MATH680 Homework 4 Fall 2015

This homework is due on Monday, November 30 at 11:59pm. Submit your solutions in a pdf document on MyCourses. Please also submit text files with your R code, which must be commented and properly indented.

1. Please see homework 1 for the definition of the design matrix $X \in \mathbb{R}^{n\times p}$, the vector of measured responses $y \in \mathbb{R}^n$, and the vector of regression coefficients $\beta_* = (\beta_{*1}, \ldots, \beta_{*p})' \in \mathbb{R}^p$. Consider the constrained least squares estimate of β_* defined by

$$\bar\beta \in \arg\min_{\beta\in\mathbb{R}^p} \|y - X\beta\|^2$$
$$\text{subject to } \beta_p \le 0.$$

   (a) Use the KKT optimality conditions to derive a procedure to compute $\bar\beta$ given any $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n\times p}$ with $X'X \in \mathbb{S}^p_+$. (see propositions 1 & 2 in the lecture notes on constrained optimization)

   (b) Set up and generate data from the normal linear regression cases model so that the generated unconstrained least squares estimate of β_{*p} is *positive*. Use the formulas in part a to compute $\bar\beta$. Report the KKT point and confirm that it satisfies conditions 1–4 in proposition 2 in the lecture notes on constrained optimization.

   (c) Set up and generate data from the normal linear regression cases model so that the generated unconstrained least squares estimate of β_{*p} is *negative*. Use the formulas in part a to compute $\bar\beta$. Report the KKT point and confirm that it satisfies conditions 1–4 in proposition 2 in the lecture notes on constrained optimization.

2. Create a rejection sampling algorithm to generate a realization of a random variable with the Gamma(α, β) distribution, where $\alpha > 1$. The probability density function f is defined by

$$f(x) = \frac{\beta^\alpha x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)} 1(x > 0).$$

   Be sure to use a different trial/proposal density than we used in lecture.

   (a) Write down your algorithm

   (b) Create an R function that generates a realization of $X_1, \ldots, X_n$ that are iid Gamma(α, β) distribution, where $\alpha > 1$. The only random number generator allowed is the `runif()` function.

   (c) Test your function with $n = 1000$ for a few values of α, including one large value. Report estimated acceptance probabilities.

3. Suppose that we can see a realization of the random variable X where $(X|\Theta = \theta) \sim N(\theta, 1)$ and $\Theta \sim$ Cauchy(μ, σ), where $\mu \in \mathbb{R}$ and $\sigma > 0$. The density for Cauchy(μ, σ) is defined by

$$p(\theta) = \frac{1}{\pi\sigma\left\{1 + (\frac{\theta-\mu}{\sigma})^2\right\}}.$$

   Suppose we want to sample from the posterior distribution $(\Theta|X = x)$ and we will set $\mu = 0$ and $\sigma = 1$ in our prior distribution.

(a) Write down an expression for $f(\theta|x)$, the posterior density evaluated at θ.

(b) Create a rejection sampling algorithm to generate samples from $(\Theta|X = x)$ using a Cauchy(m, s) trail density. Select m and s to roughly maximize acceptance rates.

(c) Create an R function for your algorithm in part b that generates n independent draws from $(\Theta|X = x)$. Use this to give a simulated estimate of $E(\Theta|X = x)$ for some value that you choose for x.

(d) Use the same trial density as in part b, create an importance sampling algorithm to provide a simulated estimate of $E(\Theta|X = x)$.

(e) Write an R function for the algorithm in part d and compare its performance to the method used in part c with a brief simulation study.

4. Suppose that f is a target density and g is a trial density such that $f(x) \leq cg(x)$ for some $c < \infty$. We plan to perform $n \geq 1$ iterations of the rejection sampling algorithm that draws from f. Let $X_1, \ldots, X_T$ be the accepted subsample and $R_1, \ldots, R_{n-T}$ be the rejected subsample. Let $X \sim f$ and consider the following two estimators of $E\{h(X)\}$:

$$\hat{E}_1 = \frac{1}{T}\sum_{i=1}^{T} h(X_i), \quad \hat{E}_2 = \frac{1}{n-T}\sum_{i=1}^{n-T} h(R_i)\frac{(c-1)f(R_i)}{cg(R_i) - f(R_i)}.$$

(a) Show that $\hat{E}_1$ and $\hat{E}_2$ are independent and unbiased estimators of $E\{h(X)\}$.

(b) Let $\hat{E}_3(b) = b\hat{E}_1 + (1-b)\hat{E}_2$, where b is a constant. Compute

$$\hat{b} = \underset{b\in[0,1]}{\arg\min}\, \mathrm{var}\{\hat{E}_3(b)\}.$$