Problem Set Week 4

GV4C8 Game Theory for Political Science

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Question 1

Find the mixed strategy equilibria of the following static games:

1a:

	L	R
T	1, **2**	**2**, 0
B	**2**, 1	0, **3**

We can pretty quickly see there are no Nash Equilibria. Now, let us define p and q as the following:

- p is probability player 1 plays T, $1-p$ is probability plays B. Let us call this strategy σ_1.
- q is probability player 2 plays L, $1-q$ is probability plays R . Let us call this strategy σ_2.
- $\sigma_i(s_k)$ denotes the probability that player i plays strategy s_k

What are **player 1's** expected utilities when playing either strategy?

- $u_1(T, \sigma_2) = \sigma_2(s_L)u_1(T, L) + \sigma_2(S_R)u_1(T, R) = (q)1 + (1-q)2$
- $u_1(B, \sigma_2) = \sigma_2(s_L)u_1(B, L) + \sigma_2(s_R)u_1(B, R) = (q)2 + (1-q)0$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned}
u_1(T, \sigma_2) &= u_1(B, \sigma_2) \\
(q)1 + (1-q)2 &= (q)2 + (1-q)0 \\
q + 2 - 2q &= 2q \\
-q + 2 &= 2q \\
3q &= 2 \\
q &= 2/3
\end{aligned}$$

What are **player 2's** expected utilities when playing either strategy?

- $u_2(\sigma_1, L) = \sigma_1(s_T)u_2(T, L) + \sigma_1(s_B)u_2(B, L) = (p)2 + (1-p)1$
- $u_2(\sigma_1, R) = \sigma_1(s_T)u_2(T, R) + \sigma_1(s_B)u_2(B, R) = (p)0 + (1-p)3$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$
\begin{aligned}
u_2(\sigma_1, L) &= u_2(\sigma_1, R) \\
(p)2 + (1-p)1 &= (p)0 + (1-p)3 \\
2p + 1 - p &= 3 - 3p \\
p + 1 &= 3 - 3p \\
4p &= 2 \\
p &= 1/2
\end{aligned}
$$

Thus, there is a Mixed Strategy Nash Equilibria where:

$$(\sigma_1^*, \sigma_2^*) = \Big((\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}) \Big)$$

1b:

	L	R
T	**6**, 0	0, **6**
B	3, **4**	**6**, 0

We can pretty quickly see there are no Nash Equilibria. Now, let us define p and q as the following:

- p is probability player 1 plays T, $1-p$ is probability plays B. Let us call this strategy σ_1.
- q is probability player 2 plays L, $1-q$ is probability plays R . Let us call this strategy σ_2.
- $\sigma_i(s_k)$ denotes the probability that player i plays strategy s_k

What are **player 1's** expected utilities when playing either strategy?

- $u_1(T, \sigma_2) = \sigma_2(s_L)u_1(T, L) + \sigma_2(S_R)u_1(T, R) = (q)6 + (1-q)0$
- $u_1(B, \sigma_2) = \sigma_2(s_L)u_1(B, L) + \sigma_2(s_R)u_1(B, R) = (q)3 + (1-q)6$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned} u_1(T, \sigma_2) &= u_1(B, \sigma_2) \\ (q)6 + (1-q)0 &= (q)3 + (1-q)6 \\ 6q &= 3q + 6 - 6q \\ 6q &= -3q + 6 \\ 9q &= 6 \\ q &= 2/3 \end{aligned}$$

What are **player 2's** expected utilities when playing either strategy?

- $u_2(\sigma_1, L) = \sigma_1(s_T)u_2(T, L) + \sigma_1(s_B)u_2(B, L) = (p)0 + (1-p)4$
- $u_2(\sigma_1, R) = \sigma_1(s_T)u_2(T, R) + \sigma_1(s_B)u_2(B, R) = (p)6 + (1-p)0$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned} u_2(\sigma_1, L) &= u_2(\sigma_1, R) \\ (p)0 + (1-p)4 &= (p)6 + (1-p)0 \\ 4 - 4p &= 6p \\ 10p &= 4 \\ p &= 2/5 \end{aligned}$$

Thus, there is a Mixed Strategy Nash Equilibria where:

$$(\sigma_1^*, \sigma_2^*) = \Big((\frac{2}{5}, \frac{3}{5}), (\frac{2}{3}, \frac{1}{3}) \Big)$$

1c:

	L	R
T	0, 1	**0**, **2**
B	**2**, **2**	**0**, 1

We can pretty quickly see there are two Nash Equilibria, one involving each player playing another strategy. Now, let us define p and q as the following:

- p is probability player 1 plays T, $1-p$ is probability plays B. Let us call this strategy σ_1.
- q is probability player 2 plays L, $1-q$ is probability plays R . Let us call this strategy σ_2.
- $\sigma_i(s_k)$ denotes the probability that player i plays strategy s_k

What are **player 1's** expected utilities when playing either strategy?

- $u_1(T,\sigma_2) = \sigma_2(s_L)u_1(T,L) + \sigma_2(S_R)u_1(T,R) = (q)0 + (1-q)0$
- $u_1(B,\sigma_2) = \sigma_2(s_L)u_1(B,L) + \sigma_2(s_R)u_1(B,R) = (q)2 + (1-q)0$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned} u_1(T,\sigma_2) &= u_1(B,\sigma_2) \\ (q)0 + (1-q)0 &= (q)(2) + (1-q)0 \\ 0 &= 2q \\ q &= 0 \end{aligned}$$

What are **player 2's** expected utilities when playing either strategy?

- $u_2(\sigma_1,L) = \sigma_1(s_T)u_2(T,L) + \sigma_1(s_B)u_2(B,L) = (p)1 + (1-p)2$
- $u_2(\sigma_1,R) = \sigma_1(s_T)u_2(T,R) + \sigma_1(s_B)u_2(B,R) = (p)2 + (1-p)1$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned} u_2(\sigma_1,L) &= u_2(\sigma_1,R) \\ (p)1 + (1-p)2 &= (p)2 + (1-p)1 \\ p + 2 - 2p &= 2p + 1 - p \\ -p + 2 &= p + 1 \\ 1 &= 2p \\ p &= 1/2 \end{aligned}$$

Thus, there is a Mixed Strategy Nash Equilibria where:

$$(\sigma_1^*,\sigma_2^*) = \Big((\frac{1}{2},\frac{1}{2}),(0,1)\Big)$$

1d:

Table 4: For player 1, strategy C^1 is strictly dominanted. They will never play that strategy. We know that will not be a part of the mixed strategy nash equilibrium.

	A2	B2	C2
A1	0, 0	**2**, **3**	**7**, 0
B1	**3**, **2**	0, 0	0, 0
C1	0, **7**	0, 0	6, 6

For player 2, strategy C^2 is strictly dominated. They will never play that strategy. We know that will not be a part of the mixed strategy nash equilibrium.

Thus, we can simplify the matrix to:

	A2	B2
A1	0, 0	**2**, **3**
B1	**3**, **2**	0, 0

Now, let us find the mixed strategy nash equilibriums. Let us define p and q as the following:

- p is probability player 1 plays T, $1-p$ is probability plays B. Let us call this strategy σ_1.
- q is probability player 2 plays L, $1-q$ is probability plays R . Let us call this strategy σ_2.
- $\sigma_i(s_k)$ denotes the probability that player i plays strategy s_k

What are **player 1's** expected utilities when playing either strategy?

- $u_1(A^1, \sigma_2) = \sigma_2(s_{A^2})u_1(A^1, A^2) + \sigma_2(S_{B^2})u_1(A^1, B^2) = (q)0 + (1-q)2$
- $u_1(B^1, \sigma_2) = \sigma_2(s_{A^2})u_1(B^1, A^2) + \sigma_2(s_{B^2})u_1(B^1, B^2) = (q)3 + (1-q)0$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$\begin{aligned} u_1(A^1, \sigma_2) &= u_1(B^1, \sigma_2) \\ (q)0 + (1-q)2 &= (q)3 + (1-q)0 \\ 2 - 2q &= 3q \\ 2 &= 5q \\ q &= 2/5 \end{aligned}$$

What are **player 2's** expected utilities when playing either strategy?

- $u_2(\sigma_1, A^2) = \sigma_1(s_{A^1})u_2(A^1, A^2) + \sigma_1(s_{B^1})u_2(B^1, A^2) = (p)0 + (1-p)2$
- $u_2(\sigma_1, B^2) = \sigma_1(s_{A^1})u_2(A^1, B^2) + \sigma_1(s_{B^1})u_2(B^1, B^2) = (p)3 + (1-p)0$

For mixed strategy equilibrium to exist, by definition, players must be indifferent to both strategies:

$$
\begin{aligned}
u_2(\sigma_1 A^2) &= u_2(\sigma_1, B^2) \\
(p)0 + (1-p)2 &= (p)3 + (1-p)0 \\
2 - 2p &= 3p \\
2 &= 5p \\
p &= 2/5
\end{aligned}
$$

Thus, there is a Mixed Strategy Nash Equilibria where:

$$
(\sigma_1^*, \sigma_2^*) = \Big((\frac{2}{5}, \frac{3}{5}, 0), (\frac{2}{5}, \frac{3}{5}, 0) \Big)
$$

Since the third strategy is strictly dominated for both so it is 0 probability.

Question 2

Consider the following Battle of the Sexes Game:

	Cafe	Pub
Cafe	4, 3	1, 1
Pub	0, 0	3, 4

- **a) find all pure strategy NE**
- **b) compute best responses to any mixed strategy of the opponent**
- **c) display the best responses in a two dimensional graph**
- **d) describe mutual best responses**

Question 3

Consider the following Hawk-Dove game:

	Cafe	Pub
Cafe	4, 3	1, 1
Pub	0, 0	3, 4

- **a) find all pure strategy NE**
- **b) compute best responses to any mixed strategy of the opponent**
- **c) display the best responses in a two dimensional graph**
- **d) describe mutual best responses**

Question 4

Coordination game. Two people can perform a task if and only if they both exert effort.

- **They are both better off if they both exert effort and perform the task, than if neither exerts effort (and nothing is accomplished);**
- **the worst outcome for each person is that she exerts effort and the other person does not (in which case again nothing is accomplished).**
- **Specifically, the players' preferences are represented by the expected value of the payoff functions below where c is a positive number less than 1 that can be interpreted as the cost of exerting effort.**

Find all the mixed strategy Nash equilibria of this game. How do the equilibria change as c increases? Explain the reasons for the changes.

	No Effort	Effort
No Effort	0, 0	0, -c
Effort	-c, 0	1-c, 1-c

So let us first look at the pure strategies, and see if there is any equilibria:

	No Effort	Effort
No Effort	**0**, **0**	0, -c
Effort	-c, 0	**1-c**, **1-c**

So we have two pure nash strategy equilibria: (No Effort, No Effort), and (Effort, Effort). That means players can pick either strategy and are indifferent between the two.So let us find the mixed strategy nash Equilibria!:

Definitions

For clarity, here are the definitions I am using

- I will define strategy *No Effort* as N, and Strategy *Effort* as E for ease of writing equations.
- To make it even more clear, player 1's strategies will have a subscript 1: N_1, E_1, while player 2's strategies will have a subscript 2: N_2, E_2.

Mixed strategies:

- p is the probability player 1 plays N_1, thus $1 - p$ is probability plays E_1 (cuz rules of probability) Let us call this mixed strategy profile σ_1.
- q is the probability player 2 plays N_1, $1 - q$ is probability plays E_1 . Let us call this mixed strategy profile σ_2.
- $\sigma_i(s_k)$ denotes the probability that player i plays strategy s_k. For example, $\sigma_1(N_1)$ is the probability that player 1 plays N_1 in the mixed strategy σ_1.

Player 1

Player 1's utility for playing *No Effort* N_1, while player 2 plays mixed strategy σ_2:

$$\begin{aligned} u_1(N_1, \sigma_2) &= \sigma_2(N_2) \times u_1(N_1, N_2) + \sigma_2(E_2) \times u_1(N_1, E_2) \\ &= q \times 0 + (1-q) \times 0 \\ &= 0 \end{aligned}$$

Player 1's utility for player *Effort* E_1, while player 2 plays mixed strategy σ_2:

$$\begin{aligned} u_1(E_1, \sigma_2) &= \sigma_2(N_2) \times u_1(E_1, N_2) + \sigma_2(E_2) \times u_1(E_1, E_2) \\ &= q \times (-c) + (1-q) \times (1-c) \\ &= -cq + (1-q)(1-c) \\ &= -cq + 1 - c - q + qc \\ &= 1 - c - q \end{aligned}$$

Now since player 1 is indifferent between N_1 and E_1, they should have equal utility:

$$\begin{aligned} u_1(N_1, \sigma_2) &= u_1(E_1, \sigma_2) \\ 0 &= 1 - c - q \\ q &= 1 - c \end{aligned}$$

Player 2

It is a symmetric game so in theory, the same outcome for player 2 and p.

But let us be thorough and double check.

Player 2's utility for playing *No Effort* N_2, while player 1 plays mixed strategy σ_1:

$$\begin{aligned} u_2(\sigma_1, N_2) &= \sigma_1(N_1) \times u_2(N_1, N_2) + \sigma_1(E_1) \times u_2(E_1, N_2) \\ &= p \times 0 + (1-p) \times 0 \\ &= 0 \end{aligned}$$

Player 2's utility for player *Effort* E_2, while player 1 plays mixed strategy σ_1:

$$
\begin{aligned}
u_2(\sigma_1, E_2) &= \sigma_1(N_1) \times u_2(N_1, E_2) + \sigma_1(E_1) \times u_2(E_1, E_2) \\
&= p \times (-c) + (1-p) \times (1-c) \\
&= -pc + (1-p)(1-c) \\
&= -pc + 1 - p - c + pc \\
&= 1 - p - c
\end{aligned}
$$

Now since player 2 is indifferent between N_2 and E_2, they should have equal utility:

$$
\begin{aligned}
u_2(\sigma_1, N_2) &= u_2(\sigma_1, E_2) \\
0 &= 1 - p - c \\
p &= 1 - c
\end{aligned}
$$

Conclusion

Thus, $q = 1 - c$ and $p = 1 - c$ as we have shown above.

- So, $1 - q = 1 - (1 - c) = c$, and $1 - p = 1 - (1 - c) = c$.

Thus, the mixed strategy Nash Equilibrium of this game is $(\sigma_1^*, \sigma_2^*) = ((1-c, c), (1-c, c))$

OK so what does that mean?

- As c becomes larger, q and p decrease.
- q and p represent the probabilities that a player will play No effort within the mixed strategy σ_1, σ_2.
- So in context of this game, as the cost of exerting effort increases, the probability a player will play *No Effot* will decrease.