MATH-GA 2791 Financial Securities and Market

Final Project Report

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Report

Abstract

This report presents the results of a project aimed at pricing a financial contract using two distinct methodologies: The hull-White model and the Geometric Brownian Motion. The report begins with an overview of the problem, followed by data source, data preprocessing, and key assumptions. The mathematical derivation of the models employed is discussed, and the two methodologies are implemented in Python. The resulting prices are analyzed, and their implications are interpreted for future reference. The project's objective is to apply theoretical pricing models in real-life scenarios, specifically in the context of derivative contracts. Through a rigorous approach, the project provides insights into the use of these methodologies for pricing financial contracts.

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1. Problem Statement

We intend to purchase a contract that pays an amount in USD at maturity T:

$$\max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)}\right)\right]$$

Where:

- $S(t)$ is the Nikkei-255 spot price quantoed from JPY into USD
- $L(t, T-\Delta, T)$ is the 3-month USD LIBOR rate between $T-\Delta$ and T
- Δ is a period of 3 months (0.25 years)
- T the expiration date (e.g. 3 years)
- k, k' given relative strike prices

Our objective is to develop a pricing routine that computes the value of this contract using the deal terms, such as T, k, k', and market data like volatility and correlation between the underlying assets. In subsequent sections, we will outline the assumptions made in the models and discuss the origin of the data being used.

2. Data Preparation

The date for the following data are all from 2020-01-06 to 2022-12-30, which is three years (2020, 2021, 2022).

2.1 Data Sources and Assumptions

- **Nikkei-225 Historical Stock Price Data:** Data for the Nikkei-225 stock prices, expressed in JPY, have been procured from Yahoo Finance. The monthly time series data are utilized in the Hull-White model, and the daily time series data are employed in the Geometric Brownian Motion Method for determining the correlation and underlying volatility.
- **USD to JPY Historical Exchange Rate Data:** The historical exchange rate data for USD to JPY has been obtained from Macro Trend. Both monthly and daily time series data are used in the Hull-White and Geometric Brownian Motions method respectively, to compute exchange rate volatility and correlation. Refer to Figure 1 for the historical chart, which only considers data within the specified time period.
- **3-Month LIBOR Historical Data:** The 3-month LIBOR rate time series data has been sourced from iborate, which is directly substituted for the LIBOR rate in our computations.
- **US 3-Month Interest Rate Historical Data:** Monthly time series data of US short-term interest rates has been retrieved from OECD.
- **US Treasury-Bond Yield Curve:** The yield curve for US Treasury bonds of varying maturities is sourced from US Department of Treasury. This data is used in the determination of T-bond prices.

- **Stock Dividend Rate:** To streamline the calculation process, the stock dividend rate is set at 1.50%, which corresponds to the 4-year average dividend rate as per Seeking Alpha.
- **Risk-Free Interest Rate in Japan:** The risk-free interest rate in Japan is set at -0.1%. The data has been obtained from Trading Economics and is employed in the calculation of the LIBOR rate and relative quantoed equity.
- **US 3-Month Short Rate:** The US 3-month short rate is set at 0.81%. This is calculated as the three-year average of 3-Month short rate on Market Watch.

Figure 1: Dollar-Yen Exchange Rate Historical Chart

The preceding section enumerates the key datasets and constant variables employed in our pricing algorithm. The time-series data primarily serves to compute asset correlations and individual volatilities. It's crucial to note that these fixed values may change over time, and thus it is advisable to regularly update these values when utilizing the given pricing algorithm for enhanced accuracy.

2.2 Data Preprocessing

In our data preprocessing stage, we utilized the Python programming language, along with the `pandas` library for data manipulation and analysis. The following steps were taken for each dataset (the full Python script can be found in the supplementary materials):

- **Nikkei-225 Historical Stock Price Data:** We selected only the "Date" and "Close" columns from the original dataset, and saved the modified dataframe back to a new CSV file.
- **3-Month LIBOR Rate:** Only the "Date" and "3M" columns were kept from the original dataset, and the modified dataframe was saved back to a new CSV file.

- **USD/JPY Exchange Rate:** The "date" column was converted to datetime format and filtered to include only the desired date range. The filtered dataframe was then saved to a new CSV file.
- **Daily Treasury Rates:** The datasets from 2020, 2021, and 2022 were concatenated, and the "Date" and "3 Mo" columns were selected. The "Date" column was converted to datetime format, the data was filtered for the desired date range, sorted by date, and saved to a new CSV file.
- **3-Month Short Rate:** The datasets from 2020, 2021, and 2022 were concatenated, and the "Date" column was converted to datetime format. The data was filtered for the desired date range, and the "Date" and "Close" columns were selected. The data was sorted by date, the "Close" column was converted to float and the average was calculated. The sorted dataframe was then saved to a new CSV file.

3. Derivation of Mathematical Models

The key components of the contract that need theoretical model computations are: 1. Nikkei-225 option's quanto from JPY to USD, and 2. The LIBOR. This document will highlight two methodologies for developing the necessary pricing models: A. Hull-White Model, and B. Log-Normal Model. The derivations are drawn from *Arbitrage Theory in Continuous Time* by Tomas Bjork [1] and lecture notes by professor Alireza Javaheri.

3.1 Approach A: Hull-White Model

The first approach is to employ the Hull-White model to simulate the U.S. interest rate. Given the Hull-White model under the domestic risk-neutral measure Q^d:

$$dr(t) = \{\phi(t) - ar(t)\}dt + \sigma dW^{Q^d} \tag{1}$$

Here, $\phi(t), t, \sigma$ are constants or deterministic functions that will be identified later. The Monte Carlo method can simulate the interest rate process and yield $r(T-\Delta)$. Moreover, using the affine term structure, $p(T-\Delta, T)$ can be calculated as:

$$p(T-\Delta, T) = e^{A(T-\Delta,T)-B(T-\Delta,T)r(T-\Delta)} \tag{2}$$

Where A and B are deterministic functions defined as:

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\}$$
$$A(t,T) = \int_t^T \{\frac{1}{2}\sigma^2 B^2(s,T) - \phi(s)B(s,T)\}\, ds$$

Using the definition of LIBOR, we have:

$$L(T-\Delta T, T-\Delta T, T) = -\frac{p(T-\Delta, T) - 1}{\Delta \cdot p(T-\Delta, T)} \tag{3}$$

$$L(0, T-\Delta T, T) = -\frac{p(0,T) - p(0, T-\Delta)}{\Delta \cdot p(0,T)} \tag{4}$$

Here, $p(0,T)$ and $p(0, T-\Delta)$ can be calculated using the above equations.

The subsequent step is to identify a model that illustrates the quanto equity. Under the risk-neutral measure, equities will have an average return rate equal to the risk-free rate. Hence, a stochastic differential equation can describe the stock price difference in the quanto currency for a small time interval:

$$\frac{dS}{S} = (r_f - q_d - \rho\sigma_S\sigma_X)dt + \sigma_S dV^{Q^d} \tag{5}$$

Here, r_f represents the foreign risk-free rate, q_d stands for the dividend rate, ρ symbolizes the correlation between log of exchange rate FX and log of stock price S, and σ_X and σ_S represent the volatility of exchange rate and stock price. Note that W^{Q^d} and V^{Q^d} are two normal distributions with correlation ρ, the correlation between log of exchange rate FX and log of stock price S.

Assuming the stock has quantoed drift μ, we can write:

$$\frac{dS}{S} = \mu dt + \sigma_S dW^{Q^d} \tag{6}$$

Next, considering a compo (a foreign stock traded domestically) with another Wiener's Process V^{Q^d}, we can derive a new formula for stock price $\tilde{S}$:

$$\frac{d\tilde{S}}{\tilde{S}} = r_d dt + (\sigma_S^2 + \sigma_X^2 + 2\rho\sigma_S\sigma_X)^{\frac{1}{2}} dV^{Q^d} \tag{7}$$

where r_d represents the domestic risk-free rate. The exchange rate process is represented as:

$$\frac{dFX}{FX} = (r_d - r_f)dt + dW^{Q^d}$$

Observing the definition of compo, we have:

$$\tilde{S} = S \cdot X \Longrightarrow \ln\tilde{S} = \ln S + \ln X$$

We can see that all three processes satisfy a log-normal model. Thus, their solutions take a similar exponential form:

$$\frac{Z(t)}{Z} = \alpha dt + \sigma dW \Longrightarrow Z(t) = Z(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}}$$

As all three processes have similar forms, we can equate their drift terms:

$$r_d - \frac{1}{2}(\sigma_S^2 + \sigma_X^2 + 2\rho\sigma_S\sigma_X) = (\mu - \frac{1}{2}\sigma_S^2) + r_d - r_f - \frac{1}{2}\sigma_X^2 \Longrightarrow \mu = r_f - \rho\sigma_S\sigma_X$$

Hence, the quanto price will satisfy:

$$\frac{dS}{S} = (r_f - \rho\sigma_S\sigma_X)dt + dW^{Q^d} \tag{8}$$

This indicates that equation (6) follows when considering the dividend rate.

At each time step Δt, the quantoed stock price can be simulated by:

$$S(t + \Delta t) = S(t) \cdot e^{(r_f - q_d - \rho\sigma_S\sigma_X - \frac{1}{2}\sigma_S^2)\Delta t + \Delta\sqrt{\Delta t}} \tag{9}$$

By iterating the simulation, the stock price at time T can be obtained. The final payoff is then determined by:

$$\Pi(t) = max[0, (\frac{S(T)}{S(0)} - k) \cdot (k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)})] \tag{10}$$

which can be calculated by equations (4), (5), and (8).

The discounting process can be simulated by:

$$D(T) = e^{-\sum_{i=0}^{n-1} r_i \Delta t} \tag{11}$$

where r_i is the short rate simulated at time $(i+1) \cdot \Delta t$. In this context, it's important to note that the discounting process is path-independent, which allows us to determine:

$$V(0) = \mathbb{E}[D(T)\Pi(T)] \tag{12}$$

For the Hull-White Model to fully extend its predictive capacity towards the price of the exotic rate derivative, calibration will be needed to better align the model parameters with the swaptions market. Various methodologies exist for performing this calibration. By calibrating, the model will have an adjusted time-dependent mean reversion and volatility. Then, given the market data of the chosen product, optimization over the model parameters will be carried out, yielding adjusted parameters and thus more accurate price predictions. A simple way to perform the calibration is through exact line search which aims to find the minimizing parameter that gives the smallest L_2 norm between the model price and the market price. As such calibration works efficiently with a large amount of market data, this report and the provided pricing routine do not provide a specific algorithm for performing the calibration. Instead, fixed values for parameters are given based on approximation and market data adjustment.

Other algorithms include fixing the mean reversion while optimizing the volatility; approximating mean reversion using SMM approximation and then optimizing the volatility using the implied model.

3.2 Approach B: Geometric Brownian Motion

Geometric Brownian Motion (GBM) is a common model used in financial mathematics due to its simplicity and analytical tractability. In this section, we are going to utilize GBM to model the dynamics of both equity and LIBOR. The GBM is based on the assumption that the logarithm of the underlying asset price follows a normal distribution. This leads us to model the equity and LIBOR dynamics with a log-normal distribution.

The quantoed equity can be simulated using GBM after a quanto adjustment is performed. The dynamic of the equity price S is described by the stochastic differential equation:

$$\frac{dS}{S} = (r_f - q - \rho_{SX}\sigma_S\sigma_X)dt + \sigma_S dW^{Q^d} \tag{13}$$

In this equation, the GBM is applied to capture the dynamic of the equity price, including factors like foreign risk-free rate r_f, dividend rate q, historical correlation between exchange rate process X and foreign stock price process S denoted by ρ_{SX}, and volatilities σ_S and σ_X.

Similarly, we can model the LIBOR rate L using a log-normal model:

$$\frac{dL}{L} = Adt + \sigma_L dV^{Q^d} \tag{14}$$

Here, V is a Brownian motion representing the evolution of the LIBOR rate under the domestic risk-neutral measure. After changing the measure from the original domestic risk-neutral measure to the forward measure, the LIBOR process becomes a martingale, resulting in the modified LIBOR process:

$$dL = L\sigma_L dV^{Q^T} \tag{15}$$

The drift term in the equity model can be adjusted by adding $\rho_{Sp}\sigma_S\sigma_p$, where σ_p is the historical volatility of the T-bond. After this modification and using techniques similar to the derivation of the Black-Scholes Model, the price simulation models for the quantoed equity and the LIBOR under the GBM assumption become:

$$\text{Quantoed Equity: } S(T) = S(0) \cdot e^{[(r_f - q - \rho_{SX}\sigma_S\sigma_X + \rho_{Sp}\sigma_S\sigma_p - \frac{1}{2}\sigma_S^2)T + \sigma_S\sqrt{T}\xi]} \tag{16}$$

$$\text{LIBOR: } L(T) = L(0) \cdot e^{(\sigma_L\sqrt{T}\tilde{\xi} - \frac{1}{2}\sigma_L^2 T)} \tag{17}$$

Using these models, we can compute the payoff of the contract:

$$\Pi(T) = \max\,[0, (\frac{S(T)}{S(0)} - k) \cdot (k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)})] \tag{18}$$

The final price of the contract under the forward measure is given by:

$$V(0) = p(0,T) \cdot \mathbb{E}^{Q^T}[\Pi(T)] \tag{19}$$

where $p(0,T)$ is the price of a zero-coupon bond that pays \$1 at time T. The expectation $\mathbb{E}^{Q^T}[\Pi(T)]$ is taken under the forward measure Q^T, which reflects the fact that we are pricing a derivative contract that pays off at time T.

In summary, the GBM assumption allows us to model the dynamics of the quantoed equity and LIBOR, compute the payoff of the contract, and ultimately determine the price of the contract. It is worth noting that, due to the GBM assumption, the log-returns of the equity and LIBOR are normally distributed, which justifies the use of the Black-Scholes formula in the pricing model. Moreover, this approach allows us to easily adjust for the risk factors associated with the foreign currency and the T-bond.

4. Implementation of Models in Python

The implementation of the previously discussed models, namely the Hull-White and Geometric Brownian Motion (GBM) models, is carried out using Python. Both models utilize a Monte Carlo simulation approach to simulate the dynamics of the quantoed equity and the LIBOR. Essential data is sourced and processed using Python, including the calculation of asset correlations. The Monte Carlo simulation mainly uses the Quantlib library, and follows the methodology outlined by Iliya Valchanov at Data Science Plus. In the case of the Hull-White model, we also refer to the implementation guide by Mansoor Ahmed at Hull-White Model. The presented code is adapted from these sources to address the current problem. Detailed code can be found in the supplementary material.

4.1 Hull-White Model

In the context of the Hull-White model, we first compute the correlation between the underlying assets and their individual volatilities. For every simulation iteration, a pair of random variables is drawn from a normal distribution that reflects the historical correlation between the stock and exchange rate. The simulation process involves a one-step Monte Carlo simulation for both the quantoed equity and the LIBOR, with a step size of one month. Once the simulations are complete, the simulated LIBOR rate and quantoed equity prices are discounted.

Subsequently, we compute $p(T-\Delta, T)$, $p(0,T)$, and $p(0, T-\Delta)$ based on the simulated values, and then determine $L(T-\Delta, T-\Delta, T)$ and $L(0, T-\Delta, T)$. Using these simulated results, we can calculate the contract's final payoff, discount it using the interest rate defined earlier, and obtain the average of all simulated paths in the Monte Carlo simulation to compute the contract's price.

4.2 Geometric Brownian Motion

In line with the GBM model, two normally distributed random numbers are generated considering the historical correlation between the stock and LIBOR. We then simulate the equity and LIBOR prices for steps T and $T - \Delta$, respectively. The final payoff of the contract is determined by substituting the simulated prices into Equation (19), which yields the contract price for a specific simulated path from the Monte Carlo simulation. Averaging these prices gives us the future contract price, which is then discounted to obtain the present value.

It's noteworthy to mention that the code includes a segment to accommodate the change of measure, as the GBM model does not inherently operate under the forward measure.

5. Results

5.1 Hull-White Model

The calibration process typically yields the required model parameters. In our specific model, we proceed under the assumption that these parameters are already known, specifically a mean-reversion speed of 0.05 and a volatility of 0.02. Utilizing these parameters in the Hull-White model, we perform a simulation of the short rate over a period of 3 years, with daily sampling. This outcome is depicted in Figure 2.

The simulated results show desirable characteristics - the short-rate movements are neither excessively volatile nor overly stable. In the context of financial modeling, this balance is important. If the short rate were to fluctuate too violently, it would imply an unrealistic level of volatility in the market, which could lead to overestimated risks and potential mispricing of derivatives. On the other hand, if the rate were too stable, it would not accurately reflect the inherent uncertainties and fluctuations seen in real-world financial markets, which could lead to an underestimation of risks.

Furthermore, for the purpose of contract pricing, we observe that 90% of the 200 simulated prices (with 1000 paths for each simulation) fall within the range of 0.318 to 0.396. The average simulated price comes out to be 0.358, which we employ as the benchmark contract price for our initial methodology. The distribution of these simulated contract prices is illustrated in Figure 3.

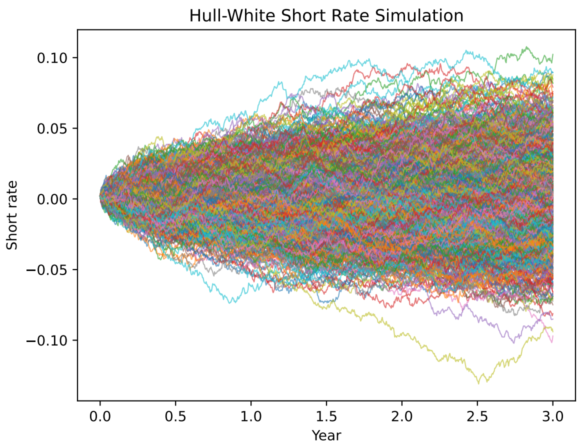

Figure 2: 3-Year Hull-White Simulation

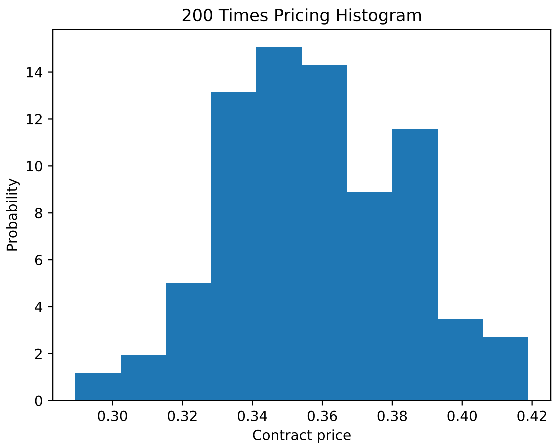

Figure 3: Hull-White Pricing Paths

5.2 Geometric Brownian Motion

In the case of the Geometric Brownian Motion model, we find that 90% of the 200 simulated contract prices, each with 1000 paths, range between 0.262 to 0.346. The mean value of these simulated prices is 0.216, which is notably lower than the benchmark contract price derived from the Hull-White model. The distribution of these prices is presented in Figure 4.

Additionally, we perform a sensitivity analysis with respect to the strike price, the results of which are demonstrated in Figure 5. The analysis reveals a direct linear relationship between the strike price and the contract price, i.e., an increase in the strike price corresponds to an increase in the contract price.

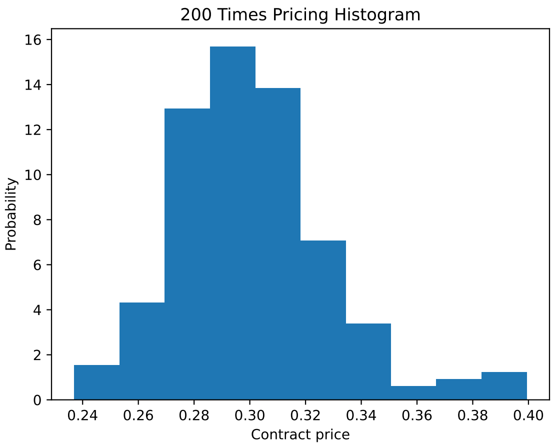

Figure 4: GBM Pricing Paths

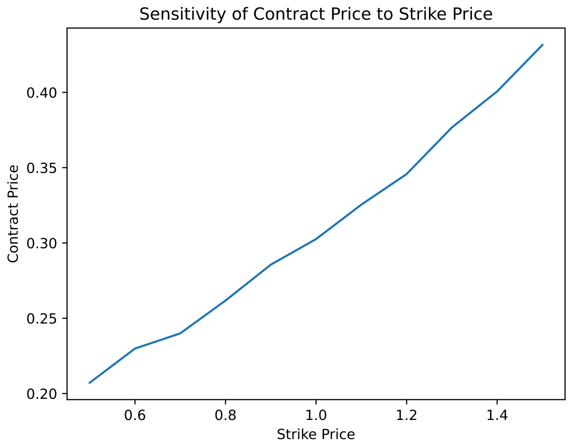

Figure 5: Sensitivity Analysis: Contract Price Versus Strike Price

6. Conclusion

The results from our study clearly demonstrate the unique advantages and limitations of the two methodologies we've employed, namely the Hull-White model and the Geometric Brownian Motion (GBM) model. Each methodology operates under a different measure and is best suited to specific circumstances, thus providing a versatile toolkit for various scenarios.

The Hull-White model, our first methodology, is leveraged for the simulation of the short rate price, which is vital for determining LIBOR. Post-simulation, we extract the discounted present value of the LIBOR rate after averaging all simulated paths. Essentially an advanced version of the Vasicek model, the Hull-White model offers an enriched feature set, including the ability to adjust simulations with a range of inputs by altering the maturity as needed. Both the 'quantoed' equity and LIBOR are subjected to the domestic risk-neutral measure in this model. Additionally, the Hull-White model's analytic and closed-form nature make it a compelling choice, despite the additional calibration effort required to fine-tune it.

Our second approach employs the Geometric Brownian Motion model for LIBOR. In this model, we assume LIBOR follows a log-normal distribution, contrasting with the normal distribution postulated by the Hull-White model. This model operates under the forward measure, thereby easing the calibration burden. The Python implementation for this methodology is relatively straightforward, as it demands less complex mathematical derivations. However, it comes with its own limitations, including a less flexible structure due to the mandatory input of real LIBOR data for maturity T. Hence, in reality, we have limited choices for LIBOR maturity.

Even though the pricing outcomes differ between the two models, the variation is relatively minor. This discrepancy could stem from the divergence between the actual bond price and the simulated bond price, or between the anticipated and simulated bond prices based on extrapolation from the imported dataset.

7. Future Work

As we continue to delve deeper into this intriguing field, there are several promising avenues for further research and development in relation to this project.

- **Enhanced Calibration Techniques:** For the Hull-White model, we have identified the need for an improved calibration process. This model's calibration is complex, and working to streamline this process will enhance its usability and efficiency. Advanced machine learning and optimization techniques might be useful in enhancing this calibration process.
- **Model Comparison and Validation:** The discrepancy between the Hull-White and GBM models in pricing outcomes, although minor, warrants further exploration. Future work can be devoted to a detailed comparative analysis of the two models under various market scenarios, including but not limited to, high volatility, low volatility, market crashes, and so forth.
- **Theoretical Exploration of Model Assumptions:** Both models have been built on certain assumptions. A deeper theoretical analysis of these assumptions, their implications, and potential improvements will add to the robustness of our models.
- **Exploration of Other Models:** While the Hull-White and GBM models are robust and widely accepted, it would be beneficial to explore other models and methodologies. By comparing these models with our current methodologies, we could potentially uncover more efficient or accurate ways to perform our simulations.
- **Real-time Data Application:** The GBM model, in particular, requires actual LIBOR data for maturity. Future projects could aim to incorporate real-time data feeds, allowing for more accurate and up-to-date simulations.

Bibliography

[1] Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2009.