

Electromagnetics & Anisotropy

Half-Waveplates, Quarter-Waveplates, and Uniaxial Media

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Figure 1: A picture of a half-waveplate (HWP) from Thorlabs [1]. HWPs rotate the polarization of LP light by 90° and quarter-waveplates (QWP) convert LP light to CP and visa versa.

[1] Thorlabs, https://www.thorlabs.com/newgroupage9.cfm?objectgroup_id=711.



The Displacement Field

The electric displacement field \mathbf{D} (with units $C \cdot m^{-2}$) is defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \quad (1)$$

where \mathbf{P} is the polarization of the material, and the red part of the equation is true given an isotropic dielectric [2].



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The perpendicular and parallel boundary conditions (BC) for the displacement field on an interface are

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}}_{12} = \sigma_f \quad \text{and} \quad (\mathbf{D}_2 - \mathbf{D}_1) \times \hat{\mathbf{n}}_{12} = (\mathbf{P}_2 - \mathbf{P}_1) \times \hat{\mathbf{n}}_{12} \quad (2)$$

where σ_f is the free (unbounded) surface charge density, and $\hat{\mathbf{n}}_{12}$ is the normal vector of the interface from medium 1 to medium 2.



The Displacement Field & Anisotropy

What is Anisotropy?

Materials are *Anisotropic* if their properties have **directional dependence**—permittivity could take two different values depending on if the E field is in the \hat{x} or \hat{y} direction [3].

[3] C. A. Balanis, *Antenna Theory: Analysis and Design* (Wiley-Interscience, 2005).

[4] P. Beyersdorf, "Coupled mode analysis,"
<https://www.sjsu.edu/faculty/beyersdorf/Archive/Phys208F07/ch%204-EM%20waves%20in%20anisotropic%20media.pdf>.

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For anisotropic media, we use the relative permittivity tensor (rank 2), which must be a Hermitian matrix [4, 5].

$$\bar{\epsilon}_r = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \quad \text{where} \quad \epsilon_{ij} = \epsilon_{ji}^* \quad (3)$$

Equation (1) becomes

$$\mathbf{D} = \epsilon_0 \bar{\epsilon}_r \mathbf{E} \quad (4)$$

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Uniaxial Media

Calcite, mica, and quartz are examples of *uniaxial media*, where the permittivity tensor takes the form

$$\bar{\epsilon}_r = \begin{bmatrix} \epsilon_r^e & 0 & 0 \\ 0 & \epsilon_r^o & 0 \\ 0 & 0 & \epsilon_r^o \end{bmatrix} \quad (5)$$



Uniaxial Media

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Here, the *x*-axis is the *extraordinary* axis, and *y/z*-axes are the *ordinary* axes [5].

Wave Propagation

- The permittivity has two values
- Waves propagate at different speeds depending on their polarization



Half-Waveplate Diagram

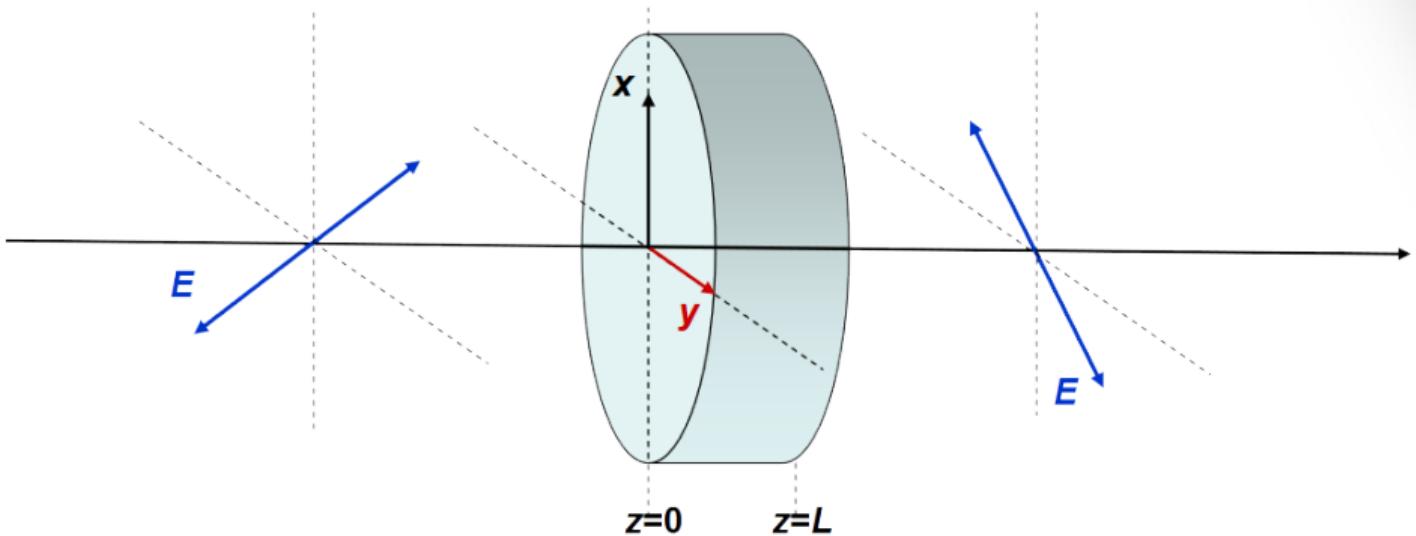


Figure 2: A half-waveplate (HWP) in the xy -plane with length L with an incident plane wave at $z = 0$ [5]. HWP s rotate the polarization of an incident plane wave 90° . We choose $z < 0$ to be region 1, $0 < z < L$ to be region 2, and $z > L$ to be region 3.



The Wave Equation

Using Faraday's Law, Ampere's Law, and Equation (4) we obtain the wave equation [3, 5]

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0\nabla \times \mathbf{H}(\mathbf{r}) = \omega^2\mu_0\mathbf{D}(\mathbf{r}) \quad (6)$$

which becomes

$$\nabla^2\mathbf{E}(\mathbf{r}) = -\omega^2\mu_0\mathbf{D}(\mathbf{r}) = -\omega^2\mu_0\epsilon_0\bar{\epsilon}_r\mathbf{E}(\mathbf{r}) \quad (7)$$

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For normal incidence, a wave polarized in \hat{x} stays in \hat{x} in the dielectric

$$\mathbf{E}_1^i(z) = \hat{\mathbf{x}}E_0e^{-j\beta_0 z} \Rightarrow \mathbf{E}_2^t(z) = \hat{\mathbf{x}}T_{12}^e E_0 e^{-j\beta^e z} \quad (8)$$

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and \hat{y} stays in \hat{y}

$$\mathbf{E}_1^i(z) = \hat{y}E_0e^{-j\beta_0 z} \Rightarrow \mathbf{E}_2^t(z) = \hat{y}T_{12}^o E_0 e^{-j\beta^o z} \quad (9)$$

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The Half-Waveplate

How do they work?

For just \hat{x} or \hat{y} incident polarization the polarization inside the dielectric does not change. How does a HWP cause the polarization to rotate?

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where (for a nonmagnetic material)

$$\beta^e = \omega \sqrt{\mu_0 \epsilon_r^e \epsilon_0} \quad \text{and} \quad \beta^o = \omega \sqrt{\mu_0 \epsilon_r^o \epsilon_0} \quad (11)$$

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The field at $z = L$ becomes

$$\mathbf{E}_2^t(z)|_{z=L} = \frac{E_0}{\sqrt{2}} \left(\hat{x} T_{12}^e e^{-j\beta^e L} + \hat{y} T_{12}^o e^{-j\beta^o L} \right) \quad (12)$$

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The Half-Waveplate Cont.

For 90° polarization change we want to flip \hat{x} or \hat{y} , so we choose Equation (13)¹ such that Equation (12) becomes Equation (14) [3, 5].

$$L(\beta^e - \beta^o) = (2m + 1)\pi \quad \text{where} \quad \forall m \in \mathbb{Z} \quad (13)$$

¹ \mathbb{Z} is defined as the set of all integers.

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$$\begin{aligned} \mathbf{E}_2^t(z)|_{z=L} &= \frac{E_0}{\sqrt{2}} \left(\hat{\mathbf{x}} T_{12}^e e^{-j[\beta^o L + (2m+1)\pi]} + \hat{\mathbf{y}} T_{12}^o e^{-j\beta^o L} \right) \\ &= \frac{E_0}{\sqrt{2}} e^{-j\beta^o L} (-\hat{\mathbf{x}} T_{12}^e + \hat{\mathbf{y}} T_{12}^o) \end{aligned} \quad (14)$$

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We have obtained the desired polarization change, with limitations...

- ϵ_r^e and ϵ_r^o are ω dependent
- The device must be a precise thickness L

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The Quarter-Waveplate

Quarter-waveplates (QWP) turn LP light into CP and visa versa, and we can use Figure 2 to design an achromatic QWP



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Hence, if $T_{12}^e \approx T_{12}^o$

LP \rightarrow CP

&

CP \rightarrow LP

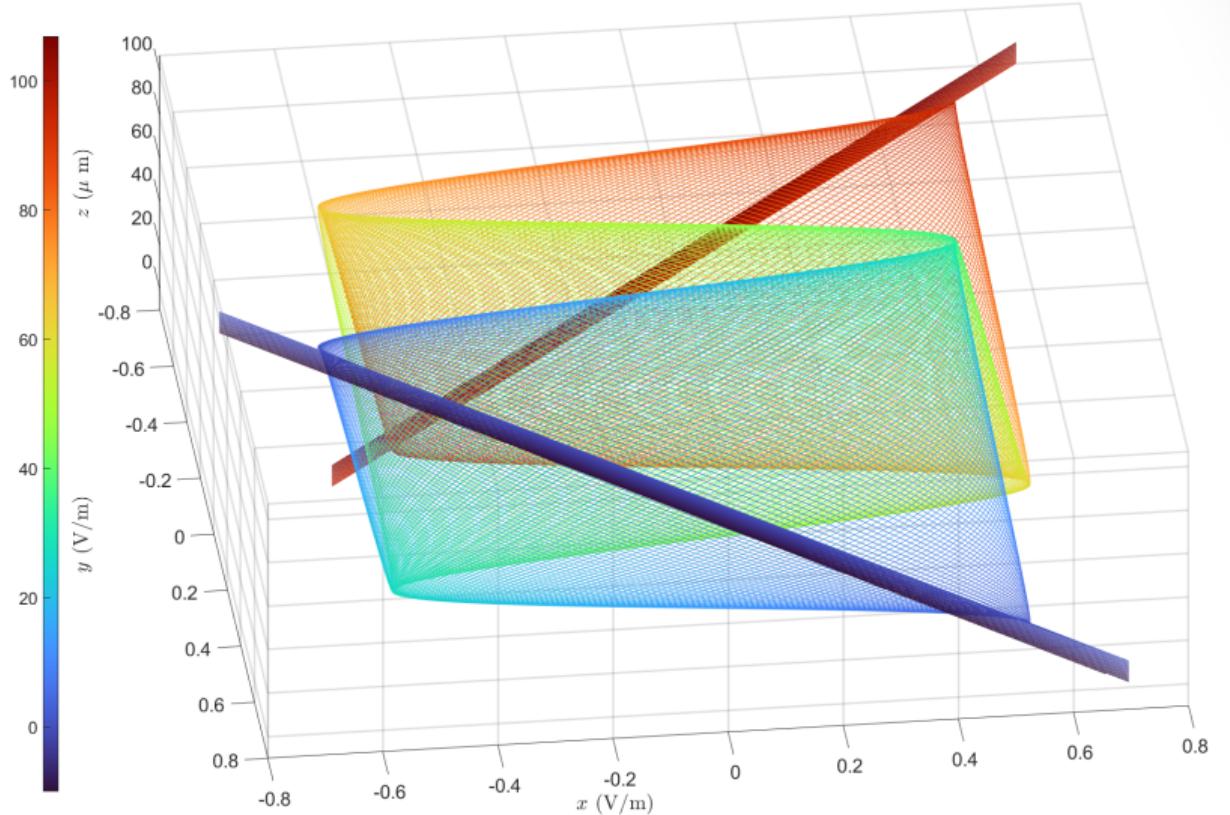


Figure 3: Forward propagating Electric Field before, inside, and after a quartz HWP, $L = 96.933\mu\text{m}$ and $m = 1$. Color bar helps visualize propagation in z . Reflections not considered, discontinuities of $\vec{\mathcal{E}}(z; t = 0)$ would vanish if they were.

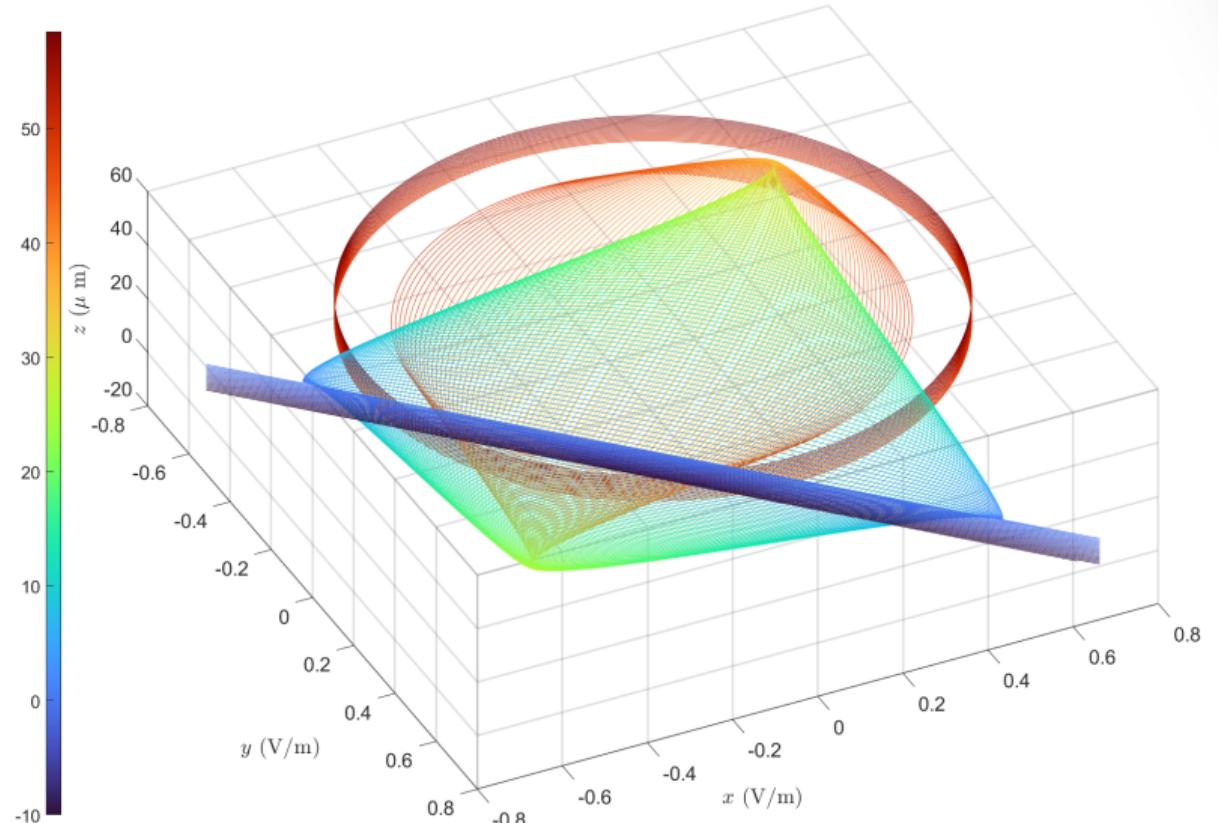


Figure 4: Forward propagating Electric Field before, inside, and after a quartz QWP, $L = 48.4667\mu\text{m}$ and $m = 1$. Color bar helps visualize propagation in z . Reflections not considered, discontinuities of $\vec{\mathcal{E}}(z; t = 0)$ would vanish they were.



Total Field after the Second Interface ($z > L$)

Half-wave plate:

$$\mathbf{E}_3(z) = \xi(z) \sum_{n=0}^{\infty} e^{-j(2n+1)\beta^o L} \begin{bmatrix} -T_{12}^e (\Gamma_{23}^e)^{2n} T_{23}^e \\ T_{12}^o (\Gamma_{23}^o)^{2n} T_{23}^o \\ 0 \end{bmatrix} \quad (17)$$

Quarter-wave plate:

$$\mathbf{E}_3(z) = \xi(z) \sum_{n=0}^{\infty} e^{-j(2n+1)\beta^o L} \begin{bmatrix} -j(-1)^n (-1)^m T_{12}^e (\Gamma_{23}^e)^{2n} T_{23}^e \\ T_{12}^o (\Gamma_{23}^o)^{2n} T_{23}^o \\ 0 \end{bmatrix} \quad (18)$$

where

$$\xi(z) \equiv \frac{E_0}{\sqrt{2}} e^{-j\beta_0(z-L)} \quad \text{and} \quad \forall m \in \mathbb{Z}$$

Figures of Merit





Figures of Merit

We define $R\%$ as the ratio of the power of first internally reflected and then transmitted wave \mathbf{E}_3^{trrt} to the power of the only forward propagating wave \mathbf{E}_3^{tt}

$$R\% = \frac{|\mathbf{E}_3^{trrt}|^2}{|\mathbf{E}_3^{tt}|^2} \cdot 100 = \frac{[T_{12}^e(\Gamma_{23}^e)^2 T_{23}^e]^2 + [T_{12}^o(\Gamma_{23}^o)^2 T_{23}^o]^2}{[T_{12}^e T_{23}^e]^2 + [T_{12}^o T_{23}^o]^2} \cdot 100 \quad (19)$$



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We define the difference in the extraordinary and ordinary transmission coefficients to be ΔT

$$\Delta T = |T_{12}^e T_{23}^e - T_{12}^o T_{23}^o| \quad (20)$$

where ideally $\Delta T \approx 0$.



Designed Quartz Waveplates

For a quartz waveplate where $\epsilon_r^e = 2.4130$ and $\epsilon_r^o = 2.3847$ at $\lambda = 590$ nm and non-magnetic $\mu = \mu_0$ [6]

Plate Type	L (μm)	m	ΔT	$R\%$
HWP	96.933	1	0.001	0.215%
HWP	226.178	3	0.001	0.215%
QWP	48.467	1	0.001	0.215%
QWP	113.089	3	0.001	0.215%

Table 1: Designed quantities and figures of merit.

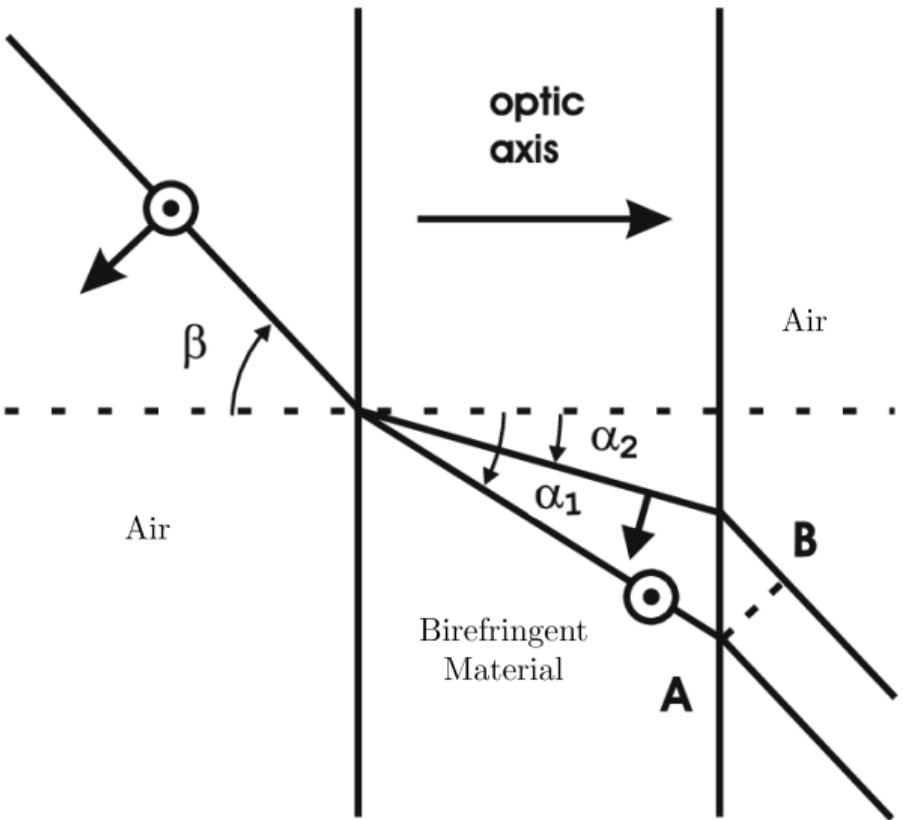


Figure 5: We obtain two spatially separated propagating fields when we consider the case of oblique incidence on a **birefringent material** [7].

[7] A. Lutich, M. Danailov, s. Volchek, V. Yakovtseva, V. Sokol, and S. Gaponenko, "Birefringence of Nanoporous Alumina: Dependence on Structure Parameters," Appl. Phys. B **84**, 327-331 (2006).



Citations

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Data Availability: \LaTeX and MATLAB code for this project and its reports can be found on the author's github:
<https://github.com/kevinlindstrom/ECE5201>.



Questions?