Problem Set Week 2

GV4C8 Game Theory for Political Science

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Question 1

Introduction to multidimensional politics (or divide-the dollar games). Three legislators need to decide how to distribute the budget among their three constituencies; the final agreement needs to be approved by majority.

- **a) Show that given any budget allocation there is always another allocation that is preferred by a majority.**
- **b) Construct three budget allocations such that aggregate preferences over these budget allocations are cyclic.**

Let us take a total pot of 1 dollar. The most obvious way to start is to divide it into thirds, so all three constituencies get 1/3. The strategy profile is thus $(1/3, 1/3, 1/3)$

However, only 2 legislators need to approve the budget. So, 2 of the legislators can "team up" and screw over the third legislator. For example, the strategy profile $(1/2, 1/2, 0)$ would be supported by the first two legislators, as they both get more - and would pass since the 2 make up a majority of the 3.

But then, the third legislator could propose to take everything away from the first legislator, give some to the second, and take the remaing bit. For example, $(0, 3/5, 2/5)$ The second legislator likes this since they are getting more. The third also likes this because they are not getting 0. Them 2 make a majority, so this passes again.

We can keep doing this: there is always a deviation.

An example of a cyclic budget allocation preferences are as follows:

1. $(1/2, 1/2, 0)$
2. $(0, 3/5, 2/5)$
3. $(2/5, 0, 3/5)$
4. And goes back to the first.

Question 2:

Electoral competition with three parties. Consider a Downsian model of electoral competition with three candidates that are rent seeking (i.e. they are driven by the wish to win the election).and plurality rule (the candidate that gets most votes wins the election). Do candidates converge to the median as with the two-candidate case?

Let us define the three parties as party A, party B, and party C. The positions each party takes will be p_A, p_B, p_C

Let us define the median voter as m

What if all parties position at m, such that $p_A = p_B = p_C$? Does any party have a profitable deviation?

- First, what is the payoff of each party at this point? It is 1/3, since they each have 1/3 chance of winning election.
- Let us define party i as any one of the three parties above
- Party i could reposition itself at $m + \epsilon$
- Now, party i has captured voters on the spectrum from $m + \epsilon/2$ to 1, giving it a total of $1 - \epsilon/2$ distance in the uniform distribution.
- The other two parties still have more overall of the spectrum: $1 + \epsilon/2$. However, the two must split these voters between the two. Thus, they occupy $1/2 + 1/4\epsilon$ each.
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  - Thus, Party i will get the most votes, since $1 - \frac{1}{2}\epsilon > \frac{1}{2} + \frac{1}{4}\epsilon$ when the deviation ϵ is relatively small.

So no, the candidates do not converge at the median, since player i has a profitable deviation.

Question 3:

Electoral competition in two districts. Consider a variant of the Downsian model of electoral competition that captures features of the US presidential election (and many other countries!).

- **Voters are divided between two districts. District 1 is worth more Electoral College votes than is district 2.**
- **The winner is the candidate who obtains the most Electoral College votes.**
- **Denote by m_i the median favourite position among the citizens of district i, for $i = 1, 2$; assume that $m_2 < m_1$. Each of two candidates chooses a single position.**
- **Each citizen votes (non-strategically) for the candidate whose position in closest to her favourite position. The candidate who wins a majority of the votes in a district obtains all the electoral college votes of that district; if the candidates obtain the same number of votes in a district, they each obtain half of the electoral college votes of that district.**
- **Find the Nash Equilibrium (equilibria?) of the strategic game that models this situation.**

This game is quite straight forward. Since we know district 2 has more electoral college votes than district 1, as long as you win district 1, you win the election.

How would you win district 1?

- Well this is just a downsian model - you would position yourself at the median.
- So each party has a 50/50 chance of winning the district. They would also have a 50/50 chance of winning district 2, since they are at the same position. Thus, 50% chance of winning the election.

Let us show there is no profitable deviation from this point:

- Let say player i deviates from m_1 towards m_2
- By deviating from m_1, player i looses district 1. Without any of district 1's votes, player i has a 0% chance of winning the election.
- Thus, there is no profitable deviation.

Question 4:

Electoral competition in heterogeneous districts. Consider a country with three regions with citizens that have (symmetric) single peaked preferences. There are two political parties. In Region 1 (the city) there is 50% of the population of the country and their preferred policy is uniformly distributed in [0,1]; in Region 2 (suburbia) there is 30% of the population of the country and their preferred policy is uniformly distributed in [1,1.5]; finally, in Region 3 (rural) there is the remaining 20% of the population of the country and their preferred policy is uniformly distributed in [1.5,2].}

- **The country elects its president by plurality considering a unique national district (i.e. the votes of all three regions are pooled together and the candidate that gets most votes wins). There are two rent-seeking parties. Find all the set of policies parties can propose that constitute an equilibria.**
- **Now assume each region is an electoral district that selects a candidate from one of the two parties by plurality. The president belongs to the party that obtains a majority of candidates (i.e. analogous system to the electoral college in the United States' presidential elections). The two parties need to announce a single policy for the whole country; which policies constitute an equilibrium of the game?**

Scenario 1 is just a big Downsian model, with a spectrum between [0,2].

- The distribution is not uniformly distributed, however, we know that the there is still a convergence at the median voter, no matter the distribution.
- What is the median voter in this scenario? Well, the median voter is simply the 50th percentile. We know region 1 has 50% of the votes, so the last voter of region 1 will be the median voter, at position 1.
- Thus, the equilibrium is at the median of the entire spectrum, thus the equilibrium is $p_A = p_B = 1$

Scenario 2 is more complicated. You want to win 2 districts, but you have to account for your opponent's move.

Initially, the points of 1 and 1.5 jump out as interesting, because they are the borders between 2 districts, and would allow you to win both

- Let us consider an equilibrium of (1, 1.5), which seems plausible. However, player i has a deviation here.
  - Player at 1.5 could deviate to 1.0000001, and win the last 2 districts.
  - Or player at 1 could deviate to 1.49999999, and win the first 2 districts.
- Now let us consider an equilibrium of (1.5, 1.5). The same problem arises, one player could deviate to 1.49 and win the first 2 districts.
- Same with an equilibrium of (1, 1). A player could deviate to 1.00000001 and win the last 2 districts.

Thus, we have to consider another equilibrium. Let us take a more radically different approach. Instead of assuming some continuous uniform distribution between 0 and 2 with several regions, let us formulate this as a small downsian model:

- Pretend there are only 3 citizens - each district is a citizen
- Your goal is to win a majority of these citizens (2/3).
- Thus, by the Downsian model, you need to win the middle citizen (or district 2)

How do you win district 2?

- You need to position yourself at the median of district 2, $m_2 = 1.25$

Thus, $(1.25, 1.25)$ is a nash equilibrium, with each player having a 50/50 chance of winning the election. We can test this with potential deviations:

- If player i deviates below 1.25, they will lose district 2, and their opponent will win district 2 and 3. Thus, they will have 0% chance of winning, and thus, not a profitable deviation
- If player i deviates above 1.25, they will lose district 2, and their opponent will win district 2 and 1. Thus, they will have 0% chance of winning, and thus, not a profitable deviation.

Question 5:

Electoral competition under the Alternative Vote. The Alternative Vote (AV)} is a preferential system where the voter has the chance to rank the candidates in order of preference. The voter puts a '1' by their first choice a '2' by their second choice, and so on. Candidates are elected outright if they gain more than half of the first preference votes. If not, the candidate who lost (the one with least first preferences) is eliminated and their votes are redistributed according to the second (or next available) preference marked on the ballot paper. This process continues until one candidate has half of the votes and is elected.}

Three rent-seeking candidates can propose a policy in a one-dimensional policy space, the interval [0,1]. The voting procedure followed is the Alternative Vote (majoritarian system in which voters cast a priority ranking). Citizens' preferred policies are uniformly distributed on the policy space. Citizens vote sincerely, i.e. a voter gives the highest rank to the candidate who is closest to his ideal point, second highest rank to the second closest candidate, and lowest rank to the last candidate; in case of indifference the placement is random (i.e. if a voter is indifferent between two top ranked candidates, say parties A and B; she places A first and B second with probability ½ and B first and A second with probability ½). Is there an equilibrium where candidates converge to the median?

Definitions

Let us define the three parties as party A, B, C, who choose some position $p_A, p_B, p_C \in [0,1]$. Let us define m as the position of the median voter in the spectrum $[0,1]$

Payoffs if Everyone was at Median

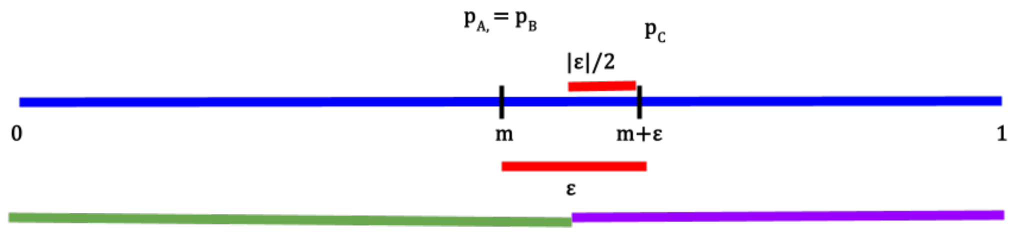
Let us see if strategy profile (m,m,m) is a Nash Equilibrium through seeing if any party i has a profitable deviation.

- Before we look at profitable deviation, what is the expected utility of (m,m,m)? We need to know this in order to see what a profitable deviation would be.
- Well at (m,m,m), all three parties would get the same amount of first, second, and third place votes: 1/3. Thus, no one has a majority, so one party by random chance will be eliminated. 2/3 chance to get to the second round.
- In the second round, at (m,m), both parties still get the same amount of votes. Thus, they have a 1/2 chance of winning the election.
- Thus, expected chance of winning the election at (m,m,m) for all three parties is $2/3 \times 1/2 = 1/3$
  - This makes sense, each party has equal chance of winning, 1/3

<u>So any profitable deviation requires a chance of winning higher than 1/3</u>

Looking for Profitable Deviation: First Round

Let us say any party i deviates from the median by some value ϵ, so it has a new position of $m + \epsilon$:



Now, let us look at the first round and how the voting will turn out. Since the distribution is uniformly distributed, we can just look at the distance each party wins on the number line, since that is proportional to the total number of votes.

Party i that deviates will win the distance $\frac{1}{2} - \frac{1}{2}|\epsilon|$

The other two parties will win distance $1/2 + |\epsilon|/2$, however, they will split the votes between themselves as they are equidistant between all the voters here, so each party gets $\frac{1/2+|\epsilon|/2}{2} = \frac{1}{4} + \frac{0.5|\epsilon|}{2} = \frac{1}{4} + \frac{1}{4}|\epsilon|$

Why? See figure above

Important thing is that 50% of votes (which would outright win the first round) would only be possible if a party won more than 50% of the distance between $[0, 1]$, which is a distance of $1/2$.

- Party i that deviated cannot win a majority, since they will win distance $\frac{1}{2} - \frac{1}{2}|\epsilon|$, which cannot be greater that $1/2$ when $|\epsilon| > 0$. Thus, party i has no chance of winning outright.
- The other two parties win $\frac{1}{4} + \frac{1}{4}|\epsilon|$, and that can only be a majority of the first round (i.e. greater than $1/2$) if $|\epsilon| > 1$. But, since ϵ is the deviation of party i in one direction, it cannot possibly exceed 0.5. Thus, there is no case where the other two parties outright win the election in the first round.

Then, that means one party with the lowest votes will be eliminated? Which party will be eliminated? That depends on ϵ. Party i will survive, if it has a higher proportion of vote than the other two parties - see the inequality: $\frac{1}{2} - \frac{1}{2}|\epsilon| > \frac{1}{4} + \frac{1}{4}|\epsilon|$

$$\frac{1}{2} - \frac{1}{4} > \frac{1}{2}|\epsilon| + \frac{1}{4}|\epsilon|$$

$$\frac{1}{4} > \frac{3}{4}|\epsilon|$$

$$\epsilon < \frac{1}{4} \times \frac{4}{3}$$

$$\epsilon < \frac{1}{3}$$

Thus, the following scenarios can occur

1. If $|\epsilon|$ deviation distance of party i is greater than 1/3, then they lose the first round, and thus have a 0 percent chance of winning the election, thus, that is not a profitable deviation.
2. If $|\epsilon|$ deviation distane of party i is exactly 1/3, then they have a 2/3 chance of getting to the second round (since they will be tied with the other two parties)
3. When $|\epsilon|$ deviation distance is less than 1/3, party i will survive to the second round, and one of the other two parties will be eliminated.

Looking for Profitable Deviation: Second Round

In the second and third scenario, we are on the the second round. Here, the eliminated party's voters will choose to vote for their 2nd choice party.

- For player i to have advanced to the second round, a party at m must have been eliminated.
- If one of the other 2 parties votes are eliminated, those voters will just jump to the one who was not eliminated (since both had same policy position m)
- Thus, team i would win $\frac{1}{2} - \frac{1}{2}|\epsilon|$ of the distance (as seen above), and the median team would win $\frac{1}{2} + \frac{1}{2}|\epsilon|$ (basically, the original distance not split between the two median teams). By simple math, the median team always has the higher vote share.

Thus, party i will have 0% chance of winning the second round, which is lower than the 33% chance they had originally at m, so there is no profitable deviation.

Alternatively, we could also view this as a Downsian Model: a two party battle between party i at position $m + \epsilon$, and the other party at m.

- Since m is the median voter, and the median voter by definition is the 50th percentile, you need to win the median voter to win the election.
- So, party i that deviated will lose the second round to the party at the median

Summary

There is no profitable deviation ϵ for any party i from strategy profile (m, m, m):

- If party i deviates by more that $|\epsilon| > 1/3$, they lose in the first round, so 0% chance, which is less than the original 33% chance. No profitable deviation.
- If party i deviates by less than $|\epsilon| \leq 1/3$, then they lose in the second round, so 0% chance, which is less than the original 33% chance. No profitable deviation.