CS70 In Simpler Terms - Note 5

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1 Counting

For counting problems, there is nothing that can replace practice. Find practice problems and practice, practice, practice. You will soon gain an intuition that is critical for your continuation into discrete and continuous probability.

- coin flipping: 2^k outcomes
- roll n dice $\Rightarrow 6^n$ choices
- k balls and n bins problem: $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$

2 Combinatorial Proofs

- $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + ...\binom{k}{k}$
  - LHS: Number of ways to choose $k+1$ from n
  - RHS: Order the items.
    (*first term*) Select first item and choose k from the remaining $n-1$
    (*second term*) Select second item and choose k from the remaining $n-2$ ... and so forth
- $\binom{n}{0} + \binom{n}{1} + ...\binom{n}{n} = 2^n$
  - LHS: Sum of ways of choosing i items
  - RHS: Total number of subsets from n items.
- $\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$
  Think of this as triplets: (i, j, k), where $i, j \leqslant k$ and $0 \leqslant k \leqslant n$
  - LHS: k options for i, k options for $j \Rightarrow k^2$
  - RHS: Take the sum of the following 2 cases.
    - case 1: $i = j$ - choose $i, k = \binom{n+1}{2}$
    - case 2: $i \neq j$ - choose $i, j, k = \binom{n+1}{3}$, but i, j can switch so you multiply your result by 2
- $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$
  - LHS: Select a team of a members from n people. Then select 1 captain from the team, and then a co-captain from the remaining $n-a$ people that have not been chosen for the team yet.

- RHS: Pick a captain from n, and then a co-captain from $n-1$. Select the remaining $a-1$ members from $n-2$

3 Probability

Probability is essentially counting, but dividing that by the sample size. If you understand counting, probability will be easy.

- Independent Events: $P(A \cap B) = P(A)P(B) \Rightarrow P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)}$
- Product Rule: $P(A \cap B) = P(A)P(B \mid A)$
- $P(B) = P(B \mid A)P(A) + P(B \mid \overline{A})(1 - P(A))$
- $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)P(A)+P(B|\overline{A})(1-P(A))}$
- $P(A_1 \cup A_2 ... A_n)$ has $2^n - 1$ intersections at most. The -1 is the $\emptyset$
- Principle of Inclusion and Exclusion: Remember to subtract the intersection, otherwise you will double add!
- For Disjoint events (*mutually exclusive*), $P[\bigcup_{i=1}^{n} A_i] = \sum_{i=1}^{n} P[A_i]$. This formula means implies that you can just sum up the probabilities of each individual event if the events are disjoint.
- Union Bound: $P[\bigcup_{i=1}^{n} A_i] \leqslant \sum_{i=1}^{n} P[A_i]$. This effectively states that the sum of the probabilities of events is bounded by the sum of the disjoint probabilities. This is because when events are not disjoint, you must subract overlapping probabilities, so the upper bound for the sum will always be the sum of its dijoint event probabilities.
- Symmetric events: There are 500 red marbles and 500 blue marbles in the bag. THe probability that the first marble is blue $\equiv$ the probability the fifth marble picked is blue $= 1/2$