Combinatorics Portfolio

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Introduction and the Pigeon Hole Principle

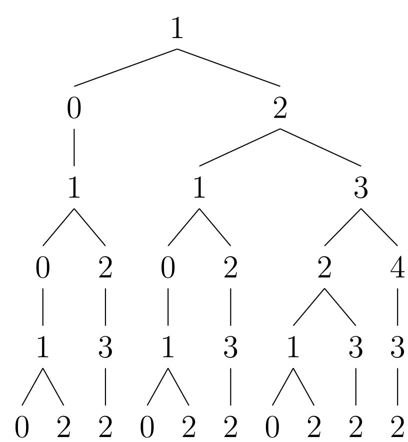
Lecture and Tutorial

Intro to Combinatorics

Jellybean Problem

Julia starts with an empty jar. Each day (for 8 total days) Julia either puts one jelly bean in the jar OR takes out one jelly bean and then records the total number of beans in the jar (to get a "jelly bean sequence" with 8 entries). At the end of the 8 days she has no beans in the jar. For example, 1,2,1,2,3,2,1,0 is a jelly bean sequence. List out all possible jelly bean sequences.

Answer: Since there are only two actions for 8 total days. We know that the first and seventh day needs to be 1 and the last day needs to be 0. Hence we only need to think about days 2-6. Accordingly, I've come up with a tree. Where the root of the tree is the 2nd day and each subsequent level is the day after:



Hence there are 10 solutions.

Tutorial Questions

Matrix Problem

Let A be an n $\times$ n matrix with 0 and 1 entries only. Let us assume that $n \geq 2$, and that at least 2n entries are equal to 1. Prove that A contains two entries equal to 1 so that one of them is strictly above and strictly on the right of the other.

Initial Approach
When first faced with this question, I drew a simple diagram. I knew that the length of the matrix is n and the width of the matrix is n-1.
I tried to place the 0s and 1s such that it look like this. However, this is incorrect as I'm predetermining where the 1s are going. Instead, I needed to be more abstract with the pigeonholes themselves.

Final Answer
Let us create pigeon holes as $f : \{(i, j) : a_{i,j}, i + 1 \notin 0, j - 1 \notin 0\}$ (All diagonals originating from the edge). Hence it will look something like this
Since the top edge has n entries and the side edge has n-1 entries. The total amount of diagonals are 2n-1. However there are 2n amount of 1's we need to put in. Hence by the pigeonhole principle, there is at least one entry that is diagonal.

Checkers Problem

On a 8x8 chessboard, can we place 32 checkers so that no two checkers are adjacent.
Final Answer
Let us create pigeon holes as adjacent non-overlapping squares. So it looks something like this. As we can see there are 32 such pigeon holes and 32 checkers, thus we can place 32 checkers that no two checkers are adjacent.

Follow up: General nxn matrices.
Essentialy, on a nxn checkerboard, how many non adjacent checkers can we place. Two cases: n is odd and n is even.

Case 1: When n is even
Let us once again pair adjacent non-overlapping square. Since n is even we can always create $\frac{n}{2}$ number of pairings. Hence, we can place that many checkers before it become adjacent.

Case 2: When n is odd
Similar to case 1 but a little bit trickier. If we pair adjacent non-overlapping squares. We would have left over. However, the last square does not need be paired or can overlap and still fulfill the non-overlapping property. Therefore, we can have n/2 rounding up amount of checkers before we are forced to overlap.

Additional Exercises

Mr. Slow's Guessing Problem

A group of seven co-workers are trying to predict the total number of points scored in a given basketball game. The first six people already took their guesses, and, curiously, they all picked distinct even numbers. Mr. Slow is the last person to guess, and he knows all previous guesses. Is there a strategy for him that assures that his guess will be better than the guesses of half of his colleagues?

Approach

When I first saw this question I immediately thought of the mean of the question. Since we know all of their guesses, we can also use their guess to calculate the mean. In addition, there are 6 unique guesses meaning the mean will be a unique number. If the actual score falls above the mean, we would have done better than at least the 3 people that guess below us. Accordingly, if the actual score falls below the mean, we would have done better than the 3 people who guessed above us.

Initial Answer

Let $2k_1, 2k_2, 2k_3, 2k_4, 2k_5, 2k_6$ represent the 6 individual guesses made. Let a represent the actual score Therefore the mean of the numbers would be

$$\frac{2k_1 + 2k_2 + 2k_3 + 2k_4 + 2k_5 + 2k_6}{6} = \frac{k_1 + k_2 + k_3 + k_4 + k_5 + k_6}{3} = \bar{x}$$

since $\bar{x} \in \mathbb{R}$ and $\bar{x}$ will be distinct as it is the mean of distinct numbers. There would always be 3 numbers above and below it as it is also the middle number.
Hence, by guessing the mean Mr.Slow will always be better than at least 3 people.

Remarks

I quickly realized that there are a few major flaws to my Initial answer.

Rounding
In a game of basketball, all points are positive integers. By calculating the mean, there is always a chance the numbers can be a rational number. In that case, it is difficult to determine to either round up or down.

Extreme Values

Another problem that I did not consider is extreme values. For example the guesses: {2,4,6,8,10,1000} would have a mean of $171.\bar{6}$, which is greater than 5 of the guess not 3.

hence, we are not actually looking for the mean of the guesses, instead, we are looking for the ***median***.

Final Answer

Let $2k_1, 2k_2, 2k_3, 2k_4, 2k_5, 2k_6$ represent the 6 individual guesses made. Let a represent the actual score. Let us order each guess such that

$$2k_1 < 2k_2 < 2k_3 < 2k_4 < 2k_5 < 2k_6$$

Let Mr.Slow's guess, x, be the median of the guess such that

$$2k_1 < 2k_2 < 2k_3 < x < 2k_4 < 2k_5 < 2k_6$$

Therefore, if $x < a$ then Mr. Slow's guess will be better than $2k_1, 2k_2, 2k_3$.
Alternatively, if $a < x$ then Mr. Slow's guess will be better than $2k_4, 2k_5, 2k_6$.
Lastly, if $a = x$ then Mr. Slow's guess will be better than everone's guesses.
Hence, there does exist a strategy that allow Mr. Slow to be better than half of his colleagues.

Followup Questions

General Formula

Since the question merely had 6 *distinct even* guesses from his 6 colleagues. Is it possible for Mr. Slow to still guess better than half of his colleagues if he had n amount of colleagues and the guesses don't need to be even nor distinct?

Case 1: he has an even number colleagues
In this case, the general formula is similar to the inital question
Let us order all of the guesses $x_1, x_2, ..., x_n$ such that:

$$x_1 < x_2 < ...x_i < x_j < ... < x_n$$

Where x_i and x_j be the two guess that are closest to the median of the set. We know that the median of guesses is not in the guesses as the set of guess is even
Let Mr. Slow's guess, x be the median such that

$$x_1 < x_2 < ... < x_i < x < x_j < ... < x_n$$

Let a be the actual score of this game.
It follows that if $a < x$ then Mr. Slow's guess will be better than $x_1, x_2, ..., x_i$
if $x < a$ then Mr. Slow's guess will be better than $x_j, x_{j+1}, ..., x_n$
if $x = a$ then Mr. Slow's guess will be better than everyone
Therefore he can still guess better than half of his colleagues

Case 2: he has an odd number of colleagues
This case is slightly more complicated. Since there an odd number of colleagues, the middle number is already in the set of guesses.
Therefore, unless the actual score is the median, there is no way for Mr. Slow to be guaranteed better than half of the people. At most, he can only be better than $\frac{n-1}{2}$ people.

Statistical Approach
So far, we have been looking at this question mathematically; however, there are real world limits to a game of basketball. Essentially, knowing the median and some statistical information. Can Mr.Slow guess even better than 50% of his colleagues most of the time using statistics?

Let us assume they are betting on NBA games, which is also the most popular basketball league in the world.
The highest scoring NBA game was 186-184. A total score of 370.
The lowest scoring NBA game was 19-18. A total score of 37.
The avg score of a NBA game (all time) is 102 per team a total of 204.

Lets discount any score above 370 and any score below 37. By taking the avg score and the median guesses and making a guess off of that. We should retain a result that is greater than more than half of the colleagues.

I am unsure of the exact math, but by using staistics I am sure we can obtain a better result.

Higher Success
A follow up question to the statistical approach. Is there a way using pure mathematics that allows Mr. Slow to guess better than 70% of his colleagues. Perhaps 65% or even 51%.

From my own conclusions, I think that it is impossible; there is no sure mathematical way. I think that even knowing if the numbers are distinct and even, it is impossible. From the median approach, we are strictly finding the middle number and "cheesing" the question. Any other method will not work 100% of the time which would then be a counter proof.

Although, another may be possible. As I do not have counter proof or evidence that the way *does not* exist. I will think about this some more and perhaps look into it another time.

Triangle-Semicircle problem

Let T be a triangle with angles of 30, 60 and 90 degrees whose hypotenuse is of length 1. We choose ten points inside T at random. Prove that there will be four points among them that can be covered by a half-circle of radius 0.42.

Approach

When I first saw this question, the immediate thing that came to my mind is that this would be somehow related to the pigeon hole principle. Following this idea, I came with a few ways that I can possibly divide the pigeon holes.

First Idea
I first attempted to divide the triangle by their angles. Since we know the triangle is 90/60/30, this is a special triangle:

Accordingly, I attempted to divide the triangle like this; by dividing each angle by 3.

However, there are a few problems with this. The first problem is that each area of the triangle are not necessarily equal. The divided triangles could have different areas. On the topic of areas, does not necessarily solve our problem at all. Nothing about this is related to half circles nor related to the number 0.42. Lastly, this is not a very general way of dividing the triangle. We have to choose which angle to divide and that violates the pigeonhole principle (I think).

Second Idea
Following this, I decided to divide them into similar triangles. Like this:

Although this solves some of the problems, it is still unrelated to half circles. This is also when I realized that I need 3 pigeonholes and not 10 as having 10 would be excessive. 3 pigeon holes with 3 pigeons each inside and forcing a 4th one is all I needed to solve this problem.

Third Idea
Now realizing that I don't need to create 10 pigeon holes, I decided to divide the triangle such that the radius of each semicircles are 0.42 and that the center of the circle is placed upon the vertices.

Graphically it would look like something like this:

I feel like this could work as long as there are overlaps between at least one of the semi-circles. According to some of my rough math and rough drawing, I felt confident that this is right answer.

Final Answer

Let T be a triangle with a hypotenuse of 1 and angles 30, 60 and 90. Let us create semicircle partitions of T such that the radius of each semicircle is 0.42 and the centre of each semicircle is placed on the verticies of the T

The arbitrary initial 9 points can be contained within the 3 subsection of the T. However, since each of the 3 subsection can only contain 3 point each and that one overlap exists. We can conclude that the final point must be within one of the subsections or the overlapped section. Therefore the original statement is True.

Fractions Problem

Find all 4-tuples (a, b, c, d) of distinct positive integers so that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$$

Answer

(2,4,5,20)
(2,4,6,12)
(2,3,7,42)
(2,3,8,24)
(2,3,9,18)
(2,3,10,15)

Follow up: General Formula

The above answer is done by guess and check. We are going to attempt to find some more general properties. Mainly, I want to find a general formula for n fractions that equates that also fulfill the property of one greater than each other and being distinct.

Initial Approach

Doing some algebra we managed to get some potentially useful properties. By the definition of the question, we know that $0 < a < b < c < d < 1$. Another property that we know is that

$$\frac{bcd + acd + abd + abc}{abcd} = 1$$

from that we can conlude

$$a + b < ab, c + d < cd, ab < cd, a + b < c + d$$

$$\frac{b-2}{2b} = \frac{1}{c} + \frac{1}{d}$$

Second Approach
I've realized that these properties although important, fiddling around with them further yielded me no results. Instead I start laying out the problem once again. Essentially, we want to find

Journal and Extra

Journal Week 1

Some of the most mathy people have two main characteristics. They first have a good memory and second good analytical skills. Often, they are able to quickly relate question to ones they seen before or quickly lay down the logic of each problem.

I think something I greatly admire from them and something I lack is being able to break problems down quickly and relating them. Even though I really enjoy solving problems in small chunks and slowly building up to a bigger and better solution. One thing I absolutely hate about math is the amount of theorems you need to memorize and how bland some of them are.

I think through out the scope of the course, I want to tackle large complicated problems that can be broken down in small cases. In addition, I want to work with interesting theorems and perhaps even conjecture some of my own.

Chapter 2

Enumeration

Lecture and Tutorial

Tutorial Questions

TA Office Hour Dilemma

A T.A. has to conduct office hours on six days of January but not on consecutive days. How many choices do they have?

Final Answer
We know that that date of each of day is ascending and at least one apart from the last. Lets define x_i as the i th TA office hour day where $x_i \in \{x_1 < x_2 < ... < x_6\}$. Hence we can define the we also know that there are 31 days in January and that days need to at least one apart. Therefore,

$$1 \leq x_1 < x_2 - 1 < ... < x_6 - 5 \leq 26$$

Hence, we are choosing 6 days out of 26 aka $\binom{26}{6}$ which equals 230230 possibilities.

Follow up Question: General formula
Now, what if the T.A. has to conduct n amount of office hours in k amount of non consecutive days.

Following what we constructed from above the non general case. We can once again define x_i as the i th TA office hour where $x_i \in \{x_1 < x_2 < ... < x_n\}$. We can also once again define how we choose the days as

$$1 \leq x_1 < x_2 - 1 < ... < x_n - (n-1) \leq k - (n-1)$$

Seeing this, we can define the general formula as $\binom{k-n+1}{n}$ where k is the amount of days and n is the amount of office hours needed.

Additional Exercises

Team Formation Question

A class is attended by n sophomores, n juniors, and n seniors. In how many ways can these students form n groups of three people each if each group is to contain a sophomore, a junior, and a senior?

Initial Approach

I thought the answer to this question is quite simple. Since each group only consists of 3 people. Hence the number of ways to choose the first person is n, the number of ways to choose the second person is n, the number of ways to choose the third person is n. Therefore, the answer is n^3. However, I quickly realized thats not the case. Since for $n > 3$. There are a 3^3 to choose the first group, then 2^3 ways to choose the second group and so on.

Final Answer

There are $\binom{n}{1} \cdot \binom{n}{1} \cdot \binom{n}{1}$ to choose the first team as we need to pick one of each category. Following this, there are $\binom{n-1}{1} \cdot \binom{n-1}{1} \cdot \binom{n-1}{1}$ to choose the second group, repeating this step over until we only 1 remaining way. Also to account for any repeats (abc def = def abc) we need to divide the whole thing by n. Hence the answer is

$$\frac{\prod_{i=1}^{n} \binom{n-i}{1}^3}{n}$$

.

4 digit Question

How many four-digit positive integers are there in which all digits are different?

Final Answer

For each digit we have 10 possible choices (0,1,2,3,4,5,6,7,8,9). The beginning of each digit also can't be zero so the first digit only has 9 possible choices, thus the possible choices are

$$\binom{9}{1} \cdot \binom{9}{1} \cdot \binom{8}{1} \cdot \binom{7}{1} = 4536$$

Follow up: repeats

Essentially what if there are 2 or 3 or 4 digits that are the same?
For 4 repeats it is obvious, There are a total of 9 ways

For 3 repeats, it is slightly trickier. Since no 4 digit integers could start with 0. So there are 9 ways to choose the first digit. There are 10 ways to choose the second, third and fourth digit. Hence the answer should be.

$$9 \cdot (\binom{8}{1} + 3\binom{9}{1}) = 315$$

With 2 repeats it gets even trickier. There are 6 ways to arrange a digit this way. However 3 of them starts with the non-repeating digit and 3 of them does not. Hence the answer should be

$$9 \cdot (3\binom{8}{1}\binom{8}{1} + 3\binom{9}{1}\binom{8}{1}) = 3672$$

Follow up: Pigeon hole

Originally, I was going to attempt to generalize this problem. However, for a digit of size 11 there a guaranteed repeats. Lets define the pigeon holes as number of non-repeating digits. Then, lets define the pigeons as the number of possible digits as in 1,2,3,4,5,6,7,8,9,0. Therefore for size 11 integers and above, by the pigeonhole principle, there are guaranteed to be 1 repeat.

Follow up: Total

How many total numbers there are that don't have repeating digits. Since we now know there are a finite number of such digits lets just sum it all up in an equation that should be

$$\binom{9}{1} + \sum_{1}^{9} 9 \cdot (10 - i)!$$

This number is very large so I wont explicitly write it out.

Math Conference Problem

A two-day mathematics conference has n participants. Some of the participants give a talk on Saturday, some others give a talk on Sunday. Nobody gives more than one talk, and there may be some people who do not give a talk at all. At the end of the conference, a few talks are selected to be included in a book. In how many different ways is this all possible if we assume that there is at least one talk selected for inclusion in the book?

Initial Approach

Since there is at least one participant who talked, there is at least one way. I think that the way we approach this question is to count how many people don't give a talk. Then, from those who did give a talk we select the talks we want.

Final Answer

Let k be the amount of talks included in the book, and j be the number of people who did not give a talk on that day. Thus the total amount of possibilities is

$$\binom{(n-j_1)(n-j_2+j_1)}{k}$$

Mentor Pairing Problem

A group organizing a faculty-student tennis match must match four faculty volunteers to four of the 13 students who volunteered to be in the match. In how many ways can they do this?

Final answer

Since there are 13 students and 4 volunteers, we are essentially choosing 4 of the students to be mentored. Therefore, the answer is $\binom{13}{4} = 715$

Follow up: General Formula

Let's say there are now k faculty volunteers and n amount of students where $k \leq n$. The answer would be $\binom{n}{k}$

Follow up: Multiple volunteers

Now what if each student can have multiple volunteers assigned. For example 1 student can now have all 4 volunteers assigned to him. Now the answer changes to

$$\sum_{1}^{4} \binom{13}{i} = 1092$$

Hence the general form of the formula following this is then

$$\sum_{1}^{k} \binom{n}{i} = 1092$$

Where k is the number of volunteers and n is the number of students.

Journal and Extra

Week 2

I think that I am definitely a mixture of both. I guess it is dependent on the topic for me. Sometimes for some topics that I find interesting, I like to theory build and make conjectures. For other topics, that I don't understand as well, I would prefer to solve problems so I can understand the topic better.

As such I don't think I identify particularly with either of the two types prescribed. I think that someone could easily be both a problem solver and a theory builder. In addition, I agree that there are more types of mathematicians out there as well. For example, one who like to model things and apply ideas to real life or a supporter type mathematician who helps build theories and problems alike.

In the end, however, I would like to be a better conjecture builder. I think that this course allows you a lot of creative freedom that many other more structured math courses don't allow. As a result, I can take leaps toward that direction by building theories, following up on questions and generalizing solutions.

Week 3

Some of my goals in MAT344 are to improve my math skills, improve the way I reason things, learn combinatorics and just become a person who could study math effectively.

I say the most "high-energy + high importance" goal is definitely becoming a person who could study effectively. While a "low-energy + high importance" goal is like learning combinatorics and understanding the theorems.

I would say some of my goals are quite big and abstract. I will makes edits. Instead of wanting to improve my math skills, I want to improve my mathematical problem-solving skills. Instead of improve the way I reason things, I want to improve my discrete mathematical skills.

I will measure all of these goals by how efficiently I can solve textbook problems and to what quality.

I think I will achieve most of these goals, by writing this portfolio, pushing questions to the max and understanding the different ways you can prove thigns.

Midterm Meeting

The midterm meeting went well overall, I think that I can do a lot more for some of my problems and display it better for the final portfolio. A few things I should've worked on is writing the journal articles and being able to articulate my questions better.

Chapter 3

Binomial Theorem

Exercises

Exercise 4

Let k, m, n be nonnegative integers such that $k + m \geq n$. Prove that

$$\binom{n}{m} \cdot \binom{n-m}{k} = \binom{n}{k} \cdot \binom{n-k}{m}$$

Mathematical Approach

LHS:

$$\binom{n}{m} \cdot \binom{n-m}{k} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{k!(n-m-k)!} = \frac{n!}{m!k!(n-m-k)!}$$

RHS:

$$\binom{n}{k} \cdot \binom{n-k}{m} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{k!(n-k-m)!} = \frac{n!}{m!k!(n-m-k)!}$$

Since LHS = RHS, the proof is complete.

Combinatorial Approach

LHS: First from a group of n we choose m objects. Then from the remaining group, choose k objects.

RHS: First from a group of n we choose k objects. Then from the remaining group, choose m objects.

Notice that right hand side and left hand side are choosing the same thing but in different orders. Hence they are equal.

Subset Comparison

How many subsets of [n] are larger than their complements?

Initial Approach

We know that there are 2^n subsets for any set. We also know that, at most, half of the subsets are larger than the other half. Then I tried brute forcing through some examples,
The subsets of {a} (one element)

$$\{\}, \{a\}$$

This answer is obviously 1.

The subsets of {a,b} (two elements)

$$\{\}, \{a\}, \{b\}, \{a, b\}$$

Here, I noticed that the answer is 1 once again since {a}'s complement {b} is the same size.

The subsets of {a,b,c} (three elements)

$$\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$$

The answer here is 4.

The subsets of {a,b,c,d} (four elements)

$$\{\}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, d\}, \{c, d\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}$$

This time I noticed once again that several of the subset's complements have the same length as their complements. Hence the answer here will be 5.

I noticed that for odd n, the amount of subsets that are larger than their complements is simply 2^{n-1}.

For even n on the other hand, it is slightly more complicated. Essentially any subset that is half of the length of the original subset will not satisfy the condition. and for the rest only half of them will satisfy the condition.

Final Answer

We know there are 2^n subsets for any set $[n]$.

If n is odd, then there are no subsets with a same length complement (otherwise, n would have to be even). Hence, 2^{n-1} subsets will have be larger than their complement.

If n is even, then there would be subsets where their complement is the same size. The subsets that satisfies those conditions are any subset that's half the length of n (their complements will also be half the size of n). Meaning from the total amount of subsets, we need git rid of all n/2 sized subsets. From the remaining subsets, only half of them will be larger than their complement.

Combining those two facts, we get $2^{n-1} - \frac{1}{2} \cdot \binom{n}{\frac{n}{2}}$.

Vandermande's Identity Question

Prove that for all positive integers n,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Initial Approach

When I first saw this question, I immediately tried expanding it. Which gave me

$$\frac{(2n)!}{n!(2n-n)!} = \sum_{k=0}^{n} \frac{(n)!}{k!(n-k)!} \cdot \frac{(n)!}{k!(n-k)!}$$

. I quickly noticed that this won't help me at all. I then decided to look for a better alternative in lecture notes and the textbook. Eventually, I found Vandermande's Identity which states that

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \cdot \binom{m}{k-i}$$

Changing a few variables, I quickly noticed the answer.

Final Answer

Let $m = n, k = nandi = k$ in Vandermande's Identiy, then we get:

$$\binom{n+n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

This is close to our desired answer, however $\binom{n}{n-k}$ is a problem. Expanding it we get

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-n+k)!} = \frac{n!}{(n-k)!(k)!} = \binom{n}{k}$$

Hence:

$$\binom{n+n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k} \implies \binom{2n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{k}$$

as desired.

Exercise 44

Prove that for all positive integers $n > 1$,

$$\sum_{k=0}^{n} \frac{1}{k+1} \cdot \binom{n}{k} \cdot (-1)^{k+1} = \frac{-1}{n+1}$$

Initial Approach

First I attempted to expand the binomial coefficient and reorganize.

$$\sum_{k=0}^{n} \frac{1}{k+1} \cdot \binom{n}{k} \cdot (-1)^{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \cdot \frac{n!}{k!(n-k)!} \cdot (-1)^{k+1}$$

$$= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} \cdot (-1)^{k+1} = \sum_{k=0}^{n} \binom{n}{k+1} \cdot (n-k)^{-1} \cdot (-1)^{k+1}$$

After getting this result, I attempted to find an appropriate x and y to fit the binomial expansion theorem to no avail.

Second Approach

On my second attempt, I looked more closely at some of the steps I did initially. Mainly, I was pondering what $\frac{1}{k+1} \cdot \binom{n}{k}$ can be manipulated into. After doing some rough work, I noticed that:

$$\frac{1}{k+1} \cdot \binom{n}{k} = \frac{1}{n+1} \cdot \binom{n+1}{k+1}$$

This could be useful as in the sum $\frac{1}{n+1}$ is a constant now. Combining this with the fact that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. I was able to get the final answer.

Final Answer

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k+1} \cdot \binom{n}{k} \cdot (-1)^{k+1} &= \sum_{k=0}^{n} \frac{1}{n+1} \cdot \binom{n+1}{k+1} \cdot (-1)^{k+1} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} \cdot (-1)^{k+1} \\
&= \frac{1}{n+1} (\sum_{k=0}^{n} \binom{n}{k+1} \cdot (-1)^{k+1} + \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{k+1}) \\
&= \frac{1}{n+1} (\sum_{k=0}^{n} \binom{n}{k+1} \cdot (-1)^{k+1} + (-1) \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{k}) \\
&= \frac{1}{n+1} (\sum_{k=0}^{n} \binom{n}{k+1} \cdot (-1)^{k+1}) \\
&= \frac{1}{n+1} ((\sum_{k=1}^{n} \binom{n}{k} \cdot (-1)^{k}) + \binom{n}{n+1} (-1)^{n+1}) \\
&= \frac{1}{n+1} ((\sum_{k=1}^{n} \binom{n}{k} \cdot (-1)^{k}) + \binom{n}{0} (-1)^{0} - \binom{n}{0} (-1)^{0}) \\
&= \frac{1}{n+1} ((\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{k}) - \binom{n}{0} (-1)^{0}) \\
&= \frac{1}{n+1} (-\binom{n}{0} (-1)^{0}) \\
&= \frac{-1}{n+1}
\end{aligned}$$

as desired.

Follow Up: General Formula/Exercise 45

Now the question is instead of -1 what if its any constant t. plugging in t instead of -1 in the original equation we get.

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k+1}\cdot\binom{n}{k}\cdot(t)^{k+1} &= \sum_{k=0}^{n}\frac{1}{n+1}\cdot\binom{n+1}{k+1}\cdot t^{k+1}\\ &= \frac{t}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\cdot t^{k}\\ &= \frac{t}{n+1}(\sum_{k=0}^{n}\binom{n}{k+1}\cdot t^{k}+\sum_{k=0}^{n}\binom{n}{k}\cdot t^{k})\end{aligned}$$

We know that $\sum_{k=0}^{n}\binom{n}{k}\cdot t^k$ is simply the binomial expansion where x = t and y = 1. Hence is is equal to $(t+1)^n$. $\sum_{k=0}^{n}\binom{n}{k+1}\cdot t^k$ is a little more tricky but similar to the -1 case.

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k+1}\cdot t^{k} &= \frac{1}{t}\sum_{k=0}^{n}\binom{n}{k+1}\cdot t^{k+1}\\ &= \frac{1}{t}((\sum_{k=1}^{n}\binom{n}{k}\cdot t^{k})+\binom{n}{0}(t)^{0}-\binom{n}{0}(t)^{0})\\ &= \frac{1}{t}(\sum_{k=0}^{n}\binom{n}{k}\cdot t^{k}-\binom{n}{0}(t)^{0})\\ &= \frac{(t+1)^{n}-1}{t}\end{aligned}$$

Going back to the original equation we get

$$\begin{aligned}\frac{t}{n+1}(\sum_{k=0}^{n}\binom{n}{k+1}\cdot t^{k}+\sum_{k=0}^{n}\binom{n}{k}\cdot t^{k}) &= \frac{t}{n+1}(\frac{(t+1)^{n}-1}{t}+(t+1)^{n})\\ &= \frac{t}{n+1}(\frac{(t+1)(t+1)^{n}-1}{t})\\ &= \frac{(t+1)^{n+1}-1}{n+1}\end{aligned}$$

as our general formula.

A Combinatorial Argument

Final Answer

Essentially we are choosing 3 non repeating sets of size n from a set of size 3n. Let's call the 3 non repeating sets S_1, S_2, S_3 and the original set S_n.

There are many ways to choose the elements for each set, let's call the elements in S_1, S_2, S_3, a,b,c respectively. Other than the original way, we can also swap its formation with another set. There 6 possible ways to do this. Hence there are 6 total possible different arrangements for S_1, S_2, S_3. Hence no matter what size n is and which elements we pick, it will always be divisible by 6.

Follow Up: Mathematical Approach

$$\begin{aligned}\binom{3n}{n,n,n} &= \frac{(3n)!}{n!n!n!} \\ &= \frac{(3n)!(2n)!}{n!2n!n!n!} \\ &= \binom{3n}{n}\binom{2n}{n}\end{aligned}$$

We know that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. Hence

$$\begin{aligned}\binom{3n}{n} &= \binom{3n-1}{n-1} + \binom{3n-1}{n} \\ &= \frac{(3n-1)!}{(n-1)!(2n)!} + \frac{(3n-1)!}{n!(3n-1-n)!} \\ &= \frac{(3n-1)!}{(n-1)!(2n)!} + \frac{(3n-1)!}{n!(2n-1)!} \\ &= \frac{(3n-1)!}{(n-1)!(2n-1)!}(\frac{1}{2n} + \frac{1}{n}) \\ &= \frac{3(3n-1)!}{(n-1)!(2n)!} \\ &= 3\binom{3n-1}{n-1}\end{aligned}$$

$$\begin{aligned}\binom{2n}{n} &= \binom{2n-1}{n-1} + \binom{2n-1}{n} \\ &= \frac{(2n-1)!}{(n-1)!(2n-1-n+1)!} + \frac{(2n-1)!}{n!(2n-1-n)!} \\ &= \frac{(2n-1)!}{(n-1)!n!} + \frac{(2n-1)!}{n!(n-1)!} \\ &= \frac{2(2n-1)!}{(n-1)!n!} \\ &= 2\binom{2n-1}{n-1}\end{aligned}$$

Combining them we get

$$\begin{aligned}\binom{3n}{n}\binom{2n}{n} &= 2\binom{2n-1}{n-1}3\binom{3n-1}{n-1} \\ &= 6\binom{2n-1}{n-1}\binom{3n-1}{n-1}\end{aligned}$$

Which is divisible by 6 as required.

Journal and Extra

Week 5

I got a score of 64 even though I tend to think that I don't have imposter syndrome. I guess in MAT344 the imposter syndrome often leads to my lack of participation in class and group discussion. The more I think about it, I am afraid of being judged sometimes.

I think the reason why so many MCS students have imposter syndrome is because we are in a very competitive and high ceiling space. We often see so many highly successful and highly skilled people around us that for ordinary or average people it become a liitle hard to accept. Even though the average could actually be reasonable and be at our level, seeing other people do significantly better feel really bad.

For me, talking to someone and changing your thinking helps a lot. I think that I have developed a more chill and relaxed approach to school to avoid these imposter syndrome. I'm also an advocate for suffering together (ex. going through exam period or being in the same difficult course) with a friend.

MAT344 on how its structured actually prevents these syndrome quite well. You don't feel pressure to achieve as you are comparing against yourself instead of hundreds of other students. You get to keep track of your own learning and progress and overall it can help mitagate these effects.

Chapter 4

Partitions

Lecture and Tutorial

Subset Partition Code

Essentially, the goal was to use a recursive formula to find $S(n, k)$ and make the subsets that are comprised of it.

Listing 4.1: Code for Subset Partitions

```
# Set partition for set n and amount of partitions k
# Precondition: len(n) >= k
def set_partition(n, k):
    # base case only k elements in the set
    if len(n) == k:
        # returning each element of n as a individual set
        return [[[x] for x in n]]
    # recursive step
    else:
        x = n[0]
        y = n[-1]
        # all the subsets in this level will be stored here
        final = []
        # Constructing the previous sets
        current = set_partition(n[1:], k)
        other = set_partition(n[:len(n) - 1], k)
        # using the previous set to construct the next set by adding one of the
        for partition in current:
            for i in range(len(partition)):
                # copying partition to avoid aliasing
                temp = copy.deepcopy(partition)
                temp[i].append(x)
                # sorting to check for duplicates
                temp[i].sort()
                temp.sort()
                final.append(copy.deepcopy(temp))
        # covering missing cases but essentially the same as the first loop
        for partition in other:
            for i in range(len(partition)):
                # copying partition to avoid aliasing
                temp = copy.deepcopy(partition)
```

```
            temp[i].append(y)
            # sorting to check for duplicates
            temp[i].sort()
            temp.sort()
            # duplicate checking
            if temp not in final:
                final.append(copy.deepcopy(temp))
    return final


l = ["a", "b", "c", "d"]

print(set_partition(l, 2))
# result
# [[['a', 'b', 'c'], ['d']],
# [['a', 'd'], ['b', 'c']],
# [['a', 'b', 'd'], ['c']],
# [['a', 'c'], ['b', 'd']],
# [['a', 'b'], ['c', 'd']],
# [['a', 'c', 'd'], ['b']],
# [['a'], ['b', 'c', 'd']]]

print(set_partition(l,3))
# [[['a', 'b'], ['c'], ['d']],
# [['a', 'c'], ['b'], ['d']],
# [['a', 'd'], ['b'], ['c']],
# [['a'], ['b', 'd'], ['c']],
# [['a'], ['b'], ['c', 'd']]]
```

We are essentially first constructing all of the possible base cases. Then from the base cases, add the missing elements until all of the possible combinations are added.

Additional Exercises

Closed Form Finder

Find a closed formula for $S(n, 2)$ if $n \leq 2$.

Final Answer

We know that

$$S(n,k) = S(n-1,k-1) + k * S(n-1,k)$$

Plugging in the parameters given in the question we get

$$\begin{aligned} S(n,2) &= S(n-1,2-1) + 2 * S(n-1,2) \\ &= S(n-1,1) + 2 * S(n-1,2) \\ &= 1 + 2 * S(n-1,2) \\ &= 1 + 2 * (1 + 2 * S(n-2,2)) \\ &= 1 + 2 + 4 * S(n-2,2) \end{aligned}$$

Expanding Further we eventually get

$$1 + 2 + 4 + 8 + 16.... + 2^{n-2} * S(2,2) = \sum_{k=0}^{n-2} 2^k$$

Therefore,

$$S(n,k) = \sum_{k=0}^{n-2} 2^k = 2^{n-1} - 1$$

Similar Problem: $S(n, n-2)$

Find a closed formula for $S(n, n-2)$ if $n \leq 2$.

Initial Approach
When I first saw this question, I immediately tried to expand the recurrence relation. However, I quickly noticed that this expansion is very tedious and hard to compute. Noticing this, I started to scower the textbook and lecture notes for a better approach and method.

Second Approach
Although I found no theorems that could've helped me in this problem directly, reading the textbook did allow me to explore the problem combinatorially.

Translating the question to English, it essentially is asking how many n balls into n-2 bins. This leaves us with 2 options on what the bins could potentially look like.

Option 1

$$2|2|1|1....|1|$$

Option 2

$$3|1|1|1....|1|$$

For option 2, we are essentially choosing a 3-subset from n. Hence, $\binom{n}{3}$. For option one, we are choosing from all of the 2-subset from n and then splitting them into any 2 boxes in n-2. To avoid any duplicates (ie, putting one of the two subset into one and the other into another one then switching them) we must also cut this in half. Therefore combining those two facts we should get and answer of

$$\binom{n}{3} + \frac{1}{2}\binom{n-2}{2}\binom{n}{2}$$

Similar Problem: S(n,n-3)

Find a closed formula for $S(n, n-3)$, for all $n \leq 3$.

Similar to last part, we must once again think about this problem combinatorially. We essentially have 3 options:

Option 1

$$4|1|1|1....|1|$$

Option 2

$$3|2|1|1....|1|$$

Option 3

$$2|2|2|1....|1|$$

The first option is simple, it is choosing any 4 elements from n and then putting them into one bin. Hence it would be $\binom{n}{4}$. The second option involves choosing 5 elements then splitting the 5 elements to all possible 2 and 3 subsets. Hence, it would be $\binom{n}{5}\binom{5}{2}$. Option 3 is the most complex, as there could be repeats. First choose 6 elements from n, then from those 6 elements, divide into 3 subsets of 2. After, to account for any repeats when distributing we must divide by 6. Combining all 3 facts we get the final answer of

$$\binom{n}{4} + \binom{n}{5}\binom{5}{2} + \frac{1}{6}\binom{n}{6}\binom{6}{2,2,2}$$

Follow up: General Formula

Find a closed formula for $S(n, n-k)$ if $n \leq k$.

We essentially have made "small" cases with the last 2 parts. Now for the general solution, we must have a similar approach.

First of all we know we have k options. For the first option it is always putting all k balls into one bin. Hence, it would always be $\binom{n}{k}$. The next options it is a little more difficult to write. Essentially, there will be no closed form for $S(n, n-k)$ as it is case dependent. However, there are some general rules of thumb we can follow to make it easier.

If we are splitting k into partitions of different sizes (ie 3,2,4). It will be

$$\binom{n}{k + amountofparitions}\binom{k + amountofpartitions}{individualpartionsizes}$$

If there are any repeats in partition sizes, we need to divide the result by k+amount of partitions or else there will be repeats.

Following the above two rules will make finding partitions a lot simpler.

Even Compositions

Find the number of compositions of n into an even number of parts.

Final Answer

We know that the number of of compositions of n into k parts is $\binom{n-1}{k-1}$. Let $k = 2a$ be an even number, where a is a positive natural number. Hence, replacing it within the function we get $\binom{n-1}{2a-1}$. Notice how this is just the number of n compositions into odd parts. Therefore, they must have the same amount of compositions.

We also know that there are 2^{n-1} total compositions. Combining both the facts, we can see that there are 2^{n-2} even number of compositions.

Follow Up: Even parts

Find the number of compositions of n into a number of even parts.

First of all, n must be even as an odd number will always have a odd number within its composition. So an odd n will have 0 compositions with only even numbers.

Now let's think about what compositions really mean, it is essentially saying what group of numbers when added together form n. Since we are only looking for n's even parts, we know that every number will be divisible by 2. When we do that, we are essentially finding every composition of $\frac{n}{2}$.

Therefore, we can conclude that the answer is $2^{\frac{n}{2}} - 1$.

Follow Up: Odd parts

Find the number of compositions of n into a number of odd parts.

This is much more difficult as odd numbers often don't share the same property. However we know that odd numbers are just some even number plus one. Hence let a_i be some odd number,

$$2a_1 + 1 + 2a_2 + 1 + ... + 2a_m + 1 = n$$

We know that there are at most k parts for n, hence after some algebraic manipulation we can get:

$$\sum_{i=0}^{m} a_i = \frac{n-k}{2}$$

Essentially, the number of odd compositions n can have is tally of all k that satisfies this to be an integer plus one (for all 1s).

Integer Partition Inequality

Prove that for all positive integers n,

$$p(1) + p(2) + ... + p(n) < p(2n)$$

Initial Approach

Initially, I thought that this question was quite simple, as I thought the integer partition are just sizes of n. However, I quickly realized that it would disprove the inequality instead. I then decided to more in depth into the definition of p(2n) then I quickly realized the answer.

Final Answer

Let's analyze this problem combinatorially. We know that a integer partition of 2n can contain any of the integer partitions on the LHS. For example, one partition of 2n could be

$$2n = n + partitions(n)$$

This is also p(n). Hence, we can eventually map every integer partition on the LHS and add with its complement. However, at the end of all this matching, notice

$$(2n-1) + (2n-1) = 2n$$

is never matched or utilized by the LHS. Therefore, we can conclude that p(2n) must be larger.

Journal and Extra

Week 6

I often reflect on what I have did at the end of every month. I think about everything I learned and how far I am along my goals. I tend to look back on the past month and think about how I could have improved myself. Many times, such reflections help me better understand my goals and how I should proceed with them. For example, if I noticed that I am not doing enough for my portfolio, I would try to work on it slight more.

The lion-thorn is trying to tell us that the problem is not always where we think it is. We must really search within ourselves to find the root of the problem.

Currently in MAT344, the largest piece of reflection I have done is the Midterm Reflection. I had to look through all of my work and the journey up to that point. I had to reflect on the goals I was achieving and the goals I should set for myself in the future. It was of great value as it allowed me to reflect and see what I could have been doing better. Going forward, I will attempt to look at my last weeks work a week later.

Chapter 5

Permutation Cycles

Lecture and Tutorial

Airplane Seat Riddle

All (105) students in MAT344 line to board a plane that seats 105 people.

1. The first person in line (Ali) realizes they lost their boarding pass, so when they board they decide to take a random seat instead.

2. Every person that boards the plane after Ali will either take their "proper" seat, or if that seat is taken, a random seat instead.

What is the probability that the last person that boards (Yushan) will end up in their proper seat?

Final Answer

When I first saw this question, I immediately thought of a few scenarios.

We know that the probability of Ali choosing the right seat is $\frac{1}{105}$. In addition, if Ali chooses the right seat then Yushan will definietly choose the right seat as everyone else after will choose their proper seat.

If Ali somehow chooses Yushan's seat, then the probability is 0 as Yushan cannot end up in his seat. The probability of this occurring is $\frac{1}{105}$

The above scenarios "cancels each other out" probability wise.

The last scenario is the most interesting, if Ali chooses someone else seat, then scenerio one and two will occur but with the next person. The probability of Ali choosing someone else's seat and not Yushan's seat is $\frac{103}{105}$

Hence, the process will repeat until the second last person before Yushan. He would have two choices, either choosing a seat (could be right or wrong) or choosing Yushan's seat. Hence this would result Yushan's seat being correct $\frac{1}{2}$ times.

Observation: General Solution

The more I think about the problem, the more I realize that it doesn't matter. Let's say there are n people instead.

The choice that each person has remains the same choosing the last person's seat, choosing the correct seat and choosing some random person's seat.

This will result in having the second last person in choosing between a random seat and the last person seat. Which will result in the last person picking their own seat $\frac{1}{2}$ times.

Evil Instructor Riddle

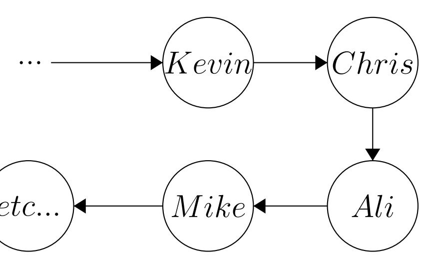
An evil instructor ("Ike") has set up a challenge for his (105) MAT344 students.

1. There are 105 boxes inside IB330, and each box has the name of a different student on the outside.
2. On 105 different pieces of paper, Ike writes the names of each student.
3. Ike randomly distributes the names into the boxes (one paper per box) and closes the box.
4. One by one the students enter the class, and may open, look into, then close, any 53 boxes. Then they leave and sit quietly in Davis.
5. The whole class wins if every single student finds their own name in a box.
6. The whole class loses if one student fails to find their own name.
7. The students may only discuss their strategy before they enter IB330. Once a student enters, they can no longer communicate with any students.

Find a strategy that gives the class the best chance of winning the game.

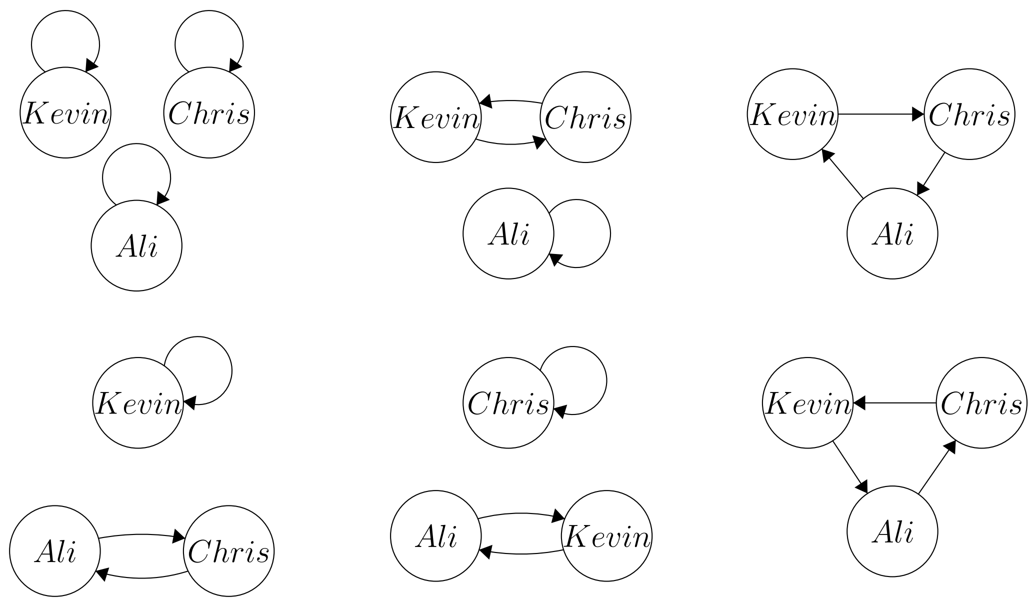
Final Answer

We can think of each box with the paper as a permutation cycle. Something like this.



Where f(name) = name in box. We know that there are at most 105 permutation cycles and at minimum one. However, this can be hard to visualize. Let's break this down into 3 students (Kevin, Chris, and Ali) and you can open 2 boxes.

Here are all the possible permutation cycles for 3 boxes.



Notice that for all 6 of these scenerios, you can find your own name in the box. Now instead what if you can only one box?

Now the scenarios on the right will still work as you can assume the box you haven't opened is yours as there are 3 names and 3 boxes. Although this only applies to this case.

Hence once we scale it back up, we can see that 53 boxes will allow everyone to choose the right name for all cycles of size 54 and under (since you start at your own name). Hence, this method should work $\frac{54}{105}$ times.

Follow Up: General Success

Now what if there are n names and you can choose k times with the same rules.

Following the same logic as the previous question, we can see if you start at your own name and if the cycle has k + 1 elements you will eventually find your name in a box. Therefore, the success rate for an n number of boxes and ability to choose k times is $min(\frac{k+1}{n}, 100\%)$ (maximum of 100%).

Additional Exercises

Longest Cycle

What is the number of (2n)-permutations whose longest cycle is of length n?

Similar Question

This is a a similar question that will act as a small case for the general question.

What is the number of permutations of length 20 whose longest cycle is of length 11?

For this question, we must first ask how many different cycles of length 11 we can for from 20? Essentially, we pick 11 numbers from 20, find all the ways to arrange them. We arrange this by finding all bijections where a number these lead to itself. Then we get the remaining 9 numbers and find all the different ways to arrange them. We do this by finding all bijections. Following this we get

$$\binom{20}{11} * 10! * 9!$$

Final Answer

Seeing the logic from the smaller case, we can attempt to apply to the larger case. We are essentially choosing n numbers, then finding all the ways to arrange that. Then arranging the rest of the numbers. Hence similarly we get

$$\binom{2n}{n}(n-1)!(2n-n)!$$

Follow Up: Even More General

What is the number of (n)-permutations whose longest cycle is of length k?

Following the last two parts, the answer for this is simply

$$\binom{n}{k}(k-1)!(n-k)!$$

Card Multiplication

We write each element of [n - 1] on a separate card, then randomly select any number of cards, and take the product of the numbers of written on them. Then we do this for all 2^{n-1} possible subsets of the set of n - 1 cards. (The empty product is taken to be 1.) Finally, we take the sum of the 2^{n-1} products we obtained. What is this sum?

Initial Approach

Let's first run through what this will like with just 2 elements:

$$[2-1] = [1] = \{1, \{\}\}$$

Since the empty set's product is taken to be 1, the sum of the products is 1+1 = 2.

Now let's try 3

$$[3-1] = [2] = \{1, 2, \{1, 2\}, \{\}\}$$

The sum of products is 1+2+2+1 = 6.
Now let's try 4

$$[4-1] = [3] = \{1, 2, \{1, 2\}, \{\}\, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, 3\}$$

The sum is 1 + 2 + 2 + 1 + 3 + 6 + 6 + 3 = 24.

Notice how you can recursively form the next set by multiplying the previous sum with n-1. Hence we can define a recurrence definition as :

$$F(n) = (n-1)F(n-1) + F(n-1)$$

This is simply equals to

$$F(n) = nF(n-1)$$

Final Answer

Let's define the sum asked in the question as F(n), we know to construct F(n) we need to find all 2^{n-1} subsets. We can recursively construct subsets by adding the new element k to the previous set s_{n-1}. Hence, we can define s_n as

$$s_n = \begin{cases} \{\} & \text{for } n = 1 \\ \{s_{n-1}, s_{n-1} + k\}, & \text{for } n \geq 1 \end{cases}$$

Hence adapting this to our question,

$$F(n) = \begin{cases} 1 & \text{for } n = 1 \\ nF(n-1), & \text{for } n \geq 2 \end{cases}$$

Solving the recurrence relation, we will end up with n!.

Follow up: Extension

Modify the previous exercise so that instead of considering all 2^{n-1} subsets, we only consider all k-element subsets of the n - 1 cards. What is the sum of all $\binom{n-1}{k}$ products we obtain in that scenario?

Once again lets draw some smaller cases

Let's start with n = 4 k = 2:

$$[4-1] = [3] = \{\{1,2\},\{1,3\},\{2,3\}\}$$

This is equal to 2 + 3 + 6 = 11.

n = 4, k = 3 would just be k! = 6.

Now how about n = 3 k = 2:

$$[3-1] = [2] = \{\{1,2\}\}$$

Which is once again k!, which is equal to 2.

n = 3 k= 1: This would be 1 + 2.

At this point, I noticed that you are finding all the possible combinations of k elements. To find the k elements we simply add the current element to the the n - 1, k- 1 set along with the k! (which is the current element).

$$F(n,k) = (n-1) * F(n-1,k-1) + k! = Sum$$

Another thing I noticed was that if k = 1, you are essentially summing n -1:

$$F(n,1) = \sum_{k=1}^{n-1} k = \frac{n(n+1)}{2} - n = \frac{n^2-n}{2} = Sum$$

Last thing I noticed was if k is only one smaller than n, than the sum is only F(n,k) can be constructed with k!

Continuation and Corrections

I noticed that the first part where I said k! is wrong. After listing out more examples I noticed that it is instead constructed using the F(n-1,k) (the subsets of the same length from the previous n.

Putting this in a recurrence relation we get:

$$F(n,k) = (n-1) * F(n-1,k-1) + F(n-1,k)$$
$$F(1,0) = 1$$
$$F(1,1) = 0$$

For all $n > 0, n > k, n, k \in \mathbb{R}$

Proof of Correctness
Base Case:
$F(2,1) = (2-1) * F(1,0) + F(1,1) = 1$. The subset for F(2,1) is 1, the sum of which is 1.

Induction Hypothesis: Assume for some $i, j \in N$ F(2,1),F(3,2),F(2,2).... F(i,j) holds.

Induction Step: Want to show F(i+1, j+1), F(i,j+1), F(i+1,j) holds.

Case 1: F(i, j+1)

$$F(i,j+1) = (i-1) * F(i-1,j) + F(i-1,j+1)$$

We know that (i-1)*F(i-1,j) holds by I.H. F(i - 1, j + 1) expanded will be:

$$F(i-1,j+1) = (i-2) * F(i-2,j) + (i-3) * F(i-3,j)..... + 0$$

Which is the sum of all the subsets of F(i-1,j+1). Therefore this case holds. Case 2: F(i+1,j)

$$F(i+1,j) = i * F(i,j-1) + F(i,j)$$

F(i,j) is the proper sum as required. We know that F(i,j-1) is the correct sum. We also know that to form all subsets excluding F(i,j), we simply need to multiply i to the previous sums. Hence this case will result in the proper sum and holds. Case 3: F(i+1,j+1)

$$F(i+1,j+1) = i * F(i,j) + F(i,j+1)$$

F(i,j+1) is the proper sum by Case 1. i*F(i,j) holds by Case 2.
Hence the proof is complete.

Inversion Problem

Let $p = p_1 p_2 ... p_n$ be a permutation. An inversion of p is a pair of entries (p_i, p_j) so that $i < j$ but $p_i > p_j$.

Let us call a permutation even (resp. odd) if it has an even (resp. odd) number of inversions.

Prove that the permutation consisting of the one cycle $(a_1 a_2 ... a_k)$ is even if k is odd, and is odd if k is even.

Initial Approach

I started by thinking how the cycle is formed. We know since there is only one cycle every element will lead to a different element. Following this, if we started from a_1 we would need to go through k elements before we end back on a_1. In addition we know there are $\binom{k}{2}$ possible pairs in this permutation.

Another observation is that we know a_1 is the largest element in the list. Hence, when we convert this cycle back into its permutation form, every term after it that forms a pair with a_1, will be inverted. And every term before it, not inverted.

The second largest term would then have all the terms in before after it not-inverted and every term before it inverted except for a_1.

Apply above logic to every term and the smallest term will have every term before it not-inverted and every term after it inverted.

We can think of every term as a "pivot point" that divides the list. With this logic we can get our final answer.

Final answer

First lets assume k is odd and hence k elements in our cycle.

We know in canonical form a_1 is the largest element in our permutation. Hence in permutation form, every number after a_1 will be inverted. Since, there are k is odd, k - 1 will be even. Hence there will be an even number of pairs that will satisfy the conditions.

Now lets think about the second largest term i, every number after i except for a_1 will be inverted and every number before it except for a_1 will not be inverted. Excluding a_1 from the list, we can see that the permutation must be odd as the number of elements that are left is even.

Do this for every single pair (acting as pivot point, hence even, then having an odd even number of elements thus making it odd) until we are left with the smallest term. Now, let's pair every odd and even permutation. Since we do this k times, we are left with k - 1 odd terms. Summing it all together will yield us an even term. We are now left with the smallest term and the largest term. Since there are no elements left included in the list, smallest term will have 0 new additional pairs to add. Hence we are left with the largest term which we know its even.

Therefore, since even + even is even, the cycle will be even.

Now lets assume k is even.

Similar to k is odd case, a_1 will form an odd number of pairs that are inverted. Hence the second largest term will form an even number, and so on. This will occur k-1 times, which is an even number of times. Pairing them once again, we will find the it will be even + odd, odd times which will yield an even number. Since a_1 is odd, odd + even is odd, we can conclude that the cycle is odd.

Journal and Extra

Week 7

Problem solving according to Polya can be divided into 4 phases.

1. Understand the problem
2. Come up with a plan
3. Execute the plan
4. Reflect

Essentially, the first step is to understand what the question is asking and what should be the final goal of the problem. The second step is coming up with small cases, try to reason with it or anything to get you started. The next step is more formal and tries to execute the rough plan. The last step is to see what you done is effective.

I think I agree with Polya and his way of problem solving. I essentially do the same thing by breaking down the problem and coming up with a plan in my "Initial Approach" phases and executing them in my "final answer" phases. I do the reflections outside of the portfolio.

Chapter 6

Inclusion/Exclusion

Exercises

Relatively Prime Problem

How many positive integers $k \leq 210$ are relatively prime to 210?

Initial Approach

210 seems like quite a large number to start, so let's start by cutting the number down to 21. Hence lets find all positive integers $k \leq 21$ are relatively prime to 21.

By brute force, we can see that this is 2,4,5,8,10,11,13,16,17,19,20. A total of 11 numbers. We are essentially not counting any number with that has prime factors of 3 or 7.

Using the inclusion-exclusion principle, we are counting every number that is divisble by 3 excluding numbers that are visible by 7, then every number that is divisible by 7 then excluding numbers that are visible by 3, then adding numbers that are divisible by 3 and 7.

Hence with this we can work on our final answer.

Final Answer

We know that there the prime factorization of 210 is 2,3,5,7. Hence let's use the sieve formula.

Let A_n denote all numbers divisible by n. Hence, we using inclusion exclusion formula we want to find

$$A_2 + A_3 + A_5 + A_7 - A_2 \cap A_3 - A_2 \cap A_5 - A_2 \cap A_7 - A_3 \cap A_5 - A_3 \cap A_7 - A_5 \cap A_7$$
$$+ A_2 \cap A_3 \cap A_5 + A_2 \cap A_3 \cap A_7 + A_2 \cap A_5 \cap A_7 + +A_3 \cap A_5 \cap A_7 - A_2 \cap A_3 \cap A_5 \cap A_7$$

Which will yield us the answer of 48.

Follow up: General Formula

How many positive integers $k \leq n$ are relatively prime to n?

First lets define the set A_i where i is a prime factor of n. Hence according to the inclusion-exclusion theorem the k satisfying the condition would be

$$n - A_{i1} \cup A_{i2} ... A_{ij} = n - \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} A_{i1} \cap A_{i2} \cap .... \cap A_{ij}$$

Cycle Length Exclusion Problem

How many n-permutations are there that contain exactly one cycle of length one?

Initial Approach

We know that there are n! permutations. From there, we just need to remove all permutations with that contain more than one cycle of length 1 and all permutations with less than 1 cycle of length one.

We know there are n ways to choose a fixed point. And D(n-1) ways to choose a the rest. Hence combining those facts we should obtain the final answer.

Final Answer

We know there are n ways to choose a one-cycle. According to the textbook, there are D(n) ways to have permutations of n with no fixed points. Hence, we want to choose the one-cycle and have no fixed points in the rest, therefore:

$$n * D(n-1) = n * \sum_{i=0}^{n-1} (-1)^i \frac{(n-1)!}{i!} = n! * \sum_{i=0}^{n-1} \frac{(-1)^i}{i!}$$

Follow Up: Any Length

How many n-permutations are there that contain exactly one cycle of length k?

This is trickier as we do not have the derangement formula to help us. However we know that there are $\binom{n}{k}$ ways to to choose a k cycle. In addition we know that there are n! permutations for any given n.

Combining those facts the answer is actually quite simple

$$\binom{n}{k} * (n-k)!$$

Follow Up: All One-Cycles

How many n-permutations are there that contain exactly a cycle of length 1?

This is quite simple, we are essentially looking for all cycles with a fixed point. We do this by taking all possible n-permutations then taking away cycles with a fixed point. Hence the answer should be

$$n! - D(n)$$

Follow Up: k One-Cycles

How many n-permutations are there that contain exactly k cycle of length 1?

This is an extension of the previous problem, albeit not as difficult as it seems. Similar to the logic of the original problem, we know there are $\binom{n}{k}$ ways of choosing a cycle of length 1. Then we just need to find the all the derangement for the rest. Hence,

$$\binom{n}{k} * D(n-k)$$

Perfect Squares Problem

How many positive integers are there that are not larger than 1000 and are neither perfect squares nor perfect cubes?

Initial Approach

First, let's break down the problem to a smaller case.

How many positive integers are there that are not larger than 10 and are neither perfect squares nor perfect cubes?

By brute force, we can see that there are only 2 perfect cubes (1,8) and 3 perfect squares(1,4,9) with 1 overlap. Hence using inclusion-exclusion theorem, we know there are 10-2-3+1 = 6 that fulfill the conditions.

From this, we can scale to the final answer.

Final Answer

By brute force, we know that the largest perfect square is $31^2 = 961$. Therefore, 1000 includes 31 perfect squares.

By brute force, we can also find that the largest perfect cube is $10^3 = 1000$. Hence there are also 10 perfect cubes.

Now the tricky part is to find numbers are that are both a perfect square and a perfect cube. We can see that a number that is both, will have to take the form of $(n^2)^3$. Hence the largest of which that is below 1000 is $(3^2)^3 = 729$.

Now applying the inclusion exclusion theorem we can get our final answer of 1000 - 31 - 10 + 3 = 962.

Follow Up: General Form

How many positive integers are there that are not larger than k and are neither perfect squares nor perfect cubes?

Following the logic from the previous section, we are finding the largest n that satisfy $n^2 \leq k$, let's call this number a. Then we find the largest n that satisfy $n^3 \leq k$, let's call this number b. Lastly subtracting all the numbers we counted twice, which is $(n^2)^3 \leq k$, let's call this number c. Hence yielding us the formula n-a-b+c.

Follow Up: Other Powers

How many positive integers are there that are not larger than k and are neither perfect squares nor perfect cubes nor perfect quads etc.?

Let's call A_n the largest number that satisfy $a^n \leq k$, $a \in \mathbb{N}$. Hence we can construct this using the inclusion-exclusion theorem.

$$k - A_2 \cup A_3 ... A_n = k - \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, ..., i_j} A_{i1} \cap A_{i2} \cap .... \cap A_{ij}$$

Journal and Extra

Week 8

So far in this course, I have improved a lot in pushing questions beyond their scope and improving my combinatorial conjecture making skills. I think I still have room to grow as a conjecture proover, but I am making a lot of progress.

Typically, I measure my progress in terms of how comfortable I am with the topic. I have achieved my learning objectives when I feel comfortable solving problems for that concept and also using that concept in other areas as well.

I generally disagree with the grades and quantity of problems solved as measurement of progress. I think that it produces an atmosphere where student stress and worry about trivial things while they should really be focused on themselves and how to improve. Unfortunately, I also think it is the best and easiest choice to model how much you understand. It is much difficult for yourself, let alone others, to track your progress on other metrics.

Using grades and having a healthy mindset with them could actually be a good tool for you to create dynamic improvement plans and achievable goals.

I think the main way I can measure my progress is by showing that I can ask meaningful questions about the problems and concepts. I can develop skills such as proof reading and writing, theory, and conjecture making just by doing that.

I think I won't change what I have been doing because I feel like I've been making really good progress with my learning. From my humble beginnings, I feel like I am much comfortable now asking questions beyond the original. Some minor things I would change is to attempt to apply this idea to other courses or try to build a healthier mindset.

Chapter 7

Generating Functions

Lecture and Tutorial

Return of Jelly Bean Sequence

Explanation

Essentially, the question is asking for how many ways we can order positive sequence of numbers starting and ending with 0 where the difference between each consecutive number is 1.

To start off, we are given 0 on the first day. We know that it is impossible to have a valid sequence for an odd number of days following. The reason being that we are forced to add or subtract 1 jellybean each day. For example if the length of day is 1, we would have the possible sequences 00, 01 yet none would satisfy the sequence

Let's call the c_n the number of ways we can have this sequence. We know that smallest sequence of length 2 is "10", which satisfies our condition.

The next length which could have strings that satisfy the sequence is length 4. which will have 2 possible options 1210, 1010.

Let's observe how the above sequences are formed. We know that at some point other than the very first day, the jar will be empty. Let's call this day the 2i (we know that this needs be even from the question). We also know that the ending day is 2n.

Now the question is how do we form such sequences from length of 2i to 2n. We know that from the 1st day to the 2i th day, every number must be positive (or else 2i as we defined will be a different day). Hence we will have a valid sequence where all of the partial sums will be positive. Let's called this sort of sequence a valid subsequence.

For every subsequence we can remove the first and last operation. For example, 1210 is +1, +1, -1, -1. Removing the first at last operation we get 10 which is +1, -1. By doing this we created a valid sequence of 2(i-1) from the sequence 2i. So we can form sequences of both size 2i and 2(n-i) for some i in n.

We do not necessarily need to use generating functions to define this recurrence relation. We can simply define it as

$$c_0 = 1$$

$$c_{n+1} = \sum_{i=0}^{n} c_i * c_{n-i}$$

However, using a generating function we can find a closed form much easier. It is as follows

Lets define the $C(x) = \sum_{n\geq 0} c_n x^n$ as the ordinary generating function for the sequence c_n. From above, we can deduce that

$$C(x) - 1 = xC(x) * C(x)$$

The reason we are subtracting 1 from the LHS, is because for n = 0, we cannot form valid subesequences. Therefore:

$$xC(x)^2 - C(x) + 1 = 0$$

This a quadratic equation, solving it we get

$$\frac{1 - \sqrt{1-4x}}{2x}$$

Simplifying we get

$$C(x) = \sum_{n\geq 0} \frac{\binom{2n}{n}}{n+1} x^n$$

Solving for c_n we get:

$$c_n = \frac{\binom{2n}{n}}{n+1}$$

Feedback

I got some feedback from Chris:

A very concise and answer. It's much easier to read than the textbook and the examples help.

Additional Exercises

Recurrence Problems

Exercise 25

Find an explicit formula for an if $a_0 = 1$ and $a_{n+1} = 3a_n + 2^n$ if $n \geq 0$.

Let us define A(x) as

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Let us also rewrite the original relation as

$$a_n = 3a_{n-1} + 2^{n-1}$$

Hence combining the above two, we get

$$A(x) = a_0 + \sum_{n=1}^{\infty}(3a_{n-1} + 2^{n-1})x^n$$

$$= 1 + \sum_{n=1}^{\infty} 3a_{n-1}x^n + \sum_{n=1}^{\infty} 2^{n-1}x^n$$

$$= 1 + 3x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 2^n x^n$$

By geometric series formula

$$= 1 + 3xA(x) + \frac{2x^2}{1-2x}$$

$$A(x) - 3xA(x) = 1 + \frac{2x^2}{1-2x}$$

$$A(x) = \frac{1-2x+2x^2}{(1-2x)(1-3x)}$$

Partial fraction decomposition will yield us

$$A(x) = \frac{1}{3} + \frac{1}{2x-1} - \frac{5}{3\,(3x-1)}$$

From here we can get our final answer:

$$a_n = (\frac{1}{4})^n - 2^n - 5 * 3^{n-1} - 1$$

Correction of Exercise 25

The above was my original answer, however after plugging in some values I quickly realized that it is incorrect. Hence I looked over my work once again to spot for errors. Upon further reading, I noticed that I made multiple mistake with geometric series formula and also with algebra. Therefore, I decided to redo the geometric formula conversion.

By geometric series formula

$$= 1 + 3xA(x) + \frac{x}{1-2x}$$

$$A(x) - 3xA(x) = 1 + \frac{x}{1-2x}$$

$$A(x) = \frac{1-x}{(1-2x)(1-3x)}$$

$$A(x) = -\frac{2}{3x-1} + \frac{1}{2x-1}$$

From here we can get our final answer:

$$a_n = 2 * 3^n - 2^n$$

Exercise 26

Find an explicit formula for an if $a_0 = 1$, $a_1 = 4$, and $a_{n+2} = 8a_{n+1} - 16a_n$ for $n \leq 0$.

Similar to last question let's rewrite and redefine a few equations.

Let's define A(x) as

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Let's rewrite the original equation as

$$a_n = 8a_{n-1} - 16a_{n-2}$$

Once again, let's combine both those definitions

$$A(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 1 + 4x + \sum_{n=2}^{\infty} (8a_{n-1} - 16a_{n-2})x^n$$

$$= 1 + 4x + \sum_{n=2}^{\infty} 8a_{n-1}x^n - \sum_{n=2}^{\infty} 16a_{n-2}x^n$$

$$= 1 + 4x + 8x\sum_{n=1}^{\infty} a_n x^n - 16x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= 1 + 4x + 8x\sum_{n=1}^{\infty} a_n x^n + 8xa_0 - 8xa_0 - 16x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= 1 - 4x + 8x\sum_{n=0}^{\infty} a_n x^n - 16x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$A(x) - 8xA(x) + 16x^2(Ax) = 1 - 4x$$

$$A(x) = \frac{1-4x}{1-8x+16x^2}$$

$$A(x) = \frac{1}{1-4x}$$

$$a_n = 4^n$$

Remarks

When I started this section, I was struggling to understand how to solve recurrence relations. Hence, I chose these two medium difficulty questions to help me understand better.

Excercise 25 was quite the struggle for me and so was excercise 26. However, after doing these 2 questions, I feel like I understand generating function a lot better. Therefore, I feel more comfortable and more prepared tackling harder questions.

Tiling Problem

Let h_n be the number of ways to tile a 1 × n rectangle with 1 × 1 tiles that are red or blue and 1 × 2 tiles that are green, yellow, or white. Find a closed formula for $H(x) = \sum_{n\geq 0} h_n x^n$.

Initial Approach

The question seems quite daunting and weird at first. However, let's start by laying out the first few cases.

The first case is a 1x0 rectangle. There is only 1 way to tile it (nothing). Hence let's define h_0 =1

Next there is 1x1 rectangle. In a 1x1 rectangle, we can only tile it using the red or blue tiles. Hence there are 2 ways to tile it. h_1 =2

The next case is interesting. First we know that we can tile it using everything in the previous case if we scaled everything by 2. More specifically we can tile it like RR,BB,BR,RB. In addition, there are also 3 new ways of tiling it as well. Hence we can define this as $h_3 = 2h_1 + 3$

Thinking about the 1x3 rectangle case. We can tile it how we did h_2 but just swapping the colour each time. Hence it will be $2h_2$ new ways. We can also construct it using a combination of 1x2 tiles and 1x1 tiles. There are 3*2 total different ways of constructing it. Hence with this we can get $h_3 = 2h_2{+}3h_1$.

Hence with this we can define our recurrence relation as

$$h_n = 2h_{n-1} + 3h_{n-2}$$

With our recurrence relation, solving this problem will be simple.

Final Answer

Let our recurrence relation be

$$h_n = 2h_{n-1} + 3h_{n-2}$$
$$h_0 = 1$$
$$h_1 = 2$$

for all $n \geq 2$

Let's plug in our recurrence relation into our H(x)

$$H(x) = \sum_{n\geq 0} h_n x^n$$

$$= h_0 + h_1 x + \sum_{n\geq 2} h_n x^n$$

$$= 1 + 2x + \sum_{n\geq 2} (2h_{n-1} + 3h_{n-2})x^n$$

$$= 1 + 2x + \sum_{n\geq 2} x^n 2h_{n-1} + \sum_{n\geq 2} x^n 3h_{n-2}$$

$$= 1 + 2x + 2x \sum_{n\geq 1} x^n h_n + 3x^2 \sum_{n\geq 0} x^n h_n$$

$$= 1 + 2x + 2x \sum_{n \geq 1} x^n h_n + 2xh_0 - 2xh_0 + 3x^2 H(x)$$

$$= 1 + 2xH(x) + 3x^2 H(x)$$

$$H(x) - 2xH(x) - 3x^2 H(x) = 1$$

$$H(x) = \frac{1}{1 - 2x - 3x^2}$$

$$H(x) = -\frac{3}{4(3x-1)} + \frac{1}{4(x+1)}$$

$$h_n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n$$

Follow Up: n x n Square

Let h_n be the number of ways to tile a n $\times$ n rectangle with 1 $\times$ 1 tiles that are red or blue and 1 $\times$ 2 tiles that are green, yellow, or white.

Find a closed formula for $H(x) = \sum_{n \geq 0} h_n x^n$.

One again, let's create a few small cases for n = 0. There is once again only 1 way (nothing).

For n = 1, still no big changes, there are still 2 ways of tiling.

For n = 2, this is were it diverges. There are now many ways. First we can tile it with only 1x2 tiles, which will yield 9 ways (GY,GW,GG, YG, YW, YY, WY, WG, WW). We could also tile with only 1x1 squares, this will have 16 ways of choosing (2^4 since there is 2 ways of choosing each square). We could also have a mixture of 1x1 and 1x2, there is 24 ways of choosing this (3 choices for the 1x2 column, 2 choices for the other 2 and 2 times this for swapping of orders).

For n = 3, this became much more interesting. Let's attempt to construct this recursively. we know that there are 4 possible positions for a 2x2 square in a 3x3 square. We can define this part as $4h_{n-1}$. In addition, there will be enough space for either a 1x2 column and 3 1x1 squares or 5 1x1 squares. For the rest of the space it is either $3 * 2 * 2 = 12$ or $2^5 = 32$. This will be the case for the rest of the rest of the squares of nxn size. Hence we can construct our recurrence relation as

$$h_n = 4(h_{n-1} + 12 + 32) = 4h_n + 176$$

$$h_0 = 1$$

$$h_1 = 2$$

$$h_2 = 49$$

For all $h_n \geq 3$

Now let's solve it

$$\begin{aligned}
H(x) &= \sum_{n\geq 0} h_n x^n \\
&= h_0 + h_1 x + h_2 x^2 + \sum_{n\geq 3} h_n x^n \\
&= h_0 + h_1 x + h_2 x^2 + \sum_{n\geq 3} (4h_n + 176) x^n \\
&= 1 + 2x + 49x^2 + \sum_{n\geq 3} 4h_n x^n + \sum_{n\geq 3} 176 x^n \\
&= 1 + 2x + 49x^2 + 4\sum_{n\geq 3} h_n x^n + 176x^3 \sum_{n\geq 0} x^n \\
&= -3 - 6x - 147x^2 + 4\sum_{n\geq 0} h_n x^n + 176x^3 \sum_{n\geq 0} x^n \\
&= -3 - 6x - 147x^2 + 4H(x) + \frac{176x^3}{1-x} \\
H(x) &= 1 + 2x + 49x^2 + \frac{176x^3}{3(1-x)}
\end{aligned}$$

Line Problem

We divide a group of people into subgroups A, B, and C, and ask each subgroup to form a line. We also require that A have an odd number of people, and that B have an even number of people. How many ways are there to do this?

Initial Approach

Let's attempt to create a generating function for A. Since A needs to be odd we can make create A as

$$x^1 + x^3 + x^5 + x^7.... = \sum_{x \geq 0} x^{2n+1}$$

Similarly we can model B as

$$1 + x^2 + x^4 + x^6 + ... = \sum_{x \geq 0} x^{2n}$$

Lastly we can model C as

$$1 + x^2 + x^3 + x^5... = \sum_{n \geq 0} x^n$$

We can also formulate out answer as S = A*B*C. In addition, all the logic would be valid as we can form a bijection between all 3 sets.

With this, I believe I have a solution.

Final Answer

Let's model our answer S(x) = $\sum_{n \geq 0} s_n x^n$ as a composition of generating functions A(x), B(x), and C(x).

Let's C(x) as $\sum_{n \geq 0} x^n$, B(x) as $\sum_{x \geq 0} x^{2n}$, A(x) as $\sum_{x \geq 0} x^{2n+1}$.

The closed form for all three is as follows

$$C(x) = \frac{1}{1-x}, B(x) = \frac{1}{1-x^2}, A(x) = \frac{x}{(1-x^2)}$$

Hence the S(x) = C(x)B(x)A(x) =

$$S(x) = \frac{1}{1-x} * \frac{1}{1-x^2} * \frac{1}{x(1-x^2)} = \frac{x}{(1-x)(1-x^2)(1-x^2)}$$

$$S(x) = \frac{1}{16\,(x-1)} - \frac{1}{4\,(x-1)^3} - \frac{1}{16\,(x+1)} - \frac{1}{8\,(x+1)^2}$$

Hence solving this we should obtain

$$s_n = -\frac{1}{16} + \frac{1}{4}(n)(n-1) + \frac{1}{16}(-1)^n - \frac{1}{8}n(-1)^{n-1}$$

Letter Problem

Let h_n be the number of sequences of length n consisting of letters A and B in which there is no subsequence of two letters A in consecutive positions. Find a closed formula for $H(x) = \sum_{n\geq 0} h_n x^n$

Initial Approach

Once again, this question seems complicated. Let's start by listing out small cases.

$h_0 = 0$. You cannot form sequences of length 0.

$h_1 = 2$ A or B by itself

h_2 is where it gets more interesting. We can constructing h_2 by adding A or B to h_1 except for adding A onto A. Therefore there should be 3.

h_3 is similar. We can add A or B to h_2 to form h_3, however we need to be once again weary of repeats. In this case, we cannot add A to one of the strings in h_2 (BA). so $h_3 = 5$. At this point the recurrence relation is not clear but there is an idea that it could be of the form $h_n = h_{n-1} + h_{n-2}$

Let's check h_4. Once again, we form by adding A and B to h_3. However I noticed that there are 2 non eligible strings this time for A, BBA and ABA. Therefore the $h_4 = 8$ which confirms our recurrence relation.

Final Answer

Let's define our recurrenece relation as

$$h_n = h_{n-1} + h_{n-2}$$

$$h_0 = 0$$

$$h_1 = 2$$

$$h_2 = 3$$

for all $n \geq 3$

We know that

$$H(x) = \sum_{n\geq 0} h_n x^n$$

Continuing from this

$$H(x) = h_0 + h_1x + h_2x^2 + \sum_{n\geq 3} h_nx^n$$

$$= 2x + 3x^2 + \sum_{n\geq 3}(h_{n-1} + h_{n-2})x^n$$

$$= 2x + 3x^2 + \sum_{n\geq 3} x^nh_{n-1} + \sum_{n\geq 2} x^nh_{n-2}$$

$$= 2x + 3x^2 + \sum_{n\geq 3} x^nh_{n-1} + \sum_{n\geq 3} x^nh_{n-2}$$

$$= 2x + 3x^2 + x\sum_{n\geq 2} x^nh_n + x^2\sum_{n\geq 1} x^nh_n$$

$$= 2x + x^2 + x\sum_{n\geq 0} x^nh_n + x^2\sum_{n\geq 0} x^nh_n$$

$$H(x) = 2x + x^2 + xH(x) + x^2H(x)$$

$$H(x) = \frac{2x + 3x^2}{1 - x - x^2}$$

As required.

Follow Up: Repeats allowed

Now what if A is allowed to repeat?

This is quite simple, since A is allowed to repeat, we are doubling every single time since we are adding A and B to all the sequences already formed. Hence the recurrence relation would be $h_n = 2h_{n-1}$, $h_0 = 0$, $h_1 = 2$.

Finding the closed form for $H(n)$ is quite simple.

$$H(x) = \sum_{n\geq 0} h_nx^n$$

$$= 0 + 2x + \sum_{n\geq 2} 2h_{n-1}x^n$$

$$= 0 + 2x + 2x\sum_{n\geq 1} h_nx^n$$

$$H(x) = 2x + 2xH(x)$$
$$H(x) = \frac{2x}{1-2x}$$

Follow Up: No Repeats

Now both A and B can't occur in the same sequence. At first, it might seem difficult, however I quickly realized the answer is quite trivial. Since both A and B cannot repeat, there will only be 2 possible sequences, one where A starts and one where B starts.

Journal and Extra

Week 9

Most of the feedback has been given through my friend and colleague Chris. We often discuss and sometimes work through smaller problems together in class. I haven't submitted too much for Fridays, so unfortunately I cannot comment too much on that. However, I should try to set an alarm for myself to submit meaningful work on Fridays to attempt to make my portfolio more professional.

Chapter 8

Graph Theory

Exercises

Simple Graphs Counting Question

How many different simple graphs are there on the vertex set [n]?

Final Answer

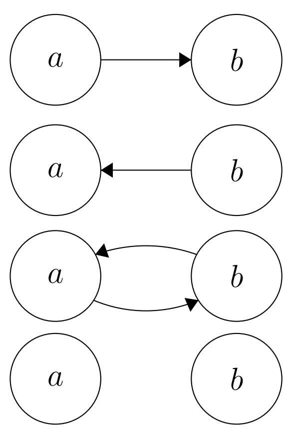
We know that a graph is essentially displaying the relation between each vertex. Hence, we need explore what are the possible relationships between each vertex. In this case, the vertex could either be connected or not connected, hence two possible choices.

The combination of every single vertex pair is $\binom{n}{2}$. Combining with the fact that there are 2 possible choices, we get our final answer of $2^{\binom{n}{2}}$

Follow Up: Directed Graphs

Instead of just plain simple graphs, what if they were directed graphs instead.

The logic is similar to non-directed graphs. First we must explore what are the possibilities between any two vertices, a and b. In this case, a can have a one way connection to b, b can have a one way connection to a, they could be simply not connected, and they could also have a two way connection.

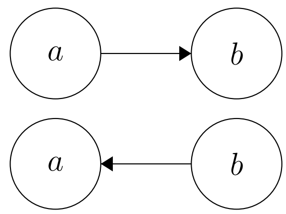


With 4 possible choices between each pair of nodes, we can reasonably conclude that the final answer for number of directed graphs is $4^{\binom{n}{2}}$.

Follow Up: Tournaments

How many tournaments are there on vertex set [n]?

Once again, the logic is quite similar to the original question. We know that, from the definition of a tournament, each vertices has 1 edge. However, since the a tournament is a directed graph, there are two possible choices for any two nodes: a, b.



Therefore we can conclude that the answer is $2^{\binom{n}{2}}$

Simple Graphs Vertex Problem

Prove that in any simple graph, there are two vertices with the same degree.

Final Answer

We know that for any simple graph with n vertexes, each vertex could have the degree of {0,1,2,3, ... n-1}. However notice that a graph cannot have n-1 degree and 0 degree simultaneously.

The justification for that is any node with a degree of 0 means there are no other nodes connected to them. While a node with n-1 degrees will have every other node connected to them.

Hence there are only n-1 number of degrees a node can have, while there are n nodes. By the pigeon hole principle, for any simple graph, there are two vertices with the same degree.

Connections Problem

Is there a simple graph G on seven vertices such that it is not connected, and each vertex of G has degree at least three?

Inital Approach

We know that in order for a graph to be connected, there would need to exists a path between any two nodes in the graph.

The formal definition of this is for a graph G = (V,E), G is connected if for all $v, w \in V$ there is a path (or trail) from v to w.

By the handshake lemma, we know the sum of all degrees of all vertices is at least 21. However, by the same lemma, there shouldn't be any odd sums. Hence the minimum degree is sum is 22. Therefore, the minimum edges is 11. Therefore, there would be at least one vertex with 4 degrees.

For the graph to be simple and complete there will exist an edge between any two nodes. Hence the number edges in a simple graph should not be greater than $\binom{n}{2}$ where n is the number of vertices.

Following this logic, a simple subgraph with $\binom{n-1}{2}$ edges will be largest amount of edges possible for n-1 vertices. By adding a new vertex (now size n) and adding any edge to it, the graph will become connected. This is true by the pigeonhole principle.

Hence with this we can construct our final answer.

Final Answer

By the handshake lemma, we know that there are at least 11 edges in this simple graph G.

To see if this is connected we can attempt to create the largest simple graph with 6 vertices. Notice, for an graph with 6 vertices we can create $\binom{6}{2}$ edges.

Notice we can for a graph to be connected it must have greater than 15 edges by the pigeonhole principle. The contra positive of the the graph can be disconnected if it has less than 15 edges.
Therefore G exists.

Follow Up: General Formula

Is there a simple graph G on n vertices such that it is not connected, and each vertex of G has degree at least k?

Lets follow the same logic as the original question. Let's first construct the possible edges. By the handshake lemma, there should be $\lceil \frac{nk}{2} \rceil$ number of edges.

Once again, let's construct the maximum size graph with n-1 vertices. The maximum number of edges for n-1 while maintaining the simple property is $\binom{n-1}{2}$. Let's call this graph G_1.

We know that a property of G_1 is that every node is connected, by adding a new vertex to G_1 and a new edge, we would ensure any 2 nodes are connected to each other by the pigeon hole principle.

Therefore, for a graph to be diconnected it must have less than $\binom{n-1}{2}$. Hence if the number of edges,

$$\lceil \frac{nk}{2} \rceil < \binom{n-1}{2}$$

. There exists a simple graph G, with each vertex of G having degree of at least k, that is not connected.

Hamiltonian Cycle on a Complete Graph

How many Hamiltonian cycles does Kn have?

Initial Approach

We know the complete graph K_n has $\binom{n}{2}$ amount of edges.

Now, let's think about the definition of the Hamiltonian cycle. It is a cycle that uses every vertex once and ends in the same at the same vertex it started with.

We also know that every node is connected to every other node.

With this information, I can conclude a final answer.

Final Answer

Since every node is connceted to every other node, we can construct an Hamiltonian cycles such that

$$n- > n-1- > n-2- > ...- > 1- > n$$

Hence the nth node will have n-1 nodes to choose from, n-1 will have n-2 so on. Therefore, we would have (n-1)! Hamiltonian cycles on a complete graph.

Follow Up: Distinction

Notice that the cycle

$$n- > n-1- > ..- > 1- > n$$

is the same as

$$1- > 2- > 3- > ..- > n- > 1$$

In our solution both of these cycles are counted even though they are the same cycle just reversed in direction. Now what if we want to only count distinct cycles? The solution is quite simple, it should be $\frac{1}{2}(n-1)!$ as each cycle has only one direction reversal.

Fun Word Problems

I decided to include some interesting word problems here because they seem fun.

Chess (Not) Tournament

Ten players participate at a chess tournament. Eleven games have already been played. Prove that there is a player who has played at least three games.

Final Answer

Let this be a simple graph (since players can't play themselves), where the nodes represent the players and each game represents an edge.

Since there are 11 games that have been played, 10 players, we can say there are 10 vertices and 11 edges.

We define the the sum of all degrees as the pigeonholes, and the edges as pigeons. Since we there are n pigeons holes and n+1 pigeons. There is at least one player who has played at least 3 games.

Handshake Di-lemma

Prove that the number of people who have shaken hands at an odd number of times (in their life so far) is even.

Final Answer

Let the number of people represent n nodes, and the number of people they shaked hands with as their degree. We know that each handshake involves 2 poeple. Hence the sum of of all degrees must be even, as the number of edges (connections for handshake) is even.

High School Connections

A high school has 90 alumni, each of whom has ten friends among the other alumni. Prove that each alumni can invite three people for lunch so that each of the four people at the lunch table will know at least two of the other three.

Final Answer

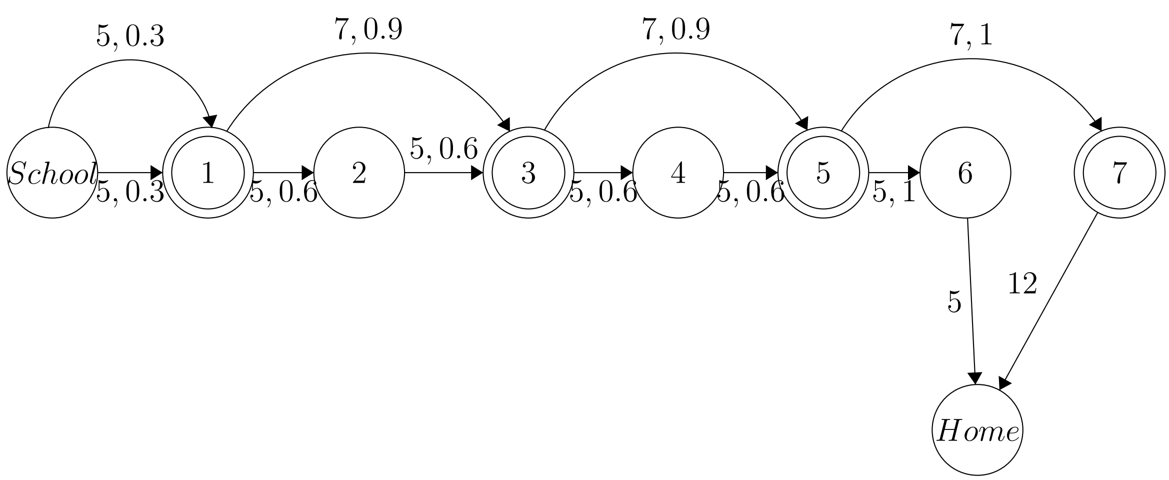
Let each person be a node with a degree of 10 (they know 10 people).

Now let's pick any random node. That node will have a degree of 10, and every node that it is will also have a degree of 10. If we exclude the picked node (since you are invited to your own lunch), there would be 89 available choice for 90 nodes. Hence by the pigeonhole principle, they must be connected to the same vertex j.

Hence invite the two who is connected to j, j himself and yourself, then we would satisfy the conditions of the question.

Commuter Problem

In the MiWay bus system, there is the 1 and 101. 1 is a normal bus, goes to every stop. 101 on the other hand, is a weekday express bus. It skips a couple of stops and supposedly drives faster (it doesn't need to stop and regain momentum each time). Here is my attempt to model the question as a directed graph problem.



This is slightly smaller than the actual number of stops and also not quite accurate to how the 101 and 1 functions as well. We know that the bus also does not stop 100% of the time. Hence, the first number is time, the second the chance of stoppage time (2 min).

Which bus is faster to go home the 101 (Stops at all double circles) or the 1 (Stops at all circles).

Final Answer

Since nothing is concrete, let's model this as a statistical question. What is the expected value of 101 vs the expected value of the 1.

Expected value of 101:
5+0.3*2+7+0.9*2+7+0.9*2 + 7 + 2 + 12= 44.2

Expected value of 1:
5+0.3*2 + 5 +0.6*2 + 5 + 0.6*2 + 5+0.6*2 + 5+0.6*2
+5 + 2+ 5 = 42.4
Hence the 1 should be marginally better on the average case.

Follow up: Worst Case

Now let's assume the bus stops at every stop, this could happen during rush hour for example. Hence the 101 time would be 46. 1 time would be 47, which would then be marginally worse.

Remarks

This is a very interesting problem. Right now the model is simple as there a lot of other factors unaccounted for. For example, traffic lights, weather condition, momentum after not stopping for one stop. There is so much more we can do with graphing this out and using real world data to determine commute time.

Journal and Extra

Week 10

This week, we talk about learning objectives and how I've accomplished them.

LO1

I have demonstrated some strong evidence in many of these topics. I have solved problems where I also went beyond the scope of the question. I have cases where I made conjectures and proved them. I even have some pieces of remarks, observations and even code. Overall I would be very happy to present this at a job interview.

LO2

I think the combinatorial objects that I ended up being most interested in is permutation cycles and inclusion-exclusion theorem. I was able to make some interesting conjectures, some I was able to prove others I wasn't. I also like generalizing my results a lot especially in the inclusion-exclusion chapter.

LO3

I think some of the solutions that I'm most proud of includes that I wrote creating subsets of k length. That was really difficult and I went through a lot of trial and error and hypothesis testing. Another one would be writing code for subset partitions. Those questions really do reflect my style of being more hands on and applied instead of hand-wavy.

LO4

The problems I found most interesting was honestly anything before graph theory. I like how the problems allowed you to generalize them and try to figure out solutions on your own. I think I adapted a mathematical proof to a combinatorial one the partitions section was really cool.

LO5

I definitely have grown a lot as mathematician. I have now gotten a lot better at some of the skills I wanted to improve at the beginning of the course. I did have to course correct a little bit especially when I was losing motivation towards the end, I had to remind myself what makes this course so fun.

I think the main idea that I learned was to go beyond what the question is asking you for. I realized that problems are their to start your thinking and not finishing it. Some of the best concepts I learned, I was able to explain to my friends and use in other courses. These include the pigeonhole principle, inclusion-exclusion, partitions and generating functions.

If I to take this course again, I would definitely try to include some questions outside of the textbook. The riddles from TEDx was quite enjoyable to do and the there is so much more in the world of combinatorics.

Chapter 9

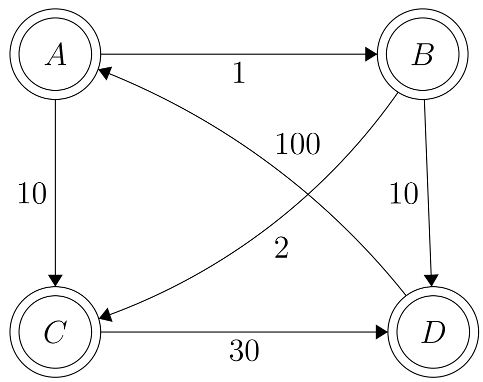
Trees

Lecture and Tutorial

Greedy Hamilton

Let's construct a greedy algorithm for finding minimum weight Hamiltonian cycle

1. Choose the edge of least weight
2. From there choose the vertex such that the vertex contains the edge of least weight and unvisited.
3. Stop after choosing every vertex at least once.



The greedy algorithim will choose ABCD while the most efficient is ABDC. The reason why the greedy algorithim does

Exercises

More Leaves Question

Prove that a tree always has more leaves than vertices of degree at least three.

Initial Approach

The question seems confusing, let's first figure out what it actually means.

Essentially, from my understanding it is saying that.

$$\#leaves \geq \#deg_3(vertex)$$

Intuitively this is true as the for every degree 3 vertex, there must be a corresponding 2 leaves.

We can model this question using sums.

Final Answer

We know the $\sum_v deg(v)$ is the total number of degrees in this tree. Let x_i be the number of vertices for the degree i. Hence:

$$\sum_{i=1}^{n} x_i * i$$

where n is amount of nodes (Note: x_n can equal to 0). We know a tree has n-1 edges, hence by the handshake lemma, this tree has 2n-2 degrees. We also know that by summing all of the number of nodes, we get n.
Combining those two facts:

$$\sum_{i=1}^{n} x_i * i = 2n - 2 = 2 * \sum_{i=1}^{n} x_i * i - 2$$

$$\sum_{i=1}^{n} x_i * i - 2 * \sum_{i=1}^{n} x_i * i = -2$$

$$\sum_{i=1}^{n} x_i * (i - 2) = -2$$

Expanding this we can see that

$$= -x_1 + x_3 + ... = -2$$

$$= 2 + x_3 + ... = x_1$$

Hence there are more leaves in the tree than number of verticies of degree 3 as required.

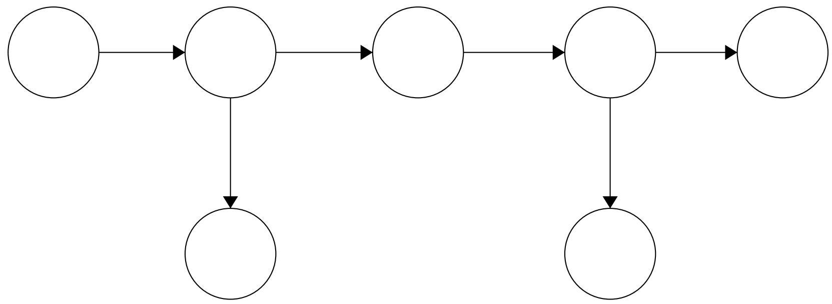
Crossing Paths

Prove that in any tree T , any two longest paths cross each other.

Inital Approach

Let's call the longest path in the tree k and the second longest path in the tree j.

Since the tree is connected we can pick a node in j and node in k and form a path.



With this we can formulate our final answer.

Final Answer

Let's assume the 2 longest paths in T, k,j, has the length l, do not cross each other. Hence they have both have a unique path. Since T is connected, we can form a new path p where it has a starting node and ending node in half way into k, j respectively. However we can also connect some part of k and j onto this path as well.

In addition, since k and j are unique, we need some nodes to act as a bridge for the path between these two paths.

Hence a contradiction as our path will have the length $\frac{l}{2} + \frac{l}{2} + c$, where c is the length of the bridge. Making this our new largest path. Hence our original assumption is wrong and k,j must cross each other.

Follow up: Common Vertex

Prove that in any tree T , all longest paths cross one another in one vertex.

We know that any two longest paths share a vertex from our previous section. Let's once again for the sake of contradiction assume the longest path k,j doesn't contain the common vertex. However, by adding that common vertex and forming a path half way through j and k shows that it is once again contradicting.

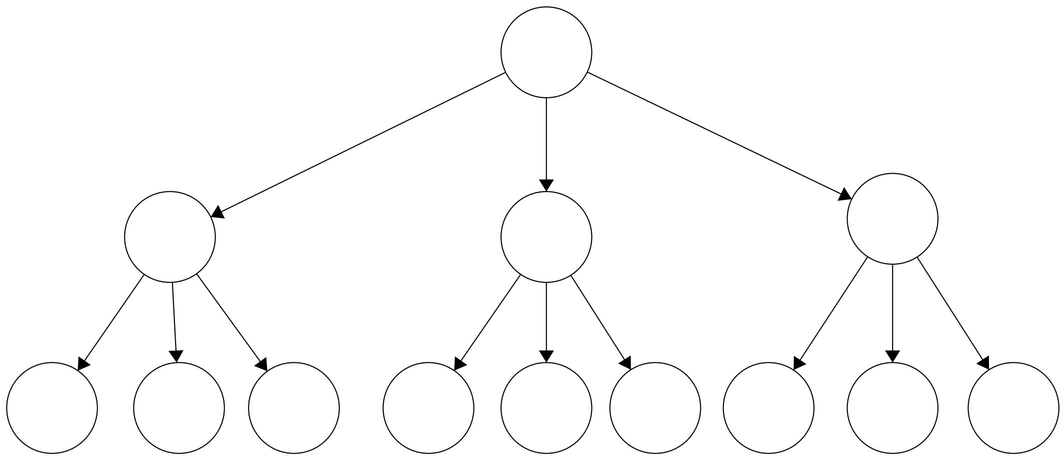
Therefore, by contradiction, any k,j longest path must share one common vertex.

k-ary Tree

A complete k-ary tree is a rooted tree in which every vertex has either k or 0 children. Let T be such a tree with m non-leaf vertices. How many leaves does T have?

Initial Approach

Translating from the formal definition, a k-ary tree is essentially a tree where everything is either a leaf or a vertex containing the same number of leaves. An example would be



We know there are m number of non leaf vertices.

Let's start from the top. If there are m vertices, there is at least going to be mk number of leaves. If m is greater than 1. Then we need to subtract (m-1) as that is the number of vertices that have non-leaves as children. Hence, combining the 2 facts together, we hace mk-(m-1) = mk-m+1 = (k-1)m+1 number of leaves.

This can be proven using induction.

Final Answer

I conjecture (k-1)m+1 as the number of leaves in T.

Proof
Base Case: m =1 then there are k number of leaves as required.
Induction Hypothesis: Assume this is true for some $n \in m$.
WTS (k-1)(n+1)+1 is True

Let T have n+1 non-leaf vertices. Pick any of the vertices whose children are leaves. Delete all of its children and now T is size of n and has (k-1)n+1 leaves. Now return the children and T now has kn+1 leaves.

Journal and Extra

Week 11

In the beginning of my combinatorics journey, I was curious and excited about the course. Learning about a combinatorics proof changed me greatly and I was excited for whats to come.

In the middle of the class, I was learning about all these interesting concepts where I feel like I could innovate and improve upon. There were times where I pushed a question too hard and fell flat. However, I feel like there was always something I could do and something I could add on. I changed my thinking from attempting to attend it as a hardcore math course where I need to achieve the best grade possible to a thinking thats better for learning and improving.

At the end of the course, I struggled a little bit. Graph theory and trees were a tougher opponent then I expected. However I was able to use other concepts that I learned in other courses. I found great joy telling people about my journey in this course and all the interesting things that I was able to do.