USAMTS 2019 -2020 Set 1

Kevin Lu

September 2019

Section 1

Section 2

Lemma 1 (Particular Case of the Rearrangement Inequality). *If $a_1 \geq a_2 \geq a_3$ and $b_1 \leq b_2 \leq b_3$ then $a_1b_1 + a_2b_2 + a_3b_3 \leq a_{1b2} + a_2b_3 + a_3b_1$ with equality iff ($a_1 = a_2$ or $b_1 = b_2$) and ($a_1 = a_3$ or $b_2 = b_3$).*

Proof. The inequality case is trivialized by the rearrangement inequality but the equality case may require a bit more work. Since $(a_1 - a_2)(b_1 - b_2) \leq 0 \implies a_1b_1 + a_2b_2 + a_3b_3 \leq a_1b_2 + a_2b_1 + a_3b_3$ with equality when $a_1 = a_2$ or $b_1 = b_2$. Since $(a_1 - a_3)(b_3 - b_2) \geq 0 \implies a_1b_2 + a_2b_1 + a_3b_3 \leq a_1b_3 + a_2b_1 + a_3b_2$ with equality when $a_1 = a_3$ or $b_2 = b_3$. Thus, we have $a_1b_1 + a_2b_2 + a_3b_3 \leq a_1b_2 + a_2b_1 + a_3b_3 \leq a_1b_3 + a_2b_1 + a_3b_2$ with equality iff ($a_1 = a_2$ or $b_1 = b_2$) and ($a_1 = a_3$ or $b_2 = b_3$). $\square$

Rearranging the given yields the following

$$\begin{aligned}\sqrt[y]{z} &= \sqrt[x]{x}\\ \sqrt[z]{x} &= \sqrt[y]{y}\\ \sqrt[x]{y} &= \sqrt[z]{z}\end{aligned}$$

Multiplying the three yields

$$\sqrt[z]{x}\,\sqrt[x]{y}\,\sqrt[y]{z} = \sqrt[x]{x}\,\sqrt[y]{y}\,\sqrt[z]{z} \tag{1}$$

WLOG let $x \geq y \geq z$. Lemma 1 with $\log x \geq \log y \geq \log z$ and $\frac{1}{x} \leq \frac{1}{y} \leq \frac{1}{z}$ yields

$$\begin{aligned}\frac{1}{x}\log x + \frac{1}{y}\log y + \frac{1}{z}\log z &\leq \frac{1}{z}\log x + \frac{1}{x}\log y + \frac{1}{y}\log z\\ log(\sqrt[z]{x}\,\sqrt[x]{y}\,\sqrt[y]{z}) &\leq \log(\sqrt[x]{x}\,\sqrt[y]{y}\,\sqrt[z]{z})\\ \sqrt[z]{x}\,\sqrt[x]{y}\,\sqrt[y]{z} &\leq \sqrt[x]{x}\,\sqrt[y]{y}\,\sqrt[z]{z}\end{aligned}$$

with equality iff $\frac{1}{x} = \frac{1}{y}$ or $\log x = \log y$ (ignoring the second requirement of the "and" since the first is sufficient), either of which implies $x = y$. Since $x^y = z^x$ and $x, y, z \geq 1$, this implies $x^x = z^x \implies x = z \implies x = y = z$, as required.

Section 3

Let $[X_1 X_2 \cdots X_n]$ be the area of n-gon $X_1 X_2 \cdots X_n$. Let O be the center of ω and X and Y the respective projections of O and P on line $\overline{IE}$. Since $[EOI] = \frac{EI \cdot OX}{2}$, $[EPI] = \frac{EI \cdot PY}{2}$ and $UOE \sim UPY$ by AA with angles $\angle OXU = 90^\circ = \angle UPY$ and $\angle OUE = \angle PUY$,

$$\frac{[EOI]}{[EPI]} = \frac{OX}{PY} = \frac{UO}{UP} = \frac{1}{2} \implies [EOI] = \frac{[EPI]}{2}. \tag{2}$$

Let $\theta = \angle EOU$ and $r = 1/2$ be the radius of ω. We have

$$[EOI] = \frac{EI \cdot OX}{2} = EX \cdot OX = r \sin\theta \cdot r \cos\theta = \frac{r^2 \sin 2\theta}{2} = \frac{\sin 2\theta}{8} \tag{3}$$

Since $\sin 2\theta \leq 1$ with equality when $2\theta = 90^\circ \implies \theta = 45^\circ$, we have $[EPI] = 2[EOI] \leq \frac{1}{4}$, with equality when $\angle EOI = 45^\circ$.

Section 4

Section 5