Y_{1i} as expected. Similarly, plug in $D_i = 0$, and you will get outcome Y_{0i} as expected. This is simply a mathematical way to express that in the real world, people who undergo treatment have the treatment outcome, and people who do not undergo treatment have the control outcome.

2.3 Causal Estimands

The fundamental problem of causal inference is that we only can observe one of the two parallel universes at the same time. For example, if you get treatment D, we cannot observe the world where you do not get treatment D.

Thus, we cannot estimate individual effects of the causal treatment. However, there are other estimands we can use.

- Note: **Estimand** is the quantity we are trying to estimate.

Since we cannot observe the individual treatment effects, we can change out estimand to the **average treatment effect**. This is exactly what it sounds like - the treatment effect of all units averaged:

$$E[\tau_i] = E[Y_{1i} - Y_{0i}] = \frac{1}{n}(\sum Y_{1i} - \sum Y_{0i})$$

There are also other treatment effects we can use to estimate.

The **average treatment effect on the treated (ATT)** is the treatment effect of only units who recieved the treatment $D_i = 1$

$$\tau_{ATT} = E[Y_{1i} - Y_{0i}|D_i = 1]$$

The **average treatment effect on the controls (ATC)** is the treatment effect of units who only did not recieve the treatment $D_i = 0$

$$\tau_{ATC} = E[Y_{1i} - Y_{0i}|D_i = 0]$$

The conditional average treatment effect (CATE) is the treatment effect of units, given they have some other variable X value. For example, if X is gender, the CATE could be the treatment effect on only females.

$$\tau_{CATE} = E[Y_{1i} - Y_{0i}|X = x]$$

2.4 Assumptions for Estimating Causal Estimands

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2.5 Naive Estimator and Sample Bias

Chapter 3

Randomised Experiments

Part II

The Multiple Linear Regression Model

Chapter 4

Introduction to the Linear Regression Model

4.1 Introduction to Regression

As we discussed in the previous sections, a randomised experiment is the best way of establishing causal relationships. This is because randomise treatments can get rid of the effect of confounding variables. However, in the social sciences, randomisation is often not possible.

Luckily, while we often cannot randomise treatment, we can account for the affect of confounding variables with a new tool: linear regression.

Linear regression allows us to estimate a model with both our explanatory and outcome variables, as well as a series of **control** variables. By including confounding variables as control variables in our regression model, we can (in theory), isolate the effect of our explanatory variable on our outcome variable.

- In reality, as we will discuss in the later parts of this book, there are often thousands of control variables, many that are not possible to control for. We will introduce extensions to the linear model to deal with these.

Regression is also a powerful tool for both descriptive and predictive inference. In fact, it forms much of the building blocks of more complicated data science and machine learning methods.

Before we dive into the linear model, here are some conventional notation that is important:

- The **response variable** (dependent variable) is notated Y. In this book, we will only have one response variable.
- The **explanatory variable** (independent variable) is notated X. There is often more than one explanatory variable, so we denote them with subscripts $X_1, X_2, ..., X_k$. We sometimes also denote all explanatory variables as the vector $\overrightarrow{X}$
- Note: our treatment variable (for causal inference) D is considered one of the explanatory variables $\overrightarrow{X}$. We typically define the first of these explanatory variables X_1 as the treatment variable, and all others $X_2, ..., X_k$ as control variables.

4.2 Specification of the Linear Model

A regression model is the specification of the conditional distribution of Y, given $\overrightarrow{X}$. Essentially, it is stating that the distribution of possible Y outcomes depends on the value of $\overrightarrow{X}$.

- I say **distribution** because there are often a range of Y outcomes, each with their own probabilities, for any given X. For example, if X was age and Y was income, at age $X = 30$, not every single 30 year old makes the same amount of money. There is some distribution of incomes Y at age $X = 30$.

The linear regression model focuses on the **expected value** or mean of the conditional distribution of Y given $\overrightarrow{X}$.

Suppose we have a set of observed data, with response variable Y, and a number of X variables for n number of observations. Thus, we will have n number of pairs of (X_i, Y_i) observations. The linear model takes the following form:

$$E[Y_i|\overrightarrow{X_i}] = \alpha + \beta_1 X_{1i} + ... + \beta_k X_{ki}$$

- Where $E[Y_i|\overrightarrow{X_i}]$ is the expected value of the conditional distribution $Y_i|\overrightarrow{X_i}$
- The distribution of $Y_i|\overrightarrow{X_i}$ has a variance $Var(Y_i|\overrightarrow{X_i}) = \sigma^2$.
- The parameters of the model are the denoted by the vector $\overrightarrow{\beta}$, and contain $\alpha, \beta_1, ..., \beta_k$

We can also write the linear model for the value of any point Y_i in our data:

$$Y_i = \alpha + \beta_1 X_{1i} + ... + \beta_k X_{ki} + \epsilon_i$$

- Where ϵ_i is the error term function - that determines the error for each point. We will go into detail on this later.
- A key assumption (that we will discuss later) is that the error function overall is normally distributed: $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$
- Essentially, ϵ_i is another way to think about the conditional distribution of $Y_i|\overrightarrow{X_i}$, and how not every 30 year old makes the exact same income - there is some variation (and error).

4.3 Estimation of Parameters

In our model, we have parameters $\alpha, \beta_1, ..., \beta_k$ that need to be estimated in order to create a best-fit line we can actually use. We estimate the parameters and fit the model by using our observed data points $(Y_i, \overrightarrow{X_i})$, and fitting a best fit line to these points. Our result should take the following form:

$$\hat{Y}_i = \hat{\alpha} + \hat{\beta}_1 X_{1i} + ... + \hat{\beta}_k X_{ki}$$

- Where $\hat{Y}_i$ is our prediction of the value of Y, given any set of $\overrightarrow{X}$ values.
- Notice the error term ϵ_i is not present. This is because of our prior assumption that the error term $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$, which says the expected value of ϵ_i is $E[\epsilon_i] = 0$.

However, how do we determine the estimates of our parameters $\alpha, \beta_1, ..., \beta_k$? **The Ordinary Least Squares Estimator (OLS).** OLS estimation attempts to **minimise the sum of squared errors** of our predicted line to our actual observed data. The estimation processes is as follows.

1. Propose some coefficient values to test. Let $\tilde{\beta}$ represent the vector of our proposed coefficient values $\tilde{\alpha}, \tilde{\beta}_1, ..., \tilde{\beta}_k$
2. Use these proposed coefficients in our prediction line: $\hat{Y}_i(\tilde{\beta}) = \tilde{\alpha} + \tilde{\beta}_1 X_{1i}, ..., \tilde{\beta}_k X_{ki}$
3. Calculate the residuals e_i of our predictions of Y using our proposed coefficients, compared to the actual values of Y_i : $e_i(\tilde{\beta}) = Y_i - \hat{Y}_i(\tilde{\beta})$
4. Calculate the sum of squared errors (SSE) for all residuals: $SSE(\tilde{\beta}) = \sum(Y_i - \hat{Y}_i(\tilde{\beta}))^2$

The set of proposed $\tilde{\beta}$ coefficients that produces the lowest SSE is chosen as our estimates.

However, in reality, we do not do all this work - testing all different $\tilde{\beta}$ and seeing which one has a lower SSE. Instead, we can mathematically solve this problem. We will discuss this in the next chapter.

4.4 Interpretations of Coefficients

Simple Linear Regression

Simple Linear Regression is a special case of the linear model, when there is only one explanatory variable X. How do we interpret parameters $\hat{\alpha}$ and $\hat{\beta}$ that we have calculated?

$\hat{\beta}$ is the slope of the the linear model. $\hat{\beta}$ is the expected change in Y, given a one-unit increase in X.

- A positive $\hat{\beta}$ means a positive relationship, a negative $\hat{\beta}$ means a negative relationship, and $\hat{\beta} = 0$ means no relationship.
- This is only for continuous X explanatory variables. See Chapter 5 for categorical/binary explanatory variables.

$\hat{\alpha}$ is the y-intercept of the linear model. $\hat{\alpha}$ is the expected value of $\hat{Y}$, given $X = 0$.

Multiple Linear Regression

Multiple linear regression is when there are multiple explanatory variables $X_1, X_2, ..., X_k$. How do we interpret these parameters $\hat{\alpha}$ and $\hat{\beta}_1, \hat{\beta}_2, ..., \hat{\beta}_k$ that we have calculated?

- Note: I will define $\hat{\beta}_j$ as any one of $\hat{\beta}_1, ..., \hat{\beta}_k$ (the interpretation is all the same). In the model, $\hat{\beta}_j$ will be multiplied to variable X_j.

Formally, any coefficient $\hat{\beta}_j$ is the expected change in Y, corresponding to a one unit increase in X_j, holding all other explanatory variables $X_1, ..., X_k$ that are not X_j constant.

- Essentially, we do the same interpretation as the single linear regression, but adding the phrase "**holding all other explanatory variables constant**".

$\hat{\alpha}$ is the expected value of $\hat{Y}$, given all explanatory variables $X_1, ..., X_k$ equal 0.

Interpreting in Terms of Standard Deviation

Sometimes, it is hard to understand what changes in Y and X mean in terms of units. For example, if democracy is measured on a 100 point scale, what does a 5 point change in democracy mean? Is it a big change, or a small change?

We can add more relevant detail by expressing the change of Y and X in standard deviations.

So, instead of the expected change of Y given one unit increase of X, we instead do the expected standard deviation change of Y, given a one standard deviation increase in X.

How do we calculate this? There is a formula! Y changes by $(SD_{X_j} \times \hat{\beta}_j)/SD_Y$, where SD represents standard deviation, and X_j is the variable whose coefficient we are interpreting.

Binary Y Variable Interpretation

If our response variable is binary, (i.e. Y only has two values, 0 and 1), then our interpretation differs slightly. We treat Y as a variable of two categories, $Y = 0$ and $Y = 1$.

$\hat{\beta}_j$ is the expected change in the probability of getting category $Y = 1$, given a one-unit increase in X_j. We could multiply this by 100 to get the expected percentage point change.

$\hat{\alpha}$ is the expected probability of getting category $Y = 1$, when $X_j = 0$.

An important note is that, in theory, for a linear regression model, Y should be continuous, not binary. In theory, we should be using logistic regression instead of linear regression (which is covered in the Advanced Statistical Methods book).

- However, in practice, researchers often do use linear regression for binary Y, simply because linear regression is easier to interpret, and easier to incorporate into more causal inference methods.

4.5 Model Summary Statistics

Estimated Residual Standard Deviation

We can derive the estimate of the **residual variance** σ^2 with this formula:

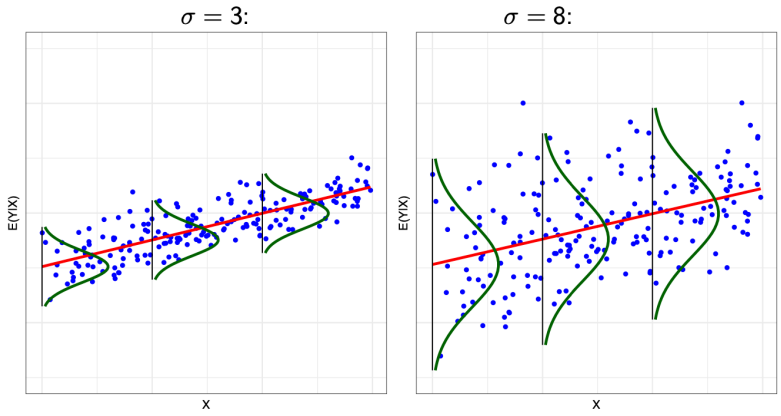
$$\hat{\sigma}^2 = \frac{\sum(Y_i - \hat{Y}_i)^2}{n - k - 1}$$

But what is the residual variance? Recall the way we write our regression model:

$$Y_i = \alpha + \beta_1 X_{1i} + ... + \beta_k X_{ki} + \epsilon_i$$

We know that $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$. Our estimate of the residual variance $\hat{\sigma}^2$ is our estimate of the variance of the error term ϵ_i's variance. More intuitively, it explains how spread out observed values of Y are from our prediction value $\hat{Y} = E(Y|X)$.

The figure below better showcases this in 2 different models. The red lines are our predicted regression line, and the green lines represent the distribution of our error term ϵ_i:



The residual standard deviation $\hat{\sigma}$ (square root of variance) is consistent throughout a model. This is one of the assumptions of the linear regression model - that errors are consistently distributed, no matter the value of X. This assumption is called **homoscedasticity**.

If $\hat{\sigma}$ varies depending on the value of X, then that is called **heteroscedasticity**. When this occurs, it is often a suggestion that our relationship may not be linear - and we perhaps need to try a few transformations. We will get into transformations in a later chapter.

Total Sum of Squares

The total sum of squares is the total amount of sample variation in Y:

$$\begin{aligned} TSS &= \sum (Y_i - \bar{Y})^2 \\ &= \sum (\hat{Y}_i - \bar{Y})^2 + \sum (Y_i - \hat{Y}_i)^2 \end{aligned}$$

Where TSS is the total sum of squares, SSM $\sum(\hat{Y}_i - \bar{Y})^2$ is the model sum of squares, and SSE $\sum(Y_i - \hat{Y}_i)^2$ is the sum of squared errors (that we used to fit the model).

SSM (model sum of squares) represents the part of the variation of Y that is explained by the model, while SSE (sum of squared errors) represents the part of the variation of Y that is not explained by the model (hence, why it is called error).

R-Squared Statistic

R-squared R^2 is a measure of the percentage of variation in Y, that is explained by our model (with our chosen explanatory variables). The percentage of variation in Y explained by our model would be:

$$R^2 = \frac{SSM}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2}$$

Since R^2 shows how much of the variation in Y our model explains, it is often used as a metric for how good our model is - however, don't overly focus on R^2, it is just one metric with its benefits and drawbacks.

4.6 Linear Regression in R

As for all of the examples in this book, we will use the "tidyverse" package. Let us also load our dataset that we will be using.

```
# if you haven't installed tidyverse, do: install.packages('tidyverse')
library(tidyverse)
democracy_data <- read_csv("data/democracy.csv")
```

We use the *lm()* function to run a regression: The general syntax is as follows:

- Replace *model_name* with your model name, *Y* with the name of your response variable, *X1, X2…* with the name of your explanatory variable, and *mydata* with the name of your dataset.
- Add additional explanatory variables with more + signs, and you can remove them down to a minimum of one *X*

```
model_name <- lm(Y ~ X1 + X2 + X3, data = mydata)
summary(model_name)
```

For example, let us run a multiple linear regression with two explanatory variables:

```
model2 <- lm(polity_2 ~ GDP_Per_Cap_Haber_Men_2 + Total_Oil_Income_PC,
    data = democracy_data)
summary(model2)
```

```
Coefficients:
                          Estimate Std. Error t value Pr(>|t|)
(Intercept)             -1.775e+00  8.554e-02  -20.75   <2e-16 ***
GDP_Per_Cap_Haber_Men_2  4.764e-04  1.084e-05   43.93   <2e-16 ***
Total_Oil_Income_PC     -1.103e-03  2.850e-05  -38.69   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.639 on 10267 degrees of freedom
  (6936 observations deleted due to missingness)
Multiple R-squared:  0.1729,    Adjusted R-squared:  0.1727
F-statistic:  1073 on 2 and 10267 DF,  p-value: < 2.2e-16
```

We can see the output. In the coefficients table:

- The *Intercept - Estimate* is $\hat{\alpha}$
- The *GDP_Per_Cap_Haber_Men_2 - Estimate* is the coefficient $\hat{\beta}_1$.
- The *Total_Oil_Income_PC - Estimate* is the coefficient $\hat{\beta}_2$.
- For interpretation, see the preceding section on interpretation.

Underneath, the *Residual Standard Error* is the Residual Standard Deviation $\hat{\sigma}^2$, and the *Multiple R-Squared* is the R^2 value.

Chapter 5

Analysis of the Ordinary Least Squares Estimator

This chapter goes into the mathematics of how we obtained our estimates of $\hat{\alpha}, \hat{\beta}_1, ..., \hat{\beta}_k$ using the ordinary least squares (OLS) estimator. This section is not essential if you are only interested in an applied approach to econometrics, however, it can foster a deeper understanding of the regression process.

5.1 Bivariate Regression Estimation

Minimising Sum of Squared Errors

A bivariate regression (also called a simple linear regression) is a linear regression model with only one explanatory variable X. It takes the following form:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

We have some fitted model to our data (best-fit line) with estimated parameters:

$$\hat{Y}_1 = \hat{\alpha} + \hat{\beta} X_i$$

We noted how the **ordinary least squares** (OLS) estimator is concerned with the sum of squared errors. We can replace $\hat{Y}_i$

$$\begin{aligned} SSE &= \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 \end{aligned}$$

The OLS estimator wants to find the parameters that **minimise the sum of squared errors**. We can mathematically express this:

$$(\hat{\alpha}, \hat{\beta}) = \arg\min_{\hat{\alpha},\hat{\beta}} \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

For notation simplicity, let us define our sum of squared errors as a function S with inputs $\hat{\alpha}$ and $\hat{\beta}$:

$$S(\hat{\alpha}, \hat{\beta}) = \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

Parameter $\hat{\alpha}$

Let us first look at the parameter $\hat{\alpha}$. How do we find what value $\hat{\alpha}$ minimises the sum of squared errors? We know through calculus, that deriving the function to find the first order condition can accomplish this:

$$\frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} = \frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} \left[\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 \right]$$

First, ignore the summation. The partial derivative of the internal section is:

$$\begin{aligned}
&\text{Chain Rule: } [f(g(x))]' = f'[g(x)] \times g'(x) \\
&\text{if } f(\hat{x}) = x^2, \text{ then } f'(x) = 2x \\
&\text{if } g(\hat{\alpha}) = Y_i - \hat{\alpha} - \hat{\beta} X_1, \text{ then } g'(\hat{\alpha}) = -1 \\
&\text{thus } [f(g(\hat{\alpha}))]' = 2(Y_i - \hat{\alpha} - \hat{\beta} X_i) \times -1 \\
&= -2(Y_i - \hat{\alpha} - \hat{\beta} X_i)
\end{aligned}$$

That is the derivative of one of the items in the summation. But how do we deal with the summation? Well, what is summation:

$$\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 = (Y_1 - \hat{\alpha} - \hat{\beta} X_i)^2 + (Y_2 - \hat{\alpha} - \hat{\beta} X_i)^2 + ... + (Y_n - \hat{\alpha} - \hat{\beta} X_i)^2$$

We know that there is the sum rule of derivatives $[f(x) + g(x)]' = f'(x) + g'(x)$. Thus, we know:

$$\frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} \left[\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2 \right] = \frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} \left[(Y_1 - \hat{\alpha} - \hat{\beta} X_i)^2 \right] + ... + \frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\alpha}} \left[(Y_n - \hat{\alpha} - \hat{\beta} X_i)^2 \right]$$

Thus, since we know the partial derivative of one term of the summation already, we can generalise to (and simplify with property of summations):

$$\begin{aligned}\frac{\partial S(\hat{\alpha},\hat{\beta})}{\partial \hat{\alpha}} &= \sum_{-i=1}^{n}\left[-2(Y_i-\hat{\alpha}-\hat{\beta}X_i)\right] \\ &= -2\sum_{i=1}^{n}(Y_i-\hat{\alpha}-\hat{\beta}X_i)\end{aligned}$$

Now that we have the partial derivative of the SSE function in respect to $\hat{\alpha}$, to find the value of $\hat{\alpha}$ that creates the minimum value of the SSE, we set the first order derivative equal to 0.

$$-2\sum_{i=1}^{n}(Y_i-\hat{\alpha}-\hat{\beta}X_i)=0$$

We can ignore the -2, since if the sum is equal to 0, then the -2 will have no effect.

$$\sum_{i=1}^{n}(Y_i-\hat{\alpha}-\hat{\beta}X_i)=0$$

Now, we use the property of summation to simplify to:

$$\sum_{i=1}^{n}Y_i-n\hat{\alpha}-\hat{\beta}\sum_{i=1}^{n}X_i=0$$

Now, isolate $\hat{\alpha}$ as follows:

$$\begin{aligned}\sum_{i=1}^{n}Y_i-n\hat{\alpha}-\hat{\beta}\sum_{i=1}^{n}X_i &= 0 \\ -n\hat{\alpha} &= -\sum_{i=1}^{n}Y_i+\hat{\beta}\sum_{i=1}^{n}X_i \\ \hat{\alpha} &= -\frac{1}{n}\left[-\sum_{i=1}^{n}Y_i+\hat{\beta}\sum_{i=1}^{n}X_i\right] \\ \hat{\alpha} &= \frac{1}{n}\sum_{i=1}^{n}Y_i-\hat{\beta}\sum_{i=1}^{n}X_i \\ \hat{\alpha} &= \bar{Y}-\hat{\beta}\bar{X}\end{aligned}$$

The final step converting to $\bar{Y}$ and $\bar{X}$ is because of the mathematical definition of average. Thus:

$$\hat{\alpha}=\bar{Y}-\hat{\beta}\bar{X}$$

We will plug this into our solution for $\hat{\beta}$ to solve that. Once we solve $\hat{\beta}$, we will come back and calculate $\hat{\alpha}$'s value.

Parameter $\hat{\beta}$

Now with $\hat{\alpha}$, we can calculate $\hat{\beta}$. Remember our original SSE equation:

$$S(\hat{\alpha}, \hat{\beta}) = \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

Now, let us find the minimum $\hat{\beta}$ value by taking the partial derivative in respect to $\hat{\beta}$ and setting it equal to 0. Once again, chain rule (ignoring summation):

$$\begin{aligned}
&\text{Chain Rule: } [f(g(x))]' = f'[g(x)] \times g'(x) \\
&\text{if } f(\hat{x}) = x^2, \text{ then } f'(x) = 2x \\
&\text{if } g(\hat{\beta}) = Y_i - \hat{\alpha} - \hat{\beta} X_1, \text{ then } g'(\hat{\beta}) = -X_i \\
&\text{thus } [f(g(\hat{\alpha}))]' = 2(Y_i - \hat{\alpha} - \hat{\beta} X_i) \times -X_i \\
&= -2X_i(Y_i - \hat{\alpha} - \hat{\beta} X_i)
\end{aligned}$$

Using the same process as above, we know the sum rule of derivatives means we can just add the sum back. We get:

$$\begin{aligned}
\frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}} &= \sum_{i=1}^{n} \left[-2X_i(Y_i - \hat{\alpha} - \hat{\beta} X_i) \right] \\
&= -2 \sum_{i=1}^{n} X_i(Y_i - \hat{\alpha} - \hat{\beta} X_i)
\end{aligned}$$

Now, let us plug in our previously solved $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$, and we get:

$$\frac{\partial S(\hat{\alpha}, \hat{\beta})}{\partial \hat{\beta}} = -2 \sum_{i=1}^{n} \left[X_i(Y_i - [\bar{Y} - \hat{\beta}\bar{X}] - \hat{\beta} X_i) \right]$$

Once again, the -2 does not matter as same reason as before. Set equal to 0 to solve for the value of $\hat{\beta}$ that minimises SSE:

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \left[X_i(Y_i - [\bar{Y} - \hat{\beta}\bar{X}] - \hat{\beta} X_i) \right] \\
0 &= \sum_{i=1}^{n} \left[X_i(Y_i - \bar{Y} - \hat{\beta}(X_i - \bar{X})) \right] \\
0 &= \sum_{i=1}^{n} \left[X_i(Y_i - \bar{Y}) - X_i\hat{\beta}(X_i - \bar{X}) \right] \\
0 &= \sum_{i=1}^{n} X_i(Y_i - \bar{Y}) - \hat{\beta}_1 \sum_{i=1}^{n} X_i(X_i - \bar{X})
\end{aligned}$$

We could solve for $\hat{\beta}$, however, we can use a few facts about summation notation to create a more "nice looking" solution that is easier to interpret. The facts on summation are:

$$\sum_{i=1}^{n}(X_i - \bar{X}) = 0$$
$$\sum_{i=1}^{n} X_i(Y_i - \bar{Y}) = \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$$
$$\sum_{i=1}^{n} X_i(X_i - \bar{X}) = \sum_{i=1}^{n}(X_i - \bar{X})^2$$

With these rules, we can transform what we had before into:

$$0 = \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) - \hat{\beta}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Then solve for $\hat{\beta}$ to get:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

If we remember the introductory statistics classes prior to this (in the Quantitative Methods sequence), we know that:

- Covariance: $\sigma_{XY} = \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$
- Variance: $\sigma_X^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$

Thus, $\hat{\beta}$ could also be rewritten as (but only for bivariate linear regression):

$$\hat{\beta}_1 = \frac{Cov(X, Y)}{Var(X)}$$

Thus, with this solution, we have estimated $\hat{\beta}$. If we recall, $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$, so we can quickly solve for that as well.

With $\hat{\beta}$ and $\hat{\alpha}$ estimated, we can now plug them into our bivariate regression and complete our fitted model:

$$\hat{Y}_i = \hat{\alpha} + \hat{\beta}X_i$$

5.2 Algebraic Properties

Residuals expected = 0

Homoscedasticity

Explanatory variable has variance.

5.3 Linear Algebra Depiction of Multivaraite Regression

5.4 Multivariate Regression Estimation

5.5 Geometric Properties

Chapter 6

Confidence Intervals and Hypothesis Testing

6.1 Samples and Population

In the social sciences, we are often interested in studying large groups of people and entities. However, it is often impossible to ask every single individual in the population. For example, if we wanted to study the effect of educational level on voter turnout in the UK, we would need to ask nearly 70 million people.

A **sample** is a subset of a population, which ideally, can tell us something about the population. For example, we could use a sample to estimate the relationship between two features X_j and Y with a coefficient β_j. If our sample can reflect the population, then we can use the sample to learn about the true population β_j

The quality of a sample depends on two major factors: the sampling procedure, and luck.

- Sampling procedure can create **bias**: systematic reasons why our samples are not representative of the population.
- Luck in sampling is called **noise** or **variance**: essentially, even if we have a perfect sampling procedure, every sample will differ slightly just due to luck.

The gold standard of sampling procedure is a **random sample** - where individuals in the sample are selected at random from the population. In a random sample, every possible individual has an equal chance of being selected, and thus, the resulting sample is likely to be reflective of the population.

6.2 Sampling Distributions and Standard Error

A sampling distribution is a hypothetical construct that is useful to understanding how many of our statistical techniques work. A sampling distribution is as follows.

- Imagine that we take a sample from a population. Then, we find the $\widehat{\beta}_j$ coefficient between some X_j and Y. That is a **sample estimate**.

- Then, let us take another sample from the same population, and find the sample estimate. This will be slightly different than the first sample, since we are randomly sampling. That is another sample estimate. We keep taking samples from the same population, and getting more and more sample estimates.
- Now, let us plot all our sample estimates (different $\hat{\beta}_j$ values) into a "histogram" or density plot. The x axis labels the possible β_j values, and the y axis is how frequently a specific sample mean occurs. We will get a distribution, just like a random variable distribution.
- That distribution is the **sampling distribution**

A Sampling distribution is the imaginary distribution of estimates, if we repeated the sampling and estimation process many, many times.

The **standard error** is the standard deviation of the sampling distribution. It is often notated $se(\hat{\beta}_j)$. Our software will calculate an estimate of this value, which will be notated as $\hat{se}(\hat{\beta}_j)$

- A smaller standard error generally means a more accurate estimate, and is usually due to a larger sample size n or lower residual standard deviation $\hat{\sigma}^2$

6.3 Confidence Intervals

Our OLS estimation (from chapter 3) has produced a $\hat{\beta}_j$ based on the data in one sample. However, remember, the sample is just a fraction of the population - and the true $\hat{\beta}_j$ is unlikely to be exactly the one in our sample, due to sampling variation.

Thus, we have to create an interval around our estimate $\hat{\beta}_j$ to account for this uncertainty. We assume our estimated $\hat{\beta}_j$ is the centre of this distribution, then add some "buffer" to both sides. The confidence interval's lower and upper bounds are defined as, given a confidence level of 95% (the standard confidence level):

$$\hat{\beta}_j \pm 1.96 \times \hat{se}(\hat{\beta}_j)$$

- $\hat{se}(\hat{\beta}_j)$ is the standard error of our estimate of how precisely we have estimated the true value of β_j, introduced in the previous section.
- The 1.96 can slightly deviate depending on the sample size and number of variables (do not worry, the computer will solve this for us).
- Why 1.96? It is because in a normal distribution, 95% of the data is contained within 1.96 standard deviations (see section 2.4), and Central Limit Theorem states that sampling distributions are normally distributed.

What does the confidence interval represent? Essentially, it means if we repeated the sampling and estimation process many many times (like we did for our sampling distribution), 95% of the confidence intervals we construct from our samples, would correctly contain the true β_j.

Every value in a given confidence interval is a plausible value of the true β_j in the population.

The most important thing is if 0 is included within the confidence interval. $\beta_j = 0$ means that there is no relationship between X_j and Y. If our confidence interval contains 0, that means we cannot be confident that there is no relationship between X_j and Y.

6.4 Hypothesis Testing of Parameters

In academia, we are conservative - this means we do not claim we have found a new theory, unless we are quite confident that the old theory was not true.

The old theory is called our **null hypothesis**, often notated H_0. This is the old theory that we are trying to disprove.

- Most often, the "old theory" we are trying to disprove is that *there is no relationship between variables X_j and Y* (since it is very rare we are studying something that has already been proven). No relationship means that $\beta_j = 0$

The new theory we have come up with, and are trying to prove, is called the **alternate hypothesis**, often notated H_1 or H_a.

- In general, our new hypothesis is that t*here is a relationship between variables X_j and Y*, or $\beta_j \neq 0$

We assume that the null hypothesis is true, unless we are 95% confident that we can reject the null hypothesis, and only then, can we accept the alternative hypothesis (the new theory we proposed). Why 95% confidence? It is just tradition - there have been several studies showing there is nothing special about this value.

Generally, for regressions, our hypotheses that we test are:

- $H_0 : \beta_j = 0$ - i.e. there is no relationship between X_j and Y
- $H_1 : \beta_j \neq 0$ - i.e. there is a relationship between X_j and Y

How do we actually test these hypotheses? First, we have to calculate a t-test statistic. The formula for such a statistic is the parameter divided by its standard error (calculated by the computer):

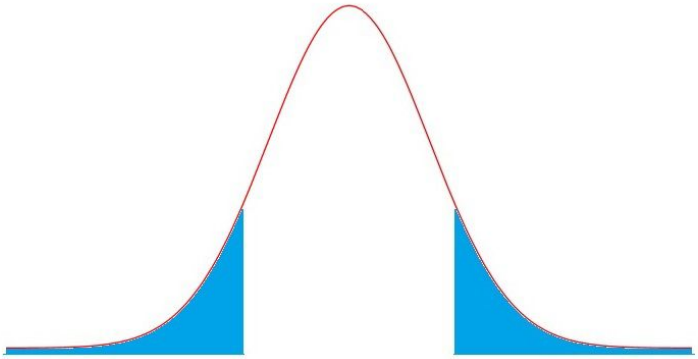
$$t = \frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$$

The t-test statistic basically tells us how far the parameter we tested is from 0, in terms of standard errors of the parameter.

Then, we have to consult a t-distribution (see section 2.4). T-distributions only has one parameter - degrees of freedom, which is calculated by the number of observations n, minus the number of variables k, then minus 1: $DF = n - k - 1$

With the degrees of freedom, we can find the corresponding t-distribution, labeled t_{n-k-1}. Then, we start from the middle of that t distribution, and go the *number of standard errors* away based on the t-test statistic. We do this on both sides from the middle of the t-distribution.

Once we have found that point, we find the probability (area under the distribution) of a t-test statistic of ours, or more extreme, could occur. The figure below, with its blue highlighted area, shows this probability:



The area highlighted is our p-value. Essentially, a p-value is how likely we are to get a test statistic at or more extreme than the one we got for our estimated β_j, given the null hypothesis is true.

- So if the p-value is very high, there is a high chance that the null hypothesis is true.
- If the p-value is very low, then there is a low chance that the null hypothesis is true

Generally, in the social sciences, if the p-value is less that 0.05 (5%), we can **reject the null hypothesis**, and conclude the alternate hypothesis.

6.5 F-Tests of Nested Models

An F-test is a variation on the coefficient significance tests. The standard F-test is quite simple - it tests for the significance of a model across multiple coefficients:

$$H_0 : \beta_1 = \beta_2 = ... = \beta_k = 0$$
$$H_1 : \text{at least one of } \beta_1, \beta_2, ..., \beta_k \neq 0$$

Thus, we assume that all explanatory variables in the model have no relationship to the outcome variable, unless we can reject the null hypothesis and show that at least one coefficient is statistically significant.

For a standard Linear Regression, if we have any significant individual coefficients, then the F-test will also become significant. So what is the point of F-tests? Well, for some models that we will see later, such as categorical explanatory variables, the coefficient's significance levels mean something else. Thus, to see if the model is statistically significant, we need to use the F-test.

The **F-test of Nested Models** is an extension of the standard F-test, that allows us to compare different regression models. We use a smaller model as our null hypothesis, and a larger model (containing the smaller model) as our alternative hypothesis. More mathematically:

$$M_0 : E[Y] = \alpha + \beta_1 X_1 + ... + \beta_g X_g$$
$$M_a : E[Y] = \alpha + \beta_1 X_1 + ... + \beta_g X_g + \beta_{g+1} X_{g+1} + ... + \beta_k X_k$$

Importantly, note how the null hypothesis model is entirely contained within the alternate hypothesis model. This must be the case - all explanatory variables in model M_0 must also be in M_a, along with additional explanatory variables in M_a.

The F-test uses the F-test statistic. This statistic compared the R^2 values of the two models. Let us say the R^2 value of M_0 is notated R_0^2, and the R^2 value of M_a is notated as R_a^2. The F-test statistic essentially measures the difference $R_a^2 - R_0^2$. If the difference is sufficiently large, that means the M_a model has significantly more explanatory power than M_0.

Let SSE_a and R_a^2 denote the sum of squared errors and R^2 values of model M_a, and SSE_0 and R_0^2 for model M_0. The total number of coefficients of M_a are k_a, and for M_0 is k_0. Mathematically, the F-test statistic is as follows:

$$F = \frac{(SSE_0 - SSE_a)/(k_a - k_0)}{SSE_a/[n - (k_a + 1)]}$$

$$F = \frac{\frac{R_a^2 = R_0^2}{[n-(k_0+1)]-[n-(k_a+1)]}}{(1 - R_a^2)/[n - (k_a + 1)]}$$

$$F = \frac{R^2_{\text{change}}/df_{\text{change}}}{(1 - R_a^2)/[n - (k_a + 1)]}$$

The sampling distribution of the F-statistic is the F distribution with parameters $k - a - k_0$ and $n - (k_a + 1)$ degrees of freedom. We then obtain the p-value from this distribution.

The p-values of the F-statistic show the following:

- If the p-value is very small, that means R_a^2 is significantly larger than R_0^2. This is evidence against model M_0, and in favour of the larger model M_a
- If the p-value is large, that means R_a^2 is not much larger than R_0^2. This means there is no evidence against M_0, and M_a is not the statistically significantly better model.

F-tests of nested models can help us determine if we should include certain extra explanatory variables. If the addition of the explanatory variables does not statistically significantly improve the performance of the model, there is little reason to include them, unless we have some other theoretical reason to include them.

6.6 Hypothesis Testing in R

For hypothesis testing, we just run the regression like we would normally (see section 4.5). For example, let us run a model prediction *polity_2* with 2 explanatory variables

```
model <- lm(polity_2 ~ GDP_Per_Cap_Haber_Men_2 + Total_Oil_Income_PC,
    data = democracy_data)
summary(model)
```

```
Call:
lm(formula = polity_2 ~ GDP_Per_Cap_Haber_Men_2 + Total_Oil_Income_PC,
    data = democracy_data)

Residuals:
```

```
    Min      1Q  Median      3Q     Max
-51.604  -5.790  -0.162   6.246  40.458

Coefficients:
                          Estimate Std. Error t value Pr(>|t|)
(Intercept)             -1.775e+00  8.554e-02  -20.75   <2e-16 ***
GDP_Per_Cap_Haber_Men_2  4.764e-04  1.084e-05   43.93   <2e-16 ***
Total_Oil_Income_PC     -1.103e-03  2.850e-05  -38.69   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.639 on 10267 degrees of freedom
  (6936 observations deleted due to missingness)
Multiple R-squared:  0.1729,    Adjusted R-squared:  0.1727
F-statistic:  1073 on 2 and 10267 DF,  p-value: < 2.2e-16
```

Now, let us look at the coefficients table:

- *GPD_Per_Cap_Haber_Men_2* is our first explanatory variable, *Total_Oil_Income_PC* is our second explanatory variable
- Under the *Std. Error* column is the respective standard errors for both β coefficients. This is used to calculate the t-statistic
- Under the *t value* column is the t-test statistics for both β coefficients. - Under the *Pr(>|t|)* column is the p-values for each test statistic.
- The stars after the p-values are the significance level. One star * means $p < 0.05$, two stars ** means $p < 0.01$, and three stars *** means $p < 0.001$
- For more details of interpretation, see the previous sections in this chapter.

To calculate confidence intervals, we can use the *confint()* command, and simply input the name of our model within:

```
confint(model)
```

```
                              2.5 %        97.5 %
(Intercept)             -1.9422959405 -1.6069289543
GDP_Per_Cap_Haber_Men_2  0.0004551738  0.0004976894
Total_Oil_Income_PC     -0.0011586240 -0.0010468834
```

As we can see, this command gives us 95% confidence intervals (both lower and upper bounds), for all parameters in our model.

Chapter 7

Explanatory Variable Analysis

7.1 Polynomial Transformations

Sometimes, a linear (straight-line) best-fit line is a poor description of a relationship. For example, the relationship between two variables could be curved, not straight.

We can model more flexible relationships that are not straight lines, by including a transformation of the variable X that we are interested in. The most common transformations for non-linearity are polynomial transformations, including quadratic and cubic transformations.

Quadratic Transformations

Quadratic transformations of X take the following form:

$$E[Y] = \alpha + \beta_1 X + \beta_2 X^2$$

If you recall from high-school algebra, an equation that takes the form of $y = ax^2 + bx + c$ creates a *parabola*. Indeed, this transformation fits a parabola as the best-fit line. However, there are a few things to consider:

- A true parabola has a domain of $(-\infty, \infty)$. However, our model often does not need to do this. The best-fit parabola is only used for the range of plausible X values, given the nature of our explanatory variable. For example, if X was age, a negative number would make no sense.
- Because the parabola's domain often exceeds our plausible range of X values, the vertex of the parabola (where it changes directions) may not be in our data.
- We always include lower degree terms in our model. For example, in this quadratic (power 2) model, we also include the X term without the square.

To fit a model like this, we simply do the same process of minimising the sum of squared errors.

When we run a quadratic model in R, we get two coefficients: β_1 is attached to the X term, while β_2 is attached to the X^2 term. How do we interpret these coefficients?

- β_1's value is no longer directly interpretable. This is because we cannot "hold all other coefficients constant", since β_2 also contains the same X variable. Thus, we cannot isolate the effect of X and β_1.
- β_2's value also cannot be directly interpreted. However, β_2 can tell us two things. First, if the coefficient of β_2 is statistically significant, we can conclude that there is a non-linear relationship between X and Y. Second, if β_2 is negative, the best-fit parabola will open downwards, and if β_2 is positive, the best-fit parabola will open upwards.

If we want to interpret the magnitude of the model, we are best off using predicted values of Y (obtained using the model equation above).

There is one more thing we can interpret with the quadratic transformation: the **vertex** of the best-fit parabola. The vertex, if we remember our algebra, is either the maximum or minimum point of a parabola. Thus, if we remember from calculus and optimisation, we can find the maximum and minimums through setting the derivative equal to 0. For the quadratic model, this is as follows - we first find the derivative, then set the derivative equal to 0:

$$\begin{aligned}
\hat{Y} &= \hat{\alpha} + \hat{\beta}_1 X + \hat{\beta}_2 X^2 \\
\frac{d\hat{Y}}{dX} &= 0 + \hat{\beta}_1 + 2\hat{\beta}_2 X \\
0 &= \hat{\beta}_1 + 2\hat{\beta}_2 X \\
-\hat{\beta}_1 &= 2\hat{\beta}_2 X \\
X &= \frac{-\hat{\beta}_1}{2\hat{\beta}_2}
\end{aligned}$$

This point is useful, as it is either the maximum or minimum of our best-fit parabola. This means that at the X value we calculate from this equation, we will either see the highest or lowest expected Y value.

General Polynomial Models

While quadratic models are the most common polynomial transformation, we do not have to stop there. We can continue to add further polynomials (although anything beyond cubic is exceedingly rare):

- Cubic: $E[Y] = \alpha + \beta_1 X + \beta_2 X^2 + \beta_3 X^3$
- Quartic: $E[Y] = \alpha + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \beta_4 x^4$

Each higher order coefficient, if statistically significant, indicates that the relationship between X and Y, is not of the previous highest power.

- For example, if the cubic term β_3 is statistically significant, we can reject a quadratic relationship between X and Y

Remember to always include the lower power monomials within our polynomial model. For example, if you have a quartic transformation, you must also have the linear, quadratic, and cubic terms.

7.2 Logarithmic Transformations

Logarithmic transformations are another form of non-linear transformations. Logarithmic transformations are commonly used for heavily skewed variables, such as when the explanatory variable is income, wealth, and so on.

In situations with heavily skewed variables, we often replace X in our models with $\log(X)$. Note that in statistics, when we refer to logarithms, we are referring to natural logarithms, such that $\log(X) = \ln(X)$.

Thus, the logarithmic transformation takes the following form:

$$E[Y] = \alpha + \beta \log(X)$$

Interpretation of the β coefficient can be a little bit trickier for logarithmic transformations. We could interpret it in the same way we interpret linear regressions: given a one unit increase in the log of X, there is an expected β change in Y.

However, this issue is that this does not really say much - I mean, who knows what a *one unit increase in the log of* X even means?

However, with some properties of logarithms, we can actually create a more useful interpretation. Based on logarithm rules, we know the following to be true:

$$\begin{aligned} \log(X) + A &= \log(X) + \log(e^A) \\ &= \log(e^A \times X) \end{aligned}$$

Now, let us plug this into our original regression model:

$$\begin{aligned} E[Y|X] &= \alpha + \beta \log(X) \\ E[Y|e^A \times X] &= \alpha + \beta \log(e^A \times X) \\ &= \alpha + \beta[\log(X) + A] \\ &= \alpha + \beta A + \beta \log(X) \end{aligned}$$

Now find the difference between $E[Y|e^A \times X]$ and $E[Y|X]$:

$$\begin{aligned} E[Y|e^A \times X] - E[Y|X] &= [\alpha + \beta A + \beta \log(X)] - [\alpha + \beta \log(X)] \\ E[Y|e^A \times X] - E[Y|X] &= \beta A \end{aligned}$$

Thus, we can seen that when we multiply X by e^A, we get an expected βA change in Y.

We can make this interpretation more useful by purposely choosing some value A that makes e^A make more sense. For example, if $A = 0.095$, then $e^A = 1.1$. Why is that A value useful? Well, that means when we multiply $X \times e^A$, we are actually doing $X \times 1.1$, which if you remember your percentages, means a 10% increase in X. Thus, increasing X by 10% is associated with an expected change of 0.095β units of Y.

7.3 Binary Explanatory Variable

Binary explanatory variables will change the interpretations of our coefficients. We can "solve" for these interpretations given the standard linear model $E[Y] = \alpha + \beta X$, given X has two categories $X = 0, X = 1$:

$$E[Y|X = 0] = \alpha + \beta(0) = \alpha$$
$$E[Y|X = 1] = \alpha + \beta(1) = \alpha + \beta$$
$$E[Y|X = 1] - E[Y|X = 0] = (\alpha + \beta) - \alpha = \beta$$

From the above, we can see the following interpretations of our coefficients:

- α is the expected value of Y given an observation in category $X = 0$
- $\alpha + \beta$ is the expected value of Y given an observation in category $X = 1$
- β is the expected difference in Y between the categories $X = 1$ and $X = 0$

Thus, we can see that β is measuring the difference between the two categories. In fact, β actually becomes a difference-in-means test, meaning that if β is statistically significant, we can conclude a significant difference in the mean Y between the two categories.

Because of the unique coefficient meanings in a regression with binary explanatory variables, the OLS estimator also takes a shortcut when estimated the coefficients α and β:

- Coefficient α is estimated as $\hat{\alpha} = \bar{Y}_0$
- Coefficient β is estimated as $\hat{\beta} = \bar{Y}_1 - \bar{Y}_0$

Where $\bar{Y}_1$ is the sample mean of Y for observations in category $X = 1$, and $\bar{Y}_0$ is the sample mean of Y for observations in category $X = 0$.

7.4 Polytomous Explanatory Variable

A **polytomous** variable is one with 3 or more categories that are unranked. A classic example is the variable *country*, which is a categorical variable with all the different countries included in a dataset such as Argentina, France, Mexico, etc.

How do we run a regression with polytomous explanatory variables? What happens is that we divide the variables into a set of dummy binary variables.

- Dummy binary variables are created for all except one of the categories in our variable. Each dummy variable has two values - 1 meaning the observation is in the category, and 0 meaning the observation is not in that category.
- The category without a dummy variable is the **reference/baseline** category. Essentially, when all other dummy variables are equal to 0, that is referring to the reference/baseline category.

Thus, for the n number of categories in X, we would create $n - 1$ dummy variables, and input it into a regression equation as follows:

$$E[Y] = \alpha + \beta_{x=0}X_{x=0} + \beta_{x=1}X_{x=1} + ... + \beta_{x=n-1}X_{x=n-1}$$

For example, take the following polytomous variable: *company*, which contains the categories *microsoft, google,* and *apple*.

- Let us create dummy variables for 2 of the 3 categories.
- *Google* will become the first dummy variable X_g. When $X_g = 1$, that observation is part of the *google* category. When $X_g = 0$, that observation is NOT a part of the *google* category.
- *Apple* will become the second dummy variable X_a. When $X_a = 1$, that observation is part of the *apple* category. When $X_a = 0$, that observation is NOT a part of the *apple* category.
- *Microsoft* will not get its own dummy variable. This is because when both *apple* and *microsoft* $X_g = X_a = 0$ that is referring to the *microsoft* category (since these are the only observations not a part of either previous category).

Mathematically, this is how it would be represented in a regression equation:

$$E[Y] = \alpha + \beta_g X_g + \beta_a X_a$$

To find the expected value of each category, we would do the following:

$$\begin{aligned} E[Y|X = \text{Google}] &= E[Y|X_g = 1, X_a = 0] = \alpha + \beta_g(1) + \beta_a(0) = \alpha + \beta_g \\ E[Y|X = \text{Apple}] &= E[Y|X_g = 0, X_a = 1] = \alpha + \beta_g(0) + \beta_a(1) = \alpha + \beta_a \\ E[Y|X = \text{Microsoft}] &= E[Y|X_g = 0, X_a = 0] = \alpha + \beta_g(0) + \beta_a(0) = \alpha \end{aligned}$$

Thus, from these above equations, we can see the interpretation of the coefficients:

- α is the expected value of the reference category, in this case, *microsoft*.
- β_g is the expected Y difference between the *google* category and the reference category *microsoft*. The statistical significance of this coefficient would be a difference of means test between the two categories.
- β_a is the expected Y difference between the *apple* category and the reference category *microsoft*. The statistical significance of this coefficient would be a difference of means test between the two categories.

More generally, the β_j of category j's dummy variable, represents the expected difference in Y between category j and the reference/baseline category.

Notice how the coefficient p-values are a difference-of-means test between two categories, and not a statistical significance test of the entire categorical variable *company* that has 3 different categories. To test the statistical significance of the entire categorical variable, we use an F-test.

An F-test, simply, tests a model's explanatory power against the explanatory power of a model where all coefficients $\alpha, \beta_1, ..., \beta_n$ all are equal to 0. Thus, an F-test would allow us to test the significance of the effect of the variable *company* on Y, which our individual β coefficients do not tell us.

7.5 Interaction Effects

Interactions, also called moderating effects, means that the effect of some X_j on Y is not constant, and depends on some third variable X_k. Essentially, X_k's value changes the relationship between X_j and Y.

This is quite common in the real world. For example, imagine outcome variable Y to be the severity of a car crash. X_1 can represent the darkness of the road at the time of the car crash. X_2 can represent the slipperiness of the road at the time of the car crash. We could quite reasonably expect that as the road is more slippery, i.e. X_2 increases, the darkness of the road X_1's effect on the severity of a car crash Y might be stronger, since slippery further enhances the danger of dark roads.

Or for a more political example, Y could be the chance of a civil war occurring, X_1 is the severity of an economic crash, and X_2 is the development level of a country. We could quite reasonably expect that in the effect of a economic crash on a chance of civil war would be significantly higher in developing nations rather than developed. Or in other words, the chance that a civil war occurs due to a economic crash is higher in countries like Venezuela, North Korea and Eritrea, compared to the relationship in Norway, Switzerland, and Denmark.

Interaction effects are represented by two variables being multiplied together in a regression equation. In the model below, X_1 and X_2 are interacting with each other:

$$E[Y] = \alpha + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$$

We can mathematically show that the effect of X_1 on Y is not constant - and varies due to the value of X_2. We show this through finding the partial derivative of X_1 on Y, since the derivative is, by definition, the function of the rate of change between X_1 and Y.

$$\begin{aligned}
\hat{Y} &= \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_1 X_2 \\
\frac{\partial \hat{Y}}{\partial X_1} &= 0 + \hat{\beta}_1 + 0 + \hat{\beta}_3 X_2 \\
\frac{\partial \hat{Y}}{\partial X_1} &= \hat{\beta}_1 + \hat{\beta}_3 X_2
\end{aligned}$$

As you can see, the relationship between X_1 and Y here depends on the value of X_2. In more intuitive words, given a one unit increase in X_1, there is an expected $\hat{\beta}_1 + \hat{\beta}_3 X_2$ increase in Y.

We can also find the effect of X_2 on Y:

$$\begin{aligned}
\hat{Y} &= \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_1 X_2 \\
\frac{\partial \hat{Y}}{\partial X_2} &= \hat{\beta}_2 + \hat{\beta}_3 X_1
\end{aligned}$$

With these equations, we can interpret the coefficients of our model:

- $\hat{\beta}_1$ is the relationship between X_1 and Y, given $X_2 = 0$.
- $\hat{\beta}_2$ is the relationship between X_2 and Y, given $X_1 = 0$.

- $\hat{\beta}_3$ represents two things. For every one unit increase of X_2, the magnitude of the relationship between X_1 and Y changes by $\hat{\beta}_3$. Similarly, for every one unit increase of X_1, the magnitude of the relationship between X_2 and Y changes by $\hat{\beta}_3$
- α is still the expected value of Y when all explanatory variables equal 0.

The coefficient β_3's significance level tells us if there is a statistically significant interaction. If β_3 is not statistically significant, we can often remove the interaction term. However, if β_3 is statistically significant, that means we have found two terms that interact.

Often times, our moderating effect X_2 is a binary variable (for example, developed/developing country, true/false, yes/no). In this scenario:

- $\hat{\beta}_1$ is the relationship between X_1 and Y when X_2 is in the category $X = 0$.
- $\hat{\beta}_1 + \beta_3$ is the relationship between X_1 and Y when X_2 is in the category $X = 1$.
- $\hat{\beta}_3$ is the difference in the magnitude of the relationship between X_1 and Y, between the categories $X = 1$ and $X = 0$.

You can get all of these interpretations above simply by plugging in $X_2 = 0$ and $X_2 = 1$ into the previous equations we have found.

7.6 Explanatory Variables in R

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Part III

Extending the Linear Model for Causal Inference

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