CS70 In Simpler Terms - Note 6

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1 Random Variables and Expectations

There are a few main distributions that we will be focusing on in this course, but understanding the probability aspect and how to calculate their Expectations will be key to your success.

Random Variabe: a function $X : \omega \Rightarrow R$ that assigns a real number to every outcome ω in the probability space. (ω is a sample point)

Example: $X(\omega) = numberofheads$, *then* $X(THH) = 2$

Expectation and Linearity:

- Expectation is the mean/average of the r.v.: $E(x) = \sum_a xP(x = a)$
- Linearity:
  - For 2 r.v. x, y on the same probability space: $E(X + Y) = E(X) + E(Y)$
  - For any constant c, $E(cX) = cE(X)$
  - **Note: Linearity of Expectation holds even when the r.v. is NOT Independent**
- $E(XY) = E(X)E(Y)$ *iff* X, Y are independent
- The expected number of fixed points in a random permutation of n items is always 1 regardless of n (e.g. $E(X) = 1$)
- Indicator Variables general form:
  - Let X_i be the indicator of of whether...
  - $X = X_1 + X_2 + ...X_n \Rightarrow E(X) = \sum_{i=1}^{n} E(X_i)$
  - $E(X^2) = E([X_1 + X_2 + ..X_n]^2) = \sum_{i=1}^{n} E(X_i)^2 + \sum_{i \neq j}^{n(n-1)} E(X_iX_j)$
    Explanation: *there are* $n(n-1)$ *unlike terms in the expansion of* $(X_1 + X_2 + ..X_n)(X_1 + X_2 + ..X_n)$*(after selecting* 1 *of the* n *terms in the first set, you have* $n-1$ *other terms to choose from in the second of the pair)*
  - $E(X_i) = E(X_i^2)$ - since you only care about $Pr(X_i = 1) \Rightarrow$
    $E(X_i) = 1 * Pr(X_i = 1) + 0 * Pr(X_i = 0)$

2 Variance and Covariance

Definition: $Var(x) = E((x-\mu)^2) = \sigma^2$, where σ is standard deviation. So $Var(x) = (\sqrt{\sigma})$

- For r.v. with $E(X) = \mu$, $Var(X) = E(X^2) - \mu^2 = E(X^2) - E(X)^2$
  - Proof: $Var(X) = E((x-\mu)^2) = E(X^2 - 2\mu X + \mu^2)$
    $= E(X^2) - 2\mu E(x) + \mu^2$ and substituting $\mu for E(X)$, we get
    $Var(X) = E(X^2) - \mu^2 = E(X^2) - E(X)^2$
- $Var(cX) = c^2 Var(X)$
- Independent r.v.: $P(X = a, Y = b) = P(X = a)P(Y = b)\ \forall a, b$
- Covariance
  - Covariance of $X, Y = E(XY) - E(X)(Y)$, and if X, Y are independent, the covariance is 0
  - $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$ and $Cov(X,Y) = 0$ if X, Y are independent
  - $Cov(a + bX, c + dY) = bdCov(X,Y)$
  - $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$
- Note that the value of $Var(X) \geqslant 0$ so $E(X^2) \geqslant E(X)^2$