Math Background Review

Basic Analysis:

A. Norm: A fn $f: \mathbb{R}^n \to \mathbb{R}$ is called norm if:

* (non-neg.): $f(\underline{x}) \geq 0$, $\forall \underline{x} \in \mathbb{R}^n$. $f(\underline{x}) = 0$ iff $\underline{x} = \underline{0}$
* (homogeneity): $f(t\underline{x}) = |t| f(\underline{x})$, $\forall \underline{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$.
* (triangle ineq.): $f(\underline{x}+\underline{y}) \leq f(\underline{x}) + f(\underline{y})$, $\forall \underline{x}, \underline{y} \in \mathbb{R}^n$

If $f(\underline{x})$ is a norm, denote it as $\|\underline{x}\|$.

2. Norm $\|\underline{x}\|$'s meaning:
   * $\|\underline{x}\|$: length of $\underline{x}$.
   * $\|\underline{x}-\underline{y}\|$: dist. btwn $\underline{x}$ & $\underline{y}$.

3. Unit ball: Set of vectors with $\|\underline{x}\| \leq 1$.

$$B = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| \leq 1\}.$$

Ex: * l_2-norm (Euclidean Norm): $\|\underline{x}\|_2 \triangleq (\underline{x}^T\underline{x})^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

* l_1-norm (sum-abs-val.): $\|\underline{x}\|_1 \triangleq |x_1| + \cdots + |x_n|$ (Manhatten dist.)

* l_∞-norm (chebyshev): $\|\underline{x}\|_\infty \triangleq \max\{|x_1|, \cdots, |x_n|\}$.

* l_p-norm: $\|\underline{x}\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$ Q: $\|\underline{x}\|_\infty \overset{?}{=} \lim_{p\to\infty} \|\underline{x}\|_p$

Proof.
$$\|\underline{x}\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} = \left(\frac{|x_1|^p}{\|\underline{x}\|_\infty^p} + \cdots + \frac{|x_n|^p}{\|\underline{x}\|_\infty^p}\right)^{\frac{1}{p}} \|\underline{x}\|_\infty$$
$$\leq (1 + \cdots + 1)^{\frac{1}{p}} \|\underline{x}\|_\infty = \sqrt[p]{n}\, \|\underline{x}\|_\infty$$

Let $i^* \in \arg\max_{\forall i} \{|x_i|\}$

$$(|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \geq (|x_{i^*}|^p)^{\frac{1}{p}} = |x_{i^*}| = \|\underline{x}\|_\infty$$

Let $p \to \infty$, $\sqrt[p]{n} \to 1$ (squeez thm)

$\blacksquare$

Equivalence of Norm:

Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$, then $\exists\, \alpha, \beta > 0$

s.t. $\forall\, \underline{x} \in \mathbb{R}^n$, $\alpha \|\underline{x}\|_a \le \|\underline{x}\|_b \le \beta \|\underline{x}\|_a$.

Ex: $\|\underline{x}\|_2 \le \|\underline{x}\|_1 \le \sqrt{n}\, \|\underline{x}\|_2$.

$\|\underline{x}\|_\infty \le \|\underline{x}\|_2 \le \sqrt{n}\, \|\underline{x}\|_\infty$

$\|\underline{x}\|_\infty \le \|\underline{x}\|_1 \le n \|\underline{x}\|_\infty$

4. Convergen Sequence & Limits

1° Def (Convergence): A seq. of vectors $\underline{x}_1, \underline{x}_2 \cdots$ are said to be convergent to a limit pt. $\bar{\underline{x}}$, if $\forall \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$

s.t. $\|\underline{x}_k - \bar{\underline{x}}\| < \varepsilon$, $\forall\, k \ge N_\varepsilon$ ($\{\underline{x}_k\} \to \bar{\underline{x}}$ as $k \to \infty$, $\lim_{k\to\infty} \underline{x}_k = \bar{\underline{x}}$).

2° Def (Cauchy Seq.): A seq. $\{\underline{x}_k\}$ is Cauchy if

$\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$, s.t. $\|\underline{x}_m - \underline{x}_n\| < \varepsilon$, $\forall\, m, n > N$

Thm: A seq. in $\mathbb{R}^n$ has a limit iff it's Cauchy.

Ex: (p-series): $a_n = \frac{1}{n^p}$. Show $\{b_n\} = \{\sum_{k=1}^{n} a_k\}$ has a limit for $p=2$.

Also, $\{b_n\}$ doesn't converge for $p=1$.

Proof. W.l.o.g let $m, n \in \mathbb{N}$ and $m < n$.

1. $p=2$: $b_n - b_m = \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{m} \frac{1}{k^2} = \sum_{k=m+1}^{n} \frac{1}{k^2} < \sum_{k=m+1}^{n} \frac{1}{k(k-1)}$

$$= \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} \cdots - \frac{1}{n} = \frac{1}{m} - \frac{1}{n}$$

$< \frac{1}{m} < \varepsilon$ I can always find suff. large m s.t. $b_n - b_m < \varepsilon$

harmonic series

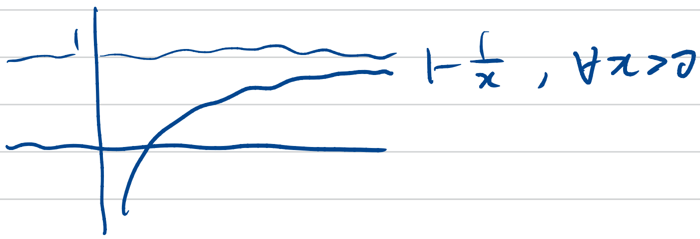
2. $p=1$: $b_n - b_m = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} = \sum_{k=m+1}^{n} \frac{1}{k} = \underbrace{\frac{1}{m+1}}_{>\frac{1}{n}} + \cdots + \underbrace{\frac{1}{n}}_{\geq\frac{1}{n}}$

$$> \frac{n-m}{n} = 1 - \frac{m}{n}$$

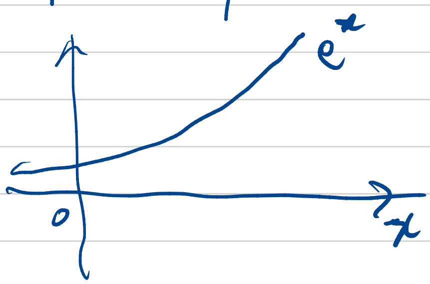
$\Rightarrow$ For any $\varepsilon > 0$, for any m (no matter how large m is).

can choose $n \geq \left[\frac{m}{1-\varepsilon}\right]$, s.t. $|b_n - b_m| > \varepsilon$.

5. Supremum of S (least UB): Smallest possible α: $\alpha \geq x, \forall \in S$.



Infimum of S (largest LB): largest possible value $\alpha \leq x, \forall x \in S$.



6. Maximum, Minimum: (achievable).

* The limit supremum $\limsup_{k\to\infty} x_k$ is the infimum of all

$q \in \mathbb{R}$ for which all but a finite # of elements in $\{x_k\}$

exceed q.

$$\limsup_{n\to\infty} x_n \triangleq \lim_{n\to\infty} \left\{ \sup_{m \geq n} x_m \right\}$$