

CS70 In Simpler Terms - Note 13

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1 Continuous Probability

Continuous probability is not that much different from discrete probability. Now, instead of dealing with the probability of distinct events, we are dealing with the probability of a range of events. This usually requires the use of integration.

There are 2 main functions that you need to be familiar with:

- Cumulative distribution function (CDF) - $F_x(x) = P(X \leq x)$
- Probability density function (PDF) - $f_x(x) = \frac{dF_x(x)}{dx}$
 - The PDF f must be nonnegative and $\int_{-\infty}^{\infty} f(x) dx = 1$
 - **Note:** Density \neq probability. This implies that the density at any specific point can be greater than 1. So don't be surprised!

Recall that in discrete probability, the way to calculate the expectation of a r.v. is to take the summation of the probability of an event multiplied by the outcome of that event. Similarly in continuous probability, replace the summation with an integral:

- $E(x) = \int_{-\infty}^{\infty} x f(x) dx$
- $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

Tail Sum is an important trick for calculating the expectation of nonnegative r.v.:

- $E(x) = \int_0^{\infty} (1 - F_x(x)) dx = \int_0^{\infty} Pr(X > x) dx$

2 Joint Distribution

- $Pr(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x, y) dx dy$
 - $f(x, y) = f_x(x)f_y(y)$
 - f must be nonnegative, and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$
- To obtain the marginal distribution of x :
 - $f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x, y) dy$ (integrate over all values of y)

- **Independent r.v.:**

- $Pr(a \leq X \leq b, c \leq Y \leq d) = Pr(a \leq X \leq b) * Pr(c \leq Y \leq d)$
- Joint density of i.r.v. = product of marginal densities:
 $f(x, y) = f_x(x) * f_y(y)$

For the following distributions, I will not go through the derivations, but rather just list out the results, which you should be really familiar with.

3 Distributions and their Properties

- **Uniform Distribution**

- $Unif[0, 1]: E(X) = \frac{1}{2}, Var(X) = \frac{1}{12}$
- $Unif[a, b]: E(X) = \frac{a+b}{2}, Var(X) = \frac{1}{12}(b-a)^2$

- **Exponential Distribution**

Analagous to the geometric distribution - *memoryless*

The parameter $\lambda > 0$

- $f(x) = \lambda e^{-\lambda x}, x \geq 0$
- $F(x) = 1 - e^{-\lambda x} = Pr(X \leq x)$
- $E(X) = \frac{1}{\lambda}, Var(X) = \frac{1}{\lambda^2}$
- **Important result:** $Pr(X > t) = \int_t^{\infty} \lambda e^{-\lambda x} dx$

Examples:

- Find the median of an exponential distribution:
 - $Pr(T \leq t) = \frac{1}{2} = Pr(T \geq t)$
 - $1 - e^{-\lambda t} = \frac{1}{2} \Rightarrow \frac{1}{2} = e^{-\lambda t} \Rightarrow t = \frac{\ln 2}{\lambda}$
- $Pr(\min(X, Y) = X)$
 $= Pr(Y > X) = \int_0^{\infty} Pr(Y > y | X = x) f_x(x) dx$
- Let $W = \max\{X, Y\}$
 - $F_w(w) = Pr(W \leq w) = Pr(X \leq w) * Pr(Y \leq w) = F_x(w) * F_y(w)$
- Let $V = \min\{X, Y\}$
 - $1 - F_v(v) = Pr(V > v) = Pr(X > v) * Pr(Y > v) = (1 - F_x(v)) * (1 - F_y(v))$
- Let $X_i \sim Exp(\lambda)$
 - $Z = \min\{X_1, \dots, X_n\} \sim Exp(\lambda_1 + \dots, \lambda_n) \sim Exp(n\lambda)$
 - The sum is still exponential!
 - $E(Z) = \frac{1}{n\lambda}$

- **Normal/Gaussian Distribution**

- $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$

- $\mathbb{P}[X \leq a] = \mathbb{P}\left[Y \leq \frac{a-\mu}{\sigma}\right]$, where Y is standard normal
a.k.a. $Y = \frac{X-\mu}{\sigma}$
 - * From the above, we get $X = \sigma Y + \mu$
 - * $Pr(a \leq Y \leq b) = Pr(\sigma a + \mu \leq X \leq \sigma b + \mu)$
- Standard normal = $\mu = 0, \sigma = 1$
- 68 – 95 – 99.7 Rule: 68% of events in a normal distribution are within 1 s.d. (± 1), 95% of events are within 2 s.d. (± 2), and 99.7% are within 3 s.d..
- Let $Z = aX + bY$, where $X \sim (\mu_x, \sigma_x^2), Y \sim (\mu_y, \sigma_y^2)$
 - * $\mu_z = a\mu_x + b\mu_y$
 - * $\sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2$