Problem Set2

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Question1

(a)
Let $n \in \mathbb{N}$
Assume $n > 15$
Then we have $n^2 > 225, n - 10 > 5$
Therefore $n^2(n - 10) > 225 \times 5 = 1125$
Then $n^3 - 10n^2 + 3 > 1128$
Hence $n^3 - 10n^2 + 3 \geq 165$
I can conclude that $\forall n \in \mathbb{N}, n > 5 \Rightarrow n^3 - 10n^2 + 3 \geq 165$

(b)
Disprove the statement $\forall n \in \mathbb{N}, n > 15 \wedge n^3 - 10n^2 + 3 \geq 165$
We prove the negation: $\exists n \in \mathbb{N}, n \leq 15 \vee n^3 - 10n^2 + 3 < 165$
Let n = 10
Then we have $n \leq 15$
Therefore $n \leq 15 \vee n^3 - 10n^2 + 3 < 165$ is True
I can conclude that $\forall n \in \mathbb{N}, n > 5 \wedge n^3 - 10n^2 + 3 \geq 165$ is False.

Question2

(a)
Let $n \in \mathbb{N}$, Let $m \in \mathbb{N}$
Assume $n < m$
Since $m > n \geq 0$
We have $m > 0$
Therefore $\frac{n}{m} < 1, -\frac{n}{m} > -1$
By the definition of Ceiling,
$\lceil -\frac{n}{m} \rceil \geq -\frac{n}{m} > -1$ $(*)$
Since $m > n \geq 0$,
We have $0 \geq -\frac{n}{m}$
Since $-\frac{n}{m} \in \mathbb{R}, 0 \in \mathbb{Z}$,
By the Fact 1 of ceiling function,

$0 \geq \left\lceil -\frac{n}{m} \right\rceil$ $(\star)$
From $(*)$ and $(\star)$, We have $-1 < \left\lceil -\frac{n}{m} \right\rceil \leq 0$
Since $\left\lceil -\frac{n}{m} \right\rceil \in \mathbb{Z}$,
We have $\left\lceil -\frac{n}{m} \right\rceil = 0$
Then $\left\lceil -\frac{n}{m} \right\rceil = \left\lceil n(\frac{m-1-m}{m}) \right\rceil = \left\lceil \frac{n(m-1)}{m} - n \right\rceil = 0$
Since $\frac{n(m-1)}{m} \in \mathbb{R}, -n \in \mathbb{Z}$
By the Fact2 of Ceiling,
$\left\lceil \frac{n(m-1)}{m} + (-n) \right\rceil = \left\lceil \frac{n(m-1)}{m} \right\rceil - n = 0,$
Therefore $\left\lceil \frac{n(m-1)}{m} \right\rceil = n$
Therefore $\left\lceil n\frac{(m-1)}{m} \right\rceil = n$

(b)
Translate the statement into predicate logic:
WTS: $\forall n \in \mathbb{N}, nextFifty(n) \geq n \wedge \forall m \in \mathbb{N}, ((m \geq n \wedge 50 \mid m) \Rightarrow m \geq nextFifty(n))$
Let $n \in \mathbb{N}$
By the definition of function nextFifty,
$nextFifty(n) = 50\lceil \frac{n}{50} \rceil$
By the definition of Ceiling,
we have $\lceil \frac{n}{50} \rceil \geq \frac{n}{50}$
Then $nextFifty(n) \geq 50 \times \frac{n}{50} = n$
Let $m \in \mathbb{N}$
Assume $m \geq n$ and $50 \mid m$ which is equivalent to assume $m \geq n$ and $\exists k \in \mathbb{Z}, 50k = m$
Then we get $50k \geq n, 0 \geq \frac{n}{50} - k$
Since $0 \in \mathbb{Z}, \frac{n}{50} - k \in \mathbb{R}$
By Fact1 of Ceiling,
$0 \geq \lceil \frac{n}{50} - k \rceil$
Since $\frac{n}{50} \in \mathbb{R}, -k \in \mathbb{Z}$
By Fact2 of Ceiling,
$\left\lceil \frac{n}{50} - k \right\rceil = \left\lceil \frac{n}{50} \right\rceil - k$
$\left\lceil \frac{n}{50} - k \right\rceil = \left\lceil \frac{n}{50} \right\rceil - \frac{m}{50} \leq 0$
Therefore $50\left\lceil \frac{n}{50} \right\rceil \leq m$
Hence $m \geq nextFifty(n)$
I 've proven that $\forall n \in \mathbb{N}, nextFifty(n) \geq n \wedge \forall m \in \mathbb{N}, ((m \geq n \wedge 50 \mid m) \Rightarrow m \geq nextFifty(n))$

Question3

(a)
Let $n \in \mathbb{N}$
Assume $n \leq 2300$
WTS : $[49 \mid n \Rightarrow 50(nextFifty - n) = nextFifty(n)] \wedge [50(nextFifty(n) - n) = nextFifty(n) \Rightarrow 49 \mid n]$

I first want to prove $49 \mid n \Rightarrow [50(nextFifty(n) - n) = nextFifty(n)]$ $(*)$
Assume $49 \mid n$,
By definition of divides, we have $\exists k \in \mathbb{Z}, 49k = n$
Since $n \leq 2300$, Then we have $n < 2450$
Therefore we get $k = \frac{n}{49} < 50$
By conclusion of 2(a),

$$\begin{aligned} \left\lceil \frac{49k}{50} \right\rceil &= k \\ \left\lceil \frac{n}{50} \right\rceil &= k \\ 49\left\lceil \frac{n}{50} \right\rceil &= n \\ 50\left\lceil \frac{n}{50} \right\rceil - n &= \left\lceil \frac{n}{50} \right\rceil \\ 50(nextFifty(n) - n) &= nextFifty(n) \end{aligned}$$

I have shown that $49 \mid n \Rightarrow 50(nextFifty(n) - n) = nextFifty(n)$

Then I want to prove $50(nextFifty(n) - n) = nextFifty(n) \Rightarrow 49 \mid n$ $(\star)$
Assume $50(nextFifty(n) - n) = nextFifty(n)$
Let $k = \lceil \frac{n}{50} \rceil$
Since $50(50\lceil \frac{n}{50} \rceil - n) = 50\lceil \frac{n}{50} \rceil$
$49\lceil \frac{n}{50} \rceil = n$
$n = 49k$
I 've shown that $\exists k \in \mathbb{Z}, 49k = n$
Therefore I 've proven $50(nextFifty(n) - n) = nextFifty(n) \Rightarrow 49 \mid n$
By $(*)$ and $(\star)$, I can get $\forall n \in \mathbb{N}, n \leq 2300 \Rightarrow (49 \mid n \iff 50(nextFifty(n) - n) = nextFifty(n))$

(b)
We have the negation:
$\exists n \in \mathbb{N}, [49 \mid n \wedge 50(nextFifty(n) - n) \neq nextFifty(n)] \vee [[50(nextFifty(n) - n) = nextFifty(n)] \wedge 49 \not\mid n]$
Let $n = 4900$
WTS:$[\exists k \in \mathbb{Z}, 49k = n] \wedge [50(nextFifty(n) - n) \neq nextFifty(n)]$
Take $k = 100$
Then $49k = 49 \times 100 = 4900 = n$
I have shown that $\exists k \in \mathbb{Z}, 49k = n$
Then $50(nextFifty(n) - n) = 50(50\lceil \frac{4900}{50} \rceil - 4900)$
$= 50 \times (4900 - 4900) = 0$
However, $nextFifty(n) = 50\lceil \frac{4900}{50} \rceil = 4900$
$50(nextFifty(n) - n) \neq nextFifty(n)$
I have shown that $50(nextFifty(n) - n) \neq nextFifty(n)$
Since I have shown that $[49 \mid n \wedge 50(nextFifty(n) - n) \neq nextFifty(n)]$ is True

I have $[49 \mid n \wedge 50(nextFifty(n) - n) \neq nextFifty(n)] \vee [[50(nextFifty(n) - n) = nextFifty(n)] \wedge 49 \not\mid n]]$ is True
I have proven the negation is True.
Therefore, $\forall n \in \mathbb{N}, 49 \mid n \iff 50(nextFifty(n) - n = nextFifty(n)$ is False.

Question4

(a)
$\exists k \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f(x) \leq k.$

(b)
Define $Bounded(f)$: "$\exists k \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f(x) \leq k.$", where $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Let function $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume f_1 and f_2 are bounded.
By the definition of Bounded,
$\exists k_1 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f_1(x) \leq k_1$
$\exists k_2 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f_2(x) \leq k_2$
WTS: $\exists k \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, (f_1 + f_2)(x) \leq k$
Let $k = k_1 + k_2$. Let $x \in \mathbb{N}$
Then $(f_1 + f_2)(x) = f_1(x) + f_2(x) \leq k_1 + k_2 = k$
Therefore $(f_1 + f_2)(x) \leq k$
I have shown that for all functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if f_1 and f_2 are bounded, then $f_1 + f_2$ is bounded.

(c)
I want to prove $\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1 + f_2) \Rightarrow Bounded(f_1) \wedge Bounded(f_2)$ is True
Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume $f_1 + f_2$ is bounded which is equivalent to assume $Bounded(f_1 + f_2)$
By definition of Bounded,
$\exists k \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f_1(x) + f_2(x) \leq k$
First I want to show: $\exists k_1 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f_1(x) \leq k_1$ $(*)$
Take $k_1 = k$. Let $x \in \mathbb{N}$.
Since $f_1(x) + f_2(x) \leq k, f_1(x) \geq 0, f_2(x) \geq 0$
We have $f_1(x) \leq k - f_2(x) \leq k = k_1$
Therefore $f_1(x) \leq k_1$
Second I want to show: $\exists k_2 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{N}, f_2(x) \leq k_2$ $(\star)$
Take $k_2 = k$. Let $x \in \mathbb{N}$.
Since $f_1(x) + f_2(x) \leq k, f_1(x) \geq 0, f_2(x) \geq 0$
We have $f_2(x) \leq k - f_1(x) \leq k = k_2$
Therefore $f_2(x) \leq k_2$
Hence by $(*)$ and $(\star)$, I can get $Bounded(f_1) \wedge Bounded(f_2)$
I can conclude that $\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1 + f_2) \Rightarrow Bounded(f_1) \wedge Bounded(f_2)$