Problem Set Week 2

GV4C8 Game Theory for Political Science

Kevin Lingfeng Li

Question 1

Introduction to multidimensional politics (or divide-the dollar games). Three legislators need to decide how to distribute the budget among their three constituencies; the final agreement needs to be approved by majority.

- **a) Show that given any budget allocation there is always another allocation that is preferred by a majority.**
- **b) Construct three budget allocations such that aggregate preferences over these budget allocations are cyclic.**

Let us take a total pot of 1 dollar. The most obvious way to start is to divide it into thirds, so all three constituencies get 1/3. The strategy profile is thus $(1/3, 1/3, 1/3)$

However, only 2 legislators need to approve the budget. So, 2 of the legislators can "team up" and screw over the third legislator. For example, the strategy profile $(1/2, 1/2, 0)$ would be supported by the first two legislators, as they both get more - and would pass since the 2 make up a majority of the 3.

But then, the third legislator could propose to take everything away from the first legislator, give some to the second, and take the remaing bit. For example, $(0, 3/5, 2/5)$ The second legislator likes this since they are getting more. The third also likes this because they are not getting 0. Them 2 make a majority, so this passes again.

We can keep doing this: there is always a deviation.

An example of a cyclic budget allocation preferences are as follows: 1. $(1/2, 1/2, 0)$ 2. $(0, 3/5, 2/5)$ 3. $(2/5, 0, 3/5)$ 4. And goes back to the first.

Question 2:

Electoral competition with three parties. Consider a Downsian model of electoral competition with three candidates that are rent seeking (i.e. they are driven by the wish to win the election).and plurality rule (the candidate that gets most votes wins the election). Do candidates converge to the median as with the two-candidate case?

Let us define the three parties as party A, party B, and party C. The positions each party takes will be p_A, p_B, p_C

Let us define the median voter as m

What if all parties position at m, such that $p_A = p_B = p_C$? Does any party have a profitable deviation? - First, what is the payoff of each party at this point? It is 1/3, since they each have 1/3 chance of winning election. - Let us define party i as any one of the three parties above - Party i could reposition itself at $m + \epsilon$ - Now, party i has captured voters on the spectrum from $m + \epsilon/2$ to 1, giving it a total of $1 - \epsilon/2$ distance in the uniform distribution. - The other two parties still have more overall of the spectrum: $1 + \epsilon/2$. However, the two must split these voters between the two. - Thus, Party i will get the most votes, since $1 - 0.5\epsilon > \frac{1+0.5\epsilon}{2}$ when the deviation ϵ is relatively small.

So no, the candidates do not converge at the median. But where would they converge?

Question 3:

Electoral competition in two districts. Consider a variant of the Downsian model of electoral competition that captures features of the US presidential election (and many other countries!).

- **Voters are divided between two districts. District 1 is worth more Electoral College votes than is district 2.**
- **The winner is the candidate who obtains the most Electoral College votes.**
- **Denote by m_i the median favourite position among the citizens of district i, for $i = 1, 2$; assume that $m2 < m1$. Each of two candidates chooses a single position.**
- **Each citizen votes (non-strategically) for the candidate whose position in closest to her favourite position. The candidate who wins a majority of the votes in a district obtains all the electoral college votes of that district; if the candidates obtain the same number of votes in a district, they each obtain half of the electoral college votes of that district.**
- **Find the Nash Equilibrium (equilibria?) of the strategic game that models this situation.**

Question 4:

Electoral competition in heterogeneous districts. Consider a country with three regions with citizens that have (symmetric) single peaked preferences. There are two political parties. In Region 1 (the city) there is 50% of the population of the country and their preferred policy is uniformly distributed in [0,1]; in Region 2 (suburbia) there is 30% of the population of the country and their preferred policy is uniformly distributed in [1,1.5]; finally, in Region 3 (rural) there is the remaining 20% of the population of the country and their preferred policy is uniformly distributed in [1.5,2].}

- **The country elects its president by plurality considering a unique national district (i.e. the votes of all three regions are pooled together and the candidate that gets most votes wins). There are two rent-seeking parties. Find all the set of policies parties can propose that constitute an equilibria.**
- **Now assume each region is an electoral district that selects a candidate from one of the two parties by plurality. The president belongs to the party that obtains a majority of candidates (i.e. analogous system to the electoral college in the United States' presidential elections). The two parties need to announce a single policy for the whole country; which policies constitute an equilibrium of the game?**

Question 5:

Electoral competition under the Alternative Vote. The Alternative Vote (AV)} is a preferential system where the voter has the chance to rank the candidates in order of preference. The voter puts a '1' by their first choice a '2' by their second choice, and so on. Candidates are elected outright if they gain more than half of the first preference votes. If not, the candidate who lost (the one with least first preferences) is eliminated and their votes are redistributed according to the second (or next available) preference marked on the ballot paper. This process continues until one candidate has half of the votes and is elected.}

Three rent-seeking candidates can propose a policy in a one-dimensional policy space, the interval [0,1]. The voting procedure followed is the Alternative Vote (majoritarian system in which voters cast a priority ranking). Citizens' preferred policies are uniformly distributed on the policy space. Citizens vote sincerely, i.e. a voter gives the highest rank to the candidate who is closest to his ideal point, second highest rank to the second closest candidate, and lowest rank to the last candidate; in case of indifference the placement is random (i.e. if a voter is indifferent between two top ranked candidates, say parties A and B; she places A first and B second with probability ½ and B first and A second with probability ½). Is there an equilibrium where candidates converge to the median?

Definitions

Let us define the three parties as party A, B, C, who choose some position $p_A, p_B, p_C \in [0, 1]$. Let us define m as the position of the median voter in the spectrum $[0, 1]$

Payoffs if Everyone was at Median

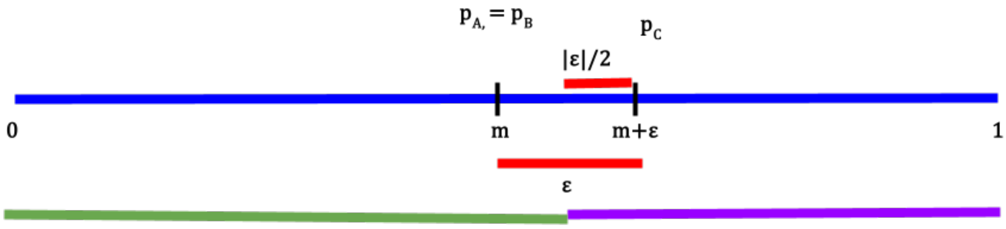
Let us see if strategy profile (m, m, m) is a Nash Equilibrium through seeing if any party i has a profitable deviation.

- Before we look at profitable deviation, what is the expected utility of (m, m, m)? We need to know this in order to see what a profitable deviation would be.
- Well at (m, m, m), all three parties would get the same amount of first, second, and third place votes: 1/3. Thus, no one has a majority, so one party by random chance will be eliminated. 2/3 chance to get to the second round.
- In the second round, at (m, m), both parties still get the same amount of votes. Thus, they have a 1/2 chance of winning the election.
- Thus, expected chance of winning the election at (m, m, m) for all three parties is $2/3 \times 1/2 = 1/3$
  - This makes sense, each party has equal chance of winning, 1/3

<u>So any profitable deviation requires a chance of winning higher than 1/3</u>

Looking for Profitable Deviation: First Round

Let us say any party i deviates from the median by some value ϵ, so it has a new position of $m + \epsilon$:



Now, let us look at the first round and how the voting will turn out. Since the distribution is uniformly distributed, we can just look at the distance each party wins on the number line, since that is proportional to the total number of votes.

Party i that deviates will win the distance $\frac{1}{2} - \frac{1}{2}|\epsilon|$

The other two parties will win distance $1/2 + |\epsilon|/2$, however, they will split the votes between themselves as they are equidistant between all the voters here, so each party gets $\frac{1/2+|\epsilon|/2}{2} = \frac{1}{4} + \frac{0.5|\epsilon|}{2} = \frac{1}{4} + \frac{1}{4}|\epsilon|$

Why? See figure above

Important thing is that 50% of votes (which would outright win the first round) would only be possible if a party won more than 50% of the distance between $[0, 1]$, which is a distance of $1/2$.

- Party i that deviated cannot win a majority, since they will win distance $\frac{1}{2} - \frac{1}{2}|\epsilon|$, which cannot be greater that $1/2$ when $|\epsilon| < 0$. Thus, party i has no chance of winning outright.
- The other two parties win $\frac{1}{4} + \frac{1}{4}|\epsilon|$, and that can only be a majority of the first round (i.e. greater than $1/2$) if $|\epsilon| > 1$. But, since ϵ is the deviation of party i in one direction, it cannot possibly exceed 0.5. Thus, there is no case where the other two parties outright win the election in the first round.

Then, that means one party with the lowest votes will be eliminated? Which party will be eliminated? That depends on ϵ. Party i will survive, if it has a higher proportion of vote than the other two parties - see the inequality: $\frac{1}{2} - \frac{1}{2}|\epsilon| > \frac{1}{4} + \frac{1}{4}|\epsilon|$

$$\frac{1}{2} - \frac{1}{4} > \frac{1}{2}|\epsilon| + \frac{1}{4}|\epsilon|$$

$$\frac{1}{4} > \frac{3}{4}|\epsilon|$$

$$\epsilon < \frac{1}{4} \times \frac{4}{3}$$

$$\epsilon < \frac{1}{3}$$

Thus, the following scenarios can occur

1. If $|\epsilon|$ deviation distance of party i is greater than 1/3, then they lose the first round, and thus have a 0 percent chance of winning the election, thus, that is not a profitable deviation.
2. If $|\epsilon|$ deviation distane of party i is exactly 1/3, then they have a 2/3 chance of getting to the second round (since they will be tied with the other two parties)
3. When $|\epsilon|$ deviation distance is less than 1/3, party i will survive to the second round, and one of the other two parties will be eliminated.

Looking for Profitable Deviation: Second Round

In the second and third scenario, we are onto the second round - a two party battle between party i at position $m + \epsilon$, and the other party at m

- This is just a downsian model game!
- Since m is the median voter, and the median voter by definition is the 50th percentile, you need to win the median voter to win the election.
- So, party i that deviated will lose the second round to the party at the median

Thus, party i will have 0% chance of winning the second round, which is lower than the 33% chance they had originally at m, so there is no profitable deviation.

We could also think of it in another way (more fitting of alternative vote):

- If one of the other 2 parties votes are eliminated, those voters will just jump to the one who was not eliminated (since both had same policy position m)
- Thus, team i would win $\frac{1}{2} - \frac{1}{2}|\epsilon|$ of the distance (as seen above), and the median team would win $\frac{1}{2} + \frac{1}{2}|\epsilon|$ (basically, the original distance not split between the two median teams). By simple math, the median team always has the higher vote share.

Summary

There is no profitable deviation ϵ for any party i from strategy profile (m, m, m):

- If party i deviates by more that $|\epsilon| > 1/3$, they lose in the first round, so 0% chance, which is less than the original 33% chance. No profitable deviation.
- If party i deviates by less than $|\epsilon| \leq 1/3$, then they lose in the second round, so 0% chance, which is less than the original 33% chance. No profitable deviation.