to solve the resulting linear system $\boldsymbol{H x} = \boldsymbol{b}$, obtaining an approximate solution $\hat{\boldsymbol{x}}$. Compute the ∞-norm of the residual $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{H}\hat{\boldsymbol{x}}$ and of the error $\Delta\boldsymbol{x} = \hat{\boldsymbol{x}} - \boldsymbol{x}$, where $\boldsymbol{x}$ is the vector of all ones. How large can you take n before the error is 100 percent (i.e., there are no significant digits in the solution)? Also use a condition estimator to obtain cond($\boldsymbol{H}$) for each value of n. Try to characterize the condition number as a function of n. As n varies, how does the number of correct digits in the components of the computed solution relate to the condition number of the matrix?

2.7. (*a*) What happens when Gaussian elimination with partial pivoting is used on a matrix of the following form?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Do the entries of the transformed matrix grow? What happens if complete pivoting is used instead? (Note that part *a* does not require a computer.)

(*b*) Use a library routine for Gaussian elimination with partial pivoting to solve various sizes of linear systems of this form, using right-hand-side vectors chosen so that the solution is known. How do the error, residual, and condition number behave as the systems become larger? This artificially contrived system illustrates the worst-case growth factor cited in Section 2.4.5 and is not indicative of the usual behavior of Gaussian elimination with partial pivoting.

2.8. Multiplying both sides of a linear system $\boldsymbol{A x} = \boldsymbol{b}$ by a nonsingular diagonal matrix $\boldsymbol{D}$ to obtain a new system $\boldsymbol{D A x} = \boldsymbol{D b}$ simply rescales the rows of the system and in theory does not change the solution. Such scaling does affect the condition number of the matrix and the choice of pivots in Gaussian elimination, however, so it may affect the accuracy of the solution in finite-precision arithmetic. Note that scaling can introduce some rounding error in the matrix unless the entries of $\boldsymbol{D}$ are powers of the base of the floating-point arithmetic system being used (why?).

Using a linear system with randomly chosen matrix $\boldsymbol{A}$, and right-hand-side vector $\boldsymbol{b}$ chosen so that the solution is known, experiment with various scaling matrices $\boldsymbol{D}$ to see what effect they have on the condition number of the matrix $\boldsymbol{DA}$ and the solution given by a library routine for solving the linear system $\boldsymbol{D A x} = \boldsymbol{D b}$. Be sure to try some fairly skewed scalings, where the magnitudes of the diagonal entries of $\boldsymbol{D}$ vary widely (the purpose is to simulate a system with badly chosen units). Compare both the relative residuals and the error given by the various scalings. Can you find a scaling that gives very poor accuracy? Is the residual still small in this case?

2.9. (*a*) Use Gaussian elimination *without* pivoting to solve the linear system

$$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+\epsilon \\ 2 \end{bmatrix}$$

for $\epsilon = 10^{-2k}$, $k = 1, \ldots, 10$. The exact solution is $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, independent of the value of ϵ. How does the accuracy of the computed solution behave as the value of ϵ decreases?

(*b*) Repeat part *a*, still using Gaussian elimination without pivoting, but this time use one iteration of iterative refinement to improve the solution, computing the residual in the same precision as the rest of the computations. Now how does the accuracy of the computed solution behave as the value of ϵ decreases?

2.10. Consider the linear system

$$\begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+(1+\epsilon)\epsilon \\ 1 \end{bmatrix},$$

where ϵ is a small parameter to be specified. The exact solution is obviously

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

for any value of ϵ.

Use a library routine based on Gaussian elimination to solve this system. Experiment with various values for ϵ, especially values near $\sqrt{\epsilon_{\text{mach}}}$ for your computer. For each value of ϵ you try, compute an estimate of the condition number of the matrix and the relative error in each component of the solution. How accurately is each component determined? How does the accuracy attained for each component compare with expectations based on the condition number of the matrix and the error bounds given in Section 2.3.4? What conclusions can you draw from this experiment?