Problem Set 1

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Question1

(a)
Yes!
Define predicate $P(n)$: $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \frac{n^2}{4}$
We know $P(234)$. That is $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = 234 \Rightarrow |E| \leq \frac{234^2}{4}$
I want to show $P(235)$.
That is to show $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = 235 \Rightarrow |E| \leq \frac{235^2}{4}$
Let $G = (V, E)$.
Assume G is a bipartite graph and $|V| = 235$.
By definition of bipartite graph,
we know $\exists V_1, V_2$ such that $V = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset$, and every edge in E has one endpoint in V_1 and the other in V_2.
Since one of V_1 and V_2 have less than or equal to 117 vertices and one of them have greater than or equal to 117 vertices, we can assume $|V_1| \leq 117 \leq |V_2|$ without loss of generality.
Consider what happen when we remove one vertex in V_2.
Let $G' = (V', E')$ be the new bipartite graph after removing one vertex in V_2 and $|V'| = 234$.
By our assumption, we have $|E'| \leq \frac{234^2}{4}$.
After removing one vertex in V_2, the total number of edges will at most decrease $|V_1|$.
Therefore we have $|E| - |E'| \leq |V_1|$

$$\begin{aligned}
|E| &\leq |E'| + |V_1| \\
&\leq \frac{234^2}{4} + 117 \\
&= 13689 + 117 \\
&= 13806 \\
&\leq \frac{235^2}{4}
\end{aligned}$$

I've shown that $P(235)$.
(b)
Yes!
Define predicate $P(n)$: $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \frac{n^2}{4}$

We know $P(235)$. That is $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = 235 \Rightarrow |E| \leq \frac{235^2}{4}$
I want to show $P(236)$.
That is to show $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = 236 \Rightarrow |E| \leq \frac{236^2}{4}$
Let $G = (V, E)$.
Assume G is a bipartite graph and $|V| = 236$.
By definition of bipartite graph,
we know $\exists V_1, V_2$ such that $V = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset$, and every edge in E has one endpoint in V_1 and the other in V_2.
Since one of V_1 and V_2 have less than or equal to 118 vertices and one of them have greater than or equal to 118 vertices, we can assume $|V_1| \leq 118 \leq |V_2|$ without loss of generality.
Consider what happen when we remove one vertex in V_2.
Let $G' = (V', E')$ be the new bipartite graph after removing one vertex in V_2 and $|V'| = 235$.
By our assumption, we have $|E'| \leq \frac{235^2}{4}$.
Since $|E'|$ is an integer, $|E'| \leq 13806$.
After removing one vertex in V_2, the total number of edges will at most decrease $|V_1|$.
Therefore we have $|E| - |E'| \leq |V_1|$

$$\begin{aligned}
|E| &\leq |E'| + |V_1| \\
&\leq 13806 + 118 \\
&= 13924 \\
&\leq \frac{236^2}{4}
\end{aligned}$$

I've shown that $P(236)$.
(c)
Define predicate $P(n)$: $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \frac{n^2}{4}$.
I want to prove $\forall n \in \mathbb{N}, P(n)$.
Define predicate $Q(n)$: $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \lfloor \frac{n^2}{4} \rfloor$.
Let $n \in \mathbb{N}$. Assume $Q(m)$ holds for all natural number m such that $0 \leq m < n - 1$.
I want to show $Q(n)$. That is $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \lfloor \frac{n^2}{4} \rfloor$.

Base Case: n = 0
Obviously, we have $|E| = 0$.
Since $\lfloor \frac{0^2}{4} \rfloor = 0, 0 \leq 0$.
We have $Q(n)$ holds when $n = 0$.

Inductive step: $n \geq 1$
Let $n \in \mathbb{N}$ and assume $n \geq 1$. Assume $Q(n)$.
That is assume $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n \Rightarrow |E| \leq \lfloor \frac{n^2}{4} \rfloor$.
I want to show $Q(n + 1)$.
That is to show $\forall G = (V, E)$, G is a bipartite graph $\wedge |V| = n + 1 \Rightarrow |E| \leq \lfloor \frac{(n+1)^2}{4} \rfloor$.
Since one of V_1 and V_2 have less than or equal to $\lfloor \frac{n+1}{2} \rfloor$ vertices and one of them have greater than or equal to $\lfloor \frac{n+1}{2} \rfloor$ vertices,
we can assume $|V_1| \leq \lfloor \frac{n+1}{2} \rfloor \leq |V_2|$ without loss of generality.

Consider what happen when we remove one vertex in V_2.
Let $G' = (V', E')$ be the new bipartite graph after removing one vertex in V_2 and we have $0 \leq |V'| = n$.
By induction hypothesis, we have $|E'| \leq \lfloor \frac{n^2}{4} \rfloor$.
After removing one vertex in V_2, the total number of edges will at most decrease $|V_1|$.
Therefore we have $|E| - |E'| \leq |V_1|$
Hence, $|E| \leq |E'| + |V_1| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor$.
Then, I want to show $\lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{(n+1)^2}{4} \rfloor$.
Case α: n is odd
Therefore we have $\exists k_1 \in \mathbb{Z}$ such that $n = 2k_1 + 1$. $(\star)$
Left hand side:

$$\begin{aligned}
\lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor &= \lfloor \frac{(2k_1+1)^2}{4} \rfloor + \lfloor \frac{2k_1+1+1}{2} \rfloor \\
&= \lfloor k_1^2 + k_1 + \frac{1}{4} \rfloor + \lfloor k_1 + 1 \rfloor \\
&= k_1^2 + 2k_1 + 1 \\
&= (k_1+1)^2
\end{aligned}$$

Right hand side:

$$\begin{aligned}
\lfloor \frac{(n+1)^2}{4} \rfloor &= \lfloor \frac{(2k_1+1+1)^2}{4} \rfloor \\
&= \lfloor \frac{(2k_1+2)^2}{4} \rfloor \\
&= \lfloor (k_1+1)^2 \rfloor \\
&= (k_1+1)^2
\end{aligned}$$

Therefore $\lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{(n+1)^2}{4} \rfloor$ holds in this case.
Case β: n is even
Therefore we have $\exists k_2 \in \mathbb{Z}$ such that $n = 2k_2$. $(*)$
Left hand side:

$$\begin{aligned}
\lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor &= \lfloor \frac{(2k_2)^2}{4} \rfloor + \lfloor \frac{2k_2+1}{2} \rfloor \\
&= \lfloor k_2^2 \rfloor + \lfloor k_2 + \frac{1}{2} \rfloor \\
&= k_2^2 + k_2
\end{aligned}$$

Right hand side:

$$\begin{aligned}
\lfloor \frac{(n+1)^2}{4} \rfloor &= \lfloor \frac{(2k_2+1)^2}{4} \rfloor \\
&= \lfloor k_2^2 + k_2 + \frac{1}{4} \rfloor \\
&= k_2^2 + k_2
\end{aligned}$$

Therefore $\lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{(n+1)^2}{4} \rfloor$ holds in this case.
By Case α and Case β,
we have $|E| \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{(n+1)^2}{4} \rfloor$.
I can conclude that $|E| \leq \lfloor \frac{(n+1)^2}{4} \rfloor$.
Hence $Q(n+1)$ holds in this case.
By Base case and Inductive step, I can conclude that $\forall n \in \mathbb{N}, Q(n)$.
Since $\lfloor \frac{n^2}{4} \rfloor \leq \frac{n^2}{4}$, it's obvious that $Q(n) \Rightarrow P(n)$.
Therefore we have $\forall n \in \mathbb{N}, P(n)$.
I can conclude that every bipartite graph on n vertices has no more than $\frac{n^2}{4}$ edges.

Question2

(a)
Yes.
Since we assume $P(3)$, we know $f(3)$ is a multiple of 4.
Then we get $\exists k \in \mathbb{Z}, f(3) = 4k$.
I want to show $P(29)$.
That is to show $\exists m \in \mathbb{Z}, f(29) = 4m$.
Let $m = 4k^2 + k$.

$$\begin{aligned} f(29) &= [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \\ &= [f(3)]^2 + f(3) \\ &= 16k^2 + 4k \\ &= 4(4k^2 + k) \\ &= 4m \end{aligned}$$

I've shown that $P(29)$ holds.
(b)
No.
Since we assume $P(4)$, we know $f(4)$ is a multiple of 4.
We want to show $P(29)$.
That is to show $\exists m \in \mathbb{Z}, f(29) = 4m$.

$$\begin{aligned} f(29) &= [f(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor) \\ &= [f(3)]^2 + f(3) \end{aligned}$$

However we only know that $f(4)$ is a multiple of 4, but know nothing about $f(3)$. Therefore we cannot prove $P(29)$ is true directly by using $P(4)$.
(Although we can show that $f(3) = f(4)$ since $f(3) = [f(\lfloor \log_3 3 \rfloor)]^2 + f(\lfloor \log_3 3 \rfloor) = [f(1)]^2 + f(1)$ and $f(4) = [f(\lfloor \log_3 4 \rfloor)]^2 + f(\lfloor \log_3 4 \rfloor) = [f(1)]^2 + f(1)$, the process does not directly use the assumption $P(4)$.)
(c)
I want to prove $\forall n \in \mathbb{N}, n > 0 \Rightarrow P(n)$ by complete induction.
Let $n \in \mathbb{N}$ and assume $n \geq 1$.
Assume $P(m)$ holds for all natural number m such that $1 \leq m < n$.
I want to show $P(n)$.
That is to show $\exists k \in \mathbb{N}, f(n) = 4k$.

Base case: n = 1
Let $k = 3$.

Then we have

$$\begin{aligned} f(1) &= [f(\lfloor \log_3 1 \rfloor)]^2 + f(\lfloor \log_3 1 \rfloor) && \text{(Since n > 0)} \\ &= [f(0)]^2 + f(0) \\ &= 3^2 + 3 \\ &= 12 \\ &= 4k \end{aligned}$$

Hence, $P(n)$ holds when $n = 1$.

Base case: $n = 2$
Let $k = 3$. Then we have

$$\begin{aligned} f(2) &= [f(\lfloor \log_3 2 \rfloor)]^2 + f(\lfloor \log_3 2 \rfloor) && \text{(Since n > 0)} \\ &= [f(0)]^2 + f(0) \\ &= 3^2 + 3 \\ &= 12 \\ &= 4k \end{aligned}$$

Hence, $P(n)$ holds when $n = 1$.

Inductive step: $n \geq 3$
Since $1 \leq \lfloor \log_3 n \rfloor \leq \log_3 n < n$ for all $n \geq 3$.
We have $P(\lfloor \log_3 n \rfloor)$ holds by induction hypothesis.
Therefore $\exists k_2 \in \mathbb{Z}, f(\lfloor \log_3 n \rfloor) = 4k_2$.
Let $k = 4k_2^2 + k_2$.
Then

$$\begin{aligned} f(n) &= [f(\lfloor \log_3 n \rfloor)]^2 + f(\lfloor \log_3 2 \rfloor) && \text{(Since n > 0)} \\ &= (4k_2)^2 + 4k_2 \\ &= 16k_2^2 + 4k_2 \\ &= 4(4k_2^2 + k_2) \\ &= 4k \end{aligned}$$

Hence, $P(n)$ holds in this case.
By Base case 1, base case 2 and inductive step,
I can conclude that $\forall n \in \mathbb{N}, n > 0 \Rightarrow P(n)$.

Question3

WTS: There are no positive integers x, y, z such that $5x^3 + 50y^3 = 3z^3$.
I will prove it by using contradiction.
Assume $\exists x, y, z \in \mathbb{N} - \{0\}, 5x^3 + 50y^3 = 3z^3$.
Let $s = \{x_0 \in \mathbb{N} - \{0\} \mid \exists y_0, z_0 \in \mathbb{N} - \{0\}, x_0^3 = \frac{3}{5}z_0^3 - 10y_0^3\}$
Since $\exists x, y, z \in \mathbb{N} - \{0\}, x^3 = \frac{3}{5}z^3 - 10y^3$,
we have $x \in S$.
Therefore S is non-empty.
Since S is a non-empty subset of $\mathbb{N}$,
by Principle of Well-Ordering, S has a smallest element. Let's call it x_{min}.
Therefore $\exists y_1, z_1 \in \mathbb{N} - \{0\}, x_{min}^3 = \frac{3}{5}z_1^3 - 10y_1^3$ $\qquad (*)$
Then we have $3z_1^3 = 5x_{min}^3 + 50y_1^3 = 5(x_{min}^3 + 10y_1^3)$
It follows that $5|3z_1^3$.
Now I want to prove a theorem: (let's call it theorem 1)
If p is prime and $p|ab$, then $p|a$ or $p|b$.
Proof: Assume p is prime and $p|ab$.
Case 1 : $p|a$
Then $p|a$ or $p|b$ is obviously true.
Case 2: $p \nmid a$
We have $gcd(p, a) = 1$.
Therefore $\exists x, y$ such that $px + ay = 1$.
Then $pbx + aby = b$
Since $p|pbx, p|(ab)y$, we have $p|b$.
By Case 1 and Case 2, I can conclude that theorem 1 is true.
Since 5 is a prime and $5|3z_1^3$,
we know $5 \nmid 3$.
Therefore by theorem 1, $5|z_1^3$. Then we have $5|z_1$.
It follows that $\exists k \in \mathbb{Z}, z_1 = 5k$.
Since $z_1 > 0, k > 0$,we have $k \in \mathbb{N} - \{0\}$.

Now I want to show $5|x_{min}$.
Substitute z_1 with 5k in the equation $(*)$:
We can get $75k^3 = x_{min}^3 + 10y_1^3$ $\qquad (\star)$
Hence $x_{min}^3 = 5(15k^3 - 2y_1^3)$
Then we have $5|x_{min}^3$,
Since 5 is a prime number, we know $5|x_{min}$
I have shown that $5|x_{min}$.

It follows that $\exists m \in \mathbb{Z}, x_{min} = 5m$.
Since $x_{min} > 0$, we know $m > 0$.
Therefore $m \in \mathbb{N} - \{0\}$

Now I want to show $5|y_1$.
Substitute x_{min} with 5m in the equation $(\star)$:
We get $75k^3 = 125m^3 + 10y_13$
Therefore $2y_1^3 = 15k^3 - 25m^3 = 5(3k^3 - 5m^3)$ $\qquad (@)$

Then we have $5|2y_1^3$.
Since 5 is prime and $5|2y_1^3$,
we know $5 \nmid 2$, by theorem 1, $5|y_1^3$.
Then we have $5|y_1$.
I have shown that $5|y_1$.

It follows that $\exists n \in \mathbb{Z}, 5n = y_1$.
Since $y_1 > 0$, we know $n > 0$.
Therefore $n \in \mathbb{N} - \{0\}$.
Substitute y_1 with 5n in the equation (@):
We get $2(5n)^3 = 15k^3 - 25m^3$.
Then we have $m^3 = \frac{3}{5}k^3 - 10n^3$.
Since $x_{min} = 5m$, we have $m < x_{min}$.
Since x_{min} is the smallest element in S, $m \notin S$.
However, we have shown that $\exists k, n \in \mathbb{N} - \{0\}$ such that $m^3 = \frac{3}{5}k^3 - 10n^3$ which means that $m \in S$.
There is a contradiction since $m \in S$ and $m \notin S$ can not both satisfy.
Hence, the assumption is false.
By contradiction, I can conclude that there are no positive integers x, y, z such that $5x^3 + 50y^3 = 3z^3$.

Question4

(a)
Let $P(t)$ be the following predicate:

$$P(t): \qquad \text{left\_count}(t) \leq 2^{\text{max\_left\_surplus}(t)} - 1$$

I'll prove $\forall t \in \tau, P(t)$ by using structural induction on τ.

verify the bases:
I want to show $P("*")$.
That is to show left_count$(t) \leq 2^{\text{max_left_surplus}(t)} - 1$
By the definition of left_count and max_left_surplus,
Since the number of "(" in "$*$" is 0, we have left_count("$*$") $= 0$
Since the only one left surplus of "$*$" is 0, we have max_left_surplus("$*$") $= 0$
Hence, $2^{\text{max\_left\_surplus}(t)-1} = 0$
It follows that left_count$(t) \leq 2^{\text{max_left_surplus}(t)} - 1$
I've shown that $P("*")$ holds.
Inductive step:
Let $t_1, t_2 \in T$. Assume $P(t_1)$ and $P(t_2)$ hold.
Therefore we have left_count$(t_1) \leq 2^{\text{max_left_surplus}(t_1)} - 1$ (1)
we also have left_count$(t_2) \leq 2^{\text{max_left_surplus}(t_2)} - 1$ (2)
I'll show that $P((t_1t_2))$ holds.
That is to show left_count$((t_1t_2)) \leq 2^{\text{max_left_surplus}((t_1,t_2))} - 1$
The number of "(" in $((t_1t_2))$ is one more than the number of "(" in $t_1$plus number of "(" in t_2, since the leftmost "(" in (t_1t_2) should be counted.
Hence left_count$((t_1t_2))$ = left_count(t_1) + left_count(t_2) + 1 (3)
Then let's consider what's the relation among max_left_surplus$((t_1t_2))$, max_left_surplus(t_1) and max_left_surplus(t_2).
I first want to show max_left_surplus$((t_1t_2)) \geq$ max_left_surplus(t_1) + 1.
We know max_left_surplus(t_1) is the maximum left surplus for all prefixes of t_1.
Let i_1 be the index such that we can get the maximum left surplus in $t_1[: i_1]$
Then we have max_left_surplus(t_1) = left_surplus(t_1, i_1) (4)
Let's consider (t_1, t_2) now.
Since the leftmost "(" is also counted in $(t_1, t_2)[: i_1 + 1]$,
We have left_surplus$((t_1, t_2),\, i_1 + 1)$ = left_surplus$(t_1,\, i_1)$ + 1, (5)
Hence,

$$\begin{aligned}
\text{max_left_surplus}((t_1, t_2)) &\geq \text{left_surplus}((t_1t_2), i_1 + 1) && \text{(by definition of max_left_surplus)}\\
&= \text{left_surplus}(t_1, i_1) + 1 && \text{(by (5))}\\
&= \text{max_left_surplus}(t_1) + 1 && \text{(by (4))}
\end{aligned}$$

I've shown that max_left_surplus$((t_1t_2)) \geq$ max_left_surplus(t_1) + 1 (6)
Then I want to show max_left_surplus$((t_1t_2)) \geq$ max_left_surplus(t_2) + 1
We know max_left_surplus(t_2) is the maximum left surplus for all prefixes of t_2.
Let i_2 be the index such that we can get the maximum left surplus in $t_2[: i_2]$

Then we have max_left_surplus(t_2) = left_surplus(t_2, i_2)   (7)
Let's consider (t_1t_2) now
Since every "(" and ")" in t_1 appear in pairs and there is only one extra "(" which appears at index 0,
we know left_surplus((t_1t_2), len(t_1) + 1) = 1.   (8)
By definition of left_surplus, we also have :
left_surplus((t_1t_2), len(t_1) +i_2 + 1)
= left_surplus((t_1t_2), len(t_1) +1) + left_surplus(t_2, i_2)   (9)
Therefore, we can get

$$\begin{aligned}
\text{max_left_surplus}((t_1, t_2)) &\geq \text{left_surplus}((t_1t_2), \text{len}(t_1) + i_2 + 1) \qquad \text{(by definition of max_left_surplus)} \\
&= \text{left_surplus}((t_1t_2), \text{len}(t_1) + 1) + \text{left_surplus}(t_2, i_2) \qquad \text{(by (9))} \\
&= 1 + \text{left_surplus}(t_2, i_2) \qquad \text{(by (8))} \\
&= 1 + \text{max_left_surplus}(t_2) \qquad \text{(by (7))}
\end{aligned}$$

I've shown that max_left_surplus((t_1t_2)) $\geq$ max_left_surplus(t_2) + 1   (10)
By what we have derived,

$$\begin{aligned}
\text{left_count}((t_1t_2)) &= \text{left_count}(t_1) + \text{left_count}(t_2) + 1 \qquad \text{(by (3))} \\
&\leq 2^{\text{max_left_surplus}(t_1)} - 1 + 2^{\text{max_left_surplus}(t_2)} - 1 + 1 \qquad \text{(by (1) and (2))} \\
&= 2^{\text{max_left_surplus}(t_1)} + 2^{\text{max_left_surplus}(t_2)} - 1 \\
&\leq 2^{\text{max_left_surplus}((t_1, t_2)) - 1} + 2^{\text{max_left_surplus}((t_1, t_2)) - 1} - 1 \qquad \text{(by (6) and (10))} \\
&= 2^{\text{max_left_surplus}((t_1, t_2))} - 1
\end{aligned}$$

By base case and inductive step, I can conclude that $\forall t \in T, P(t)$.
Hence, $\forall t \in \tau$, left_count(t) $\leq 2^{\text{max\_left\_surplus}(t)} - 1$.

(b)
Let $Q(t)$ be the following predicate

$$Q(t): \qquad \text{double\_count}(t) = \begin{cases} 0 & \text{if } t = \text{"}*\text{"} \\ \text{left\_count}(t) - 1 & \text{otherwise} \end{cases} \tag{1}$$

I'll prove $\forall t \in T, Q(t)$ by using structural induction.

verify the basis:
I want to show $Q(\text{"}*\text{"})$
Since there is no "((" and no "))" in "*", we have double_count("*") = 0
Therefore I've shown that $Q(\text{"}*\text{"})$ holds.

Inductive step:
Assume $Q(t_1)$ and $Q(t_2)$ hold.
That is assume

$$\text{double\_count}(t_1) = \begin{cases} 0 & \text{if } t_1 = \text{"}*\text{"} \\ \text{left\_count}(t_1) - 1 & \text{otherwise} \end{cases} \tag{2}$$

$$\text{double\_count}(t_2) = \begin{cases} 0 & \text{if } t_2 = "*" \\ \text{left\_count}(t_2) - 1 & \text{otherwise} \end{cases} \tag{3}$$

I want to show $Q((t_1 t_2))$.
That is to show

$$\text{double\_count}((t_1 t_2)) = \begin{cases} 0 & \text{if } (t_1 t_2) = "*" \\ \text{left\_count}((t_1 t_2)) - 1 & \text{otherwise} \end{cases} \tag{4}$$

Since $(t_1 t_2)$ can never be "$*$",
we only need to show double_count($(t_1 t_2)$) = left_count($(t_1 t_2)$) - 1
Let's consider 4 cases:

Case 1: $t_1 = "*"$ and $t_2 = "*"$
Then $(t_1 t_2) =$ "$(**)$"
Therefore the number of "((" plus number of "))" in $(t_1 t_2)$ is certainly 0,
hence double_count($(t_1 t_2)$) = 0
Since there is only one "(" in "$(**)$",
we also have left_count($(t_1 t_2)$) = 1.
Then we get double_count($(t_1 t_2)$) = left_count($(t_1 t_2)$) - 1.
Hence, $Q((t_1 t_2))$ holds in this case.

Case 2: $t_1 = "*"$ and $t_2 \,!= "*"$
We have double_count(t_1) = 0 and double_count(t_2) = left_count(t_2) - 1 by $Q(t_1)$ and $Q(t_2)$
Then double_count($(t_1 t_2)$) = 0 + double_count(t_2) + 1, since the ")" at the tail of $(t_1 t_2)$ increases the number of "))" by 1 while the "(" at the beginning does not influence the number of "((".
Since the number of "(" in $(t_1 t_2)$ is one more than the number of "(" in t_1 plus the number of "(" in t_2,
we have left_count($(t_1 t_2)$) = 1 + left_count(t_2) + left_count(t_1).
Since left_count(t_1) = 0, left_count($(t_1 t_2)$) = 1 + left_count(t_2).
Therefore

$$\begin{aligned} \text{double\_count}((t_1 t_2)) &= \text{double\_count}(t_2) + 1 \\ &= \text{left\_count}(t_2) - 1 + 1 \qquad \text{(by induction hypothesis)} \\ &= \text{left\_count}(t_2) \\ &= \text{left\_count}((t_1 t_2)) - 1 \end{aligned}$$

Hence, $Q((t_1 t_2))$ holds in this case.

Case 3: $t_1 \,!= "*"$ and $t_2 = "*"$
We have double_count(t_1) = left_count(t_2) - 1 and double_count(t_2) = 0 by $Q(t_1)$ and $Q(t_2)$.
Since the "(" at the beginning of $(t_1 t_2)$ increases the number of "((" by 1 while the ")" at

the tail does not influence the number of "))",
we have double_count((t_1t_2)) = double_count(t_1) + 1 + 0.
Moreover, since the number of "(" in (t_1t_2) is one more than the number of "(" in t_1 plus the number of "(" in t_2,
We have left_count((t_1t_2)) = 1 + left_count(t_1) + left_count(t_2) = 1 + left_count(t_1)
Therefore

$$\begin{aligned}
\text{double_count}((t_1t_2)) &= \text{double_count}(t_1) + 1 \\
&= \text{left_count}(t_1) - 1 + 1 \qquad \text{(by induction hypothesis)} \\
&= \text{left_count}(t_1) \\
&= \text{left_count}((t_1t_2)) - 1
\end{aligned}$$

Hence, $Q((t_1t_2))$ holds in this case.

Case 4: $t_1\ != "*"$ and $t_2\ != "*"$
We have double_count(t_1) = left_count(t_1) - 1 and double_count(t_2) = left_count(t_2) - 1 by $Q(t_1)$ and $Q(t_2)$.
Since "(" at the beginning of (t_1t_2) increases the number of "((" by 1 and the ")" at the tail of (t_1t_2) increases the number of "))" by 1,
we have double_count((t_1t_2)) = double_count(t_1) + 1 + double_count(t_2) + 1 = double_count(t_1) + double_count(t_2) + 2.
Since the number of "(" in (t_1t_2) is one more than the number of "(" in t_1 plus the number of "(" in t_2,
We have left_count((t_1t_2)) = 1 + left_count(t_1) + left_count(t_2)
Therefore

$$\begin{aligned}
\text{double_count}((t_1t_2)) &= \text{double_count}(t_1) + \text{double_count}(t_2) + 2 \\
&= \text{left_count}(t_1) - 1 + \text{left_count}(t_2) - 1 + 2 \qquad \text{(by induction hypothesis)} \\
&= \text{left_count}(t_1) + \text{left_count}(t_2) \\
&= \text{left_count}((t_1t_2)) - 1
\end{aligned}$$

Hence, $Q((t_1t_2))$ also holds in this case.

By Case 1, Case 2, Case 3 and Case 4, I can conclude that $Q((t_1t_2))$ holds.
By base case and inductive step, I can conclude that $\forall t \in \tau, Q(t)$