Problem Set Week 7

GV4C8 Game Theory for Political Science

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Question 3

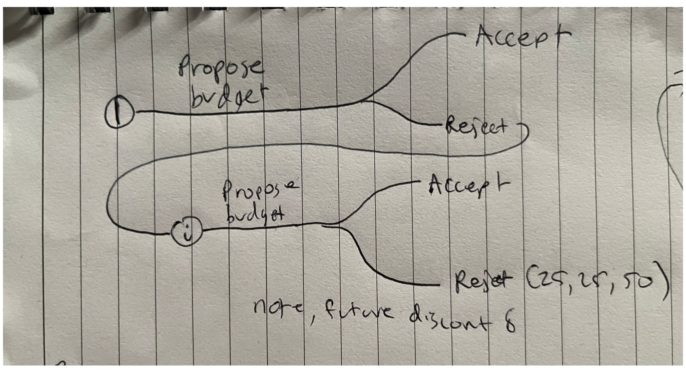
Distributive politics (bargaining over the division of a pie with more than 2 players). Three legislators need to distribute a budget of 100. There are two rounds of negotiations.

- **In the first round player 1 proposes a distribution of the budget among the 3 legislators. If a majority accepts the budget the game ends and the budget distributed accordingly. If the budget is rejected, the game continues to the second round.**
- **In the second round, one of the 3 players is randomly selected to propose a distribution. If a majority accepts the budget the game ends and the budget distributed accordingly. If the budget is rejected, the status quo budget (25, 25, 50) is implemented.**

Assume that players discount the future by and that the payoff of the players in each period is equal to the money they get.

- **a) Model this situation as an extensive form game**
- **b) Solve the game by backwards induction. What is the SPNE of the game? (Assume that legislators vote in favour of a budget when they are indifferent)**
- **c) Assume now that unanimity is required for a proposal to be accepted (i.e. all legislators need to vote in favour). How would your responses to a and b change?**

Part a) Model



Part b) Majority Rule

2nd round: What budget would player i (being rational) propose here?

- i must propose some budget that themselves, and one other player want to accept (since majority rule).
- i also wants to maximise their own payoff. The other players will only accept if the proposal gives them equal or more payoff.
- Thus, i will select a coalition partner of either player 1 or player 2 (since it makes no sense to partner with player 3, since you would need to give them 50 or more payoff).

Thus, player i will propose a budget such that they get 75, one other player (player 1 or 2) gets 25, and the third player always gets 0.

Now, let us calculate the expected utilities of each player in this second round. Note that all of second round is discounted by δ.

Thus, expected utilities of the second round are:

1. $\mathbb{E}[u_1] = \left[\frac{1}{3}(75) + \frac{1}{3}(25) + \frac{1}{3}(\frac{1}{2})(25)\right]\delta = (\frac{100}{3} + \frac{25}{6})\delta = \frac{225}{6} = \frac{75}{2}\delta$
   - Why? if P1 is selected as proposer (1/3 chance), she get 75
   - If P2 is selected as proposer (1/3 chance), P1 is in coalition, get 25
   - If P3 is selected as proposer (1/3 chance), there is 1/2 chance they are in coalition.
2. $\mathbb{E}[U_2] = \left[\frac{1}{3}(25) + \frac{1}{3}(75) + \frac{1}{3}(\frac{1}{2})(25)\right]\delta = \frac{75}{2}\delta$
   - If P1 is selected as proposer (1/3 chance), P2 is in coalition, get 25
   - If P2 is selected as proposer (1/3 chance), she gets 75
   - If P3 is selected as proposer (1/3 chance), there is 1/2 chance they are in coalition.
3. $\mathbb{E}[u_3] = \left[\frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{3}(75)\right]\delta = 25\delta$
   - If P1 is selected as proposer (1/3 chance), they never choose P3 as coalition partner
   - If P2 is selected as proposer (1/3 chance), they never choose P3 as coalition partner
   - If P3 is selected as proposer (1/3 chance), they get 75

1st round:

Player 1 is proposing. They must get one other player on board with their proposal. Which player should they form the coalition with?

- Well, since $25\delta < \frac{75}{2}\delta$, then player 1 should include player 3 in the coalition since they are less costly to convince.

Thus, player 1 (maximising their own payoff) would propose: $(100 - 25\delta,\ 0,\ 25\delta)$.

My Answer for SPNE Outcome

Thus SPNE outcome is:
- P1 proposes budget $(100 - 25\delta,\ 0,\ 25\delta)$
- P3 and P1 accepts that budget, P2 rejects. That is a majority. Thus that is final budget.

Formal SPNE

There are 5 different nodes of decisions for each player, where they have to make a decision (accept/reject). Why 5:

1. 1st round, when player 1 proposes $(100 - 25\delta,\ 0,\ 25\delta)$. We know in SPNE, player 1 and 3 accept, player 2 rejects.
2. 2nd round, when player 1 proposes $(75, 25, 0)$. We know that player 1 and 2 will accept, player 3 rejects.
3. 2nd round, when player 2 proposes $(25, 75, 0)$. We know that player 2 and 1 will accept, player 3 rejects.
4. 2nd round, when player 3 proposes and chooses P1 as coalition partner $(25, 0, 75)$. We know player 3 and 1 accept, player 2 rejects.
5. 2nd round, when player 3 proposes and chooses P2 as coalition partner $(0, 25, 75)$. We know that player 3 and 2 accept, player 1 rejects.

> My Answer for SPNE
>
> Thus, the SPNE of this game is:
>
> $$((AAAAR), (RAARA), (ARRAA))$$
>
> Where each player's strategies follow the orders of the 5 nodes listed above.

Part c) Unanimity

2nd round: What budget would player i propose here?

- If they want proposal to be accepted, they must propose $(25, 25, 50)$, since that is the only way to not make anyone else worse off (compared to status quo) given the limited budget.

Thus, expected utilities are (doesn't matter proposer or not, since status quo and proposed budget are the same):

- $\mathbb{E}[u_1] = 25\delta$
- $\mathbb{E}[u_2] = 25\delta$
- $\mathbb{E}[u_3] = 50\delta$

1st round: player 1 is proposing. Must make all players better off than the expected 2nd round utilities in order to pass budget.

Thus, P1 will give P2 and P3 their expected 2nd round utilities (since indifference means acceptance), and P1 will take the remainder - $100 - 25\delta - 25\delta = 100 - 75\delta$.

Thus, proposal will be: $(100 - 75\delta,\ 25\delta,\ 50\delta)$.

My Answer

Thus SPNE outcome is:

- Player 1 proposes budget $(100 - 75\delta,\ 25\delta,\ 50\delta)$$
- All 3 players agree, budget is passed.

Formally, SPNE is (given 1st strategy is 1st round, 2nd strategy is 2nd round):

$$(AA, AA, AA)$$

Assuming players rationally propose budget proposals (i.e. give themselves the most possible budget while ensuring enough are on board to pass the budget).

Question 5

Part a)

Game is as follows:

1st round:

1. Comittee proposes $\tilde{x}_1$, committee median preference is $x_c = 10$
2. Legislator chooses accept $\tilde{x}_1$, or reject and take status quo $x_0 = 0$. Median preference is $X_L = 2$
   - If accept, go to 2nd round part a
   - If reject, go to 2nd round part b

2nd round a (after accepting in round 1)

10% Chance that:

1. Comittee proposes $\tilde{x}_{a1}$, committee median preference is $x_c = 10$.
2. Legislator chooses accept $\tilde{x}_{a1}$, or reject and take new status quo $\tilde{x}_1$. Median preference is $X_L = 2$.

90% chance that:

1. Comittee proposes $\tilde{x}_{a2}$, committee median preference is $x_c = 10$.
2. Legislator chooses accept $\tilde{x}_{a2}$, or reject and take new status quo $\tilde{x}_1$. Median preference is $X_L = 5$.

2nd Round b (after rejecting 1st round)

10% Chance that:

1. Comittee proposes $\tilde{x}_{b1}$, committee median preference is $x_c = 10$.
2. Legislator chooses accept $\tilde{x}_{b1}$, or reject and take status quo $x_0 = 0$. Median preference is $X_L = 2$.

90% chance that:

1. Comittee proposes $\tilde{x}_{b2}$, committee median preference is $x_c = 10$.
2. Legislator chooses accept $\tilde{x}_{b2}$, or reject and take status quo $x_0 = 0$. Median preference is $X_L = 5$.

Utility functions: $u_i(x) = -|x - x_i|$.

Backwards Induction

Start with **2ND ROUND B (when 1st round reject)**. let us consider legislator

<u>10% chance utilities:</u>

- Proposal utility $u_L|\tilde{x}_{b1}| = -|\tilde{x}_{b1} - 2|$
- Status quo utility: $u_L(0) = -|0 - 2| = -4$.

Player 2 only accepts proposal if $u_L(\tilde{x}_{b1}) \geq u_L(0)$, thus accept when:

$$-|\tilde{x}_{b1} - 2| \geq -2 \implies \tilde{x}_{b1} \in [0, 4]$$

Which policy does player 1 propose?

- Player 1 preference is $x_c = 10$. Thus, 4 is the proposal that maximizes their payoff.
- They also prefer 4 over status quo 0. Thus $\tilde{x}_{b1} = 4$ is proposed.

Calculate utilities of this situation $\tilde{x}_{b1} = 4$:

- $u_L(4) = -|4 - 2| = -2$
- $u_c(4) = -|4 - 10| = -6$

<u>90% chance utilities</u>

- Proposal utility $u_L(\tilde{x}_{b2}) = -|\tilde{x}_{b2} - 5|$
- Status quo utility: $u_L(0) = -|0 - 5| = -5$.

Player 2 only accepts proposal if $u_L(\tilde{x}_{b2}) \geq u_L(0)$, thus accept when:

$$-|\tilde{x}_{b1} - 5| \geq -5 \implies \tilde{x}_{b2} \in [0, 10]$$

Which policy does player 1 propose? $\tilde{x}_{b2} = 10$ since closest to their preference.

Thus utilities of situation $\tilde{x}_{b2} = 10$ are:

- $u_L(10) = -|10 - 5| = -5$
- $u_c(10) = -|10 - 10| = 0$

Expected Utility in 2nd round, given 1st round rejected

Now, we know both 10% and 90% chance utilities, calculate expected utility of situation b:

- $\mathbb{E}[u_L] = 0.1(-2) + 0.9(-5) = -4.7$
- $\mathbb{E}[u_c] = 0.1(-6) + 0.9(0) = -0.6$

2nd ROUND A (when first round accept): let us consider legislator

<u>10% chance utilities:</u>

- Proposal utility $u_L(\tilde{x}_{a1}) = -|\tilde{x}_{a1} - 2|$
- new status quo utility: $u_L(\tilde{x}_1) = -|\tilde{x}_1 - 2|$.

Player 2 only accepts proposal if $u_L(\tilde{x}_{a1}) \geq u_L(\tilde{x}_1)$, thus accept when:

$$|\tilde{x}_{a1} - 2| \leq |\tilde{x}_{a1} - 2|$$

two conditions:

$$\tilde{x}_{a1} - 2 \leq \tilde{x}_1 - 2 \Longrightarrow \tilde{x}_{a1} \leq \tilde{x}_1$$

$$\tilde{x}_{a1} - 2 \leq -\tilde{x}_1 + 2 \Longrightarrow \tilde{x}_{a1} \leq -\tilde{x} + 4$$

Which policy does player 1 propose?

- So player 1 proposes $\tilde{x}_{a1} = \tilde{x}_1$ which is the closest to preference of $x_c = 10$

Calculate utilities of this situation $\tilde{x}_{a1} = \tilde{x}_1$:

- $u_L(\tilde{x}_1) = -|\tilde{x}_1 - 2|$
- $u_c(\tilde{x}) = -|\tilde{x}_1 - 10|$

<u>90% chance utilities:</u>

- Proposal utility $u_L(\tilde{x}_{a2}) = -|\tilde{x}_{a2} - 5|$
- new status quo utility: $u_L(\tilde{x}_1) = -|\tilde{x}_1 - 5|$.

Player 2 only accepts proposal if $u_L(\tilde{x}_{a2}) \geq u_L(\tilde{x}_1)$, thus accept when:

$$|\tilde{x}_{a2} - 5| \leq |\tilde{x}_1 - 5|$$

two scenarios:

$$\tilde{x}_{a2} - 5 \leq \tilde{x}_1 - 5 \Longrightarrow \tilde{x}_{a2} \leq \tilde{x}_1$$

$$\tilde{x}_{a2} - 5 \leq -\tilde{x}_1 + 5 \Longrightarrow \tilde{x}_{a2} \leq -\tilde{x} + 5$$

Which policy does player 1 propose?

- Player 1 preference is $x_c = 10$. Thus, $\tilde{x}_{a2} = \tilde{x}_1$ is optimal positioning.

Calculate utilities of this situation $\tilde{x}_{a2} = \tilde{x}_1$:

- $u_L(\tilde{x}_1) = -|\tilde{x}_1 - 5|$
- $u_c(4) = -|\tilde{x}_1 - 10|$

Expected Utility in 2nd round, given 1st round rejected

Now, we know both 10% and 90% chance utilities, calculate expected utility of situation a:

- $\mathbb{E}[u_L] = -0.1|\tilde{x}_1 - 2| - 0.9|\tilde{x}_1 - 5|$
- $\mathbb{E}[u_c] = -0.1|\tilde{x}_1 - 10| - 0.9|\tilde{x}_1 - 10| = -|\tilde{x}_1 - 10|$

NOW MOVE TO ROUND 1

let us consider legislator when comittee propose $\tilde{x}_1$.

- Proposal + what happens in round 2 when accepting: $u_L(\tilde{x}_1) = -|\tilde{x}_1 - 2| - 0.1|\tilde{x}_1 - 2| - 0.9|\tilde{x}_1 - 5|$
- Status quo + what happens when they reject: $u_L(0) = -|0 - 2| - 4.7 = -2.7$

When would player legislator accept? $u_L(\tilde{x}1) \geq uL(0)$. Thus:

$$u_L(\tilde{x}_1) = -|\tilde{x}_1 - 2| - 0.1|\tilde{x}_1 - 2| - 0.9|\tilde{x}_1 - 5| \geq -2.7$$

This is only true at $\tilde{x}_1 = 2$. So legislature will only accept 2.

Would committee be willing to propose $\tilde{x}_1 = 2$?

- Proposal + what happens in round 2 when accepted: $u_c(2) = -|2 - 10| - |2 - 10| = -16$
- Status quo in round 1 when rejected: $u_c(0) = -|0 - 10| - 0.6 = -10.6$

Thus, committee does not propose outcome that is accepted (since only 2 is accepted)

SPNE Outcome

First round:

1. Committee proposes any policy not equal to 2
2. Legislature rejects proposal, payoff committee = -10.6, payoff legislature = -2

Second round:

- If legislature preference is 2 (10%), comittee proposes 4, and it is accepted.
  - Payoff (2nd round) committee = -6, payoff legislature = -2
- If legislature preference is 5 (90%) committee proposes 10. payoff (-5, 0).
  - Payoff (2nd round) committee = 0, payoff committee -5.
- Expected utility of round 2 is: committee = -0.6, legislature = -4.7.