CS70 In Simpler Terms - Note 8

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1 Geometric Distribution

Problems that deal with the length of time you need to wait until an event happens

- Toss a coin until the first head shows up
- $Pr(X = i) = (1-p)^{i-1}p$, $i = 1, 2, ...$ (i starts from 1 not 0 since you have to have at least 1 trial)
- $E(X) = \sum_{i=1}^{\infty} Pr(X \geqslant i) = \frac{1}{p}$ (tail probability)
  $E(X) = p + (1-p)(1 + E(X)) = p + 1 + E(X) - p - pE(X) \Rightarrow E(X) = \frac{1}{p}$
- $Var(X) = \frac{1-p}{p^2}$
- Geometric Distributions are MEMORYLESS: If you toss a coin 5 times and they all show up tails, $E(X)$ until your first head is still 2
  $E(X|X > n) = n + \frac{1}{p}$
- Coupon Collector's Problem
  - let T_i = number of items you are required to collect until you collect the ith new coupon
  - $T = \sum_{i=1}^{n} T_i$ Once you collect $i-1$ coupons, there are $n-(i-1)$ coupons not seen $\Rightarrow Pr(\text{new coupon}) = \frac{n-i+1}{n}$
  - $E(X_i) = \frac{n}{n-i+1}$
  - $T_i \sim Geom(p) \Rightarrow E(T) = \sum_{i=1}^{n} E(T_i) = \sum_{i=1}^{n} \frac{n}{n-i+1} \approx n(\ln n + \gamma)$

2 Negative Binomial Distribution

Generalized Geometric Distribution that deals with the length of time until k successes

- Let X = total trials
- $P(X = x) = \binom{x-1}{k-1}(p)^k(1-p)^{x-k}$
- Let X_i =number of trials to obtain i successes after observing $i-1$ successes.

- $X = \sum_{i=1}^{k} X_i$,where $X_i \sim Geom(p)$
  $E(X) = \sum_{i=1}^{k} E(X_i) = k * \frac{1}{p} = \frac{k}{p}$

- $Var(X_i) \sim Geom \Rightarrow Var(X) = \sum_{i=1}^{k} Var(X_i) = \frac{k(1-p)}{p^2}$

3 Poisson Distribution

Generally for problems that deal with rare events, and is just a form of the *Binomial distribution:*

- $X \sim Bin(n,p)$ where $n \to \infty, p \to 0 \Rightarrow X \sim Bin(n, \frac{\lambda}{n})$, and $\lambda > 0$
- $E(X) = np = \lambda = constant$
- $Pr(X = i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{\lambda^i e^{-\lambda}}{i!}$ as $n \to \infty$
- $E(X) = \lambda$, $Var(X) = \lambda$

Sum of Poisson Independent r.v.:

- Let $X \sim Poiss(\lambda), Y \sim Poiss(\mu)$
- $X + Y \sim Poiss(\lambda + \mu)$ - It's still Poisson!
- $E(X + Y) = E(X) + E(Y)$

Poisson Splitting:

- $X \sim Poiss(\lambda)$, and Y conditioned on $X = x$ follows $Bin(x, p)$
  $\Rightarrow Y \sim Poiss(\lambda p)$
- Essentially, if X is "thinned" out to Y such that only a fraction $p\lambda$ of the original λ remains, Y is still Poisson!