

STABLE FUNCTIONS AND FØLNER'S THEOREM

GABRIEL CONANT

ABSTRACT. We show that if G is an amenable group and $A \subseteq G$ has positive upper Banach density, then there is an identity neighborhood B in the Bohr topology on G that is almost contained in AA^{-1} in the sense that $B \setminus AA^{-1}$ has upper Banach density 0. This generalizes the abelian case (due to Følner) and the countable case (due to Beiglböck, Bergelson, and Fish). The proof is indirectly based on local stable group theory in continuous logic. The main ingredients are Grothendieck's double-limit characterization of relatively weakly compact sets in spaces of continuous functions, along with results of Ellis and Nerurkar on the topological dynamics of weakly almost periodic flows.

1. INTRODUCTION

Let G be a (discrete) group. Following [1], a subset B of G is called a *Bohr set* if it contains a set of the form $\tau^{-1}(U)$ where $\tau: G \rightarrow K$ is a homomorphism to a compact Hausdorff group with dense image, and $U \subseteq K$ is open and nonempty. If, moreover, U contains the identity of K , then B is called a *Bohr₀* set.

Recall that G is *amenable* if there is a left-invariant finitely additive probability measure on the Boolean algebra of subsets of G . We write $\mathfrak{M}_\ell(G)$ for the collection of such measures (so G is amenable if and only if $\mathfrak{M}_\ell(G) \neq \emptyset$). If G is amenable then any set $A \subseteq G$ has an *upper Banach density*, which can be defined as

$$\text{BD}(A) = \sup\{\mu(A) : \mu \in \mathfrak{M}_\ell(G)\}.$$

(This differs from the standard definition of Banach density in terms of *Følner sequences/nets*. See Section 4.1 for details and discussion.)

A 1954 result of Følner [11, 12] shows that if G is abelian (hence amenable) and $A \subseteq G$ has positive upper Banach density, then there is a Bohr₀ set B in G that is *almost* contained in $A - A$ in the sense that $\text{BD}(B \setminus (A - A)) = 0$. This bears some analogy to Steinhaus's Theorem [28], which says that if $A \subseteq \mathbb{R}$ has positive Lebesgue measure, then $A - A$ contains an open interval around 0. See Section 4.2 for further discussion on this comparison.

In 2010, Beiglböck, Bergelson, and Fish [1] proved the analogue of Følner's result for any countable amenable group (but with a Bohr set rather than a Bohr₀ set; see [1, Corollary 5.3]). This result was actually a step along the way to their generalization of Jin's Theorem [21] to countable amenable groups (see Section 4.3). The methods of [1] leverage countability of G in order to use certain tools from ergodic theory. In this short paper, we give a proof of Følner's Theorem for arbitrary amenable groups using (relatively) classical results from functional

Date: June 17, 2025.

2020 *Mathematics Subject Classification.* Primary 43A07, 05B10, 37B05, 43A60; Secondary 03C45, 03C98.

Partially supported by NSF grant DMS-2452816.

analysis together with 1980’s topological dynamics. In particular, we prove the following theorem.

Theorem 1.1. *Let G be an amenable group and suppose $A \subseteq G$ has positive upper Banach density. Then there is a Bohr₀ set $B \subseteq G$ such that $\text{BD}(B \setminus AA^{-1}) = 0$.*

The main tools for our proof will be Grothendieck’s [17] characterization of relatively weakly compact sets in Banach spaces of continuous functions (Theorem 2.1), and structural results of Ellis and Nerurkar [10] on the topological dynamics of weakly almost periodic flows (Theorem 2.5). This draws a fundamental connection to the work of Følner [11, 12], and of Bieglböck, Bergelson, and Fish [1], where almost periodic functions play a central role.

Other than the tools from [17] and [10] mentioned above, the proof of Theorem 1.1 will be almost entirely based on elementary topological arguments. That being said, the overall strategy of the proof is heavily inspired by recent work of the author and Pillay [6] on the structure of “stable functions” on groups. For this reason, the argument has underpinnings in continuous model theory, even though this will not appear explicitly in the proof. In Section 4.4, we will provide further details on this connection for readers versed in model theory (in fact, these readers may benefit from reading this section before the proof of Theorem 1.1).

From the model-theoretic perspective, Theorem 1.1 can be seen as an application of local stable group theory in continuous logic. This makes it worthwhile to point out that Hrushovski’s Stabilizer Theorem [20] directly implies a weaker version of Theorem 1.1 in which $B \setminus AA^{-1}$ only has *lower* Banach density 0 (which is equivalent to saying AA^{-1} is “piecewise Bohr₀” in the sense of [1]). For details, see the proof of [5, Theorem 5.6]. The improvement here to upper Banach density 0 will result from “unique ergodicity” of weakly almost periodic flows (proved by Ellis and Nerurkar [10]), which a model theorist will recognize as the fact that a left-invariant stable formula in a first-order expansion of a group (in continuous logic) admits a unique left-invariant Keisler measure/functional. It could be interesting to investigate what further bearing unique ergodicity has on the Stabilizer Theorem itself.

The paper is organized as follows. In Section 2 we review preliminaries from functional analysis and topological dynamics needed for Theorem 1.1. The proof of Theorem 1.1 then follows in Section 3. Section 4 elaborates at some length on several points of discussion initiated above.

2. PRELIMINARIES

Given a topological space S , we let $\mathcal{C}_b(S)$ denote the space of bounded continuous complex-valued functions on S . Recall that $\mathcal{C}_b(S)$ is a Banach space with the sup norm, and in fact a unital C^* -algebra with the involution operation from complex conjugation. If S is compact then we omit the subscript b and write $\mathcal{C}(S)$. The *weak topology* on $\mathcal{C}_b(S)$ is the weakest topology for which all bounded linear functionals on $\mathcal{C}_b(S)$ are continuous. The following is [17, Théorème 6].

Theorem 2.1 (Grothendieck). *Let S be a topological space with a fixed dense subset $S_0 \subseteq S$. Then a set $X \subseteq \mathcal{C}_b(S)$ is relatively weakly compact if and only if X is uniformly bounded and, for any sequences $(f_i)_{i=0}^\infty$ from X and $(x_j)_{j=0}^\infty$ from S_0 ,*

$$\lim_i \lim_j f_i(x_j) = \lim_j \lim_i f_i(x_j)$$

whenever both limits exist.

Using Banach-Alaoglu and basic facts about Hilbert spaces, one obtains the following well-known consequence of the previous theorem.

Corollary 2.2. *Let H be a Hilbert space. Then for any sequences $(x_i)_{i=0}^{\infty}$ and $(y_j)_{j=0}^{\infty}$ from the unit ball of H ,*

$$\lim_i \lim_j \langle x_i, y_j \rangle = \lim_j \lim_i \langle x_i, y_j \rangle$$

whenever both limits exist.

Now let G be a (discrete) group. A G -flow is a compact Hausdorff space S together with a (left) action of G by homeomorphisms. In this case, a *subflow* of S is a nonempty closed G -invariant subset of S . A subflow of S is *minimal* if it does not properly contain another subflow of S .

Definition 2.3. Let S be a G -flow. Given $g \in G$, we use \check{g} to denote the action map by g on S . The **Ellis semigroup** of S , denoted $E(S)$, is the closure of the set $\{\check{g} : g \in G\}$ in the space of functions S^S (with the product topology).

Given a G -flow S , it is straightforward to check that $E(S)$ is a G -flow under the action $gp := \check{g} \circ p$, and a semigroup under composition of functions. Moreover, under the induced topology, $E(S)$ is a *right topological semigroup* (i.e., multiplication on the right by any fixed point in $E(S)$ is continuous). For more on the basic theory of Ellis semigroups, see [9].

Note that if S is a G -flow, then we have a natural action of G on $C(S)$ via $g\varphi(x) = \varphi(g^{-1}x)$.

Definition 2.4. Let S be a G -flow. Then a function in $C(S)$ is **weakly almost periodic** if its G -orbit is relatively weakly compact in $C(S)$. The G -flow S is called **weakly almost periodic** if every function in $C(S)$ is weakly almost periodic.

We now summarize several results from [10] on weakly almost periodic flows. It is worth noting that Theorem 2.1 is a key ingredient in the proofs of these results.

Theorem 2.5 (Ellis & Nerurkar). *Suppose S is a weakly almost periodic G -flow.*

- (a) *$E(S)$ has a unique minimal subflow K . Moreover, K is the unique left ideal in $E(S)$ and a compact group (under the semigroup operation).*
- (b) *The identity in K commutes with every element of $E(S)$.*
- (c) *$E(S)$ admits a unique G -equivariant regular Borel probability measure.*

Parts (a) and (b) are contained in [10, Proposition 11.5] (though we caution the reader that Ellis and Nerurkar use right actions rather than left). Part (c) follows from [10, Proposition 11.10], which actually says that if S has a unique minimal subflow then S admits a unique G -invariant regular Borel probability measure. So we are really applying this statement to the G -flow $E(S)$, which is also weakly almost periodic (e.g., this follows from [10, Proposition II.2] and the basic exercise that $E(S)$ and $E(E(S))$ are isomorphic).

Finally, if S is a G -flow then a subset $X \subseteq S$ is called **generic** if $S = FX$ for some finite $F \subseteq G$. A point $x \in S$ is called **generic** if every open set containing x is generic. The next fact is a straightforward topology exercise (see also Lemma 1.7 and Corollary 1.9 in [24]).

Fact 2.6. *Let S be a G -flow. The following are equivalent.*

- (i) *S has a unique minimal subflow.*
- (ii) *Every generic open subset of S contains a generic point.*
- (iii) *The set of generic points in S is the unique minimal subflow of S .*

3. PROOF OF THEOREM 1.1

Let G be an amenable group. We will make occasional use of the space βG of ultrafilters on G , i.e., the Stone-Čech compactification of G (as a discrete space). Given $X \subseteq G$, let $[X]$ denote the corresponding basic clopen set in βG consisting of the ultrafilters that contain X . Note that βG is a G -flow under the natural action. Recall also that any $\mu \in \mathfrak{M}_\ell(G)$ uniquely determines a G -invariant regular Borel probability measure μ on βG with the property that $\mu(X) = \mu([X])$ for any $X \subseteq G$.

Now fix a set $A \subseteq G$ with $\text{BD}(A) > 0$. We want to find a Bohr₀ set $B \subseteq G$ such that $\text{BD}(B \setminus AA^{-1}) = 0$. The proof proceeds in two main steps.

Step 1. In this step, we construct the group compactification $\tau: G \rightarrow K$ that will be used (in step 2) to obtain the desired Bohr₀ set B .

Since $\text{BD}(A) > 0$, there is a measure $\mu \in \mathfrak{M}_\ell(G)$ such that $\mu(A) > 0$. Define the function $\varphi: G \rightarrow [0, 1]$ so that $\varphi(x) = \mu(A \cap xA)$.

Note that $[0, 1]^G$ is a G -flow under the action $g\psi(x) = \psi(g^{-1}x)$ where $g \in G$ and $\psi: G \rightarrow [0, 1]$. Let S be the G -flow given by the orbit closure of φ in $[0, 1]^G$.

Claim 3.1. S is weakly almost periodic.

Proof. Define $\tilde{\varphi}: S \rightarrow [0, 1]$ so that $\tilde{\varphi}(\psi) = \psi(1)$. Then $\tilde{\varphi} \in \mathcal{C}(S)$ and the G -orbit of $\tilde{\varphi}$ in $\mathcal{C}(S)$ separates points in S . So by Stone-Weierstrass, $\mathcal{C}(S)$ is the (uniform) closure of the unital C^* -algebra generated by the orbit of $\tilde{\varphi}$. Also, the collection of weakly almost periodic functions in $\mathcal{C}(S)$ is a closed G -invariant unital C^* -subalgebra (see, e.g., [26, Theorem 1.2]). Hence it suffices to show that $\tilde{\varphi}$ is weakly almost periodic. Toward a contradiction, suppose $\tilde{\varphi}$ is not weakly almost periodic. We apply Theorem 2.1 to S , taking X to be the G -orbit of $\tilde{\varphi}$, and S_0 to be the G -orbit of φ . This yields sequences $(a_i)_{i=0}^\infty$ and $(b_j)_{j=0}^\infty$ from G such that $\lim_i \lim_j \varphi(a_i^{-1}b_j) \neq \lim_j \lim_i \varphi(a_i^{-1}b_j)$ (and both limits exist).

Let H be the Hilbert space $L^2_{\mathbb{C}}(\beta G, \mu)$ with inner product $\langle x, y \rangle = \int x\bar{y} d\mu$. For $i, j \geq 0$, let $x_i = \mathbf{1}_{[a_i A]}$ and $y_j = \mathbf{1}_{[b_j A]}$. Then x_i, y_j are in the unit ball of H , and

$$\varphi(a_i^{-1}b_j) = \mu(a_i A \cap b_j A) = \langle x_i, y_j \rangle.$$

So $\lim_i \lim_j \langle x_i, y_j \rangle \neq \lim_j \lim_i \langle x_i, y_j \rangle$, which contradicts Corollary 2.2. \square

Let $E = E(S)$ be the Ellis semigroup of S . By Claim 3.1 and Theorem 2.5(a), E has a unique minimal subflow K , which is also the unique left ideal in E and a compact group. Let u be the identity in K . Since K is a left ideal, we can define a map $\tau_*: E \rightarrow K$ so that $\tau_*(p) = p \circ u$.

Claim 3.2. τ_* is a continuous surjective semigroup homomorphism. Moreover, if $g \in G$ and $p \in E$ then $\tau_*(gp) = g\tau_*(p)$.

Proof. Note that τ_* is continuous since E is a right topological semigroup, and surjective since τ_* restricts to the identity on K . Also, given $p, q \in E$, we have

$$\tau_*(p \circ q) = p \circ q \circ u = p \circ q \circ u \circ u = p \circ u \circ q \circ u = \tau_*(p) \circ \tau_*(q),$$

where the third equality uses Theorem 2.5(b). Finally, given $g \in G$ and $p \in E$, we have $\tau_*(gp) = gp \circ u = \check{g} \circ p \circ u = \check{g} \circ \tau_*(p) = g\tau_*(p)$. \square

Now define $\tau: G \rightarrow K$ so that $\tau(g) = gu$. Then for $g \in G$, we have $\tau(g) = \tau_*(\check{g})$ (recall Definition 2.3). This allows us to view τ_* as an “extension” of τ factoring through $g \mapsto \check{g}$. In particular, it follows from Claim 3.2 that τ is a group

homomorphism. The image of τ is the G -orbit of u , which is dense in K since K is a minimal subflow of E . Altogether, $\tau: G \rightarrow K$ is a group compactification of G .

Step 2. In this step, we will use the compactification τ (from step 1) to obtain a Bohr₀ set B in G satisfying $\text{BD}(B \setminus AA^{-1}) = 0$.

Let \mathcal{U} be the collection of open identity neighborhoods of u in K . Given $U \in \mathcal{U}$, we write \overline{U} for the closure of U in K . The next claim is a standard exercise, but we include the proof since it is not typically presented in this form.

Claim 3.3. *If $U \in \mathcal{U}$ then $\tau_*^{-1}(U)$ is generic in E .*

Proof. Fix $U \in \mathcal{U}$. We first show $K = GU$. Fix $x \in K$. Then Ux^{-1} is a nonempty open set and hence contains gu for some $g \in G$. So $gu = vx^{-1}$ for some $v \in U$. So $gux = v$, i.e., $gx = v$, i.e., $x = g^{-1}v \in GU$.

Since K is compact, there is some finite $F \subseteq G$ such that $K = FU$. Fix $p \in E$. Then $\tau_*(p) = gv$ for some $g \in F$ and $v \in U$. So $\tau_*(g^{-1}p) = g^{-1}\tau_*(p) = v \in U$ (recall Claim 3.2), hence $g^{-1}p \in \tau_*^{-1}(U)$, i.e., $p \in g\tau_*^{-1}(U)$. This shows $E = F\tau_*^{-1}(U)$. \square

We now “extend” φ to $\hat{\varphi}: E \rightarrow [0, 1]$ by setting $\hat{\varphi}(p) = p(\varphi)(1)$. Note that $\hat{\varphi}$ is continuous and if $g \in G$ then $\hat{\varphi}(g) = \varphi(g^{-1}) = \varphi(g)$.

Claim 3.4. $\hat{\varphi}(u) > 0$.

Proof. More specifically, we show $\hat{\varphi}(u) \geq \alpha := \frac{1}{2}\mu(A)^2$. (This bound can be improved; see Section 4.5.) Let $\epsilon < \alpha$ be arbitrary. Define $X = \{p \in E : \hat{\varphi}(p) > \epsilon\}$, and note that X is open in E .

Subclaim: If $U \in \mathcal{U}$ then $X \cap \tau_*^{-1}(U)$ is generic in E .

Proof. Call a subset $F \subseteq G$ *separated* if $\mu(gA \cap hA) \leq \alpha$ for all distinct $g, h \in F$. Using a straightforward estimate based on inclusion-exclusion, one may show that any separated subset of G is finite. (Details are spelled out in the proof of Claim 1 in [6, Theorem 5.1].)

Now fix $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $\overline{V^{-1}V} \subseteq U$. Let F be a maximal separated subset of $\tau^{-1}(V)$. As noted above, F is finite. We will show that $\tau_*^{-1}(V)$ is contained in $F(X \cap \tau_*^{-1}(U))$, and hence $X \cap \tau_*^{-1}(U)$ is generic by Claim 3.3.

Define $C = \{p \in \tau_*^{-1}(\overline{V^{-1}V}) : \hat{\varphi}(p) \geq \alpha\}$, which is closed in E . Then $C \subseteq X \cap \tau_*^{-1}(U)$ by choice of ϵ and V . So it suffices to show $\tau_*^{-1}(V) \subseteq FC$. Suppose this fails. Then $\tau_*^{-1}(V) \setminus FC$ is a nonempty open set in E and hence contains \check{a} for some $a \in G$. Since $\tau(a) = \tau_*(\check{a}) \in V$, we have $a \in \tau^{-1}(V)$. By choice of F , there is some $g \in F$ such that $\mu(gA \cap aA) > \alpha$, i.e., $\varphi(g^{-1}a) > \alpha$. Recall also that F is contained in $\tau^{-1}(V)$, so $\tau(g) \in V$. Therefore $\tau(g^{-1}a) = \tau(g)^{-1}\tau(a) \in V^{-1}V$. Since $\varphi(g^{-1}a) = \hat{\varphi}(g^{-1}\check{a})$ and $\tau(g^{-1}a) = \tau_*(g^{-1}\check{a})$, it follows that $g^{-1}\check{a} \in C$. So $\check{a} \in gC \subseteq FC$, which contradicts the choice of \check{a} . \dashv_{subclaim}

By the subclaim and Fact 2.6, we have that for any $U \in \mathcal{U}$, $K \cap X \cap \tau_*^{-1}(U)$ is nonempty. Set $Y = \{p \in E : \hat{\varphi}(p) \geq \epsilon\}$ and let $\mathcal{C} = \{K \cap Y \cap \tau_*^{-1}(\overline{U}) : U \in \mathcal{U}\}$. Then \mathcal{C} is a collection of closed subsets of E with the finite intersection property, so by compactness there is some $p \in K \cap Y$ such that $\tau_*(p) \in \overline{U}$ for all $U \in \mathcal{U}$. So $\tau_*(p) = u$, i.e., $p \circ u = u$. But also $p \circ u = p$ since $p \in K$ and u is the identity in K . So $p = u$. Therefore $u \in Y$, i.e., $\hat{\varphi}(u) \geq \epsilon$. \square

Now, since $\hat{\varphi}|_K: K \rightarrow [0, 1]$ is continuous and $\hat{\varphi}(u) > 0$, there is some $U \in \mathcal{U}$ such that $\hat{\varphi}(x) > 0$ for all $x \in \overline{U}$. Let B be the Bohr₀ set $\tau^{-1}(U)$ in G .

Claim 3.5. $\text{BD}(B \setminus AA^{-1}) = 0$.

Proof. Let $\nu \in \mathfrak{M}_\ell(G)$ be arbitrary. We will show $\nu(B \setminus AA^{-1}) = 0$. The first step is to push ν from βG to a G -equivariant regular Borel probability measure on E .

Recall that E is a compactification of G via the map $g \mapsto \check{g}$. So by the universal property of βG , there is a (unique) continuous function $\rho: \beta G \rightarrow E$ such that for $g \in G$, if $[g] \in \beta G$ denotes the principal ultrafilter on g , then $\check{g} = \rho([g])$. Now let ν^* be pushforward of ν along ρ , i.e., $\nu^*(X) = \nu(\rho^{-1}(X))$ for any Borel $X \subseteq E$. We need to verify ν^* is G -invariant. Since ν is G -equivariant, it suffices to show ρ is G -equivariant. Since G acts by homeomorphisms on βG and E , and the principal ultrafilters are dense in βG , it suffices to check that ρ is G -equivariant on principal ultrafilters. For this, note that if $g, a \in G$ then $\rho(g[a]) = \rho([ga]) = \check{ga} = g\rho([a])$.

Next consider the Haar measure η on K . Recall that K is a closed subset of E . So η determines a regular Borel probability measure η^* on E supported on K , i.e., $\eta^*(X) = \eta(X \cap K)$ for any Borel $X \subseteq E$. We claim that η^* is G -equivariant. Since η is invariant under the group operation in K , it suffices to fix a Borel set $X \subseteq E$ and show that $gX \cap K = gu \circ (X \cap K)$ for any $g \in G$. But this follows from the fact K is a subgroup and $gp = gu \circ p$ for any $g \in G$ and $p \in K$.

Now, by Theorem 2.5(c), we have $\nu^* = \eta^*$. Let $D = \{p \in \tau_*^{-1}(\overline{U}): \hat{\varphi}(p) = 0\}$, which is closed in E . By choice of \overline{U} , and since τ_* restricts to the identity on K , we have $D \cap K = \emptyset$. So by definition of η^* , we have $\eta^*(D) = \eta(D \cap K) = 0$. Thus $\nu^*(D) = 0$ since $\nu^* = \eta^*$. By definition of ν^* , this yields $\nu(\rho^{-1}(D)) = 0$.

Finally, consider the clopen set $C = [B \setminus AA^{-1}]$ in βG . We want to show $\nu(C) = 0$. By the above, it suffices to show $C \subseteq \rho^{-1}(D)$. Suppose this fails. Then $C \setminus \rho^{-1}(D)$ is a nonempty open set in βG , hence contains $[g]$ for some $g \in G$. So $g \in B \setminus AA^{-1}$. Since $g \notin AA^{-1}$, we have $A \cap gA = \emptyset$, and so $\hat{\varphi}(\check{g}) = \varphi(g) = \mu(A \cap gA) = 0$. Since $g \in B$, we have $\tau_*(\check{g}) = \tau(g) \in U$. So $\check{g} \in D$, i.e., $[g] \in \rho^{-1}(D)$, contradicting the choice of g . \square

4. DISCUSSION

4.1. The definition of Banach density. Recall that for an amenable group G and a set $A \subseteq G$, we defined $\text{BD}(A) = \sup\{\mu(A) : \mu \in \mathfrak{M}_\ell(G)\}$, which differs from the standard definition using Følner nets (e.g., [18, Section 2]). The equivalence between these definitions goes back to a result of Chou [4] on locally compact σ -compact amenable topological groups, which was extended to all locally compact amenable topological groups by Paterson [25, Theorem 4.17]. In [19], Hindman and Strauss prove analogous results for discrete amenable semigroups, and explicitly derive the above identity for Banach density (see Theorems 2.12 and 2.14 in [19]).

On the other hand, note that our proof of Theorem 1.1 uses only the inequality $\text{BD}(A) \leq \sup\{\mu(A) : \mu \in \mathfrak{M}_\ell(G)\}$, which is a comparatively much easier exercise involving ultralimits of normalized counting measures obtained from Følner nets.

4.2. Følner and Steinhaus. Recall that Følner's Theorem can be viewed as a discrete analogue of Steinhaus's Theorem [28] on positive Lebesgue measure sets in \mathbb{R} (which was generalized to any locally compact group by Weil [29]). Following this analogy, it is natural to ask if Theorem 1.1 can be proved with the stronger conclusion that B is *entirely* contained in AA^{-1} . This turns out to be false due to a result of Kříž [23] that leads to a counterexample in \mathbb{Z} . The was then relaxed to ask if AA^{-1} contains some Bohr set, rather than a Bohr₀ set (see [3, Section 9.2]).

Even for \mathbb{Z} , this remained open for some time and was eventually answered in the negative by Griesmer [15] (who had previously constructed counterexamples in the case that G is an infinite elementary p -group [13]).

4.3. Følner and Jin. Beiglböck, Bergelson, and Fish [1] prove Følner's Theorem for countable amenable groups as a step along the way in their generalization of Jin's Theorem, which we now discuss. For motivation, recall that a set $X \subseteq G$ is *syndetic* if G can be covered by finitely many left translates of X . If G is amenable and $A \subseteq G$ has positive upper Banach density, then it is a standard exercise to show that AA^{-1} is syndetic. Observe also that if $X \subseteq G$ is syndetic and “almost contained” in $Y \subseteq G$ in the sense that $\text{BD}(X \setminus Y) = 0$, then Y is syndetic. Since Bohr sets are syndetic (c.f., Claim 3.3), we can thus interpret Theorem 1.1 as a strong algebraic explanation for the syndeticity of AA^{-1} when $\text{BD}(A) > 0$.

Jin's Theorem is based on the question of what happens when AA^{-1} is replaced by AA , or even AB (with A, B of positive upper Banach density). Clearly the conclusion of Theorem 1.1 fails in this case (even after the obvious concession of allowing a Bohr set in place of a Bohr_0 set) due to examples like $A = \mathbb{N}$ in \mathbb{Z} where $A + A$ isn't even syndetic. However, Jin [21] proved that if $A, B \subseteq \mathbb{Z}$ both have positive upper Banach density, then $A + B$ is *piecewise syndetic*, i.e., there is a syndetic set $X \subseteq \mathbb{Z}$ which is almost contained in $A + B$ in the weaker sense that $X \setminus (A + B)$ has *lower* Banach density 0. In [2], Bergelson, Furstenberg, and Weiss strengthen this by showing that one can take X to be a Bohr set, i.e., $A + B$ is *piecewise Bohr*. The generalization of this result to all countable amenable groups is the main result of Beiglböck, Bergelson, and Fish [1].

The next question is what happens with Jin's Theorem for an uncountable amenable group. In [7], Di Nasso and Lupini use nonstandard analysis to prove a direct generalization of Jin's Theorem in its original form. In particular, they show that if G is any amenable group and $A, B \subseteq G$ have positive upper Banach density, then AB is piecewise syndetic. However, whether this can be improved to piecewise Bohr in the uncountable case appears to be open. In light of the main result here, is then worthwhile to discuss the strategy in the countable case from [1]. From a high level, their proof consists of three steps:

- (1) [1, Corollary 5.3] Følner's Theorem for a countable amenable group.
- (2) [1, Lemma 5.4] Suppose G is countable amenable, $X \subseteq G$ is piecewise Bohr, and X is *finitely embeddable* in $Y \subseteq G$, i.e., every finite subset of X is contained in some right translate of Y . Then Y is piecewise Bohr.
- (3) [1, Proposition 4.1] Suppose G is countable amenable and $A, B \subseteq G$ have positive upper Banach density. Then there is some $C \subseteq G$ of positive upper Banach density such that CC^{-1} is finitely embeddable in AB . (Thus, since CC^{-1} is piecewise Bohr by (1), AB is piecewise Bohr by (2).)

The generalization of (1) to the uncountable case is our main result (though for these purposes one could use the weaker version of Theorem 1.1 afforded by Hrushovski's Stabilizer Theorem). As for (2), it is not hard to see that the countability assumption is inessential, and the same result holds for uncountable G by redoing the argument from [1] with nets indexed by the finite subsets of G instead of countable sequences. However, the proof of (3) uses tools from ergodic theory relying on countability of G . The extension of such tools to the uncountable setting appears to be an active area of research. For example, recent work of Durcik, Greenfeld,

Iseli, Jammeshan, and Madrid [8] could be one lead for extending the proofs of (1) and (3) in [1] to uncountable groups. It would be very interesting to develop an approach to (3) based instead on model theory and more classical topological dynamics.

4.4. The model-theoretic context. Consider an amenable group G and a subset $A \subseteq G$ with $\text{BD}(A) > 0$, i.e., $\mu(A) > 0$ for some $\mu \in \mathfrak{M}_\ell(G)$. The proof of Theorem 1.1 is based on the first-order structure in continuous logic obtained by expanding the group G with the function $\varphi(x) = \mu(A \cap xA)$ as a $[0, 1]$ -valued predicate. Here G is equipped with the discrete metric. The G -flow S in the proof then corresponds to the local type space $S_\theta(G)$ where $\theta(x, y) := \varphi(y \cdot x)$, and weak almost periodicity of S can be accounted for by stability of $\theta(x, y)$ in the sense of continuous logic. For this particular choice of θ , the role of stability was made explicit by Hrushovski [20, Proposition 2.25]. However, the general connection between stability and Hilbert spaces is a key part of the development of continuous logic going back to work of Krivine and Maurey [22] (and of course Grothendieck [17]).

The Ellis semigroup $E = E(S)$ can also be identified as a local type space, but with respect to the formula $\theta^\sharp(x; y, z) := \theta(x \cdot z, y)$; see [6, Lemma 3.1]. The model-theoretic translation of the work of Ellis and Nerurkar [10] is that E behaves like the type space of a group definable in a stable theory. In particular, E has a semigroup structure and a compact subgroup K consisting of the generic θ^\sharp -types, with principal generic u . Moreover, uniqueness of the G -invariant regular Borel probability measure on E provides the analogue of “stable compact domination” by the generic types in the sense that the measure of any θ^\sharp -formula is the Haar measure of its restriction to K (see [6, Proposition 3.6]). If we had started with a sufficiently saturated structure G , then one could see K as $G/G_{\theta^\sharp}^{00}$ where $G_{\theta^\sharp}^{00}$ is the local connected component constructed in [6, Section 5]. Thus in this case the compactification $\tau: G \rightarrow K$ (mapping g to gu) in the proof would be the canonical quotient map. In any case, τ is “ θ^\sharp -definable” in the model-theoretic sense, and the semigroup homomorphism τ_* from the proof is the unique extension of τ to a continuous map on E (viewed as $S_{\theta^\sharp}(G)$).

This brings us to Claim 3.4 which shows $\hat{\varphi}(u) > 0$ where $\hat{\varphi}: E \rightarrow [0, 1]$ maps p to $p(\varphi)(1)$. Under the model-theoretic translation, the map $p \mapsto p(\varphi)$ from E to S represents restriction from $S_{\theta^\sharp}(G)$ to $S_\theta(G)$, and thus $\hat{\varphi}$ is nothing more than the continuous function on E given by viewing φ as a θ^\sharp -formula. The idea behind the proof of Claim 3.4 is clearer when G is saturated, in which case the proof is essentially showing that for $\epsilon > 0$ small enough, $G_{\theta^\sharp}^{00} \wedge (\varphi(x) \geq \epsilon)$ is a partial generic θ^\sharp -type, hence must contain a (complete) generic θ^\sharp -type, which then must be u since $G_{\theta^\sharp}^{00}$ determines a unique generic. In any case, with Claim 3.4 in hand, we can then fix an open identity neighborhood U in K such that $\hat{\varphi}$ is positive on the closure \overline{U} . We then define $D \subseteq E$ to be the closed set of θ^\sharp -types in $\tau_*^{-1}(\overline{U})$ that are zeroes of $\hat{\varphi}$. Since D contains no generic θ^\sharp -types, it has measure 0 by stable compact domination of K . Finally, we pull back from E to the Stone-Čech compactification βG . By uniqueness of the measure on E , the preimage of D in βG has measure 0 with respect to any G -invariant regular Borel probability measure on βG . Moreover, this preimage contains the clopen set determined by $B \setminus AA^{-1}$ where B is the Bohr₀ set $\tau^{-1}(U)$. Therefore $\text{BD}(B \setminus AA^{-1}) = 0$.

4.5. Sets of popular differences. In forthcoming work (communicated to the author shortly after announcing this paper), Griesmer [16] uses the theory of positive definition functions to prove a generalization of Følner's Theorem which applies to certain “level sets” in arbitrary discrete groups obtained from unitary representations. When specialized to amenable groups and sets of positive upper Banach density, this recovers Theorem 1.1, but with the full difference set AA^{-1} replaced by a “set of popular differences” for A . In particular, let G be an amenable group and fix $A \subseteq G$ and $\mu \in \mathfrak{M}_\epsilon(G)$ with $\mu(A) > 0$. Given $\epsilon > 0$, we define the popular difference set $D_\epsilon^\mu(A) = \{x \in G : \mu(A \cap xA) > \epsilon\}$. Recall that AA^{-1} consists of all $x \in G$ for which $A \cap xA$ is nonempty, and thus $D_\epsilon^\mu(A) \subseteq AA^{-1}$ for any $\epsilon > 0$. Therefore a Bohr₀ set almost contained in $D_\epsilon^\mu(A)$ will also be almost contained in AA^{-1} (in the sense of Theorem 1.1). For details on the significance of popular difference sets in additive combinatorics, see [14, 27, 30].

In light of this forthcoming work, we point out that after some cosmetic changes, our proof of Theorem 1.1 also applies to sets of popular differences. More precisely, in the context of Section 3, one can fix any $\epsilon < \frac{1}{2}\mu(A)^2$ and find a Bohr₀ set B such that $\text{BD}(B \setminus D_\epsilon^\mu(A)) = 0$. Indeed, note that $\hat{\varphi}(u) > \epsilon$ by the proof of Claim 3.4, and thus we can instead choose U such that $\hat{\varphi}(x) > \epsilon$ for all $x \in \overline{U}$. Then, in the proof of Claim 3.5, redefine $D = \{p \in \tau_*^{-1}(\overline{U}) : \hat{\varphi}(p) \geq \epsilon\}$, and the same argument shows $\nu(\rho^{-1}(D)) = 0$. Now redefine $C = [B \setminus D_\epsilon^\mu(A)]$, and similar steps yield $C \subseteq \rho^{-1}(D)$ (using the fact that if $g \notin D_\epsilon^\mu(A)$ then $\hat{\varphi}(g) = \varphi(g) = \mu(A \cap gA) \geq \epsilon$).

In fact, one can strengthen the proof of Claim 3.4 to show $\hat{\varphi}(u) \geq \mu(A)^2$, which allows the above conclusion to work for any $\epsilon < \mu(A)^2$ (this then matches the form of the result communicated by Griesmer [16] in the amenable case). To accomplish this, we modify the first paragraph of the proof of the subclaim as follows. Fix $\alpha < \mu(A)^2$ and call $F \subseteq G$ *separated* if $\mu(gA \cap hA) \leq \alpha$ for all distinct $g, h \in F$. Then we claim that any finite separated subset of G has size at most $(\mu(A) - \alpha)/(\mu(A)^2 - \alpha)$, and hence any separated subset of G is finite. The proof is a standard Cauchy-Schwarz argument. Fix a finite separated set $F = \{g_1, \dots, g_n\}$. Define the function $f = \sum_i \mathbf{1}_{g_i A}$. Then, working with the linear functional induced by μ , we have

$$\begin{aligned} \int f \, d\mu &= \sum_i \mu(g_i A) = n\mu(A), \text{ and} \\ \int f^2 \, d\mu &= \sum_i \mu(g_i A) + \sum_{i \neq j} \mu(g_i A \cap g_j A) \leq n\mu(A) + n(n-1)\alpha. \end{aligned}$$

By Cauchy-Schwarz, $(\int f \, d\mu)^2 \leq \int f^2 \, d\mu$. Therefore $(n\mu(A))^2 \leq n\mu(A) + n(n-1)\alpha$, which simplifies to $n \leq (\mu(A) - \alpha)/(\mu(A)^2 - \alpha)$. With this modification, the proof of Claim 3.4 is now valid for any arbitrary $\alpha < \mu(A)^2$ (rather than $\alpha = \frac{1}{2}\mu(A)^2$).

Acknowledgements. Thanks to John Griesmer for several comments on the literature surrounding Følner's Theorem, and for allowing me to mention his forthcoming work in [16]. Thanks also to the referee for helpful revisions.

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS
CHICAGO

Email address: gconant@uic.edu