CSC336 A2

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Problem 1

Let n be arbitrary integer such that $n \geq 1$.
Let $x \in R^n$.
I want to show $||x||_2 \leq ||x||_1$.
Equivalently, I will show $(||x||_2)^2 \leq (||x||_1)^2$.

$$\begin{aligned}(||x||_1)^2 &= (|x_1| + |x_2| + ...|x_n|)^2 \\ &= x_1^2 + x_2^2 + ... + x_n^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |x_i||x_j| \\ &\geq x_1^2 + x_2^2 + ... + x_n^2 \\ &= (||x||_2)^2\end{aligned}$$

I have shown that $(||x||_2)^2 \leq (||x||_1)^2$.
Therefore $||x||_2 \leq ||x||_1$.

Problem 2

Yes. It is possible to have $||x||_1 > ||y||_1$ and $||x||_2 < ||y||_2$.
Example: When $x = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
Then $||x||_1 = 2 + 2 = 4$, $||y||_1 = 0 + 3 = 3$.
Therefore $||x||_1 > ||y||_1$.
We also have $||x||_2 = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, $||y||_2 = \sqrt{0^2 + 3^2} = 3$.
Since $3 > 2\sqrt{2}$, we get $||x||_2 < ||y||_2$.

Problem 3

Let $x \in R^n$. Let A be a $n \times n$ matrix.
Case 1: $x = \vec{0}$
Left hand side: Since $x = \vec{0}$,
we have $Ax = \vec{0}$.
Therefore $||Ax|| = 0$.
Right hand side: we have $||x|| = 0$,
Then $||A|| \cdot ||x|| = 0$.
Therefore $||Ax|| = 0 \leq 0 = ||A|| \cdot ||x||$.
$||Ax|| \leq ||A|| \cdot ||x||$ holds in this case.
Case 2: $x \neq \vec{0}$
Since $\frac{||Ax||}{||x||} \leq \max_{x \neq 0} \frac{||Ax||}{||x||} = ||A||$,
$||Ax|| \leq ||A|| \cdot ||x||$ also holds in this case.
By Case 1 and Case 2, I can conclude that $||Ax|| \leq ||A|| \cdot ||x||$.

Problem 4

(a)
MatLab Program:

```
function question4
a = sqrt(2)/2;
A = zeros(13, 13);
b = zeros(13, 1);

b(2, 1) = 10;
b(8, 1) = 15;
b(10, 1) = 20;

A(1,2) = 1; A(1,6) = -1;
A(2,3) = 1;
A(3,1) = a; A(3,4) = -1; A(3,5) = -a;
A(4,1) = a; A(4,3) = 1; A(4,5) = a;
A(5,4) = 1; A(5,8) = -1;
A(6,7) = 1;
A(7,5) = a; A(7,6) = 1; A(7,9) = -a; A(7,10) = -1;
A(8,5) = a; A(8,7) = 1; A(8,9) = a;
A(9,10) = 1; A(9,13) = -1;
A(10,11) = 1;
A(11,8) = 1; A(11,9) = a; A(11,12) = -a;
A(12,9) = a; A(12,11) = 1; A(12,12) = a;
A(13,12) = a; A(13,13) = 1;
f1 = A\b;

display(f1);
r = b- A * f1;
relative_residual = norm(abs(r))/norm(abs(b));
upper = cond(A) * relative_residual;
lower = relative_residual / cond(A);

fprintf(['condition number of A: %d\nrelative_residual: %d',...
'\nupper_bound: %d\nlower_bound: %d\n'], cond(A), ...
relative_residual, upper, lower);
```

Prgram Output:

```
>> question4

f1 =

  -28.2843
   20.0000
   10.0000
  -30.0000
   14.1421
   20.0000
         0
  -30.0000
    7.0711
   25.0000
   20.0000
```

```
  -35.3553
   25.0000

condition number of A: 1.041221e+01
relative residual: 2.012275e-16
upper bound: 2.095224e-15
lower bound: 1.932610e-17
```

(b)
Explanation:
We know $Af = b$, $A\hat{f} = \hat{b}$ and $Cond(A) = ||A|| \cdot ||A^{-1}||$.
First, I want to prove $\frac{||\hat{f}-f||}{||f||} \leq Cond(A) \cdot \frac{||\hat{b}-b||}{||b||}$,

$$\begin{aligned}
\frac{||\hat{f}-f||}{||f||} &= \frac{||A|| \cdot ||\hat{f}-f||}{||A|| \cdot ||f||} \\
&\leq \frac{||A|| \cdot ||\hat{f}-f||}{||b||} \\
&= \frac{||A|| \cdot ||A^{-1}b - A^{-1}\hat{b}||}{||b||} && \text{(Since } Af = b \text{ and } A\hat{f} = \hat{b}\text{)} \\
&= \frac{||A|| \cdot ||A^{-1}(b-\hat{b})||}{||b||} \\
&\leq \frac{||A|| \cdot ||A^{-1}||| \cdot ||\hat{b}-b||}{||b||} \\
&= Cond(A) \cdot \frac{||\hat{b}-b||}{||b||} && \text{(Since } Cond(A) = ||A|| \cdot ||A^{-1}||\text{)}
\end{aligned}$$

Second, I want to show $\frac{||\hat{f}-f||}{||f||} \geq \frac{1}{Cond(A)} \cdot \frac{||\hat{b}-b||}{||b||}$

$$\begin{aligned}
\frac{||\hat{f}-f||}{||f||} &= \frac{||A|| \cdot ||\hat{f}-f||}{||A|| \cdot ||f||} \\
&\geq \frac{||A\hat{f}-Af||}{||A|| \cdot ||f||} \\
&= \frac{||\hat{b}-b||}{||A|| \cdot ||f||} && \text{(Since } Af = b \text{ and } A\hat{f} = \hat{b}\text{)} \\
&= \frac{||\hat{b}-b||}{||A|| \cdot ||A^{-1}b||} \\
&\geq \frac{||\hat{b}-b||}{||A|| \cdot ||A^{-1}|| \cdot ||b||} \\
&= \frac{1}{Cond(A)} \cdot \frac{||\hat{b}-b||}{||b||} && \text{(Since } Cond(A) = ||A|| \cdot ||A^{-1}||\text{)}
\end{aligned}$$

Therefore I have shown that $\frac{1}{Cond(A)} \cdot \frac{||\hat{b}-b||}{||b||} \leq \frac{||\hat{f}-f||}{||f||} \leq Cond(A) \cdot \frac{||\hat{b}-b||}{||b||}$
Since $r = b - A\hat{f} = b - \hat{b}$,
we have $\frac{1}{Cond(A)} \cdot \frac{||r||}{||b||} \leq \frac{||\hat{f}-f||}{||f||} \leq Cond(A) \cdot \frac{||r||}{||b||}$.

By computing the condition number of A and relative residual,
we get $Cond(A) = 1.041221 \cdot 10^1$ and $\frac{||r||}{||b||} = 2.012275 \cdot 10^{-16}$.
Therefore the upper bound is $2.095224 \cdot 10^{-15}$,
The lower bound is $1.932610 \cdot 10^{-17}$.

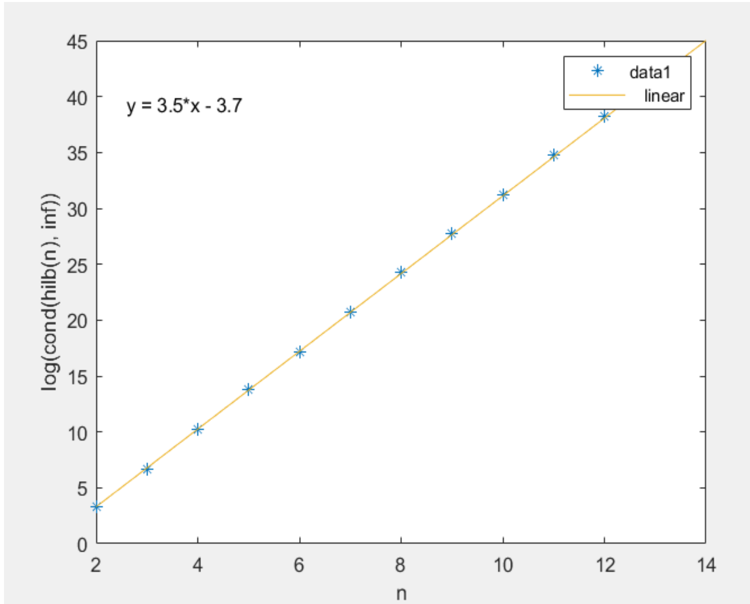
Problem 5

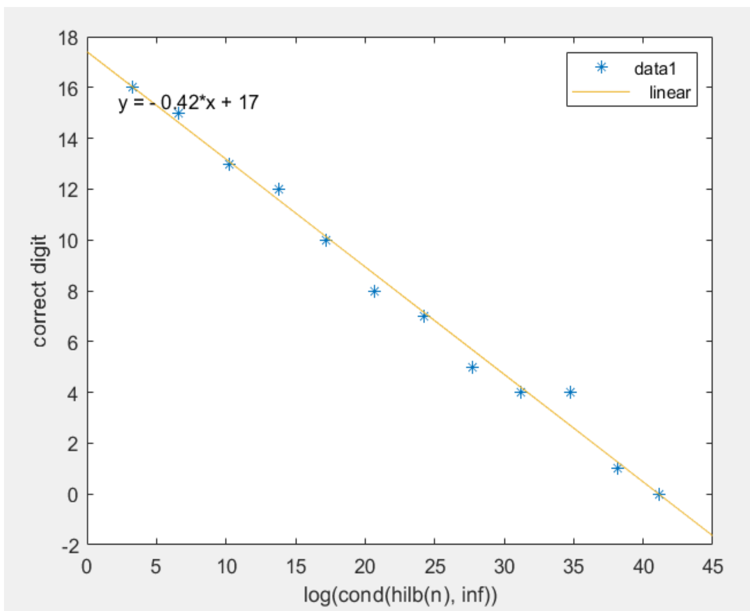
MatLab Program:

```
function hilbTest
N = (2:13)';
condition = zeros(12, 1);
logCondition = zeros(12, 1);
relative = zeros(12, 1);
correctDigit = zeros(12, 1);
for n = 2:13
    H = hilb(n);
    x = ones(n, 1);
    b = H*x;
    x1 = H\b;
    relative(n - 1) = norm(x - x1, inf)/ norm(x, inf);
    condition(n - 1) = cond(H, inf);
    logCondition(n - 1) = log(cond(H, inf));
    digit = ceil(-log10(relative(n - 1)));
    correctDigit(n - 1) = digit;
end
figure;
plot(N, logCondition, "*");
xlabel('n'); ylabel('log(cond(hilb(n), inf))');

figure;
plot(logCondition, correctDigit, "*");
xlabel('log(cond(hilb(n), inf))'); ylabel('correct digit');

Table = table(N, relative, condition, correctDigit, 'VariableNames', ...
        {'n', 'relative_error', 'condition_number', 'correct_digit'});
disp(Table);
end
```

Program Output:

```
>> hilbTest
     n     relative_error     condition_number     correct_digit
    --     --------------     ----------------     -------------

     2       7.7716e-16                   27                16
     3       7.4385e-15                  748                15
     4       4.5186e-13                28375                13
     5       4.0008e-12           9.4366e+05                12
     6        5.724e-10            2.907e+07                10
     7       2.0189e-08           9.8519e+08                 8
     8       2.0464e-07           3.3873e+10                 7
     9       1.2084e-05           1.0996e+12                 5
    10       0.00035926           3.5352e+13                 4
    11       0.00051051           1.2295e+15                 4
    12          0.49961            3.842e+16                 1
    13           2.9799           7.4907e+17                 0
```





Part1:
By the table, we can see that the largest n with relative error smaller than 1 is 12.
Part2:
By the first plot, we can see that $log(cond(hilb(n), inf))$ grows linearly based on n
and $log(cond(hilb(n), inf)) \approx 3.5n - 3.7$.
Therefore condtion number of hilb(n) grows exponentially based on n
and $cond(hilb(n), inf) \approx e^{3.5n-3.7}$.
Part3:
By the table, we can see that as n becomes larger, condition number becomes larger and the number of correct digits get fewer.
By the second plot, we can see that correct digits decreases linearly when $log(cond(hilb(n), inf))$ increases and correct digit $\approx -0.42 \cdot log(cond(hilb(n), inf)) + 17$