Polynomial Computation on GPU

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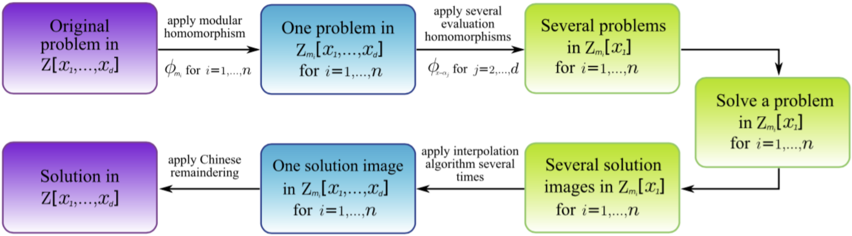
Overview of Process



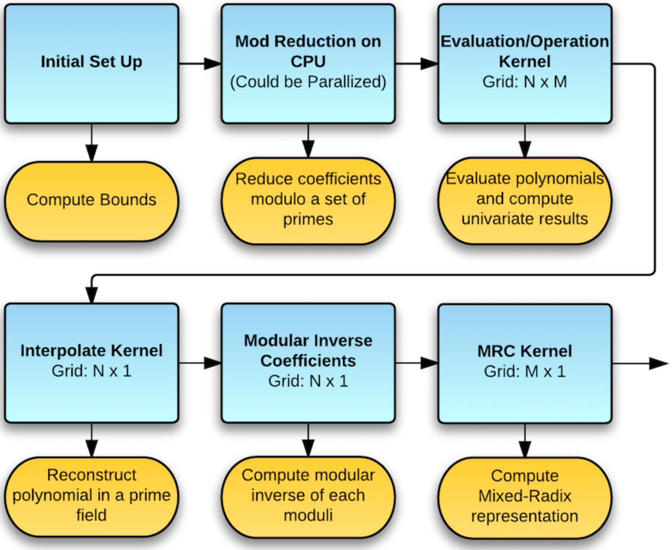


Figure: M: Number of Moduli, N: Number of Evaluation Points

Setup

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Homomorphisms

A Unique Factorization Domain (UFD) is an integral domain with the additional property that any non-zero $a \in D$ is either unit or can be expressed as a finite product of irreducible elements $a = p_1 p_2 \ldots p_n$.

This implies that the $\gcd(a, b)$ always exists.

Integers $\mathbb{Z}$ are a UFD whose only units are 1 and -1.

Closely related is the concept of relatively prime. For any two elements $a, b \in \mathbb{D}$, $\gcd(a, b) = 1$.

Evaluation

Definition 2.4.1. Let $f, g \in \mathbb{D}[x]$ be non-zero polynomials of degrees p, q, respectively. *Sylvester's matrix*[1] S associated with f and g is $(p+q) \times (p+q)$ matrix defined as:

$$S = \begin{bmatrix} f_p & f_{p-1} & \dots & f_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \ddots & & \vdots \\ 0 & \dots & 0 & f_p & f_{p-1} & \dots & f_0 \\ g_q & g_{q-1} & \dots & g_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \ddots & & \vdots \\ 0 & \dots & 0 & g_q & g_{q-1} & \dots & g_0 \end{bmatrix},$$

In general we can reduce our symbolic polynomial computations to computing greatest common denominators (GCDs),resultants and subresultants.

The resultant is an elimination tool that gives an algebraic formulation to decide whether a system of polynomial equations has a common solution expressed in terms of their coefficients.

Interpolation

Let's attempt to compute the intersection of two algebraic surfaces of degree 4.

$$
\begin{aligned}
(10y^2 + z^2 - 112)^2 - 5z^4y^4 + xy^4 - 1 &= 0 \\
50(x^2y^2 + y^2z^2 + x^2z^2) + (x^2 + y^2 + z^2 - 1)^2 &= 0
\end{aligned}
$$

Chinese Remaindering

The intersection is given by an algebraic curve in 3D which when projected onto the x-y plane satisfies the equation:

$$
\begin{aligned}
R(x,y) := {} & 1 + 100x^2 + 736x^2y^2 - 202x^4 - 46700x^4y^2 - 89220x^2y^4 + 100x^6 - 42420y^6 - \\
& 936x^6y^2 + 379384x^2y^6 + 219186x^4y^4 - 10x^8y^4 - 1102310x^4y^8 + 234160x^4y^6 + \\
& 575661x^2y^8 - 480x^6y^4 + 4160x^6y^6 - 100x^6y^8 - 5300x^4y^{10} - 2374640x^2y^{10} + \\
& 2600x^6y^{10} + 67650x^4y^{12} + 25x^8y^8 - 5300y^{12}x^2 + 2600y^{14}x^2 + 160786y^8 - \\
& 1345660y^{12} + 25y^{16} - 100y^{14} + x^8 - 13510xy^{12} - 2xy^4 - 100x^3y^4 + 1020y^{10}x + \\
& 2702x^5y^4 + 1020x^3y^8 - 100xy^6 + 1852xy^8 - 26520y^10x^3 + 100y^2 + 648y^4 + \\
& 361020y^{10} - 13510x^5y^8 + 4264x^3y^6 = 0.
\end{aligned}
$$

Analyzing the topological structure of $R(x, y)$ involves computing the resultant of R and $R'y$, which becomes a dense polynomial of 132 degrees and 500 coefficients.

Increasing the degree slightly, creates a polynomial of degree 2028 and 4500 coefficients.

Problems Encountered

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Extensions

For a more concrete example, consider the skeleton/medial axis computation in 2D. As we know points on the skeleton/medial axis correspond to singular points of the signed distance function. However the SDF is generally not available, so computing the gradient components and setting them to zero is not possible.

Geometric Viewpoint:

- The closest surface point
- the largest circle contained within the closure of the object makes contact with the surface.
- The maximal circle has at least 2 distinct points of 2nd order contact, when ignoring circles that contact the boundary at a singular point.

References

Duane Storti
appliedPolynomialComputation

Pavel Emeliyanenko
Harnessing the Power of GPUs for Problems in Real Algebraic Geometry.
PhD thesis, Universitat des Saarlandes, 2012