Problem Set3

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Question1

(a)
Proof: I will prove $\forall n \in \mathbb{N}, F_n - 2 = \prod_{i=0}^{n-1} F_i$ by induction on n

Base case: $n = 0$
Left hand side: $F_0 - 2 = 2^{2^0} + 1 - 2 = 2^1 + 1 - 2 = 1$
Right hand side: Since $0 > -1, \prod_{i=0}^{-1} F_i = 1$
Therefore we have $F_0 - 2 = \prod_{i=0}^{-1} F_i$
Hence $F_n - 2 = \prod_{i=0}^{n-1} F_i$ is True for n = 0

Inductive step: Let $k \in \mathbb{N}$. Assume $F_k - 2 = \prod_{i=0}^{k-1} F_i$
WTS: $F_{k+1} - 2 = \prod_{i=0}^{k} F_i$
Since we know $F_k = 2^{2^k} + 1$
$F_{k+1} - 2 = 2^{2^{k+1}} + 1 - 2 = 2^{2^{k+1}} - 1$

$$\begin{aligned}
\prod_{i=0}^{k} F_i &= \prod_{i=0}^{k-1} F_i * F_k \\
&= (F_k - 2) * F_k && \text{(by induction hypothesis)} \\
&= (2^{2^k} + 1 - 2)(2^{2^k} + 1) && \text{(by definition of } F_k\text{)} \\
&= (2^{2^k} - 1)(2^{2^k} + 1) \\
&= \left(2^{2^k}\right)^2 - 1 \\
&= 2^{2^{k+1}} - 1
\end{aligned}$$

Therefore we have $F_{k+1} - 2 = \prod_{i=0}^{k} F_i$
By base case and inductive step, I can conclude that $\forall n \in \mathbb{N}, F_n - 2 = \prod_{i=0}^{n-1} F_i$

Question2

(a)
$a_0 = 1$
$a_1 = \frac{1}{\frac{1}{a_0}+1} = \frac{1}{1+1} = \frac{1}{2}$
$a_2 = \frac{1}{\frac{1}{a_1}+1} = \frac{1}{\frac{1}{\frac{1}{2}}+1} = \frac{1}{3}$
$a_3 = \frac{1}{\frac{1}{a_2}+1} = \frac{1}{\frac{1}{\frac{1}{3}}+1} = \frac{1}{4}$
(b)
$\forall n \in \mathbb{N}, a_n = \frac{1}{n+1}$
Proof: I will prove $\forall n \in \mathbb{N}, a_n = \frac{1}{n+1}$ by induction on n

Base case: $n = 0$
Left hand side: We know $a_0 = 1$
Right hand side: $\frac{1}{n+1} = \frac{1}{0+1} = 1$
Therefore $a_n = \frac{1}{n+1}$ is true when $n = 0$

inductive step: Let $k \in \mathbb{N}$. Assume $a_k = \frac{1}{k+1}$
WTS: $a_{k+1} = \frac{1}{k+2}$
We will calculate starting from the left side and show that it equals the right side

$$\begin{aligned} a_{k+1} &= \frac{1}{\frac{1}{a_k}+1} && \text{(by definition of } a_{k+1}\text{)} \\ &= \frac{1}{\frac{1}{\frac{1}{k+1}}+1} && \text{(by induction hypothesis)} \\ &= \frac{1}{k+1+1} \\ &= \frac{1}{k+2} \end{aligned}$$

I have shown that $a_{k+1} = \frac{1}{k+2}$
By base case and inductive step, I can conclude that $\forall n \in \mathbb{N}, a_n = \frac{1}{n+1}$
(c)
Since we know $\forall k \in \mathbb{N}, k > 1 \Rightarrow a_{k,0} = k$
We have $a_{2,0} = 2$
Since $\forall n \in \mathbb{N}, a_{k,n+1} = \frac{k}{\frac{1}{a_{k,n}}+1}$
$a_{2,1} = \frac{2}{\frac{1}{a_{2,0}}+1} = \frac{2}{\frac{1}{2}+1} = \frac{2}{\frac{3}{2}} = 2 \times \frac{2}{3} = \frac{4}{3}$
$a_{2,2} = \frac{2}{\frac{1}{a_{2,1}}+1} = \frac{2}{\frac{1}{\frac{4}{3}}+1} = \frac{2}{\frac{3}{4}+1} = \frac{8}{7}$
$a_{2,3} = \frac{2}{\frac{1}{a_{2,2}}+1} = \frac{2}{\frac{1}{\frac{8}{7}}+1} = \frac{2}{\frac{7}{8}+1} = \frac{16}{15}$
Similarly, $a_{3,0} = 3$
$a_{3,1} = \frac{3}{\frac{1}{a_{3,0}}+1} = \frac{3}{\frac{1}{3}+1} = \frac{9}{4}$
$a_{3,2} = \frac{3}{\frac{1}{a_{3,1}}+1} = \frac{3}{\frac{4}{9}+1} = \frac{27}{13}$

$a_{3,3} = \frac{3}{\frac{1}{a_{3,2}}+1} = \frac{3}{\frac{13}{27}+1} = \frac{81}{40}$
(d)
I want to prove: $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, k > 1 \Rightarrow a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}$
proof: Let $k \in \mathbb{N}$
I will prove $\forall n \in \mathbb{N}, k > 1 \Rightarrow a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}$ by induction on n.

Base case: $n = 0$. Assume $k > 1$
Left hand side: $a_{k,0} = k$ (by definition of $a_{k,0}$)
Right hand side: $\frac{k^{n+2}-k^{n+1}}{k^{n+1}-1} = \frac{k^2-k}{k-1} = \frac{k(k-1)}{k-1} = k$
Therefore we have $a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}$ is true when $n = 0$

Inductive step: Let $n \in \mathbb{N}$ Assume $a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}, k > 1$
WTS: $a_{k,n+1} = \frac{k^{n+3}-k^{n+2}}{k^{n+2}-1}$
We will start from the left side and show that it equals the right side.

$$\begin{aligned}
a_{k,n+1} &= \frac{k}{\frac{1}{a_{k,n}}+1} && \text{(by definition of } a_{k,n+1}\text{)} \\
&= \frac{k}{\frac{1}{\frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}}+1} && \text{(by induction hypothesis)} \\
&= \frac{k}{\frac{k^{n+1}-1}{k^{n+2}-k^{n+1}}+1} \\
&= \frac{k(k^{n+2}-k^{n+1})}{k^{n+1}-1+k^{n+2}-k^{n+1}} \\
&= \frac{k^{n+3}-k^{n+2}}{k^{n+2}-1}
\end{aligned}$$

I have shown that $a_{k,n+1} = \frac{k^{n+3}-k^{n+2}}{k^{n+2}-1}$
By base case and inductive step, I can conclude that $\forall n \in \mathbb{N}, k > 1 \Rightarrow a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}$
Therefore $\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, k > 1 \Rightarrow a_{k,n} = \frac{k^{n+2}-k^{n+1}}{k^{n+1}-1}$

Question3

(a)
Let f: $\mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume $f \in \mathcal{O}(n^2)$. By definition of big-Oh, $\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 \cdot n$
WTS: $Sum_f \in \mathcal{O}(n^2)$ which is equivalent to show $\exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \sum_{i=0}^{n} f(i) \leq cn^2$
Let $c = c_1 + \sum_{i=0}^{n_1} f(i)$
Let $n_0 = n_1$

Let $n \in \mathbb{N}$
Assume $n \geq n_0$. Hence $n \geq n_1$,
Therefore we have $f(n) \leq c_1 \cdot n$

$$\begin{aligned}
\text{Then } \sum_{i=0}^{n} f(i) &= \sum_{i=0}^{n_1} f(i) + \sum_{i=n_1+1}^{n} f(i) && \text{(break up summations)} \\
&= \sum_{i=0}^{n_1} f(i) + f(n_1+1) + f(n_1+2) + \ldots + f(n) \\
&\leq \sum_{i=0}^{n_1} f(i) + c_1 \frac{(n_1+1+n)(n-n_1)}{2} \\
&= \sum_{i=0}^{n_1} f(i) + \frac{c_1}{2}(n^2 + n - {n_1}^2 - n_1) \\
&= \sum_{i=0}^{n_1} f(i) + \frac{c_1}{2}n^2 + \frac{c_1}{2}n - \frac{c_1}{2}({n_1}^2 + n_1)
\end{aligned}$$

Since $c_1 > 0, n_1 > 0, -\frac{c_1}{2}({n_1}^2 + n_1) < 0$
Since $c_1 > 0, n_1 > 0, n \geq n_0$
We get $n > 0$, therefore $n \geq 1$
Hence $\frac{c_1}{2}n(n-1) \geq 0$
$\frac{c_1}{2}n \leq \frac{c_1}{2}n^2$
Since $Sum_f(n_1) \geq 0, n^2 - 1 \geq 0$
We have $\sum_{i=0}^{n_1} f(i) \cdot (n^2 - 1) \geq 0$
Therefore $\sum_{i=0}^{n_1} f(i) \leq \sum_{i=0}^{n_1} f(i) \cdot n^2$
Hence $\sum_{i=0}^{n_1} f(i) + \frac{c_1}{2}n^2 + \frac{c_1}{2}n - \frac{c_1}{2}({n_0}^2 + n_1)$
$\leq \frac{c_1}{2}n^2 + \frac{c_1}{2}n^2 + \sum_{i=0}^{n_1} f(i) \cdot n^2$
$= (c_1 + \sum_{i=0}^{n_1} f(i)) \cdot n^2$
$= cn^2$
I have shown that $\sum_{i=0}^{n} f(i) \leq cn^2$
I can conclude that $\forall f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f \in \mathcal{O}(n)$, then $Sum_f \in \mathcal{O}(n^2)$
(b)
I will prove $\forall n \in \mathbb{N}, \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$ by induction on n

Base case: $n = 0$
Left hand side: $\sum_{i=1}^{2^n} \frac{1}{i} = \sum_{i=1}^{2^0} \frac{1}{i} = 1$
Right hand side: $\frac{n}{2} = \frac{0}{2} = 0$
Therefore $\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$ is true when n = 0

Inductive step: Let $k \in \mathbb{N}$. Assume $\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k}{2}$
WTS: $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$
I will calculate starting from the left side and show that it is greater or equal than the right hand side.

$$
\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^k \cdot 2} \frac{1}{i} && \text{(break up summations)} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} && \text{(by induction hypothesis)} \\
&= \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^k+2^k} \\
&\geq \frac{k}{2} + \frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + \ldots + \frac{1}{2^k+2^k} \\
&= \frac{k}{2} + \frac{2^k}{2^{k+1}} \\
&= \frac{k+1}{2}
\end{aligned}
$$

I have shown that $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$
Therefore I can conclude that $\forall n \in \mathbb{N}, \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$
(c)
Disprove: $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f(n) \in \mathcal{O}(g(n)) \Rightarrow Sum_f(n) \in \mathcal{O}(n \cdot g(n))$ which is equivalent to $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, [\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 g(n)] \Rightarrow [\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \sum_{i=0}^{n} f(i) \leq c \cdot n \cdot g(n)]$
To disprove the statement, I want to prove the negation.
The negation: $\exists f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, [\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 g(n)] \wedge [\forall c, n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \wedge \sum_{i=0}^{n} f(i) > c \cdot n \cdot g(n)]$
Let $f(n) = \frac{1}{n+1}$, Let $g(n) = \frac{1}{2n}$
First I want to show: $\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 \cdot g(n)$ $(*)$
Let $c_1 = 2, n_1 = 1$. Let $n \in \mathbb{N}$
Assume $n \geq n_1$, therefore $n \geq 1$
Since $n + 1 \geq n > 0$
We have $\frac{1}{n+1} \leq \frac{1}{n}$
Then $\frac{1}{n+1} \leq \frac{1}{2n} \times 2$
$f(n) \leq c_1 \cdot g(n)$
I have shown that $\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) \leq c_1 g(n)$
Second I want to show: $\forall c, n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \wedge [\sum_{i=0}^{n} f(i) > c \cdot n \cdot g(n)]$ $(\star)$
Let $c, n_0 \in \mathbb{R}^+$
Let $n = \lceil n_0 \rceil + 2^{\lceil c \rceil}$
By definition of ceiling function, we have $2^{\lceil c \rceil} \geq 2^c \geq 0$
$\lceil n_0 \rceil \geq n_0$
Therefore $2^{\lceil c \rceil} + \lceil n_0 \rceil \geq n_0$
We get $n \geq n_0$
Next, WTS: $\sum_{i=0}^{n} f(i) > c \cdot n \cdot g(n)$
I will start from the left side:
$\sum_{i=0}^{n} f(i) = \sum_{i=0}^{\lceil n_0 \rceil + 2^{\lceil c \rceil}} \frac{1}{i+1}$

$= \sum_{i=0}^{2^{\lceil c \rceil}-1} \frac{1}{i+1} + \sum_{i=2^{\lceil c \rceil}}^{\lceil n_0 \rceil + 2^{\lceil c \rceil}} \frac{1}{i+1}$(break up summations)
By conclusion of Question 3(b),
We know $\sum_{i=1}^{2^{\lceil c \rceil}} \frac{1}{i} \geq \frac{\lceil c \rceil}{2}$
Since $\sum_{i=0}^{2^{\lceil c \rceil}-1} \frac{1}{i+1} = \sum_{i=1}^{2^{\lceil c \rceil}} \frac{1}{i}$
We also have $\sum_{i=0}^{2^{\lceil c \rceil}-1} \frac{1}{i+1} \geq \frac{\lceil c \rceil}{2}$
Therefore

$$\begin{aligned}
&\sum_{i=0}^{2^{\lceil c \rceil}-1} \frac{1}{i+1} + \sum_{i=2^{\lceil c \rceil}}^{\lceil n_0 \rceil + 2^{\lceil c \rceil}} \frac{1}{i+1} \\
&\geq \frac{\lceil c \rceil}{2} + \frac{1}{2^{\lceil c \rceil}+1} + \frac{1}{2^{\lceil c \rceil}+2} + \ldots + \frac{1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} \\
&\geq \frac{\lceil c \rceil}{2} + \frac{1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} + \frac{1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} + \ldots + \frac{1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} \\
&= \frac{\lceil c \rceil}{2} + \frac{\lceil n_0 \rceil + 1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} \\
&> \frac{\lceil c \rceil}{2} \qquad\qquad \left(\text{Since } \frac{\lceil n_0 \rceil + 1}{2^{\lceil c \rceil} + \lceil n_0 \rceil + 1} > 0\right) \\
&\geq \frac{c}{2} \qquad\qquad (\text{by definition of ceiling function})
\end{aligned}$$

Then let's look at the right hand side:

$$\begin{aligned}
c \cdot n \cdot g(n) &= c \cdot (\lceil n_0 \rceil + 2^{\lceil c \rceil}) \cdot \frac{1}{2(\lceil n_0 \rceil + 2^{\lceil c \rceil})} \qquad (\text{by what we take for g(n)}) \\
&= \frac{c}{2}
\end{aligned}$$

Therefore I have shown that $\sum_{i=0}^{n} f(i) > c \cdot n \cdot g(n)$
By $*$ and $(\star)$, I can conclude that the negation is true.
Therefore the original statement is False.